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Bounds for Betti Numbers

Tim Römer

FB6 Mathematik und Informatik, Universität Essen, 45117 Essen, Germany

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In this paper we prove parts of a conjecture of Herzog giving lower bounds on the rank of the free modules appearing in the linear strand of a graded kth syzygy module over the polynomial ring. If in addition the module is $\mathbb{Z}^n$-graded we show that the conjecture holds in full generality. Furthermore, we give lower and upper bounds for the graded Betti numbers of graded ideals with a linear resolution and a fixed number of generators.

INTRODUCTION

Let $S = K[x_1, \ldots, x_n]$ be the polynomial ring over a field K equipped with the standard grading by setting $\deg(x_i) = 1$, and let M be a finitely generated graded S-module. We denote by $\beta_{i,i+j}(M) = \dim_K \operatorname{Tor}_i(M, K)_{i+j}$ the graded Betti numbers of M.

Assume that the initial degree of M is d; i.e., we have $M_i = 0$ for $i < d$ and $M_d \neq 0$. We are interested in the numbers $\beta_i^{\text{lin}}(M) = \beta_{i,i+d}(M)$ for $i \geq 0$. These numbers determine the rank of the free modules appearing in the linear strand of the minimal graded free resolution of M. Let $p = \max\{i : \beta_i^{\text{lin}}(M) \neq 0\}$ be the length of the linear strand. In [13] Herzog conjectured the following:

Conjecture. Let M be a kth syzygy module whose linear strand has length p. Then

$$\beta_i^{\text{lin}}(M) \geq \binom{p+k}{i+k}$$

for $i = 0, \ldots, p$.

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This conjecture is motivated by a result of Green [12] (see also Eisenbud and Koh [8]) that contains the case $i = 0$, $k = 1$. For $k = 0$ these lower bounds were shown by Herzog [13], and Reiner and Welker [16] proved them for $k = 1$ for the case where M is a monomial ideal.

In this paper we prove the conjecture for $k = 1$. For $k > 1$ we get the weaker result:

If $\beta_p^{\mathrm{lin}}(M) \neq 0$ for $p > 0$ and M is a kth syzygy module, then $\beta_{p-1}^{\mathrm{lin}}(M) \geq p + k$.

We also show that the conjecture holds in full generality for finitely generated $\mathbb{Z}^n$-graded S-modules. The first three sections of this paper are concerned with the question above.

In recent years many authors (see, for example, [4, 14, 15]) were interested in the following problem: Fix a possible Hilbert function H for a graded ideal. Let $\mathscr{B}(H)$ be the set of Betti sequences $\{\beta_{i,j}(I)\}$, where $I \subset S$ is a graded ideal with Hilbert function H. On $\mathscr{B}(H)$ we consider a partial order: We set $\{\beta_{i,j}(I)\} \geq \{\beta_{i,j}(J)\}$ if $\beta_{i,j}(I) \geq \beta_{i,j}(J)$ for all $i, j \in \mathbb{N}$. It is known that $\mathscr{B}(H)$ has a unique maximal element given by the Betti sequence of the lex-segment ideal in the family of considered ideals. In general there is more than one minimal element (see [6]).

In Section 4 we study a related problem. We fix an integer $d \geq 0$ and $0 \leq k \leq \binom{n+d-1}{d}$. Let $\mathscr{B}(d,k)$ be the set of Betti sequences $\{\beta_{i,j}(I)\}$, where $I \subset S$ is a graded ideal with d-linear resolution and $\beta_{0,d}(I) = k$. We show that, independent of the characteristic of the base field, there is a unique minimal and a unique maximal element in $\mathscr{B}(d,k)$.

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1. PRELIMINARIES ON KOSZUL COMPLEXES

Let K be a field, let V be an n-dimensional K-vector space with basis $\mathbf{x} = x_1, \dots, x_n$, and let $S = K[V]$ be the symmetric algebra over V equipped with the standard grading by setting $\deg(x_i) = 1$. Furthermore, let $\mathfrak{m} = (x_1, \dots, x_n)$ be the graded maximal ideal of S and let $0 \neq M$ be a finitely generated graded S-module which is generated in nonnegative degrees, i.e. $M_i = 0$ for $i < 0$.

Consider a graded free S-module L of rank j which is generated in degree 1 and let $\bigwedge L$ be the exterior algebra over L. Then $\bigwedge L$ inherits the structure of a bigraded S-module. If $z \in \bigwedge^i L$ and z has S-degree k, then we give z the bidegree (i, k). We call i the homological degree (hdeg for short) and k the internal degree (deg for short) of z.

We consider maps $\mu \in L^* = \operatorname{Hom}_S(L, S)$. Note that L^* is again a graded free S-module generated in degree -1. It is well known (see [5]) that μ defines a graded S-homomorphism $\partial_\mu\colon \bigwedge L \to \bigwedge L$ of (homological) degree -1.

Recall that if we fix a basis $e_1, \dots, e_j$ of L, then $\bigwedge^i L$ is the graded free S-module with basis consisting of all monomials $e_J = e_{j_1} \wedge \cdots \wedge e_{j_i}$ with $J = \{j_1 < \cdots < j_i\} \subseteq [j] = \{1, \dots, j\}$. One has

$$\partial_\mu(e_{j_1} \wedge \cdots \wedge e_{j_i}) = \sum_{k=1}^{i} (-1)^{\alpha(k,J)} \mu(e_{j_k}) e_{j_1} \wedge \cdots \hat{e}_{j_k} \cdots \wedge e_{j_i},$$

where for $F, G \subseteq [n]$ we set $\alpha(F,G) = |\{(f,g) : f > g, f \in F, g \in G\}|$ and where $\hat{e}_{j_k}$ indicates that e_{j_k} is to be omitted from the exterior product. Denote by $e_1^*, \dots, e_j^*$ the basis of L^* with $e_i^*(e_i) = 1$ and $e_i^*(e_k) = 0$ for $k \neq i$. To simplify notations we set $\partial_i = \partial_{e_i^*}$. Then $\partial_\mu = \sum_{k=1}^{j} \partial_\mu(e_k)\partial_k$, and we have the following:

LEMMA 1.1. *Let $z, \tilde{z} \in \bigwedge L$ be bihomogeneous elements, $f \in S$ and $\mu, \nu \in L^*$. Then*

(i) $f\partial_\mu = \partial_{f_\mu}$,

(ii) $\partial_\mu + \partial_\nu = \partial_{\mu+\nu}$,

(iii) $\partial_\mu \circ \partial_\mu = 0$,

(iv) $\partial_\mu \circ \partial_\nu = -\partial_\nu \circ \partial_\mu$,

(v) $\partial_\mu(z \wedge \tilde{z}) = \partial_\mu(z) \wedge \tilde{z} + (-1)^{\operatorname{hdeg}(z)} z \wedge \partial_\mu(\tilde{z})$.

