Nonexistence of Isochronous Centers in Planar Polynomial Hamiltonian Systems of Degree Four

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In this paper we study centers of planar polynomial Hamiltonian systems and we are interested in the isochronous ones. We prove that every center of a polynomial Hamiltonian system of degree four (that is, with its homogeneous part of degree four not identically zero) is nonisochronous. The proof uses the geometric properties of the period annulus and it requires the study of the Hamiltonian systems associated to a Hamiltonian function of the form

\[ H(x, y) = A(x) + B(x)y + C(x)y^2 + D(x)y^3. \]

1. INTRODUCTION

This paper deals with centers of planar polynomial Hamiltonian systems, i.e., differential systems of the form

\[
\begin{align*}
\dot{x} &= -H_y(x, y), \\
\dot{y} &= H_x(x, y),
\end{align*}
\]

where \( H \) is a real polynomial in \( x \) and \( y \). The period function of a center gives the period of each periodic orbit inside its period annulus. We are

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interested in the case in which the period function is constant; the center is then said to be isochronous.

Questions relating to the behaviour of the period function have been extensively studied by a number of authors. Some of these works have been motivated by the desire to find conditions under which the period function is monotone (see [1, 22] for instance). This is so because monotonicity is a nondegeneracy condition for the bifurcation of subharmonic solutions of periodically forced Hamiltonian systems and because monotonicity implies existence and uniqueness for certain boundary value problems. In fact, for these boundary value problems, the range of values of the period function determines the set of initial conditions for which the problem has at least one solution. The interested reader is referred to [2, 3, 24]. We note that in these problems isochronicity represents the worst possible setting.

On the other hand, to fully understand the properties of the period function in a family of differential systems, it is necessary to consider the bifurcation of critical points of the period function. This is done by calculating the Taylor series of the period function in a neighbourhood of the center. When this is performed on a polynomial family, then the coefficients of the period function are polynomials in the coefficients of the system. Once again the problem becomes more difficult in those cases in which the center is isochronous. Therefore a complete study of these bifurcations requires the identification of the subclass of systems that have an isochronous center. The work of Chicone and Jacobs [2] and the one of Rousseau and Toni [20] are good examples of this approach.

Concerning polynomial Hamiltonian systems, the knowledge of isochronicity is also important due to its connection with the well-known Jacobian conjecture (see [10, 21] for details). This is so because if we take a polynomial canonical mapping \((x, y) \mapsto (P(x, y), Q(x, y))\) such that \(P(0, 0) = Q(0, 0) = 0\) then the polynomial Hamiltonian system given by

\[
H(x, y) = \frac{P(x, y)^2 + Q(x, y)^2}{2}
\]

has an isochronous center at the origin. The Jacobian conjecture is equivalent to proving that the period annulus of any center constructed in this way is the whole plane. Here by a canonical mapping we mean that the determinant of its Jacobian is constant.

Note that the method described above provides only polynomial Hamiltonian systems of odd degree (that is, given by an even degree polynomial). However, not all the isochronous polynomial Hamiltonian systems can be obtained in this way (see [7]); as a matter of fact in the literature there is not any example of a polynomial Hamiltonian system of
The problem of characterizing isochronicity has attracted the attention of several authors. However there are very few families of polynomial differential systems in which a complete classification of the isochronous centers has been found. The most significant results for general polynomial systems are due to Loud [14], who classifies the quadratic ones, and Pleshkan [19], who classifies the cubic systems with homogeneous nonlinearities. In relation to Hamiltonian systems, several authors (see [4, 9, 23]) have proved that there is not any isochronous center in those systems that have homogeneous nonlinearities. In [5] it is shown that the only isochronous center among the polynomial systems associated to \( H(x, y) = F(x) + G(y) \) is the linear one. The classification of the cubic polynomial Hamiltonian isochronous centers is given in [7]. For some other results on isochronicity we refer the reader to [16, 17] and references there in. In this paper we prove the following.

**Theorem A.** Every center of a planar polynomial Hamiltonian differential system of degree four is nonisochronous.

We stress that by definition a four degree differential system has its homogeneous part of degree four not identically zero. Consequently, with the contribution of Theorem A we have the classification of the isochronous centers of the polynomial Hamiltonian systems of degree 2, 3, and 4. On the other hand, in view of Theorem A the following question arises naturally.

**Question.** Do there exist planar polynomial Hamiltonian systems of even degree having an isochronous center?

An argument in support of a negative answer is the following result (see [11, 15] for instance). An analytic Hamiltonian system (1) has an isochronous center at the origin if and only if there exists an analytic canonical mapping \((x, y) \mapsto (f(x, y), g(x, y))\) with \(f(0, 0) = g(0, 0) = 0\) such that

\[
H(x, y) = \frac{f(x, y)^2 + g(x, y)^2}{2}.
\]

In the case that \(H(x, y)\) is a polynomial of odd degree it seems that there is some kind of obstruction for the existence of such a mapping.

In order to prove Theorem A one may try to show that some period constants cannot vanish simultaneously. These are strongly related to the Lyapunov constants, which give the center conditions. However, this approach is rather hopeless because the family that we consider has too
many parameters, and even the computation of a significant number of
period constants is intractable with a computer. The strategy of the proof is
to take account of the geometric tools that a Hamiltonian system provides.
Thus, we investigate the shape of the period annulus of the center in the
Poincaré disk. In addition we study what we call the cubic-like Hamiltonian
systems, namely the ones given by

$$H(x, y) = A(x) + B(x) y + C(x) y^2 + D(x) y^3,$$

where $A, B, C,$ and $D$ are analytic functions on $\mathbb{R}$. Concerning these
systems, it is to be noted that we obtain an analytic expression of the
period function, not only in a neighbourhood of the center but in the whole
period annulus.

2. DEFINITIONS AND SETTING OF THE PROBLEM

For any center $p$ of a planar differential system, the largest neigh-
bourhood of $p$ which is entirely covered by periodic orbits is called the
period annulus of $p$ and we will denote it by $\mathcal{P}$. A center is said to be a
global center when its period annulus is the whole plane. The function
which associates to any periodic orbit $\gamma$ in $\mathcal{P}$ its period is called the period
function. The center is called an isochronous center when the period func-
tion is constant. It is well known that only nondegenerate centers can be
isochronous. A center is said to be nondegenerate when the linearized
vector field at the critical point has two nonzero eigenvalues. It can be
shown also that an isochronous center has no finite critical point in the
boundary of its period annulus. When the differential system is analytic
this implies that the period annulus of an isochronous center is unbounded.

Our goal in this paper is to prove that any center of a polynomial
Hamiltonian system of degree four is nonisochronous. For the remain-
er of this section we shall show that it suffices to prove the result for a
particular type of center. Of course this will be done in such a manner that
we do not lose generality. Simultaneously we shall introduce the notions
and definitions that will be used henceforth.

A polynomial Hamiltonian system of degree $n$ is a differential system of
the form

$$\begin{align*}
\dot{x} &= -H_n(x, y), \\
\dot{y} &= H_n(x, y),
\end{align*}
$$

(2)
where $H(x, y)$ is a polynomial of degree $n + 1$. Let us fix that $H(0, 0) = 0$ and denote the homogeneous part of degree $i$ of $H(x, y)$ by $H_i(x, y)$. Notice that, by definition, if (2) is a system of degree $n$ then $H_{n+1} \not= 0$.

Since only nondegenerate centers can be isochronous, in order to study the isochronicity of a center we can assume that it is located at the origin and that

$$H(x, y) = \frac{x^2 + y^2}{2} + H_1(x, y) + \cdots + H_{n+1}(x, y). \quad (3)$$

It is clear that this can be done without loss of generality taking an appropriate coordinate system and scaling the time by a constant amount. In this case it follows (see [9] for instance) that $H(x, y) > 0$ for any point $(x, y) \in \mathcal{P}$ different from the origin, and therefore $H(\mathcal{P}) = [0, h_0]$ for some $h_0 \in \mathbb{R}^+ \cup \{+\infty\}$. The solutions of (2) are contained in the level curves \{H(x, y) = h, h \in \mathbb{R}\}. It can be shown that when the center is nonglobal then $h_0$ is finite and the boundary of its period annulus, $\partial \mathcal{P}$, is contained in the level curve \{H(x, y) = h_0\}. We will use this notation throughout the paper.

On the other hand one can prove (see [9]) that the set of all the periodic orbits in the period annulus can be parametrized by the energy $h$. Thus, for each $h \in (0, h_0)$ we will denote the periodic orbit in $\mathcal{P}$ of energy level $h$ by $\gamma_h$. This allows us to consider the period function over $(0, h_0)$ instead of the original period function which is defined over the set of periodic orbits contained in the period annulus. Therefore in the following we will talk about the period function $T(h)$ which gives the period of the periodic orbit with energy $h \in (0, h_0)$. It can be proved that $T(h)$ can be extended analytically to $h = 0$ when the center is nondegenerate.

Notice in addition that when $n$ is even then any center of the Hamiltonian system given by (3) is nonglobal. This is so because if $H(x, y)$ is a polynomial of odd degree then it is not possible that $H(x, y) > 0$ for all $(x, y) \in \mathbb{R}^2 \setminus \{(0, 0)\}$. Consequently, since we deal with a Hamiltonian system of degree four, the period annulus of the center at the origin cannot be the whole plane, and note then that $h_0 < +\infty$. We can also assume without loss of generality that $\mathcal{P}$ is unbounded because we are interested in isochronous centers. Therefore $\mathcal{P}$ must be nonglobal and unbounded.

At this point it is convenient to recall that when we deal with a polynomial system then the Poincaré compactification provides an analytic extension of the vector field at infinity. It is then possible to study the behaviour of the flow at infinity, which corresponds to the equator of the Poincaré sphere. This fact will be a crucial item in our work because in order to study the boundary of the period annulus we shall take account of a property of the Hamiltonian systems at infinity. To state this property precisely we need an additional definition.
Definition 2.1. Let \( q \) be an infinite critical point of a planar polynomial Hamiltonian system in the Poincaré compactification, and let \( \mathcal{H} \) be a hyperbolic sector associated to \( q \). We say that \( \mathcal{H} \) does not contain straight lines if for any finite straight line \( \ell \) (in the Poincaré compactification) which passes through \( q \) there exists a neighbourhood \( V \) of \( q \) such that \( \ell \cap V \) is not contained in the interior of \( \mathcal{H} \).

We are now ready to state the above-named property. Thus, in [9] is proved the following.

Lemma 2.2. Let \( q \) be an infinite critical point of a polynomial Hamiltonian system with a hyperbolic sector \( \mathcal{H} \). Then either \( \mathcal{H} \) does not contain straight lines or its two separatrices are contained in the equator of the Poincaré sphere. In the first case both separatrices are finite and belong to the same level curve of the Hamiltonian.

Now, according to Lemma 2.2, the fact that \( \mathcal{P} \) is unbounded and nonglobal implies the existence of an infinite critical point with a hyperbolic sector that has both separatrices lying in the finite part and inside \( \partial \mathcal{P} \). Let us say that, in the Poincaré disk, this infinite critical point is given by the direction \( h^* \in [0, 2\pi) \). In this case Proposition 4.1 in [7] shows that if the Hamiltonian system associated to (3) has such an infinite critical point then

\[
g_{n+1}(\theta^*) = g'_{n+1}(\theta^*) = g_n(\theta^*) = 0,
\]

where \( g_i(\theta) = H_i(\cos \theta, \sin \theta) \) for \( i = 2, 3, \ldots, n+1 \). Notice that by means of a rotation of axis we can suppose without loss of generality that \( \theta^* = \pi/2 \). This is so because after such a change of coordinates the quadratic part of the Hamiltonian remains unchanged. In view of this it is natural to introduce the following definition.

Definition 2.3. In the following we shall say that a period annulus \( \mathcal{P} \) is admissible when \( \mathcal{P} \) is unbounded, nonglobal, and its boundary, in the Poincaré disk, contains the infinite critical point given by the direction \( \theta = \pi/2 \). We shall denote by \( \mathcal{H}^* \) the hyperbolic sector of this infinite critical point that has both separatrices inside \( \partial \mathcal{P} \).