Proof. The proof consists of straightforward calculations (most of them are done in [5]). ∎

We fix a graded free S-module L of rank n for the rest of the paper. Let $\mathbf{e} = e_1, \dots, e_n$ be a basis of L with $\deg(e_i) = 1$ for $i \in [n]$, and $\mu \in L^*$ with $\mu(e_i) = x_i$ for $i \in [n]$. For $j = 1, \dots, n$ let $L(j)$ be the graded free submodule of L generated by $e_1, \dots, e_j$. Then $(K(j, M), \partial)$ is the Koszul complex of $x_1, \dots, x_j$ with values in M where $K(j, M) = \bigwedge L(j) \otimes_S M$ and ∂ is the restriction of $\partial_\mu \otimes_S \operatorname{id}_M$ to $\bigwedge L(j) \otimes_S M$. We denote by $H(j, M)$ the homology of the complex $K(j, M)$, and the homology class of a cycle $z \in K(j, M)$ will be denoted by $[z]$. If it is clear from the context, we write $K(j)$ instead of $K(j, M)$ and $H(j)$ instead of $H(j, M)$.

Notice that $K_i(j)_{i+k} = 0$ for $k < 0$ and that $H_i(n) \cong \operatorname{Tor}_i(K, M)$ are isomorphic as graded K-vector spaces. One has the following exact sequence (see [5]):

$$\cdots \to H_{i+1}(j) \to H_i(j-1)(-1) \to H_i(j-1) \to H_i(j) \to \cdots.$$

The following observation is crucial for the rest of the paper. For a homogeneous element $z \in K_i(j)$ we can write z uniquely as $z = e_k \wedge \partial_k(z) + r_z$, and e_k divides none of the monomials of r_z.

LEMMA 1.2. *Let $z \in K_i(j)$ be a homogeneous cycle of bidegree (i, l). Then $\partial_k(z)$ is for all $k \in [n]$ a homogeneous cycle of bidegree $(i-1, l-1)$.*

Proof. The proof follows from 1.1. ∎

In the sequel we need the following:

LEMMA 1.3. *Let $p \in \{0, \ldots, n\}$ and $t \in \mathbb{N}$. Suppose that $H_p(j)_{p+l} = 0$ for $l = -1, \ldots, t-1$. Then*

(i) $H_p(j-1)_{p+l} = 0$ *for* $l = -1, \ldots, t-1$,
(ii) $H_p(j-1)_{p+t}$ *is isomorphic to a submodule of* $H_p(j)_{p+t}$,
(iii) $H_i(j)_{i+l} = 0$ *for* $l = -1, \ldots, t-1$ *and* $i = p, \ldots, j$.

Proof. We prove (i) by induction on $l \in \{-1, \ldots, t-1\}$. If $l = -1$ there is nothing to show because $H_p(j-1)_{p+l} = 0$ for $l < 0$. Now let $l > -1$ and consider the exact sequence

$$\cdots \to H_p(j-1)_{p+l-1} \to H_p(j-1)_{p+l} \to H_p(j)_{p+l} \to \cdots.$$

By the induction hypothesis $H_p(j-1)_{p+l-1} = 0$, and by the assumption $H_p(j)_{p+l} = 0$ we get that $H_p(j-1)_{p+l} = 0$.

For $l = t$ the exact sequence of the Koszul homology together with (i) yields

$$0 \to H_p(j-1)_{p+t} \to H_p(j)_{p+t} \to \cdots,$$

which proves (ii).

We show (iii) by induction on $j \in [n]$. The case $j = 1$ is trivial, and for $j > 1$ and $i = p$ the assertion is true by assumption. Now let $j > 1$, $i > p$ and consider

$$\cdots \to H_i(j-1)_{i+l} \to H_i(j)_{i+l} \to H_{i-1}(j-1)_{i-1+l} \to \cdots.$$

By (i) and the induction on j we get that $H_i(j-1)_{i+l} = H_{i-1}(j-1)_{i-1+l} = 0$. Hence $H_i(j)_{i+l} = 0$. ∎

2. LOWER BOUNDS FOR BETTI NUMBERS OF GRADED S-MODULES

In this section M is always a finitely generated graded S-module which is generated in degrees ≥ 0. For $0 \neq z \in K_i(j)$ we write

$$z = m_J e_J + \sum_{I \subseteq [n],\, I \neq J} m_I e_I$$

with coefficients in M, and where e_J is the lexicographic largest monomial of all e_I with $m_I \neq 0$. Recall that for $I, J \subseteq [n]$, $I = \{i_1 < \cdots < i_t\}$, $J = \{j_1 < \cdots < j_{t'}\}$ $e_I <_{lex} e_J$ if either $t < t'$ or $t = t'$, and there exists a number p with $i_l = j_l$ for $l < p$ and $i_p > j_p$. We call $\operatorname{in}(z) = m_J e_J$ the initial term of z. Furthermore, for $I = \{i_1 < \cdots < i_t\} \subseteq [n]$ we write $\partial_I = \partial_{i_1} \circ \cdots \circ \partial_{i_t}$.