In the preceding discussion we only used that we study isochronicity in a polynomial Hamiltonian system of even degree. We should now focus our attention on the case we shall study, the polynomial Hamiltonian systems of degree four. In this case we have shown that there is no loss of generality in assuming that

\[
H_3(x, y) = x^2(a_0 y^3 + a_1 y^2 x + a_2 y x^2 + a_3 x^3)
\]
and
\[ H_4(x, y) = x(b_0 y^3 + b_1 y^2 x + b_2 y x^2 + b_3 x^3). \]

Then, setting \( H_3(x, y) = c_0 y^3 + c_1 y^2 x + c_2 y x^2 + c_3 x^3, \) note that the Hamiltonian in (3) can be written as
\[ H(x, y) = A(x) + B(x) y + C(x) y^2 + D(x) y^3, \] (4)
where
\[ A(x) = x^2(1/2 + c_1 x + b_1 x^2 + a_3 x^3), \]
\[ B(x) = x^2(c_2 + b_2 x + a_2 x^2), \]
\[ C(x) = 1/2 + c_1 x + b_1 x^2 + a_3 x^3, \] (5)
\[ D(x) = c_0 + b_0 x + a_0 x^2. \]

In fact we have done better than this; let us gather it in short:

**Remark 2.4.** In order to prove Theorem A it suffices to see that any center at the origin of the Hamiltonian system given by (4) and having an admissible period annulus is not isochronous. Here by “any center” we mean for any possible choice of parameters in (5), and note that \( a_0 = a_1 = a_2 = a_3 = 0 \) is not allowed because then \( H_5 \equiv 0. \)

There is another result in [7] that should be mentioned. Thus, Section 3 in that paper is devoted to studying the so-called quadratic-like Hamiltonian systems which correspond to the ones given by (4) taking \( D \equiv 0. \) In this case, Proposition 3.19 in [7] shows that if \( A, B, \) and \( C \) are polynomials (of arbitrary degree) and the origin is an isochronous nonglobal center then \( \deg(C) \geq 4. \) This result has the following immediate consequence for the case we are studying.

**Remark 2.5.** If \( a_0 = b_0 = c_0 = 0 \) then the center at the origin of the Hamiltonian system given by (4) is not isochronous.

Consequently, in order to prove Theorem A we can restrict our study to the following three cases: \( a_0 \neq 0, a_0 = 0 \) with \( b_0 \neq 0, \) and \( a_0 = b_0 = 0 \) with \( c_0 \neq 0. \) We shall study each one in a different section, and our goal will be to show that isochronicity is not possible in any case.

The paper is organized as follows. In Section 3 we develop some general tools that will be used to prove Theorem A. It is to be noted that the results that we obtain can be applied to a more general situation than the
one given by the polynomials in (5). Section 4 is devoted to studying the case \( a_0 \neq 0 \), in which we prove that \( T(h) \to +\infty \) as \( h \to h_0 \) (see Theorem 4.1). In Section 5 we show that in fact the case \( a_0 = 0 \) with \( h_0 \neq 0 \) implies \( H_x \equiv 0 \) (see Proposition 5.1). The third case, \( a_0 = b_0 = 0 \) with \( c_0 \neq 0 \), is considered in Section 6 and we show that the first period constant of the center is different from zero (see Proposition 6.1). We stress that the results that we obtain in Sections 4, 5, and 6 follow from using that the center at the origin has an admissible period annulus. Finally, in the Appendix we describe some general notions of the Poincaré compactification and the blow-up method to study a nonelementary critical point. We also show another important consequence of Lemma 2.2, namely Lemma 7.2, that will provide us some necessary conditions for the existence of the hyperbolic sector \( \mathcal{H}^* \).

3. CUBIC-LIKE HAMILTONIAN SYSTEMS

In this section we assume that \( A, B, C \) and \( D \) are any analytic functions on \( \mathbb{R} \) with \( A(x) = x^2/2 + O(x^3) \) and \( C(0) = 1/2 \), and we study the center at the origin of the Hamiltonian system given by
\[
H(x, y) = A(x) + B(x) y + C(x)^2 + D(x) y^3.
\]
Thus
\[
\dot{x} = -B(x) - 2C(x) y - 3D(x) y^2,
\]
\[
\dot{y} = A'(x) + B'(x) y + C'(x) y^2 + D'(x) y^3.
\]
This is a more general situation than the one we shall later deal with but we find it interesting in itself. The main result that we obtain in this section is Theorem 3.6, which provides an expression to compute \( T(h) \) for any \( h \in (0, h_0) \).

Recall that \( \mathcal{P} \) denotes the period annulus of the center at the origin. From now on we also define \((x_L, x_R)\) to be the projection of \( \mathcal{P} \) to the \( x \)-axis; that is,
\[
(x_L, x_R) = \{ x \in \mathbb{R} : \text{there exists } y \in \mathbb{R} \text{ such that } (x, y) \in \mathcal{P} \}.
\]
Then \( x_L < 0 < x_R \), and note that \( x_L \) or \( x_R \) may not be finite. The key point in the proofs is that, for each fixed \( x \), \( H(x, y) \) is a polynomial of degree three with respect to \( y \). This is the reason why we shall always assume that \( D(x) \neq 0 \) for all \( x \in (x_L, x_R) \).
Lemma 3.1. If \( D(x) \neq 0 \) for all \( x \in (x_L, x_R) \) then \( C(x)^2 - 3D(x) B(x) > 0 \) and
\[
\left( x, \frac{-C(x) + \sqrt{C(x)^2 - 3D(x) B(x)}}{3D(x)} \right) \in \mathcal{P} \quad \text{for any} \quad x \in (x_L, x_R).
\]

Proof. Assume for instance that \( D(x) > 0 \) for all \( x \in (x_L, x_R) \) and consider any \( \bar{x} \) in this interval. Then there exists \( \bar{h} \in (0, h_0) \) such that the periodic orbit \( \gamma_h \) has two intersection points with the straight line \( x = \bar{x} \), say \((\bar{x}, y_1)\) and \((\bar{x}, y_2)\) with \( y_1 < y_2 \). Notice that the discriminant of
\[
H_y(x, y) = B(x) + 2C(x)y + 3D(x)y^2
\]
with respect to \( y \) is precisely \( 4(C(x)^2 - 3D(x) B(x)) \). So, if \( C(x)^2 - 3D(x) B(x) \leq 0 \) then on account of \( D(x) > 0 \) it follows that \( \dot{x} \leq 0 \) on \( x = \bar{x} \). This clearly contradicts that \( \gamma_h \) intersects twice with \( x = \bar{x} \), and accordingly \( C(x)^2 - 3D(x) B(x) > 0 \).

Note now that \((\bar{x}, y) \in \mathcal{P}\) for all \( y \in (y_1, y_2) \) because otherwise there exist at least four different values of \( y \) such that \( H(x, y) = \bar{h} \). In addition one can show that \( y_1 \) and \( y_2 \) are located as in Fig. 1. This is a consequence of the last observation and the fact that, as \( h \) increases, the two intersection points of \( \gamma_h \) with \( x = \bar{x} \) must go away from each other. The minimum of the function \( y \mapsto H(x, y) \) is given precisely by
\[
\bar{y} = \frac{-C(x) + \sqrt{C(x)^2 - 3D(x) B(x)}}{3D(x)}
\]
and therefore in view of Fig. 1 we conclude that \((\bar{x}, \bar{y}) \in \mathcal{P}\). \( \blacksquare \)

FIG. 1. The graph of \( H(\bar{x}, y) \) in the case that \( D(\bar{x}) > 0 \).
At this point, given \( x \in (x_L, x_R) \) and \( h \in (0, h_0) \), we need to make precise how many real roots have the equation \( H(x, y) = h \). If \( D(x) \neq 0 \) then

\[
A(x) - h + B(x) y + C(x) y^2 + D(x) y^3 = 0
\]

is a third degree equation with respect to \( y \) and its discriminant is given by

\[
A(x, h) = \frac{1}{108D(x)^3} \left( 18(A(x) - h) B(x) C(x) D(x) - 4D(x) B(x)^3 - 4C(x)^3 (A(x) - h) + C(x)^2 B(x)^2 - 27D(x)^2 (A(x) - h)^2 \right).
\]

It is well known that the signum of \( A \) provides us the information that we need. Thus, if \( A < 0 \) then one root is real and the other two are complex, if \( A > 0 \) then the three roots are real and different, and finally if \( A = 0 \) then the three roots are real and at least two of them are equal.

Notice in addition that the numerator of \( A(x, h) \) is a second degree polynomial with respect to \( h \), the coefficient of \( h^2 \) is \(-27D(x)^2\), and on the other hand one can check that its discriminant is given by \((C(x)^2 - 3D(x) B(x))^3\). Thus, for each fixed \( x \in (x_L, x_R) \), Lemma 3.1 shows that the equation \( A(x, h) = 0 \) has two real solutions, say \( h = F(x) \) and \( h = G(x) \). From now on we fix that

\[
F(x) = \frac{2C(x)^3 + 27A(x) D(x)^2 - 9D(x) C(x) B(x)}{2C(x)^2 - 3D(x) B(x)},
\]

and

\[
G(x) = \frac{-2C(x)^3 + 27A(x) D(x)^2 - 9D(x) C(x) B(x)}{2C(x)^2 - 3D(x) B(x)}.
\]

and consequently this allows us to write

\[
A(x, h) = \frac{1}{4D(x)^2} (F(x) - h)(h - G(x)) \quad \text{for all} \quad x \in (x_L, x_R).
\]

These functions will play an important role in order to compute the period of each periodic orbit in \( \mathcal{P} \).
Lemma 3.2. If \( D(x) \neq 0 \) for all \( x \in (x_L, x_R) \) then

(a) \( G(x) = \frac{1}{2} x^2 + O(x^3) \) and \( G'(x) \neq 0 \) for all \( x \in (x_L, x_R) \setminus \{0\} \).

(b) \( F(x) \geq h_0 \) for all \( x \in (x_L, x_R) \).

Proof. Consider any \( \bar{x} \in (x_L, x_R) \setminus \{0\} \), and note that if we denote

\[
\bar{y} = \frac{-C(\bar{x}) + \sqrt{C(\bar{x})^2 - 3D(\bar{x}) B(\bar{x})}}{3D(\bar{x})}
\]

then \( H_r(\bar{x}, \bar{y}) = 0 \). Thus, taking into account that \( (\bar{x}, \bar{y}) \in P \) by Lemma 3.1, it follows that \( H_r(\bar{x}, \bar{y}) \neq 0 \) (otherwise \( P \) would contain a critical point different than the origin). On the other hand a computation shows that

\[
G'(x) = H_x \left( x, \frac{-C(x) + \sqrt{C(x)^2 - 3D(x) B(x)}}{3D(x)} \right),
\]

and therefore we can assert that \( G'(\bar{x}) \neq 0 \). This proves (a) because one can easily see that \( G(x) = \frac{1}{2} x^2 + O(x^3) \).

Part (b) will be proved by contradiction. So assume that we have \( F(\bar{x}) < h_0 \) for some \( \bar{x} \in (x_L, x_R) \). Then there exists \( \epsilon > 0 \) so that \( F(\bar{x}) < h \) for all \( h \in (h_0 - \epsilon, h_0) \). Notice also that \( G(\bar{x}) \neq h \) for all \( h \in (h_0 - \epsilon, h_0) \) because from Lemma 3.1 it follows that \( F(\bar{x}) > G(\bar{x}) \). Hence

\[
A(\bar{x}, h) = \frac{1}{4D(\bar{x})} (F(\bar{x}) - h)(h - G(\bar{x})) < 0,
\]

and therefore \( H(\bar{x}, y) = h \) has only one real solution for all \( h \in (h_0 - \epsilon, h_0) \). This obviously contradicts that \( \bar{x} \in (x_L, x_R) \), and so (b) follows.