LEMMA 2.1. *Let $p \in \{0, \dots, j\}$, $r \in \{0, \dots, p\}$, and $0 \neq z \in K_p(j)$ be homogeneous with $\operatorname{in}(z) = m_J e_J$. Then for all $I \subseteq J$ with $|I| = r$ the elements $\partial_I(z)$ are K-linearly independent in $K_{p-r}(j)$. In particular, if z is a cycle, then $\{\partial_I(z) : I \subseteq J, |I| = r\}$ is a set of K-linearly independent cycles.*

Proof. This follows from the fact that $\operatorname{in}(\partial_I(z)) = m_J e_{J-I}$. Induction on $r \in \{0, \dots, p\}$ proves that all $\partial_I(z)$ are cycles if z is one. ∎

LEMMA 2.2. *Let $p \in [j]$, $t \in \mathbb{N}$, and $z \in K_p(j)_{p+t}$. Assume that $H_{p-1}(j)_{p-1+l} = 0$ for $l = -1, \dots, t-1$.*

(i) *If $p < j$ and $\partial_j(z) = \partial(y)$ for some y, then there exists $\tilde{z}$ such that $\tilde{z} = z + \partial(r)$ and $\partial_j(\tilde{z}) = 0$. In particular, $\tilde{z} \in K_p(j-1)$, and if z is a cycle, then $[z] = [\tilde{z}]$.*

(ii) *If $p = j$ and $\partial_j(z) = \partial(y)$ for some y, then $z = 0$. In particular, if $z \neq 0$ is a cycle, then we always have $0 \neq [\partial_j(z)] \in H_{p-1}(j)_{p-1+t}$.*

Proof. We proceed by induction on $t \in \mathbb{N}$ to prove (i). If $t = 0$, then $y \in K_p(j)_{p+t-1} = 0$, and so $\partial_j(z) = 0$. Thus we choose $\tilde{z} = z$.

Let $t > 0$ and assume that $\partial_j(z) = \partial(y)$. We see that $\partial_j(y)$ is a cycle because

$$0 = \partial_j\big(\partial_j(z)\big) = \partial_j(\partial(y)) = -\partial\big(\partial_j(y)\big).$$

But $\partial_j(y) \in K_{p-1}(j)_{p-1+t-1}$. Since $H_{p-1}(j)_{p-1+t-1} = 0$, it follows that $\partial_j(y) = \partial(y')$ is a boundary for some element y'. By the induction hypothesis we get $\tilde{y} = y + \partial(r')$ such that $\partial_j(\tilde{y}) = 0$. Note that $\partial(\tilde{y}) = \partial(y) = \partial_j(z)$. We define

$$\tilde{z} = z + \partial\big(e_j \wedge \tilde{y}\big) = z + x_j \tilde{y} - e_j \wedge \partial_j(z).$$

Then

$$\begin{aligned}\partial_j(\tilde{z}) &= \partial_j(z) + x_j \partial_j(\tilde{y}) - \partial_j(e_j) \wedge \partial_j(z) + e_j \wedge \partial_j \circ \partial_j(z) \\ &= \partial_j(z) - \partial_j(z) = 0,\end{aligned}$$

and this proves (i).

If $p = j$, we see that $z = me_{[j]}$ for some $m \in M$, and therefore $\partial_j(z) \neq 0$ if and only if $z \neq 0$.

We prove (ii) by induction on $t \in \mathbb{N}$. For $t = 0$ there is nothing to show. Let $t > 0$ and assume $\partial_j(z) = \partial(y)$. By the same argument as in the proof of (i) we get $\partial_j(y) = \partial(y')$ for some y'. The induction hypothesis implies $y = 0$, and then $z = 0$. ∎

LEMMA 2.3. *Let $p \in \{0, \dots, j\}$, $t \in \mathbb{N}$, and $0 \neq z \in K_p(j)_{p+t}$. Assume that $H_p(j)_{p+l} = 0$ for $l = -1, \dots, t-1$ and let $q \in \{0, \dots, j\}$. If $z \in K_p(j - q)_{p+t} \subseteq K_p(j)_{p+t}$ and $\partial(y) = z$ in $K(j)$ for some element y, then there exists an element $\tilde{y} \in K_{p+1}(j-q)_{p+1+t-1}$ such that $\partial(\tilde{y}) = z$ in $K(j-q)$.*

Proof. We prove the assertion by induction on j for all $q \in \{0, \dots, j\}$. For $j = 0$ and $j > 0$, $q = 0$, there is nothing to show. Let $j \geq q > 0$. Write $y = e_j \wedge \partial_j(y) + r_y$. Since $0 = \partial_j(z)$ and

$$z = \partial(y) = \partial\big(e_j \wedge \partial_j(y) + r_y\big) = x_j \partial_j(y) - e_j \wedge \partial\big(\partial_j(y)\big) + \partial(r_y),$$

we see that $\partial_j(y)$ is a cycle and therefore a boundary by the assumption that $H_p(j)_{p+t-1} = 0$. By Lemma 2.2 we may assume that $y \in K(j-1)$. By the induction hypothesis we find the desired $\tilde{y}$ in $K(j-q) = K(j-1-(q-1))$. ∎

LEMMA 2.4. *Let $t \in \mathbb{N}$. If $\beta_{n-1,n-1+l}(M) = 0$ for $l = -1, \dots, t-1$ and $\beta_{n,n+t}(M) \neq 0$, then there exists a basis $\mathbf{e}$ of L and a cycle $z \in K_n(n)_{n+t}$ such that*

(i) *$[z] \in H_n(n)_{n+t}$ is not zero,*

(ii) *$[\partial_i(z)] \in H_{n-1}(n)_{n-1+t}$ are K-linearly independent for $i = 1, \dots, n$.*

In particular, $\beta_{n-1,n-1+t}(M) \geq n$.

Proof. Let $\mathbf{e}$ be an arbitrary basis of L. Since $\beta_{n,n+t}(M) \neq 0$ there exists a cycle $z \in K_n(n)_{n+t}$ with $0 \neq [z] \in H_n(n)_{n+t}$. Furthermore, $H_{n-1}(n)_{n-1+l} = 0$ for $l = -1, \dots, t-1$. In this situation we have $z = me_{[n]}$ for some socle element m of M, and we want to show that every equation

$$0 = \sum_{i=1}^{n} \mu_i [\partial_i(z)] = \left[\sum_{i=1}^{n} \mu_i \partial_i(z)\right] \qquad \text{with } \mu_i \in K$$

implies $\mu_i = 0$ for all $i \in [n]$. Assume there is such an equation where not all μ_i are zero. After a base change we may assume that $\sum_{i=1}^{n} \mu_i \partial_i = \partial_n$. We get

$$0 = [\partial_n(z)],$$

contradicting Lemma 2.2(ii). ∎

THEOREM 2.5. *Let $t \in \mathbb{N}$ and $p \in [n]$. If $\beta_{p-1,p-1+l}(M) = 0$ for $l = -1,\dots,t-1$ and $\beta_{p,p+t}(M) \neq 0$, then there exists a basis $\mathbf{e}$ of L and a cycle $z \in K_p(n)_{p+t}$ such that*

(i) *$[z] \in H_p(n)_{p+t}$ is not zero,*

(ii) *$[\partial_i(z)] \in H_{p-1}(n)_{p-1+t}$ are K-linearly independent for $i = 1,\dots,p$.*

In particular, $\beta_{p-1,p-1+t}(M) \geq p$.