Remark 3.3. One can check that

\[
F'(x) = H_x \left( x, \frac{-C(x) - \sqrt{C(x)^2 - 3D(x) B(x)}}{3D(x)} \right)
\]

is also satisfied. We shall take account of this fact later.

For each \( h \in (0, h_0) \), let \( [x_r(h), x_l(h)] \) be the projection of the periodic orbit \( \gamma_h \) to the \( x \)-axis, that is

\[
[x_r(h), x_l(h)] = \{x \in \mathbb{R} : \text{there exists } y \in \mathbb{R} \text{ such that } (x, y) \in \gamma_h \}.
\]
We have thus a geometrical definition of the endpoints \( x_a(h) \) and \( x_r(h) \) in terms \( \gamma_h \) (see Fig. 2). We shall next obtain a result that provides an analytic way to compute them. To this end we define

\[
g(x) = \text{sgn}(x) \sqrt{G(x)} = x \sqrt{\frac{G(x)}{x^2}},
\]

which by (a) in Lemma 3.2 is an analytic function on \((x_L, x_R)\) satisfying \( g'(x) > 0 \).

**Proposition 3.4.** If \( D(x) \neq 0 \) for all \( x \in (x_L, x_R) \) then

(a) For each \( h \in (0, h_0) \), \( A(\cdot, h) \) is strictly positive in the interior of \([x_a(h), x_r(h)]\) and zero at the endpoints. In addition

\[
x_a(h) = g^{-1}(-\sqrt{h}) \quad \text{and} \quad x_r(h) = g^{-1}(\sqrt{h}).
\]

(b) \( G(x) \to h_0 \) as \( x \searrow x_L \) or \( x \nearrow x_R \).

**Proof.** Let us fix some \( h \in (0, h_0) \) and note that

\[
x_L < x_a(h) < 0 < x_r(h) < x_R. \tag{6}
\]

In view of Fig. 2 it is clear that \( H(x_a(h), y) = h \) has a double root for some value of \( y \) and therefore \( A(x_a(h), h) = 0 \). Now, since

\[
A(x, h) = \frac{1}{4D(x)^2} (F(x) - h)(h - G(x)), \tag{7}
\]
we conclude that \( G(x_r(h)) = h \) because, taking (6) into account, (b) in Lemma 3.2 shows that \( F(x_r(h)) \geq h_0 > h \). Thus \( x_r(h) = g^{-1}(\sqrt[3]{h}) \). The fact that \( x_r(h) = g^{-1}(\sqrt[3]{h}) \) follows exactly the same way. Consider next any \( \bar{x} \in (x_l(h), x_r(h)) \) and notice that according to (6) then \( \bar{x} \in (x_L, x_R) \). In this case by applying Lemma 3.2 we can assert that \( F(\bar{x}) \geq h_0 > h \) and, due to

\[
G(x_r(h)) = G(x_r(h)) = h, \tag{8}
\]

that \( G(\bar{x}) < h \). Therefore from (7) it turns out that \( A(\bar{x}, h) > 0 \), and this proves (a). Finally (b) follows readily from (8) using that \( x_l(h) \wedge x_r \) and \( x_r(h) \not\wedge x_r \) as \( h \not\geq h_0 \).

We are now almost in position to prove the main result of this section. First we have to describe precisely the three roots of the equation \( H(x, y) = h \) (see [25] for instance). Thus, setting

\[
\Psi(x, h) = \frac{9B(x)C(x)D(x) - 27(A(x) - h)D(x)^2 - 2C(x)^3}{54D(x)^3},
\tag{9}
\]

we define

\[
R(x, h) = (\Psi(x, h) - iA(x, h)^{1/2})^{1/3}
\]

and

\[
S(x, h) = (\Psi(x, h) + iA(x, h)^{1/2})^{1/3}.
\]

Then the three roots are given by

\[
y_1(x, h) = R(x, h) + S(x, h) - \frac{1}{3} C(x),
\]

\[
y_2(x, h) = -\frac{1}{2} (R(x, h) + S(x, h)) - \frac{1}{3} C(x) + \frac{i\sqrt{3}}{2} (S(x, h) - R(x, h)),
\]

and

\[
y_3(x, h) = -\frac{1}{2} (R(x, h) + S(x, h)) - \frac{1}{3} C(x) + \frac{i\sqrt{3}}{2} (R(x, h) - S(x, h)).
\]

In the following when we refer to some root we shall use the above notation. In particular notice that \( y_1(x, h) \) is the one that is always real. Here by “always” we mean independent of which the signum of \( A(x, h) \) is. From now on, given \( z \in \mathbb{C} \), Re\( (z) \) and Im\( (z) \) will denote respectively the real and the imaginary part of \( z \).

The straight lines \( x = x_l(h) \) and \( x = x_r(h) \) split the periodic orbit \( \gamma_h \) into two components (see Fig. 2). The following result determines them, in case that \( D < 0 \).
Lemma 3.5. For each $h \in (0, h_0)$, the upper component of $y_h$ is given by the graphic of $y_3(\cdot, h): (x_a(h), x_r(h)) \to \mathbb{R}$ and the lower one by $y_2(\cdot, h): (x_a(h), x_r(h)) \to \mathbb{R}$.

Proof. Fix some $h \in (0, h_0)$ and notice that from (a) in Proposition 3.4 it follows that $A(x, h) > 0$ for all $x \in (x_a(h), x_r(h))$. Consequently, given any $x \in (x_a(h), x_r(h))$, the three roots of $H(x, y) = h$ are different and real. The ones which are contained in $y_a$ are precisely $y_2(x, h)$ and $y_3(x, h)$ because $y_1(x, h)$ is real for all $x \in (x_a, x_r)$. Finally, taking $A(x, h) > 0$ into account, we get
\[ y_3(x, h) - y_2(x, h) = \sqrt{3} \text{Im}(S(x, h)) > 0 \]
and this completes the proof of the result. 

Now, the main result of this section is the following.

Theorem 3.6. If $D(x) < 0$ for all $x \in (x_L, x_R)$ then, for each $h \in (0, h_0)$, the period of the periodic orbit $\gamma_h$ is given by
\[
T(h) = \int_{e^{-\sqrt{h}}}^{e^{\sqrt{h}}} \left( \frac{1}{H_y(x, y_3(x, h))} - \frac{1}{H_y(x, y_2(x, h))} \right) dx.
\]

Proof. The result follows from Lemma 3.5 using that $x = -H_y(x, y)$ to compute the time and taking into account that, according to (a) in Proposition 3.4, $x_a(h) = g^{-1}(-\sqrt{h})$ and $x_r(h) = g^{-1}(\sqrt{h})$.

It should be pointed out that Theorem 3.6 allows us to compute the period of any periodic orbit in the period annulus, not only in a neighbourhood of the center. It is also to be noted that none of the functions appearing in the expression of $T(h)$ is defined in a geometric way, all of them are defined analytically. We think that these two facts make Theorem 3.6 a useful tool.

The next result will be used to compute $T(h)$. It follows from a straightforward computation using the fact that $R(x, h)$ is the complex conjugate of $S(x, h)$. This is so because, for any given $h \in (0, h_0)$, (a) in Proposition 3.4 asserts that $A(x, h) > 0$ for all $x \in (x_l(h), x_r(h))$.

Lemma 3.7. If $h \in (0, h_0)$ and $x \in (g^{-1}(-\sqrt{h}), g^{-1}(\sqrt{h}))$ then
\[
H_y(x, y_2(x, h)) = 3D(x)\left( Q(x) + (\text{Re}(S(x, h)) + \sqrt{3} \text{Im}(S(x, h)))^2 \right)
\]
and
\[
H_y(x, y_3(x, h)) = 3D(x)\left( Q(x) + (\text{Re}(S(x, h)) - \sqrt{3} \text{Im}(S(x, h)))^2 \right),
\]
where
\[ Q(x) = \frac{3D(x) B(x) - C(x)^3}{9D(x)^2}. \]

We conclude this section with a result, namely Proposition 3.9, in which
it is necessary that \( A, B, C, \) and \( D \) are polynomials. This is so because its
proof requires the following easy consequence of Lemma 2.2.

**Lemma 3.8.** Let \( \mathcal{P} \) be the period annulus of a center of a polynomial
Hamiltonian system. If \( \mathcal{P} \) is nonglobal then for any half a straight line \( \ell \) there
exists a compact set \( K \) satisfying that \( \ell \cap (\mathbb{R}^2 \setminus K) \) is not contained in \( \mathcal{P} \).

**Proof.** Assume that \( \ell \cap \mathcal{P} \neq \emptyset \) because otherwise the result is obvious.
We shall prove it by contradiction. If such a compact set does not exist then, on account of \( \ell \cap \mathcal{P} \neq \emptyset \), one can easily see that \( \ell \subset \mathcal{P} \). It follows then that \( \mathcal{P} \) is unbounded and that, in the Poincaré disk, there exists an
infinite critical point \( q \) in \( \partial \mathcal{P} \). It is also clear that \( q \) has an hyperbolic sector
\( \mathcal{H} \) with both separatrices inside \( \partial \mathcal{P} \) and that, in a neighbourhood \( V \) of
\( q \), \( \ell \cap V \) is contained in the interior of \( \mathcal{H} \). This contradicts Lemma 2.2 and
therefore the result follows. \( \square \)

**Proposition 3.9.** Assume that \( A, B, C, \) and \( D \) are polynomials with
\( D(x) \neq 0 \) for all \( x \in (x_L, x_R) \). If \( A(x, h_0) = 0 \) for some \( x \in (x_L, x_R) \) then \( \partial \mathcal{P} \)
has a finite critical point.

**Proof.** If \( A(x, h_0) = 0 \) then from the combination of Lemma 3.2 and (b)
in Proposition 3.4 it follows that \( F(x) = h_0 \) and \( F'(x) = 0 \). On the other
hand it implies the existence of \( \bar{y} \in \mathbb{R} \) such that \( H(x, \bar{y}) = h_0 \) and \( H_x(x, \bar{y}) = 0 \).
Since \( D(x) \neq 0 \), there exist two possible values of \( \bar{y} \) so that \( H_y(x, \bar{y}) = 0 \).
By applying Lemma 3.1 we can assert that
\[ \bar{y} = -\frac{C(x) - \sqrt{C(x)^2 - 3D(x) B(x)}}{3D(x)} \]
because otherwise \((x, \bar{y}) \in \mathcal{P}\) and this is not possible since \( H(x, \bar{y}) = h_0 \).
In addition, taking account of \( F'(x) = 0 \), Remark 3.3 shows that \( H_x(x, \bar{y}) = 0 \). Thus \((x, \bar{y})\) is a critical point with \( H(x, \bar{y}) = h_0 \). It remains
only to prove that \((x, \bar{y}) \in \partial \mathcal{P} \). To this end notice first that, due to
\( x \in (x_L, x_R) \), by applying Lemma 3.8 we can assert that there exist two
points such that \((x, y_1), (x, y_2) \in \partial \mathcal{P} \). Note then that \( y_1 \) and \( y_2 \) are two
different roots of \( H(x, y) = h_0 \). Consequently, if \((x, \bar{y}) \neq \partial \mathcal{P} \) then \( \bar{y} \neq y_1 \)
and \( \bar{y} \neq y_2 \), and in this case the sum of the multiplicities of the roots of
\( H(x, y) = h_0 \) would be at least four. This is not possible, so \((x, \bar{y}) \in \partial \mathcal{P} \). \( \square \)
It is worthwhile noting that in order to prove the results in this section it is not essential that $A(x) = x^2/2 + O(x^3)$ and $C(0) = 1/2$. As a matter of fact the same results follow assuming only that the center at the origin is non-degenerate.

4. THE CASE $a_0 \neq 0$

For the remainder of the paper we shall focus our attention on Theorem A. So recall that in Section 2 (see Remark 2.4) we reduce its proof to study the cubic-like Hamiltonian systems given by (4). Note in particular that $D(x) = a_0 x^2 + b_0 x + c_0$. This section is devoted to the case $a_0 \neq 0$ and the following will be shown.