Proof. We have $H_p(n)_{p+t} \neq 0$ because $\beta_{p,p+t}(M) \neq 0$. Choose $0 \neq h \in H_p(n)_{p+t}$. We prove by induction on n that we can find a basis $\mathbf{e}$ of $L(n)$ and a cycle $z \in K_p(n)_{p+t}$ representing h such that every equation

$$0 = \sum_{i=1}^{p} \mu_i [\partial_i(z)] = \left[\sum_{i=1}^{p} \mu_i \partial_i(z)\right] \qquad \text{with } \mu_i \in K$$

implies $\mu_i = 0$ for all i. The cases $n = 1$ and $n > 1$, $p = n$ were shown in Lemma 2.4.

Let $n > 1$ and $p < n$. Assume that there is a basis $\mathbf{e}$ and such an equation for a cycle z with $[z] = h$ where not all μ_i are zero. After a base change of $L(n)$ we may assume that $\sum_{i=1}^{p} \mu_i \partial_i = \partial_n$. Then $0 = [\partial_n(z)]$, and therefore $\partial_n(z) = \partial(y)$ for some element y. By Lemma 2.2 we can find an element $\tilde{y}$ such that $[\tilde{y}] = [z]$ and $\tilde{y} \in K_p(n-1)_{p+t}$. Now Lemma 1.3 guarantees that we can apply our induction hypothesis to $\tilde{y}$, and we find a base change $\mathbf{l} = l_1,\dots,l_{n-1}$ of $e_1,\dots,e_{n-1}$, $[\tilde{z}] = [\tilde{y}]$ in $H_p(n-1)_{p+t}$ (with respect to the new basis) such that $[\partial_i(\tilde{z})] \in H_{p-1}(n-1)_{p-1+t}$ are K-linearly independent for $i = 1,\dots,p$. By Lemma 1.3 we have $H_i(n-1)_{i+t} \subseteq H_i(n)_{i+t}$ for $i = p-1, p$. Then $\tilde{z}$ is the desired cycle because $[\tilde{z}] = [z]$ in $H_p(n)_{p+t}$. ∎

The Castelnuovo–Mumford regularity for a finitely generated graded S-module $0 \neq M$ is defined as $\operatorname{reg}(M) = \max\{j \in \mathbb{Z} : \beta_{i,i+j}(M) \neq 0 \text{ for some } i \in \mathbb{N}\}$. For $k \in \{0,\dots,n\}$ we define $d_k(M) = \min(\{j \in \mathbb{Z} : \beta_{k,k+j}(M) \neq 0\} \cup \{\operatorname{reg}(M)\})$. We are interested in the numbers $\beta_i^{k,\mathrm{lin}}(M) = \beta_{i,i+d_k(M)}(M)$ for $i \geq k$. Note that $\beta_i^{0,\mathrm{lin}}(M) = \beta_i^{\mathrm{lin}}(M)$. If $0 \neq \Omega_k(M)$ is the kth syzygy module in the minimal graded free resolution of M (see [7] for details), then we always have $\beta_{i,i+j}(M) = \beta_{i-k,i-k+j+k}(\Omega_k(M))$ for $i \geq k$. Therefore $\beta_i^{k,\mathrm{lin}}(M) = \beta_{i-k}^{\mathrm{lin}}(\Omega_k(M))$ for these i. Observe that $d_0(\Omega_k(M)) = d_k(M) + k$.

COROLLARY 2.6. *Let $k \in \{0,\dots,n\}$. If $\beta_p^{k,\mathrm{lin}}(M) \neq 0$ for some $p > k$, then*

$$\beta_{p-1}^{k,\mathrm{lin}}(M) \geq p.$$

For the numbers $\beta_i^{\text{lin}}(M)$ and $\beta_i^{1,\text{lin}}(M)$ we get more precise results. The next result was first discovered in [13].

THEOREM 2.7. *Let $p \in \{0, \dots, n\}$. If $\beta_p^{\text{lin}}(M) \neq 0$, then*

$$\beta_i^{\text{lin}}(M) \geq \binom{p}{i}$$

for $i = 0, \dots, p$.

Proof. The proof follows from Lemma 2.1 and the fact that there are no non-trivial boundaries in $K_i(n)_{i+d_0(M)}$. ∎

To prove lower bounds for $\beta_i^{1,\text{lin}}$ we use slightly different methods. Let $S = K[x_1, \dots, x_n]$. We fix a basis $\mathbf{e}$ of L such that $\partial(e_i) = x_i$ for all $i \in [n]$ for the rest of this paper. For $a \in \mathbb{N}^n$ we write $x^a = x_1^{a_1} \cdots x_n^{a_n}$ and call it a monomial in S. Let F be a graded free S-module with free homogeneous basis $g_1, \dots, g_t$. Then we call $x^a g_i$ a monomial in F for $a \in \mathbb{N}^n$ and $i \in [t]$. Let $>$ be an arbitrary degree refining term order on F with $x_1 g_i > \cdots > x_n g_i > g_i$ (see [7] for details). For a homogeneous element $f \in F$ we set $\text{in}_>(f)$ for the maximal monomial in a presentation of f. Note that we also defined in (z) for some bihomogeneous $z \in K(n)$.