**Theorem 4.1.** Assume $a_0 \neq 0$. If $\mathcal{P}$ is admissible then $T(h) \to +\infty$ as $h \nearrow h_0$.

Our first goal is to show that if $\mathcal{P}$ is admissible then $D(x) \neq 0$ for all $x \in (x_L, x_R)$ because in this case we can take account of the results of Section 3. This is the most geometric part of the paper and we shall study frequently the phase portrait of the vector field in the Poincaré disc.

**Proposition 4.2.** If $\mathcal{P}$ is admissible then $b_0^2 - 4a_0c_0 = 0$ and $b_0 \neq 0$.

**Proof.** To show this we shall use the Poincaré compactification of the vector field (see the Appendix). The characteristic polynomial of the critical point at the origin of the local chart $U_2$ is given by

$$W(u, v) = -3v(a_0 u^2 + b_0 uv + c_0 v^2).$$

Note that this critical point is precisely the one given by the direction $\theta = \pi/2$ in the Poincaré disk and recall that, since $\mathcal{P}$ is admissible, it must have the hyperbolic sector $\mathcal{H}^\ast$. According to Lemma 7.2, and taking $a_0 \neq 0$ into account, a necessary condition for the existence of $\mathcal{H}^\ast$ is that $b_0^2 - 4a_0c_0 = 0$.

Next we shall prove by contradiction that $b_0 \neq 0$. If $b_0 = 0$ then $W(u, v) = -3a_0 u^2 v$ and, by Lemma 7.2 again, the two separatrices of $\mathcal{H}^\ast$ must be tangent to $u = 0$. In the Poincaré disk this means that both separatrices, say $s_1$ and $s_2$, are tangent to the $y$-axis. On account of Lemma 2.2 we can assert that either one of them is contained in the $y$-axis or that they are located as in Fig. 3. In the first case, due the analyticity of the Poincaré compactification of the vector field, the $y$-axis should be invariant and this is obviously not possible because there is a center at the origin. The two situations shown in Fig. 3 cannot occur either. For
instance, in case of Figure 3a it is clear that \( s_1 \) should intersect the \( y \)-axis as it is shown in Fig. 4. Thus, for any \( \bar{x} \geq 0 \), the straight line \( x = \bar{x} \) has three intersection points with \( \partial \mathcal{P} \), say \((\bar{x}, y_1), (\bar{x}, y_2), \) and \((\bar{x}, y_3)\) with \( y_1 < y_2 < y_3 \). Then, since \( H(\partial \mathcal{P}) = h_0 \) and there are at most three values of \( y \) such that \( H(\bar{x}, y) = h_0 \), it follows that \((\bar{x}, y) \in \mathcal{P}\) for all \( y < y_1 \) and this contradicts Lemma 3.8. A similar argument rules out that Fig. 3b may happen. Therefore we conclude that \( b_0 \neq 0 \).

We will be in position to apply the results of Section 3 once we prove the following. Notice that in particular it asserts that either \( x_L \) or \( x_R \) is finite.

FIG. 3. Possible location of the separatrices of \( \mathcal{H}^* \) when \( h_0 = 0 \).

FIG. 4. The intersection of \( \partial \mathcal{P} \) with the straight line \( x = \bar{x} \).
Proposition 4.3. If \( \mathcal{P} \) is admissible and its boundary does not contain any finite critical point then

(a) \( D(x) = a_0(x-x_R)^2 \) and \( C(x_R) = 0 \) in the case that \( a_0b_0 < 0 \),
(b) \( D(x) = a_0(x-x_L)^2 \) and \( C(x_L) = 0 \) in the case that \( a_0b_0 > 0 \).

Proof. Note that \( b_0 \neq 0 \) and \( b_0^2 - 4a_0c_0 = 0 \) by Proposition 4.2. Consequently the characteristic polynomial of the critical point at the origin of the local chart \( U_z \) is

\[
W(u, v) = -3a_0v \left( u + \frac{b_0}{2a_0}v \right)^2.
\]

Now, according to Lemma 7.2, the two separatrices of the hyperbolic sector \( \mathcal{H}^* \) must be tangent to \( u = -\frac{b_0}{(2a_0)}v \). In the Poincaré disk this means that both separatrices are tangent to the straight line \( x = -\frac{b_0}{(2a_0)} \).

Let us consider only the case \( a_0b_0 < 0 \) because the other one follows exactly the same way. Thus, on account of Lemma 2.2, there are four possible location for the separatrices (see Fig. 5). As a matter of fact we shall see that some of them are not possible in our setting. Indeed, in Figs. 5b and 5c the whole straight line \( x = -\frac{b_0}{(2a_0)} \) must be a component of \( \partial \mathcal{P} \) because we assume that it does not contain any finite critical point. This fact rules out that Fig. 5c may occur because the two separatrices of \( \mathcal{H}^* \) are in the boundary of \( \mathcal{P} \) and the center is at the origin. In case of Fig. 5b it implies that the infinite critical point given by the direction \( \theta = -\frac{\pi}{2} \) has a hyperbolic sector with the separatrices in \( \partial \mathcal{P} \) and located as it is shown in Fig. 6a. Next it will be shown that Fig. 5d is not possible. This is so because in this case the separatrix \( s_1 \) should intersect the straight line \( x = -\frac{b_0}{(2a_0)} \). Then, exactly as in the proof of Proposition 4.2, for any fixed \( x > -\frac{b_0}{(2a_0)} \) there would exist three different values, say \( y_1 < y_2 < y_3 \), such that \( H(x, y) = h_0 \). However this implies that \( (x, y) \in \mathcal{P} \) for all \( y < y_1 \), which contradicts Lemma 3.8.

FIG. 5. Possible location of the separatrices of \( \mathcal{H}^* \) when \( a_0b_0 < 0 \).
FIG. 6. (a) Shape of $\partial \mathcal{P}$ corresponding to Fig. 5b. (b) Sketch of the proof of $x_R = -b_0/(2a_0)$.

In short, until now it has been shown that the only possible situations are Figs. 5a and 6a. We claim now that $x_R = -b_0/(2a_0)$. This is obviously true in the case of Fig. 6a, so it only remains to study Fig. 5a. Note that in order to prove the claim it suffices to see that there does not exist any point $(x, y) \in \mathcal{P}$ with $x > -b_0/2a_0$. However this is clear because otherwise the separatrix $s_i$ should intersect the straight line $x = -b_0/2a_0$ as it is shown in Figure 6b, and then again this would lead to a contradiction with Lemma 3.8. Hence the claim is true and we can assert therefore that $D(x) = a_0(x - x_R)^2$.

Finally we shall prove that $C(x_R) \neq 0$ leads to contradiction with Figs. 5a and 6a, which are the only possible ones. If $C(x_R) \neq 0$ then $H(x_R, z) - h_0$ is a second degree polynomial with respect to $z$. We consider its two zeros, say $z_1, z_2 \in \mathbb{C}$, and we take a Jordan curve $\Gamma$ in $\mathbb{C}$ containing $z_1$ and $z_2$ in its interior. In addition we choose $\varepsilon > 0$ small enough so that for any $x \in (x_R - \varepsilon, x_R + \varepsilon)$ it holds

$$\sup_{z \in \Gamma} |H(x, z) - H(x_R, z)| < \inf_{z \in \Gamma} |H(x_R, z) - h_0|.$$ 

Consequently, for any $x \in (x_R - \varepsilon, x_R + \varepsilon)$, the following is

$$|H(x, z) - h_0 - (H(x_R, z) - h_0)| < |H(x, z) - h_0|$$

$$+ |H(x_R, z) - h_0| \quad \text{for all} \quad z \in \Gamma.$$

Then, according to Rouche's theorem, in this situation we can assert that for any fixed $\bar{x} \in (x_R - \varepsilon, x_R + \varepsilon)$ there are two zeros of the polynomial $H(\bar{x}, z) - h_0$ inside $\text{Int} \ \Gamma$. Since $H(\bar{x}, z) - h_0 = 0$ has at most three roots, taking $m = \sup \{|z| : z \in \Gamma\}$, we conclude that for any fixed $\bar{x} \in (x_R - \varepsilon, x_R)$ there exists at most one real value $y$ with $|y| > m$ such that $H(\bar{x}, y) = h_0$. This obviously contradicts Figs. 5a and 6a because $H(\partial \mathcal{P}) = h_0$. Therefore $C(x_R) = 0$ holds and this completes the proof of the result. \[\square\]
From now on, in order to prove Theorem 4.1, let us focus our attention in the case \( a_0 b_0 < 0 \) because the other one follows similarly. In this case, according to Proposition 4.3, we can write \( C(x) = (x-x_R) C_1(x) \) for some polynomial \( C_1 \). The following result refers to this polynomial and in order to prove it we shall take account of the properties of the function \( G \) that we show in Section 3.

**Lemma 4.4.** The following holds in case that \( a_0 b_0 < 0 \).

(a) If \( C_1(x_R) \leq 0 \) then \( B(x_R) = 0 \) and \( A(x_R) = h_0 \).

(b) If \( C_1(x_R) > 0 \) then \( B(x) = 1/(4a_0) C_1(x)^2 + (x-x_R) C_2(x) \) for some polynomial \( C_2 \) and \( A(x_R) = 1/(2a_0) C_1(x_R) C_2(x_R) + h_0 \).

**Proof.** Note first of all that by (a) in Proposition 4.3 we have \( D(x) = a_0 (x-x_R)^2 \) and \( C(x) = (x-x_R) C_1(x) \) for some polynomial \( C_1 \). Then, since \( D(x) \neq 0 \) for all \( x \in (x_L, x_R) \), we can apply (b) in Proposition 3.4 and assert that

\[
G(x) \rightarrow h_0 \quad \text{as} \quad x \nearrow x_R.
\] (10)

In addition, taking \( x-x_R < 0 \) into account, one can check that

\[
G(x) = A(x) + \frac{2C_1(x)^3 - 9a_0 C_1(x) B(x) + 2(C_1(x)^2 - 3a_0 B(x))^{3/2}}{27a_0^2 (x-x_R)}.
\] (11)

Now, since \( h_0 \) is finite, it is clear that

\[
2C_1(x_R)^3 - 9a_0 C_1(x_R) B(x_R) + 2(C_1(x_R)^2 - 3a_0 B(x_R))^{3/2} = 0
\] (12)

is a necessary condition in order that (10) holds. An elementary manipulation of this equality yields

\[
27a_0^2 B(x_R)^2 (4a_0 B(x) - C_1(x_R)^2) = 0,
\]

which has \( B(x_R) = 0 \) and \( B(x_R) = 1/(4a_0) C_1(x_R)^2 \) as solutions. Let us consider first the case \( C_1(x_R) < 0 \). In this case one can check that (12) is satisfied only when \( B(x_R) = 0 \). Taking this into account and using that

\[
G(x) = A(x) + \frac{B(x)^2 (4a_0 B(x) - C_1(x)^2)}{(x-x_R)(2C_1(x)^3 - 9a_0 C_1(x) B(x) - 2(C_1(x)^2 - 3a_0 B(x))^{3/2})},
\]

from (10) it follows that \( A(x_R) = h_0 \). On the other hand, if \( C_1(x_R) = 0 \) then from (12) we obtain \( B(x_R) = 0 \). Now, on account of (10), from (11) we get
This proves (a). Finally, if $C_1(x_R) > 0$ then one can check that (12) is satisfied only when $B(x_R) = 1/(4a_0) C_1(x_R)^2$, and consequently it is clear that

$$B(x) = \frac{1}{4a_0} C_1(x)^2 + (x - x_R) C_2(x)$$

for some polynomial $C_2$. This relation allows us to write

$$G(x) = A(x) + \frac{4a_0 B(x)^2 C_2(x)}{2C_1(x)^3 - 9a_0 C_1(x) B(x) - 2(C_1(x)^3 - 3a_0 B(x))^{3/2}},$$

and then one can readily see that (10) implies

$$A(x_R) - \frac{1}{2a_0} C_1(x_R) C_2(x_R) = h_0.$$

This proves (b) and concludes the proof of the result.