LEMMA 2.8. *Let $M \subset F$ be a finitely generated graded S-module and let $0 \neq z$ be a homogeneous cycle of $K_1(n, M)$ with $z = \sum_{j \geq i} m_j e_j$, $\text{in}(z) = m_i e_i$. Then there exists an integer $j > i$ with $0 \neq \text{in}_>(m_j) > \text{in}_>(m_i)$. In particular, m_i and m_j are K-linearly independent.*

Proof. We have

$$0 = \partial(z) = m_i x_i + \sum_{j>i} m_j x_j.$$

Hence there exists an integer $j > i$ and a monomial a_j of m_j with $\text{in}_>(m_i) x_i = a_j x_j$ because all monomials have to cancel. Assume that

$$\text{in}_>(m_i) \geq \text{in}_>(m_j).$$

Then

$$\text{in}_>(m_i) x_i > \text{in}_>(m_i) x_j \geq \text{in}_>(m_j) x_j \geq a_j x_j$$

is a contradiction. Therefore

$$\text{in}_>(m_i) < \text{in}_>(m_j).$$

∎

LEMMA 2.9. *Let $M \subset F$ be a finitely generated graded S-module, $p \in \{0, \ldots, n\}$, and let $0 \neq z$ be a homogeneous cycle of $K_p(n, M)$ with $z = \sum_{J, |J|=p} m_J e_J$, $\operatorname{in}(z) = m_I e_I$. Assume that $I = \{1, \ldots, p\}$. Then there exist distinct numbers $j_1, \ldots, j_p \in [n]$ such that $j_k > p - k + 1$ for $k = 1, \ldots, p$ and*

$$\operatorname{in}_>(m_{J_p}) > \operatorname{in}_>(m_{J_{p-1}}) > \cdots > \operatorname{in}_>(m_{J_0}),$$

where $J_0 = I$ and $J_k = \{1, \ldots, p-k, j_1, \ldots, j_k\}$ for $k = 1, \ldots, p$.

Proof. We construct the numbers j_k and the sets J_k with the desired properties by induction on $k \in \{0, \ldots, p\}$. For $k = 0$ we set $J_0 = I$. Let $0 < k \leq p$. Assume that J_{k-1} is constructed. Then we apply Lemma 2.8 to

$$\partial_{\{1, \ldots, p-k, j_1, \ldots, j_{k-1}\}}(z) \quad \text{where } \operatorname{in}\big(\partial_{\{1, \ldots, p-k, j_1, \ldots, j_{k-1}\}}(z)\big) = m_{J_{k-1}} e_{p-k+1}$$

and find $j_k > p - k + 1$ such that $\operatorname{in}_>(m_{\{1, \ldots, p-k, j_1, \ldots, j_{k-1}\} \cup \{j_k\}}) > \operatorname{in}_>(m_{J_{k-1}})$. We see that $j_k \neq j_i$ for $i = 1, \ldots, k-1$ because these e_{j_i} do not appear with non-zero coefficients in $\partial_{\{1, \ldots, p-k, j_1, \ldots, j_{k-1}\}}(z)$. This concludes the proof. ∎

COROLLARY 2.10. *Let $M \subset F$ be a finitely generated graded S-module, $p \in \{0, \ldots, n\}$, and let $0 \neq z$ be a homogeneous cycle of $K_p(n, M)$ with $z = \sum_{J, |J|=p} m_J e_J$. Then there exist $p + 1$ coefficients m_J of z, which are K-linearly independent.*

Proof. The proof follows from Lemma 2.9 because the coefficients there have different leading terms. ∎

THEOREM 2.11. *Let $p \geq 1$ and let M be a finitely generated graded S-module with $\beta_p^{1, \operatorname{lin}}(M) \neq 0$. Then*

$$\beta_i^{1, \operatorname{lin}}(M) \geq \binom{p}{i}$$

for $i = 1, \ldots, p$.

Proof. Let

$$0 \to M' \to F \to M \to 0$$

be a presentation of M such that F is free and $M' = \Omega_1(M)$. We show that

$$\beta_i^{\operatorname{lin}}(M') \geq \binom{p'+1}{i+1} \qquad \text{for } p' = p - 1.$$

Since $\beta_i^{1, \operatorname{lin}}(M) = \beta_{i-1}^{\operatorname{lin}}(M')$, this will prove the theorem.

After a suitable shift of the grading of M we may assume that $d_0(M') = 0$. Note that M' is a submodule of a free module. Since $\beta_{p'}^{\text{lin}}(M') \neq 0$, we get a homogeneous cycle $0 \neq z = \sum_{J,|J|=p'} m_J e_J$ in $K_{p'}(n, M')_{p'}$. There are no boundaries except for zero in $K_i(n, M')_i$, and therefore we only have to construct enough K-linearly independent cycles in $K_i(n, M')_i$ to prove the assertion. Assume that $\operatorname{in}(z) = m_I e_I$ where $I = \{i_1, \dots, i_{p'}\}$. We make a base change of the basis $e_1, \dots, e_n$ that maps e_{i_j} to e_j for $j = 1, \dots, p'$. Then $\operatorname{in}(z) = m_{[p']} e_{[p']}$ with respect to the new basis. By Lemma 2.9 we obtain numbers $j_1, \dots, j_{p'}$ and the induced sets $J_0, \dots, J_{p'}$ for z such that $j_k > p - k + 1$ for $k = 1, \dots, p$ and $\operatorname{in}_>(m_{J_{p'}}) > \operatorname{in}_>(m_{J_{p'-1}}) > \cdots > \operatorname{in}_>(m_{J_0})$.

Let $t \in \{0, \dots, p\}$ and set $i = p' - t$. Consider the (by Lemma 2.1) cycles $\partial_G(z)$ with $G \in W = W_0 \dot\cup \cdots \dot\cup W_t$ where

$$W_k = \{I \cup \{j_1, \dots, j_k\} : I \subseteq [p' - k], |I| = t - k\} \qquad \text{for } k \in \{0, \dots, t\}.$$

We have

$$|W_k| = \binom{p' - k}{t - k},$$

and therefore

$$|W| = \sum_{k=0}^{t} \binom{p' - k}{t - k} = \binom{p' + 1}{t} = \binom{p' + 1}{p' - t + 1} = \binom{p' + 1}{i + 1}.$$

If we show that the cycles $\partial_G(z)$ are K-linearly independent, the assertion follows.