**Proof of Theorem 4.1.** Note that we can assume without loss of generality that $\partial \mathcal{P}$ does not contain any finite critical point because otherwise $T(h) \to +\infty$ as $h \nearrow h_0$. Thus, by applying Proposition 4.3 we can assert that $D(x) \neq 0$ for all $x \in (x_L, x_R)$. Now from Proposition 3.9 it turns out that

$$A(x, h_0) > 0 \text{ for all } x \in (x_L, x_R). \quad (13)$$

Hence $y_1(x, h_0), y_2(x, h_0)$ and $y_3(x, h_0)$ are real and different for all $x \in (x_L, x_R)$. Notice then that the graphics of $y_2(\cdot, h_0): (x_L, x_R) \to \mathbb{R}$ and $y_3(\cdot, h_0): (x_L, x_R) \to \mathbb{R}$ are components of $\partial \mathcal{P}$.

From Theorem 3.6 and applying Fatou’s lemma it follows that

$$\lim_{h \nearrow h_0} T(h) \geq \int_{x_L}^{x_R} \frac{1}{H_1(x, y_1(x, h_0))} - \frac{1}{H_2(x, y_2(x, h_0))} \, dx.$$

Here we use that, by Proposition 3.4, $g^{-1}(-\sqrt{h}) \to x_L$ and $g^{-1}(\sqrt{h}) \to x_R$ as $h \nearrow h_0$. On account of (13), one can also see that $H_1(x, y_3(x, h_0)) > 0$ and $H_2(x, y_2(x, h_0)) < 0$ for all $x \in (x_L, x_R)$. Consequently in order to prove the result it suffices to show that

$$\int_{x_L}^{x_R} \frac{dx}{H_2(x, y_2(x, h_0))} = +\infty. \quad (14)$$
Our goal is to see that $H_y(x, y(x, h_0)) = O(x - x_R)$, and to this end we shall use the relations that provide Proposition 4.3 and Lemma 4.4. That it vanishes at $x = x_R$ is very simple to see, the only technical difficulty will be to check that it does so with enough multiplicity in order that (14) follows. With this end in view, note that from Lemma 3.7 it follows that

$$H_y(x, y(x, h_0)) = 3D(x)\{Q(x) + (\text{Re}(S(x, h_0)) - \sqrt{3} \text{Im}(S(x, h_0)))^2\}, \quad (15)$$

and on the other hand by (a) in Proposition 4.3 we can assert that

$$D(x) = a_0(x - x_R)^2 \quad \text{and} \quad C(x) = (x - x_R) C_1(x)$$

for some polynomial $C_1$. We shall study the three possible cases that contemplate the value of $C_1(x_R)$, namely $C_1(x_R) > 0$, $C_1(x_R) = 0$, and $C_1(x_R) < 0$, and the result will be proved in each one.

Let us consider first the case $C_1(x_R) > 0$. In this case, by applying (b) in Lemma 4.4, we can write

$$B(x) = \frac{1}{4a_0} C_1(x)^2 + (x - x_R) C_2(x)$$

and

$$A(x) = h_0 + \frac{1}{2a_0} C_1(x) C_2(x) + (x - x_R) C_3(x)$$

for some polynomials $C_2$ and $C_3$. Then, recalling (9), a computation shows that

$$\Psi(x, h_0) = \frac{1}{(x - x_R)^2} P_1(x) \quad \text{and} \quad A(x, h_0) = \frac{1}{(x - x_R)^3} P_2(x), \quad (16)$$

where $P_1$ and $P_2$ are again polynomials with

$$P_1(x_R) = \frac{1}{216a_0} C_1(x_R)^3$$

and, on account of (13), $P_2(x) > 0$ for all $x \in (x_L, x_R)$. We claim that

$$(\text{Re}(S(x, h_0)) - \sqrt{3} \text{Im}(S(x, h_0)))^2 = \frac{P_1(x)^{2/3}}{(x - x_R)^7} (1 + O(x - x_R)). \quad (17)$$
In order to show the claim let us assume first that $a_0 > 0$. In this case $P_1(x_R) > 0$ and therefore, since $x - x_R \leq 0$, we obtain

$$S(x, h_0) = (\Psi(x, h_0) + i D(x, h_0)^{1/2})^{1/3}$$

$$= \left( \frac{-P_1(x)}{(x-x_R)^2} \right)^{1/3} \left( -1 - i \frac{(x-x_R)P_2(x)^{1/2}}{P_1(x)} \right)$$

$$= \frac{P_1(x)^{1/3}}{x_R - x} \left( \frac{1 + i \sqrt{3}}{2} + O(x-x_R) \right).$$

The first factor in the product of the last equality is real, and hence one can check that

$$\text{Re}(S(x, h_0)) - \sqrt{3} \text{Im}(S(x, h_0)) = \frac{P_1(x)^{1/3}}{x_R - x} (-1 + O(x-x_R)),$$

showing the validity of (17) for the case $a_0 > 0$. Assume now that $a_0 < 0$. Then $P_1(x_R) < 0$ and, on account of $x - x_R \leq 0$, it turns out that

$$S(x, h_0) = (\Psi(x, h_0) + i D(x, h_0)^{1/2})^{1/3}$$

$$= \left( \frac{P_1(x)}{(x-x_R)^2} \right)^{1/3} \left( 1 + i \frac{(x-x_R)P_2(x)^{1/2}}{P_1(x)} \right)$$

$$= \frac{-P_1(x)^{1/3}}{x_R - x} (1 + O(x-x_R)).$$

In this case it follows that

$$\text{Re}(S(x, h_0)) - \sqrt{3} \text{Im}(S(x, h_0)) = \frac{(-P_1(x))^{1/3}}{x_R - x} (1 + O(x-x_R)),$$

and therefore this shows that (17) also holds for $a_0 < 0$. The claim has thus been proved. On the other hand notice that

$$Q(x) = \frac{3a_0 B(x) - C_1(x)^2}{9a_0^2 (x-x_R)^2},$$

and consequently, taking account also of (15) and (17), we obtain

$$H_s(x, y_3(x, h_0)) = B(x) - \frac{1}{3a_0} C_1(x)^2 + 3a_0 P_1(x)^{2/3} (1 + O(x-x_R)).$$

(18)
At this point notice that
\[ x \mapsto B(x) - \frac{1}{3a_0} C_1(x)^2 + 3a_0 P_1(x)^{2/3} \] (19)
is an analytic function on a neighbourhood of \( x = x_R \) because \( P_1(x_R) \neq 0 \).
Note also that it vanishes at \( x = x_R \) since
\[ B(x_R) = \frac{1}{4a_0} C_1(x_R)^2 \quad \text{and} \quad P_1(x_R) = C_1(x_R)^3. \]
This shows that \( H(x, y(x, h_0)) = O(x - x_R) \) and consequently (14) follows.

Theorem 4.1 has thus been proved for the case \( C_1(x_R) > 0 \).

The case \( C_1(x_R) < 0 \) follows in a similar way. Indeed, in this case by (a) in Lemma 4.4 we can write
\[ B(x) = (x - x_R) C_2(x) \quad \text{and} \quad A(x) = h_0 + (x - x_R) C_3(x) \] (20)
for some other two polynomials \( C_2 \) and \( C_3 \). Then we obtain the same relations as in (16) but now with
\[ P_1(x_R) = \frac{-1}{27a_0^3} C_1(x_R)^3. \]
One can check in addition that (17) also holds, and thus we get (18) again.
The unique difference is that in this case the function in (19) vanishes at \( x = x_R \) due to
\[ B(x_R) = 0 \quad \text{and} \quad P_1(x_R) = \left( \frac{C_1(x_R)}{3a_0} \right)^3. \]

It only remains to study the case \( C_1(x_R) = 0 \). In this case, apart from the relations which appear in (20), we have \( C(x) = (x - x_R)^2 \tilde{C}_1(x) \) for some polynomial \( \tilde{C}_1 \). Then a computation shows that
\[ \Psi(x, h_0) = \frac{1}{x - x_R} \tilde{P}_1(x) \quad \text{and} \quad A(x, h_0) = \frac{1}{(x - x_R)^3} \tilde{P}_2(x), \]
where \( \tilde{P}_1 \) and \( \tilde{P}_2 \) are polynomials. Note also that from (13) it follows that \( \tilde{P}_2(x) > 0 \) for all \( x \in (x_L, x_R) \). Hence
\[
S(x, h_0) = (\Psi(x, h_0) + iA(x, h_0))^{1/3} = \left( \frac{\tilde{P}_1(x)}{x - x_R} + i \frac{1}{(x_R - x)^{3/2}} \tilde{P}_2(x)^{1/2} \right)^{1/3}
\]
\[ = \frac{1}{(x_R - x)^{1/2}} \left( -\tilde{P}_1(x) (x_R - x)^{1/2} + i \tilde{P}_2(x)^{1/2} \right)^{1/3} \]
and thus, denoting \( w(x) = \left( -\bar{P}_1(x) (x_R - x)^{1/2} + i \bar{P}_2(x)^{1/2} \right)^{1/3} \), we can write
\[
\text{Re}(S(x, h_0)) - \sqrt{3} \text{Im}(S(x, h_0)) = \frac{1}{(x_R - x)^{1/2}} \left( \text{Re}(w(x)) - \sqrt{3} \text{Im}(w(x)) \right).
\]
(21)

On the other hand
\[
Q(x) = \frac{3a_0 \bar{C}_2(x) - (x - x_R) \bar{C}_1(x)^2}{9a_0^2 (x - x_R)},
\]
and therefore, taking (15) and (21) into account, we obtain
\[
H_f(x, y_3(x, h_0)) = (x - x_R) \left\{ \bar{C}_2(x) - \frac{1}{3a_0} (x - x_R) \bar{C}_1(x)^2 \right.
\]
\[
- 3a_0 \left( \text{Re}(w(x)) - \sqrt{3} \text{Im}(w(x)) \right)^2 \}.
\]

This shows that \( H_f(x, y_3(x, h_0)) = O(x - x_R) \), and accordingly (14) also holds in the case that \( C_1(x_R) = 0 \). This concludes the proof of the result. 

The case \( a_0 b_0 > 0 \), which is not contemplated in the proof of Theorem 4.1, follows in a similar way. Indeed, in this case one can show that \( H_f(x, y_3(x, h_0)) = O(x - x_L) \) and hence
\[
\int_{x_L}^{x_L + \epsilon} \frac{dx}{H_f(x, y_3(x, h_0))} = +\infty.
\]

To do so, we first need to study the values of \( A(x_L) \) and \( B(x_L) \) as was done in Lemma 4.4.

5. THE CASE \( a_0 = 0 \) WITH \( b_0 \neq 0 \)

In this section we show that, under these parameter values, if \( \mathcal{P} \) is admissible then \( H_0 \equiv 0 \). To this end we shall use the blow-up method described in the Appendix and the expression of the Hamiltonian system given by (4) in the local chart \( U_2 \).

Proposition 5.1. Assume \( a_0 = 0 \) and \( b_0 \neq 0 \). If \( \mathcal{P} \) is admissible then \( a_1 = a_2 = a_3 = 0 \).
Proof. Recall that, since $\mathcal{P}$ is admissible, the critical point at the origin of the local chart $U_2$ has a hyperbolic sector $\mathcal{K}^*$. It will be shown that $a_1 = a_2 = a_3 = 0$ is a necessary condition for the existence of $\mathcal{K}^*$. The characteristic polynomial of the critical point (see system (26) in the Appendix taking $a_0 = 0$) is given by $W(u, v) = -3v^2(b_0 u + c_0 v)$. Thus $u = -c_0 v / b_0$ is a simple characteristic direction and $v = 0$ is a double one.

Let us consider first the case $c_0 \neq 0$. Then $u = 0$ is not a characteristic direction and we do the blow up $\{u = u_1, v = v_1\}$. We obtain

$$
\begin{align*}
\dot{u}_1 &= -u_1(5a_1 u_1 + 4b_0 v_1 + p(u_1, v_1)), \\
\dot{v}_1 &= 2a_1 u_1 v_1 + 3b_0 v_1^2 + 3c_0 v_1^4 + u_1 q(u_1, v_1),
\end{align*}
$$

where $p$ and $q$ are polynomials in $u_1$ and $v_1$ starting with terms of degree 2. The critical points of system (22) on $u_1 = 0$ are $(0, 0)$, which is linearly zero, and $(0, -b_0 / c_0)$, which is a hyperbolic node with linear part

$$
\begin{pmatrix}
b_0^2 / c_0 \\
* \\
3b_0^2 / c_0
\end{pmatrix}.
$$

To study the critical point at the origin we first compute its characteristic polynomial. One can check that $W(u_1, v_1) = -7u_1 v_1^2(a_1 u_1 + b_0 v_1)$. We claim that if $a_1 \neq 0$ then the critical point at the origin of $U_2$ has no hyperbolic sectors. In this case the origin has three simple characteristic directions, namely $u_1 = 0$, $v_1 = 0$, and $v_1 = -a_1 u_1 / b_0$. For the sake of simplicity let us assume that $a_1 + b_0 \neq 0$; otherwise, we should do a 2-blow up and we arrive at the same conclusion. Thus, we do the 1-blow up $\{u_2 = u_1, v_1 = v_1; u_2 = \bar{u}_1, v_1 = v_2 \bar{u}_1\}$. The critical points of system $(\dot{u}_2, \dot{v}_2)$ on $u_2 = 0$ are $(0, 0)$, $(0, -1)$, and $(0, -a_1 / (b_0 + a_1))$. Moreover one can check that the linear parts are given by

$$
\begin{pmatrix}
-5a_1 & 0 \\
0 & 7a_1
\end{pmatrix},
\begin{pmatrix}
-3b_0 & 0 \\
0 & 7b_0
\end{pmatrix}
$$

and

$$
\begin{pmatrix}
-a_1 b_0 \\
0
\end{pmatrix},
\begin{pmatrix}
0 & 0 \\
* & -7a_1 b_0
\end{pmatrix},
$$

respectively. Hence the critical points $(0, 0)$ and $(0, -1)$ are hyperbolic saddles while $(0, -a_1 / (b_0 + a_1))$ is a hyperbolic node. Figure 7 gathers the chain of blow ups and shows that the origin of the local chart $U_2$ has no hyperbolic sectors. This proves the claim and accordingly $a_1 = 0$. In this case the characteristic polynomial of the critical point at the origin of system (22) is $W(u_1, v_1) = -7b_0 u_1 v_1^2$. Thus, $u_1 = 0$ is a simple characteristic
FIG. 7. Going back through the blow ups when \( a_1 > 0, b_2 > 0, \) and \( c_3 > 0. \)

direction and \( v_1 = 0 \) is a double one. We do the 1-blow up \( \{ \bar{u}_1 = u_1 - v_1, \bar{v}_1 = v_1; u_2 = \bar{u}_1, v_2 = v_2 \bar{u}_1 \} \) and we get

\[
\begin{align*}
\dot{u}_2 &= -u_2 (5a_2 u_2 + 4b_0 v_2 + p(u_2, v_2)), \\
\dot{v}_2 &= 6a_2 u_2 v_2 + 7b_0 v_2^2 + 7b_0 v_2^2 + u_2 q(u_2, v_2),
\end{align*}
\]

where \( p \) and \( q \) are polynomials in \( u_2 \) and \( v_2 \) starting with terms of degree 2.

The critical points of system (23) on \( u_2 = 0 \) are \((0, -1)\), which is a hyperbolic saddle with linear part

\[
\begin{pmatrix}
-3b_0 & 0 \\
0 & 7b_0
\end{pmatrix},
\]

and \((0, 0)\), which is linearly zero. The characteristic polynomial of this second critical point is given by \( W(u_2, v_2) = -11u_2 v_2 (a_2 u_2 + b_0 v_2) \). We claim now that if \( a_2 \neq 0 \) then the critical point at the origin of \( U_2 \) has no hyperbolic sectors. Indeed, if \( a_2 \neq 0 \) then the origin of system (23) has three simple characteristic directions, namely \( u_2 = 0, v_2 = 0, \) and \( v_2 = -a_2 u_2 / b_0 \).

Again, for the sake of simplicity, let us assume that \( a_2 + b_0 \neq 0 \); otherwise we should do a 2-blow up and the conclusion is the same. So we perform

\[
\begin{align*}
\{ \bar{u}_2 = u_2 - v_2, \bar{v}_2 = v_2; u_3 = \bar{u}_2, v_3 = v_2 \bar{u}_2 \}.\end{align*}
\]

The critical points of system \((\bar{u}_3, \bar{v}_3)\) on \( u_3 = 0 \) are \((0, 0), (0, -1), \) and \((0, -a_2 / (b_0 + a_2))\), and one can check that their linear parts are given by

\[
\begin{align*}
\begin{pmatrix}
-5a_2 & 0 \\
0 & 11a_2
\end{pmatrix}, \\
\begin{pmatrix}
-7b_0 & 0 \\
0 & 11b_0
\end{pmatrix}, \quad \text{and} \quad \begin{pmatrix}
-a_2 b_0 \\
\frac{b_0 + a_2}{a_2} & 0 \\
0 & -11a_2 b_0 \\
\frac{b_0 + a_2}{a_2} & 0
\end{pmatrix},
\end{align*}
\]

respectively. Then the critical points \((0, 0)\) and \((0, -1)\) are hyperbolic saddles while \((0, -a_2 / (b_0 + a_2))\) is a hyperbolic node. Going back through the blow ups, Fig. 8 shows that the origin of the local chart \( U_2 \) has no
Going back through the blow ups when \(a_1 = 0\), \(a_2 > 0\), \(b_0 > 0\) and \(c_0 > 0\).

hyperbolic sectors. This proves the claim and consequently \(a_2 = 0\). In this case the characteristic polynomial of the critical point at the origin of system (23) is \(W(u_2, v_2) = -11b_0u_2v_2^2\). Thus \(u_2 = 0\) is a simple characteristic direction and \(v_2 = 0\) is a double one. Now, in order to study the origin we do the 1-blow up \(\{\bar{u}_2 = u_2 - v_2, \bar{v}_2 = v_2; u_3 = \bar{u}_2, \bar{v}_2 = v_3\bar{u}_2\}\) and we obtain

\[
\begin{align*}
u_3 &= -u_3(5a_3u_3 + 4b_0v_3 + p(u_3, v_3)), \\
v_3 &= 10a_3u_3v_3 + 11b_0v_3^2 + 11b_0v_3^3 + u_3q(u_3, v_3),
\end{align*}
\]

(24)

where \(p\) and \(q\) are polynomials in \(u_3\) and \(v_3\) starting with terms of degree 2. The critical points of system (24) on \(u_3 = 0\) are \((0, 0)\), which is linearly zero, and \((0, -1)\), which is a hyperbolic saddle with linear part

\[
\begin{pmatrix}
-7b_0 & 0 \\
0 & 11b_0
\end{pmatrix}.
\]

It only remains to show that \(a_3 = 0\), and to this end we shall study the critical point at the origin. Its characteristic polynomial is given by \(W(u_1, v_1) = -15u_1v_1(a_3u_1 + b_0v_1)\). If \(a_3 \neq 0\) then the origin has three simple characteristic directions, namely \(u_1 = 0\), \(v_1 = 0\), and \(v_1 = -a_3u_1 / b_0\). Once again let us assume that \(a_3 + b_0 \neq 0\); otherwise we should do a 2-blow up and we conclude the same result. So we perform the 1-blow up
\{ \bar{u}_3 = u_3 - v_3, \bar{v}_3 = v_3; \bar{u}_4 = \bar{u}_3, \bar{v}_4 = v_4 \bar{u}_3 \}. \) It follows that the critical points of system \((\bar{u}_4, \bar{v}_4)\) on \(u_4 = 0\) are \((0, 0), (0, -1), \) and \((0, -a_3/(b_0 + a_3))\), with linear parts given by
\[
\begin{pmatrix}
-5a_3 & 0 \\
0 & 11a_3
\end{pmatrix}, \quad \begin{pmatrix}
-11b_0 & 0 \\
0 & 15b_0
\end{pmatrix}
\quad \text{and} \quad \begin{pmatrix}
-a_3 b_0 & 0 \\
0 & -15a_3 b_0 \\
0 & b_0 + a_3
\end{pmatrix},
\]
respectively. Thus, the origin and \((0, -1)\) are hyperbolic saddles while \((0, -a_3/(b_0 + a_3))\) is a hyperbolic node. Figure 9 gathers the chain of blow ups and shows that the origin of the local chart \(U_2\) has no hyperbolic sectors. Consequently \(a_3 = 0\) and this completes the proof of the proposition for the case \(c_0 \neq 0\).

In the case that \(c_0 = 0\) then \(u = 0\) is a characteristic direction of the critical point at the origin of the local chart \(U_2\). Then we must begin the blow-up study with the 1-blow up \(\{ \bar{u}_1 = u - v, \bar{v} = v; u_1 = \bar{u}, v_1 = v, \bar{u}_1 \} \) instead of the blow up \(\{ u_1 = u, v = v_1 u \} \). Up to this change the rest of the study is analogous and so we do not include it here.

Hence Proposition 5.1 shows that, under the parameter values \(a_0 = 0 \) and \(b_0 \neq 0\), the only possibility for the Hamiltonian system given by (4) to have the hyperbolic sector \(\mathcal{H}^*\) is that \(H_5 \equiv 0\).

![FIG. 9. Going back through the blow ups when \(a_1 = a_2 = 0, a_3 > 0, b_3 > 0,\) and \(c_3 > 0\).](image-url)
6. THE CASE \( a_0 = b_0 = 0 \) WITH \( c_0 \neq 0 \)

Notice that in this case \( D(x) = c_0 \) with \( c_0 \neq 0 \), and so we can turn account of the results of Section 3. The following will be proved.

**PROPOSITION 6.1.** Assume \( a_0 = b_0 = 0 \) and \( c_0 \neq 0 \). If \( \mathcal{P} \) is admissible then the first period constant is different from zero.

With this end in view we shall first focus our attention on the function \( G \), which now can be written as

\[
G(x) = \frac{P(x) - Q(x)^{1/2}}{27c_0^2},
\]

where \( P(x) = 2C(x)^3 + 27c_0^2A(x) - 9c_0C(x)B(x) \) and \( Q(x) = 4(C(x)^2 - 3c_0B(x))^3 \). The following result refers to the degree of these polynomials \( P \) and \( Q \).

**LEMMA 6.2.** If \( \mathcal{P} \) is admissible and \( \deg(P) = n \) then \( \deg(Q) = 2n \) and \( \deg(P^2 - Q) = n \).

**Proof.** To show this we shall first use the Poincaré compactification of the vector field (see the Appendix). The characteristic polynomial of the critical point at the origin of the local chart \( U_2 \) is given by \( W(u, v) = -3c_0v^3 \). Consequently the two separatrices of the hyperbolic sector \( \mathcal{H}^* \) must be tangent to \( v = 0 \). In the Poincaré disk this means that they are tangent to infinity, and accordingly \( x_R = +\infty \) or \( x_L = -\infty \). Let us assume that it occurs in the first case. Then, by applying (b) in Proposition 3.4, we can assert that

\[
G(x) = \frac{P(x) - Q(x)^{1/2}}{27c_0^2} \to h_0 \quad \text{as} \quad x \to +\infty.
\]

On account of \( h_0 < +\infty \), this implies that \( P(x) \to +\infty \) as \( x \to +\infty \) and that if \( \deg(P) = n \) then \( \deg(Q) = 2n \). On the other hand a manipulation yields

\[
G(x) = \frac{1}{27c_0^2} \frac{P(x)^2 - Q(x)}{P(x) + Q(x)^{1/2}}
\]

and hence, since \( h_0 < +\infty \), this shows that \( \deg(P^2 - Q) = n \). The result has thus been proved.