Take $G \in W$, $G = I_G \cup \{j_1, \dots, j_{k_G}\}$ for some $I_G \subseteq [p' - k_G]$, $|I_G| = t - k_G$. Since $z = \sum_{J,|J|=p'} m_J e_j$ we get

$$\partial_G(z) = \sum_{J,|J|=p'} m_J e_{J-G} = \sum_{I_G \cup \{j_1, \dots, j_{k_G}\} \subseteq J, |J|=p'} m_J e_{J - I_G - \{j_1, \dots, j_{k_G}\}}.$$

Thus

$$\operatorname{in}(\partial_G(z)) = m_{\{1, \dots, p' - k_G\} \cup \{j_1, \dots, j_{k_G}\}} e_{\{1, \dots, p' - k_G\} - I_G}.$$

It is enough to show that the initial terms of the cycles are K-linearly independent.

If cycles have different initial monomials in the e_i, there is nothing to show. Take G, G' and assume that the corresponding cycles have the same initial monomial in the e_i. We have to consider two cases. If $k_G = k_{G'}$, then $I_G = I_{G'}$ and the cycles are the same. For $k_G < k_{G'}$ Lemma 2.9 implies

$$\operatorname{in}(m_{\{1,\dots,p'-k_G\}\cup\{j_1,\dots,j_{k_G}\}}) < \operatorname{in}(m_{\{1,\dots,p'-k_{G'}\}\cup\{j_1,\dots,j_{k_{G'}}\}}),$$

which proves the K-linear independence. ▮

The next corollary summarizes our results related to the conjecture of Herzog.

COROLLARY 2.12. *Let M be a finitely generated graded S-module and let $p \in \{0,\dots,n\}$. Suppose that $\beta_p^{\text{lin}}(M) \neq 0$ and M is the kth syzygy module in a minimal graded free resolution. Then*

(i) *If $k = 0$, then $\beta_i^{\text{lin}}(M) \geq \binom{p}{i}$ for $i = 0,\dots,p$.*

(ii) *If $k = 1$, then $\beta_i^{\text{lin}}(M) \geq \binom{p+1}{i+1}$ for $i = 0,\dots,p$.*

(iii) *If $k > 1$ and $p > 0$, then $\beta_{p-1}^{\text{lin}}(M) \geq p + k$.*

Proof. Statement (i) was shown in Theorem 2.7. In the proof of Theorem 2.11 we proved (ii) in fact. Finally, (iii) follows from Corollary 2.6 since $\beta_i^{\text{lin}}(M) = \beta_{i+k}^{k,\text{lin}}(N)$ if M is the kth syzygy module in the minimal graded free resolution of some module N. ▮

Recall that a finitely generated graded S-module M satisfies *Serre's condition $\mathscr{S}_k$* if

$$\operatorname{depth}(M_P) \geq \min(k, \dim S_P)$$

for all $P \in \operatorname{Spec}(S)$. We recall the Auslander–Bridger theorem [2]:

LEMMA 2.13. *Let M be a finitely generated graded S-module. Then M is a kth syzygy module in a graded free resolution if and only if M satisifes $\mathscr{S}_k$.*

Proof. The proof is essentially the same as in [11], where the local case is treated. ▮

COROLLARY 2.14. *Let M be a finitely generated graded S-module and $p \in \{0,\dots,n\}$. Suppose that M satisfies $\mathscr{S}_k$ and $\beta_p^{\text{lin}}(M) \neq 0$. Then*

(i) *If $k = 0$, then $\beta_i^{\text{lin}}(M) \geq \binom{p}{i}$ for $i = 0,\dots,p$.*

(ii) *If $k = 1$, then $\beta_i^{\text{lin}}(M) \geq \binom{p+1}{i+1}$ for $i = 0,\dots,p$.*

(iii) *If $k > 1$ and $p > 0$, then $\beta_{p-1}^{\text{lin}}(M) \geq p + k$.*

Proof. According to Lemma 2.13 the module M satisfies $\mathscr{S}_k$ if and only if M is a kth syzygy module in a graded free resolution $G_\bullet \to N \to 0$ of some graded S-module N. It is well known (see, for example, [3]) that $G_\bullet = F_\bullet \oplus H_\bullet$ as graded complexes where $F_\bullet$ is the minimal graded free resolution of N and $H_\bullet$ is split exact. Then M splits as a graded module into $\Omega_k(N) \oplus W$, where W is a graded free S-module. If $p = 0$, there is nothing to show. For $p > 0$ it follows that $\beta_p^{\text{lin}}(\Omega_k(N)) \neq 0$. Then Corollary 2.12 applied to $\Omega_k(N)$ proves the corollary, since $\beta_{i,j}(M) \geq \beta_{i,j}(\Omega_k(N))$ for all integers i, j. ∎

3. PROOF OF THE CONJECTURE IN THE CASE OF $\mathbb{Z}^N$-GRADED MODULES

S is $\mathbb{Z}^n$-graded with $\deg(x_i) = \varepsilon_i = (0, \dots, 1, \dots, 0)$, where the 1 is at the ith position. Let $M = \bigoplus_{a \in \mathbb{Z}^n} M_a$ be a finitely generated $\mathbb{Z}^n$-graded S-module. Recall that every $\mathbb{Z}^n$-graded S-module M is naturally $\mathbb{Z}$-graded by setting $M_i = \bigoplus_{a \in \mathbb{Z}^n, |a| = i} M_a$. Therefore all methods from the last section can be applied in the following. Furthermore, the Koszul complex and homology are $\mathbb{Z}^n$-graded if we assign the degree ε_i to e_i. For example, if $m \in M_a$ for some $a \in \mathbb{Z}^n$, then $\deg(me_I)$ is $a + \sum_{i \in I} \varepsilon_i$; or if $z \in K_i(j)$ is homogeneous of degree a, then $\deg(\partial_I(z)) = a - \sum_{i \in I} \varepsilon_i$.

We want to prove a more precise result than in the last section for this more restricted situation. Note that Lemmas 2.2 and 2.3 hold in the $\mathbb{Z}^n$-graded setting. The proofs are verbatim the same if we replace "graded" with "$\mathbb{Z}^n$-graded." We prove now a modified version of Theorem 2.5. Usually we assume that $M = \bigoplus_{a \in \mathbb{N}^n} M_a$.