**LEMMA 6.3.** If \( \mathcal{P} \) is admissible then \( a_1 = 0 \).
Proof. We shall prove that if \( a_1 \neq 0 \) then the critical point at the origin of the local chart \( U_2 \) cannot have a hyperbolic sector, which contradicts the fact that \( \mathcal{P} \) is admissible. To this end we shall use the blow-up method described in the Appendix and the expression of the Hamiltonian system given by (4) in the local chart \( U_2 \) (that is, system (26) taking \( a_0 = b_0 = 0 \)). The characteristic polynomial of the critical point is \( W(u, v) = -3c_0 v^3 \).

Since \( u = 0 \) is not a characteristic direction we do the blow up \( \{u_1 = u, v = v/u\} \) and we obtain

\[
\dot{u}_1 = -5a_1 u_1^2 + p(u_1, v_1), \\
\dot{v}_1 = 2a_1 u_1 v_1 + q(u_1, v_1),
\]

where \( p \) and \( q \) are polynomials in \( u_1 \) and \( v_1 \) starting with terms of degree 3.

Hence the origin is linearly zero and its characteristic polynomial is given by \( W(u_1, v_1) = -7a_1 u_1^2 v_1 \). Thus, \( u_1 = 0 \) is a double characteristic direction and \( v_1 = 0 \) is a simple one. We do the 1-blow up \( \{\bar{u}_1 = u_1 - v_1, \bar{v}_1 = v_1; u_2 = \bar{u}_1, v_2 = v_2 \bar{u}_1\} \). The critical points on \( u_2 = 0 \) are \( (0, 0) \), which is a hyperbolic saddle with linear part given by

\[
\begin{pmatrix}
-5a_1 & 0 \\
0 & 7a_1
\end{pmatrix},
\]

and \( (0, -1) \), which is linearly zero. In order to study this second critical point we first move it to the origin of a new system, but let us keep the name of the variables. We then obtain

\[
\dot{u}_2 = 3c_0 u_2^2 + 2a_1 a_1 u_2 v_2 + p(u_2, v_2), \\
\dot{v}_2 = -6c_0 u_2 v_2 - 7a_1 v_2^2 + q(u_2, v_2),
\]

where \( p \) and \( q \) are polynomials in \( u_2 \) and \( v_2 \) starting with terms of degree 3. Therefore the origin is linearly zero and its characteristic polynomial is

\[ W(u_2, v_2) = 9c_0 v_2 (c_0 u_2^2 + a_1 v_2). \]

Now we perform the 1-blow up \( \{\bar{u}_2 = u_2 - v_2, \bar{v}_2 = v_2; u_3 = \bar{u}_2, \bar{v}_3 = v_3 \bar{u}_2\} \). To do this we assume that \( a_1 + c_0 \neq 0 \); otherwise we should do a 2-blow up and we conclude the same result. The critical points on \( u_3 = 0 \) are \( (0, 0), (0, -1), \) and \( (0, -c_0/(a_1 + c_0)) \), and their linear parts are given by

\[
\begin{pmatrix}
3c_0 & 0 \\
0 & 0
\end{pmatrix}, \quad \begin{pmatrix}
7a_1 & 0 \\
0 & 0
\end{pmatrix}, \quad \begin{pmatrix}
ad_1 c_0 \\
0
\end{pmatrix}, \quad \begin{pmatrix}
a_1 + c_0 \\
0
\end{pmatrix}, \quad \begin{pmatrix}
9a_1 c_0 \\
0
\end{pmatrix}, \quad \begin{pmatrix}
* \\
0
\end{pmatrix}, \quad \begin{pmatrix}
9a_1 c_0 \\
0
\end{pmatrix}.
\]
respectively. Clearly the critical points (0, 0) and (0, −1) are hyperbolic saddles while (0, −c_0/(a_1 + c_0)) is a hyperbolic node. Figure 10 gathers the chain of blow ups and shows that the origin of U_2 is a topological node. □

**Proof of Proposition 6.1.** By applying Lemma 6.3 we have that \( a_1 = 0 \) is a necessary condition for the existence of the hyperbolic sector \( \mathcal{W}^s \). Then, in our situation (that is, \( a_0 = b_0 = a_1 = 0 \)), one can check that \( \deg(P) = 6 \) and \( \deg(Q) = \deg(P^2 - Q) = 12 \). Consequently, denoting

\[
P^2(x) - Q(x) = 27 \sum_{i=0}^{12} I_i x^i,
\]

from Lemma 6.2 it follows that \( I_i = 0 \) for \( i = 7, \ldots, 12 \). Our goal is to prove that the vanishing of these coefficients implies that the first period constant of the center is different from zero. To this end we can assume without loss of generality that \( c_0 = 1 \). Indeed, this is due to the homogeneity properties of the coefficients because one can check that after the substitutions \( c_i = \bar{c}_i c_0 \), \( b_i = \bar{b}_i c_0^2 \), and \( a_i = \bar{a}_i c_0^3 \), for \( i = 1, 2, 3 \), each coefficient becomes of the form \( c_0^n \) times a polynomial which does not depend on \( c_0 \). This is also the case of the first period constant. Taking this into account a computation shows that

![Figure 10. Going back through the blow ups when \( a_1 > 0 \) and \( c_0 > 0 \).](image-url)
\[ \Gamma_{12} = -a_2^3(b_1^2 - 4a_2). \]
\[ \Gamma_{11} = -2c_1 a_2^3 b_2^2 - 2b_1^2 b_2 b_3 + 4b_1^3 a_3 - 18b_1 a_2 a_3 + 12b_2 a_2^2. \]
\[ \Gamma_{10} = -b_2^2 b_3 - 4c_1 b_2 b_3 - c_1^2 a_2^2 - 2b_1 b_2 b_3 - 18 c_1 a_2 a_3 + 4b_1 b_3 - 18b_2 b_3 a_3 + 12c_1 b_2^2 a_3. \]
\[ \Gamma_9 = -4c_1 c_2 b_2 - 2c_1^2 b_1 - 2c_1 b_2 a_3 - 9a_3 c_1 - c_1^2 a_2^2 - 2b_2 b_3 a_3 - 2b_1 c_2 b_2 + 6b_1^2 a_3 - 18b_1 b_2 b_3 + 54a_3 b_3 - 18b_2 a_2 b_3 + 12b_1 b_3 a_3. \]
\[ \Gamma_8 = -18c_1 b_2 a_3 - 18 c_1 a_2 a_3 - 2c_1^2 a_3 - 2c_2 b_1 b_3 - b_1^2 c_2 + c_1 b_3 - 9a_3 b_2 + 12c_1 b_3 b_2 + 27b_2^3 - 9a_3 b_2 + 2b_2 - 1/4a_3^2 - 9b_2 a_3. \]
\[ \Gamma_7 = -c_1 b_2 - 9c_2 a_3 - 1/2b_2 a_3 - 2c_1 c_2 a_3 - 18c_1 c_2 b_3 - 2c_1 c_2 b_1 - 2c_2 a_3. \]
\[ \Gamma_6 = -2c_2 b_1 b_2 - 9b_1 b_2 - 9c_1 a_3 - 9a_3 b_2 + 4c_1 b_3 + 6b_1 c_3 - 18c_1 b_3 c_3 + 6c_2 b_2 c_3 + 54a_3 c_3 + 12c_2 b_2 + 12c_1 b_3 + 12c_2 b_1 + 27a_3. \]

On the other hand, under our assumptions (that is, \( a_0 = b_0 = a_1 = 0 \) and \( c_0 = 1 \)), the first period constant of the center (see [6]) is given by

\[ T = \frac{9}{4} c_1^2 + \frac{9}{2} c_2 + \frac{9}{4} c_1^2 + \frac{9}{2} c_1 c_3 + \frac{45}{4} c_2 + \frac{9}{2} c_1 c_2 + \frac{45}{4} c_1^2 - \frac{3}{2} b_1 - \frac{9}{2} b_2. \]

It is to be noted that if \( a_2 = 0 \) then, on account of \( \Gamma_{11} = \Gamma_{10} = 0 \), one can easily see that \( a_3 = 0 \). In this case \( H_3 = 0 \), and therefore the Hamiltonian is not a polynomial of degree five. Consequently we can assume henceforth that \( a_2 \neq 0 \). Thus, taking account of \( \Gamma_{12} = 0 \), we get \( a_2 = b_1^2 / 4 \) and \( b_1 \neq 0 \). Then, using that \( \Gamma_{11} = \Gamma_{10} = \Gamma_9 = 0 \) and making the substitutions recursively, we obtain

\[ a_3 = \frac{b_1 (2b_2 - c_1 b_1)}{4}, \]
\[ c_2 = \frac{4b_2^2 + 4c_1^2 b_1^2 + b_1^3 - 8b_2 c_1 b_1 + 8b_1 b_3}{4b_1^2}. \]
and
\[
c_3 = \frac{1}{4b_1^2} (8c_1b_1^2b_3 - 8b_1b_2b_3 + 20c_1b_2^2b_1 - 8b_2^3 - 16c_2^2b_3^2 - c_1b_1^2 + b_2b_1^3 + 4c_1b_1^3).
\]

The two remaining coefficients become more simplified if we introduce a new variable \( \eta \) performing the substitution \( b_2 = c_1b_1 - \eta \). Indeed,
\[
\begin{align*}
\Gamma_8 &= -\frac{1}{b_1^2} \left( b_1^4/4 + b_1^4/64 + 64 + 64 - b_1^4b_3/4 + b_1^2b_1^3 - b_1^2b_1^2/4 
+ c_1b_1^2\eta^2 - 4c_1b_1\eta^3 + c_1b_1\eta^3/4 + 4b_1b_3\eta^2 - 2c_1b_1^2\eta \right)
\end{align*}
\]
and one can check that
\[
\begin{align*}
\Gamma_7 &= \frac{4\eta^3 - 12b_1^2\eta - 4c_1b_1\eta^2 + 8b_1b_2\eta - b_1^2\eta + 4c_1b_1^3}{8b_1} + \frac{12\eta - 4c_1b_1}{b_1^2} \Gamma_8, 
\end{align*}
\]

(25)

Until now it has been used that \( \Gamma_{12} = \Gamma_{11} = \Gamma_{10} = \Gamma_9 = 0 \). Next we wish to show that \( \Gamma_8 = \Gamma_7 = 0 \) yields \( T \neq 0 \). To this end let us consider first the case \( \eta \neq 0 \). In this case from expression (25) we obtain
\[
b_3 = \frac{12b_1^2\eta + 4c_1b_1\eta^2 + b_1^3\eta - 4\eta^3 - 4c_1b_1^3}{8b_1\eta},
\]
and then a computation shows that
\[
\Gamma_8 = \frac{b_1^2 + \eta^2}{4\eta^2b_1^2} (6b_1^2\eta c_1 - c_1^2b_1^4 - b_1^2\eta^2 - 9b_1^2\eta^2 - c_1^2b_1^2\eta^2 + 6c_1b_1\eta^3 - 9\eta^4)
\]
and the first period constant becomes
\[
T = \frac{3}{4b_1^3\eta^2} \left( b_1^4\eta + 135b_1^4\eta^4 + 3b_1^4\eta^4 + 60b_1^4\eta^2 + 15\eta^4 + 6c_1b_1^2\eta^2 + 3b_1^4c_1^2\eta^4 \right.
- 48b_1^2c_1\eta^3 - 24b_1^2c_1\eta^5 + 3b_1^4c_1^2 + 90\eta^6b_1^2).
\]

Let us denote the second factors of \( \Gamma_8 \) and \( T \) by \( \tilde{\Gamma}_8 \) and \( \tilde{T} \), respectively. Clearly \( \tilde{\Gamma}_8 = 0 \) implies \( \tilde{T} = 0 \). We claim that then \( \tilde{T} \neq 0 \), and note that Proposition 6.1 will follow for the case \( \eta \neq 0 \) once we prove this. Assume first that \( c_1 \neq 0 \). Then \( \tilde{\Gamma}_8 \) is a quadratic polynomial with respect to \( c_1 \) (the coefficient of \( c_1^2 \) is \( b_1^2(b_1^2 + \eta^2) \neq 0 \) and one can check that its discriminant is equal to \( -b_1^2\eta^2(b_1^2 + \eta^2) \). Consequently, since all the parameters are real,
$b_1 < 0$ is a necessary condition for $\bar{F}_8 = 0$. On the other hand a computation shows that the resultant between $\bar{F}_s$ and $\bar{T}$ with respect to $c_1$ is equal to

\[
\eta^4 b_1^2 (b_1^2 + \eta^2)^2 \left( 225\eta^{12} + 1350b_1^2 \eta^{10} + 3375b_1^4 \eta^8 + (4500b_1^6 - 24b_1^6) \eta^6 \\
+ (3375b_1^8 - 72b_1^8) \eta^4 + (1350b_1^{10} - 72b_1^{10}) \eta^2 + 4b_1^{14} + 225i^2 - 24b_1^{14} \right),
\]

which is strictly positive because $\eta \neq 0$ and $b_1 < 0$. This proves that $\bar{F}_8 = 0$ implies $\bar{T} \neq 0$. Assume now that $c_1 = 0$. In this case it follows that $\bar{F}_s = -\eta^2 (b_1^2 + 9b_1^2 + 9\eta^2)$ and

\[
\bar{T} = \eta^2 b_1^2 + 60b_1^2 + (135b_1^4 + 3b_1^4) \eta^2 + 90b_1^2 \eta^4 + 15\eta^6).
\]

Since $\eta \neq 0$, from $\bar{F}_s = 0$ we obtain $\eta^2 = -b_1^2 / 9 - b_1^2$, and note that in particular this shows $b_1 < 0$. Then the substitution of $\eta^2$ in $\bar{T}$ and a computation gives

\[
\bar{T} = \frac{\eta^2 b_1^2}{243} (486 - 54b_1 + 5b_1^2),
\]

which is strictly positive because $b_1 < 0$ and $\eta \neq 0$. The claim has thus been proved.

It only remains to study the case $\eta = 0$. In this case, since $I_8 = I_s = 0$, from (25) we obtain that $c_1 = 0$, and then one can show that

\[
I_s = \frac{-1}{64} (16b_1^2 + (b_1^2 - 8b_1)^2)
\]

and

\[
T = \frac{3}{64b_1^2} (3b_1^4 - 48b_1^2 b_3 + 192b_3^2 - 8b_1^3 + 192b_1 b_3 + 240b_1^2).
\]

At this point note that $b_1 < 0$ is a necessary condition in order that $I_s = 0$. Finally, the resultant between the second factors of $I_s$ and $T$ with respect to $b_3$ is

\[
32b_1^4 (4b_1^2 - 4b_1 + 225),
\]

which, on account of $b_1 < 0$, is different from zero. Accordingly $I_s = 0$ implies $T \neq 0$, and this completes the proof of the result.  \[\square\]
APPENDIX: THE POINCARÉ COMPACTIFICATION

In this Appendix we describe the key tools of the blow-up method to study a nonelementary critical point and the Poincaré compactification of a polynomial vector field. This has been used through the paper to study the boundary of an admissible period annulus “at infinity.”

Let $X$ be a planar polynomial vector field of degree $n$. The Poincaré compactification of $X$ is an analytic vector field on the two-sphere defined as follows (see [18] for details). Let $S^2 = \{ y \in \mathbb{R}^3 : \| y \| = 1 \}$ be the two-sphere in $\mathbb{R}^3$ and consider the hyperplane $H = \{ y \in \mathbb{R}^3 : y_3 = 1 \}$. For each $p = (y_1, y_2, 1) \in H$, the straight line that joins $p$ with $(0, 0, 0)$ has an intersection point with $H_+ = \{ y \in S^2 : y_3 > 0 \}$, say $f_+(p)$, and another one with $H_- = \{ y \in S^2 : y_3 < 0 \}$, say $f_-(p)$. This defines two mappings $f_+: H \rightarrow H_+$ and $f_-: H \rightarrow H_-$ which one can easily see that are diffeomorphisms. Note in addition that $S^1 = \{ y \in S^2 : y_3 = 0 \}$ is identified with the infinity of $\mathbb{R}^2 = H_+$. Thus, $X$ induces a vector field $X^2$ on $H_+ \cup H_-$ defined by

$$X(y) = \begin{cases} (df_+(x))_* (X(x)) & \text{if } y = f_+(x), \\ (df_-)_* (X(x)) & \text{if } y = f_-(x). \end{cases}$$

We consider the vector field $y^{n-1} X(y)$, which is an analytic vector field defined on the whole $S^2$. This vector field is called the Poincaré compactification of $X$ and it is denoted by $p(X)$. The Poincaré disk is the projection of $H_+ \cup S^1$ on $\{ y \in \mathbb{R}^3 : y_3 = 0 \}$. A critical point of $p(X)$ in $S^1$ is called an infinite critical point. To study the analytic expression of $p(X)$ we consider $S^2$ a differentiable manifold. We choose six coordinate neighbourhoods given by $U_i = \{ y \in S^2 : y_i > 0 \}$ and $V_i = \{ y \in S^2 : y_i < 0 \}$, $i = 1, 2, 3$. The corresponding coordinate maps $F_i: U_i \rightarrow \mathbb{R}^2$ and $G_i: U_i \rightarrow \mathbb{R}^2$ are the inverses of convenient central projections.

In order to study the shape of an admissible period annulus we consider the local chart $U_i$. In this case $F_i(y) = (y_1/y_2, y_3/y_2)$, and we define $u = y_1/y_2$ and $v = y_3/y_2$. The expression of the Hamiltonian system given by (4) in this local chart is

$$\begin{align*}
u &= -5a_0 u^2 - 4b_0 uv - 3c_0 v^2 - 5a_1 u^3 - 4b_1 u^2 v - 3c_1 u v^2 - 5u^4 - 4b_2 u^3 v - 3c_2 u^2 v^2 - u v^3, \\
v^2 &= -v(2a_0 u + b_0 v + 3a_1 u^2 + 2b_1 u v + c_1 v^2 + 4a_2 u^3 + 3b_2 u^2 v + 2c_2 u v^2 + 5a_3 u^4) + 4b_3 u^3 v + 3c_3 u^2 v^2 + uv^3. \end{align*}$$

(26)

Note that the origin is a critical point with linear part identically zero, i.e., it is a linearly zero critical point. This is precisely the infinite critical point.
given by the direction $\theta = \pi/2$ in the Poincaré disk. Note also that, on account of Remark 2.5, the parameters $a_0$, $b_0$, and $c_0$ cannot be simultaneously zero, and accordingly system (26) starts with terms of degree two. Thus the origin is a linearly zero critical point of degree two. Concerning these critical points Ref. [12] should be mentioned because in that paper the authors give a topological classification of their local phase portraits.

To study linearly zero critical points of degree two we use the directional blow-up method that can be described as follows (see [8, 12] for details). Consider the polynomial differential system

$$
\begin{align*}
\dot{u} &= \sum_{i=2}^{n} P_i(u, v), \\
\dot{v} &= \sum_{i=2}^{n} Q_i(u, v),
\end{align*}
$$

(27)

where, for $i = 2, 3, ..., n$, $P_i$ and $Q_i$ are homogeneous polynomials of degree $i$. The characteristic polynomial of the critical point at the origin is the homogeneous polynomial $W(u, v) = vP_2(u, v) - uQ_2(u, v)$. It is well known that if $W$ is not identically zero then every orbit that tends to the origin in forward or backward time must be tangent to a straight line $au + bv = 0$, where $au + bv = 0$ is a real solution of $W(u, v) = 0$. This is precisely the case of the characteristic polynomial of the critical point at the origin of system (26), which is given by

$$W(u, v) = -3v(a_0 u^2 + b_0 u v + c_0 v^2),$$

because in view of Remark 2.5 we can assume that $a_0$, $b_0$ and $c_0$ are not simultaneously zero. The special directions determined by the real solutions of $W(u, v) = 0$ are called characteristic directions. A natural way of finding them is as follows. We divide $W$ by $u$ and, due to its homogeneity, we obtain a new polynomial $\bar{W}(v) = vP_2(1, v) - uQ_2(1, v)$. It is said then that $v = mu$ is a simple (respectively, double or triple) characteristic direction in the case that $v = m$ is a simple (respectively, double or triple) real root of $\bar{W}(v) = 0$. Of course the case in which $u = 0$ is a characteristic direction must be considered apart.

The directional blow up is the change of variables given by $\{u_1 = u, v = v/u\}$. Hence the blow up sends the critical point $(0, 0)$ of the system (27) to the axis $u_1 = 0$ of system $(u_1, v_1)$. Indeed, after performing the blow up the axis $u_1 = 0$ is full of critical points and we must change the time $t$ by a new time $\tau$ such that $u_1 \, dt = d\tau$. Due to this change we must take into account the orientation of the trajectories when we go back
through the blow up(s). One can check that system (27) in the new variables becomes

\[\begin{align*}
\dot{u}_1 &= u_1 p(u_1, v_1), \\
\dot{v}_1 &= \tilde{W}(v_1) + u_1 q(u_1, v_1),
\end{align*}\]  

(28)

where \(p\) and \(q\) are polynomials in \(u_1\) and \(v_1\). It is now clear that if there is a trajectory tending to the origin of system (27) with slope \(m\) (that is, \(\tilde{W}(m) = 0\)) then we see system (28) has a critical point at \((0, m)\) (with a simpler local flow than before the blow up). Perhaps we need more than one blow up to study completely the local phase portrait of a linearly zero critical point. However, it can be proved (see [8]) that by means of a finite number of blow ups a linearly zero critical point can be reduced to critical points with linear part not identically zero.

Remark A.1. The blow-up method introduced above has no meaning on \(u = 0\), and it cannot be used to study the (possible) characteristic direction \(u = 0\). In order to study all the characteristic directions simultaneously, before performing the blow up we first do the “rotation” \(\{\bar{u} = u - kv, \bar{v} = v\}\), where \(k \in \mathbb{N}\) is chosen so that \(u = kv\) is not a characteristic direction. In short we describe the \(k\)-blow up as the change of variables \(\{\bar{u} = u - kv, \bar{v} = v; u_1 = \bar{u}, v_1 = v_1 \bar{u}\}\).

Following Definition 2.3 and Remark 2.4, we conclude the Appendix showing some necessary conditions in order that the critical point at the origin of the local chart \(U_2\) has the hyperbolic sector \(\mathcal{H}^*\).

**Lemma 7.2.** The two separatrices of the hyperbolic sector \(\mathcal{H}^*\) are tangent to the same characteristic direction. Moreover, this characteristic direction cannot be simple.

**Proof.** The first part of the result follows from the fact that if its two separatrices were tangent to different characteristic directions then the hyperbolic sector \(\mathcal{H}^*\) would contain straight lines, which contradicts Lemma 2.2. We shall prove the second part by contradiction. For the sake of simplicity let us assume that \(u = 0\) is not a characteristic direction of the critical point at the origin of system (26). Otherwise we should do a \(k\)-blow up but we arrive to the same conclusion. Thus, we do the blow up \(\{u_1 = u, v = v_1 u\}\). Then, if the separatrices of \(\mathcal{H}^*\) are tangent to \(v = mu\), it follows that \((0, m)\) is a critical point of system (28) with linear part

\[
\begin{pmatrix}
p(0, m) & 0 \\
* & \tilde{W}'(m)
\end{pmatrix}.
\]
If the characteristic direction is simple then \( \bar{W}(m) \neq 0 \), and consequently \((0, m)\) is either a saddle, a node, or a saddle-node. Note in addition that \( u_t = 0 \) is invariant for the flow associated to system (28). Therefore, going back through the blow up, one can easily see that either the simple characteristic direction is parabolic or there is a unique orbit tending to the origin of system (26) tangent to \( v = mu \). Hence we cannot have two hyperbolic separatrices tending to the origin and being tangent to a simple characteristic direction.

REFERENCES