LEMMA 3.1. *Let $p \in [j]$ and $t \in \mathbb{N}$. Suppose that $H_i(j)_{i+l} = 0$ for $i = p-1, \dots, j$, $l = -1, \dots, t-1$ and let $z \in K_p(j)_{p+t}$ be a $\mathbb{Z}^n$-homogeneous cycle with $0 \neq [z] \in H_p(j)_{p+j}$. Then there exists a $\mathbb{Z}^n$-homogeneous cycle $\tilde{z}$ with*

(i) $[\tilde{z}] = [z] \in H_p(j)_{p+t}$,

(ii) *the elements $[\partial_{j_i}(\tilde{z})] \in H_{p-1}(j)_{p-1+t}$ are K-linearly independent for some distinct $j_1, \dots, j_p \in [j]$.*

Proof. We prove by induction on $j \in [n]$ that we find $[\tilde{z}] = [z]$ and a set $\{j_1, \dots, j_p\}$ such that the cycles $[\partial_{j_i}(\tilde{z})]$ are K-linearly independent for $i = 1, \dots, p$.

The cases $j = 1$ and $j > 1$, $p = j$, follow from Lemma 2.2(ii) because if $\deg(z) = a \in \mathbb{N}^n$, then all $\partial_k(z)$ have different degrees $a - \varepsilon_k$, and it suffices to show that these elements are not boundaries.

Let $j > 1$ and assume that $p < j$. Again it suffices to show that the cycles $\partial_{j_i}(z)$ are not boundaries for a suitable subset $\{j_1, \dots, j_p\} \subseteq [j]$. If such a set exists, then there is nothing to prove. Otherwise there exists a $k \in [j]$ with $[\partial_k(z)] = 0$, and we may assume $k = j$. By Lemma 2.2 we find $\tilde{z}$ such that $[\tilde{z}] = [z]$ in $H(j)$ and $\tilde{z} \in K(j-1)$. By Lemma 1.3 we can apply our induction hypothesis and assume that $\tilde{z}$ has the desired properties in $H(j-1)$. Again by Lemma 1.3 we have $H_{p-1}(j-1)_{p-1+t} \subseteq H_{p-1}(j)_{p-1+t}$, and $\tilde{z}$ is the desired element. ▮

We need the following simple combinatorial result. Let $p \in [n]$. Define inductively a sequence of subsets $W_t \subseteq 2^{[n]}$ for $t = 0, \dots, p$. Set $W_0 = \{\emptyset\}$. If W_{t-1} is defined, then for every set $w \in W_{t-1}$ we choose $p - t + 1$ different elements $j_1^w, \dots, j_{p-t+1}^w$ such that $j_l^w \notin w$. Define

$$W_t = \left\{ w \cup \{ j_l^w \} : w \in W_{t-1} \text{ and } l = 1, \dots, p - t + 1 \right\}.$$

LEMMA 3.2. *Let W_t be defined as above. Then for $t = 0, \dots, p$ we have that*

$$|W_t| \geq \binom{p}{t}.$$

Proof. We prove this by induction on $p \in [n]$. The case $p = 0$ is trivial, so let $p > 0$, and without loss of generality we may assume that $W_1 = \{\{1\}, \dots, \{p\}\}$. The set W_t is the disjoint union of the sets $W_t^1 = \{w \in W_t : 1 \in w\}$ and $W_t^{\hat{1}} = \{w \in W_t : 1 \notin w\}$. The induction hypothesis applied to W_t^1 and $W_t^{\hat{1}}$ implies

$$|W_t| = |W_t^1| + |W_t^{\hat{1}}| \geq \binom{p-1}{t-1} + \binom{p-1}{t} = \binom{p}{t}.$$

▮

We prove the main theorem of this section.

THEOREM 3.3. *Let $k \in [n]$ and let M be a finitely generated $\mathbb{Z}^n$-graded S-module. If $\beta_p^{k,\mathrm{lin}}(M) \neq 0$ for some $p \geq k$, then*

$$\beta_i^{k,\mathrm{lin}}(M) \geq \binom{p}{i}$$

for $i = k, \dots, p$.

Proof. Without loss of generality $M = \bigoplus_{a \in \mathbb{N}^n} M_a$. Since $\beta_p^{k,\mathrm{lin}}(M) \neq 0$, there exists a $\mathbb{Z}^n$-homogeneous cycle $z \in K_p(n)_{p+d_k(M)}$ such that $0 \neq [z]$ in $H_p(n)_{p+d_k(M)}$ and $\deg(z) = a$ for some $a \in \mathbb{N}^n$. By the definition of $d_k(M)$ and Lemma 1.3 we have $H_i(n)_{i+l} = 0$ for $i \geq k$ and $l = -1, \dots, d_k(M) - 1$.

We construct inductively W_t as above, as well as cycles z_w for each $w \in W_t$ such that $[z_w] \neq 0$, $\deg(z_w) = a - \sum_{j \in w} \varepsilon_j$, $0 \neq [\partial_{j_l^w}(z_w)]$ for $l = 1, \dots, p - t$ and suitable $j_l^w \notin w$. Furthermore, z_w is an element of the Koszul complex with respect to the variables x_j with $j \notin w$. For $i \geq k$ we take all cycles $z_w \in K_i(n)_{i+d_k(M)}$ with $w \in W_{p-i}$ which have different $\mathbb{Z}^n$-degree. They are not zero and are therefore K-linearly independent in homology. By Lemma 3.2 there are at least $\binom{p}{p-i} = \binom{p}{i}$ of them, and this concludes the proof.

Let $W_0 = \{\emptyset\}$. By Lemma 3.1 we can choose z in such a way that $[z_{j_l}] = [\partial_{j_l}(z)] \neq 0$ for $l = 1, \ldots, p$ and some $j_l \in [n]$. Choose $z_\emptyset = z$ and $j_l^\emptyset = j_l$.

If W_{t-1} and z_w for $w \in W_{t-1}$ are constructed, then define W_t with W_{t-1} and the given j_l^w for $w \in W_{t-1}$. For $w' \in W_t$ with $w' = w \cup \{j_l^w\}$, rechoose $z_{w'} = \partial_{j_l^w}(z_w)$ by Lemma 3.1 in such a way that $[\partial_{j_{l'}^{w'}}(z_{w'})] \neq 0$ for $l' = 1, \ldots, p-t$ and some $j_{l'}^{w'} \in [n]$.

Note that since z_w has no monomial which is divisible by some e_j for $j \in w$, we can use Lemmas 2.2 and 2.3 to avoid these e_j in the construction of $z_{w'}$ again. By Lemma 1.3 the cycles $\partial_{j_{l'}^{w'}}(z_{w'})$ are also not zero in $H(n)$. Clearly $j_{l'}^{w'} \notin w'$, and the assertion follows. ∎

In the $\mathbb{Z}^n$-graded setting we prove the desired results about β_i^{lin} in full generality.

COROLLARY 3.4. *Let M be a finitely generated $\mathbb{Z}^n$-graded S-module, where M is the kth syzygy module in a minimal $\mathbb{Z}^n$-graded free resolution. If $\beta_p^{\text{lin}}(M) \neq 0$ for some $p \in \mathbb{N}$, then*

$$\beta_i^{\text{lin}}(M) \geq \binom{p+k}{i+k}$$

for $i = 0, \ldots, p$.

Proof. This follows from Theorem 3.3 and the fact that

$$\beta_i^{\text{lin}}(M) = \beta_{i+k}^{k,\text{lin}}(N) \geq \binom{p+k}{i+k},$$

where M is the kth syzygy module of a $\mathbb{Z}^n$-graded S-module N. ∎

As an analogue to Lemma 2.13 we get

LEMMA 3.5. *Let M be a finitely generated $\mathbb{Z}^n$-graded S-module. Then M satisfies $\mathcal{S}_k$ if and only if M is a kth syzygy module in a $\mathbb{Z}^n$-graded free resolution.*

COROLLARY 3.6. *Let M be a finitely generated $\mathbb{Z}^n$-graded S-module satisfying $\mathcal{S}_k$. If $\beta_p^{\text{lin}}(M) \neq 0$ for some $p \in \mathbb{N}$, then*

$$\beta_i^{\text{lin}}(M) \geq \binom{p+k}{i+k}$$

for $i = 0, \ldots, p$.

Proof. The assertion follows from Corollary 3.4 with arguments similar to those for the graded case. ∎

4. BOUNDS FOR BETTI NUMBERS OF IDEALS WITH A FIXED NUMBER OF GENERATORS IN GIVEN DEGREE AND A LINEAR RESOLUTION

In this section we are interested in bounds for the graded Betti numbers of graded ideals of S. We assume that the field K is infinite and fix a basis $\mathbf{x} = x_1, \dots, x_n$ of S_1. For a monomial x^a of S with $|a| = d$ we denote by $R(x^a) = \{x^b : |b| = d, x^b \geq_{\text{rlex}} x^a\}$ the revlex-segment of x^a, where $>_{\text{rlex}}$ is defined as follows: $x^a >_{\text{rlex}} x^b$ if either $|a| > |b|$ or $|a| = |b|$ and there exists an integer r such that $a_r < b_r$ and $a_s = b_s$ for $s > r$. Note that for a given $d \in \mathbb{N}$ and $0 \leq k \leq \binom{n+d-1}{d}$ there exists a unique ideal $I(d, k)$ which is generated in degree d by a revlex-segment $R(x^a)$ for some monomial x^a with $|R(x^a)| = k$.

Following Eliahou and Kervaire [10] for a given monomial x^a, let $m(x^a)$ be the maximal i such that x_i divides x^a. An ideal is a monomial ideal if it is generated by monomials. We call a monomial ideal I stable if for all monomials $x^a \in I$ we have $x_i x^a / x_{m(x^a)} \in I$ for $i < m(x^a)$. Is it easy to see that it is enough to prove this condition for the generators of the ideal I. For example, $I(d, k)$ is stable.

For stable ideals there exist explicit formulas for the Betti numbers (see [10]). Let $I \subset S$ be a stable ideal and let $G(I)$ be the set of minimal generators of I. Then

$$\beta_{i,i+j}(I) = \sum_{x^a \in G(I), |a|=j} \binom{m(x^a) - 1}{i} \quad (*).$$

If a stable ideal I is generated in one degree d, then I has a linear resolution.

PROPOSITION 4.1. *Let $I \subset S$ be a stable ideal generated in degree $d \in \mathbb{N}$ with $\beta_{0,d}(I) = k$. Then*

$$\beta_{i,i+j}(I) \geq \beta_{i,i+j}(I(d, k))$$

for all $i, j \in \mathbb{N}$.

Proof. Fix $d \in \mathbb{N}$ and $0 \leq k \leq \binom{n+d-1}{d}$. Let $l(I)$ be the number of monomials in I_d which are not monomials in $I(d, k)_d$. We prove the statement by induction on $l(I)$. If $l(I) = 0$, then $I = I(d, k)$ and there is nothing to show.

Assume that $l(I) > 0$. Let x^a be the smallest monomial in I_d with respect to $>_{\text{rlex}}$ which is not in $I(d,k)_d$, and let x^b be the largest monomial which is in $I(d,k)_d$ but not in I_d. Define the ideal $\tilde{I}$ by $G(\tilde{I}) = (G(I) \setminus \{x^a\}) \cup \{x^b\}$. Then $l(\tilde{I}) = l(I) - 1$, and $\tilde{I}$ is also stable. Thus, by the induction hypothesis,

$$\beta_{i,i+j}(\tilde{I}) \geq \beta_{i,i+j}(I(d,k)).$$

Since $x^b >_{\text{rlex}} x^a$, the revlex order implies $m(x^b) \leq m(x^a)$. Therefore $(*)$ yields

$$\beta_{i,i+j}(I) \geq \beta_{i,i+j}(\tilde{I}),$$

which proves the assertion. ∎

For a graded ideal I, the initial ideal $\text{in}(I)$ is generated by all $\text{in}(f)$ for $f \in I$ with respect to some monomial order.

Every element g of the general linear group $\text{GL}(n)$ induces a linear automorphism of S by

$$g(x_j) = \sum_{i=1}^{n} g_{i,j} x_i \qquad \text{for } g = (g_{i,j}).$$

There is a non-empty open set $U \subset \text{GL}(n)$ and a unique monomial ideal J with $J = \text{in}(g(I))$ for every $g \in U$ with respect to the revlex order (for details see [7]). We call J the generic initial ideal of I and denote it by $\text{Gin}(I)$. A nice property is that $\text{Gin}(I)$ is Borel-fixed, i.e., $\text{Gin}(I) = b\,\text{Gin}(I)$ for all $b \in B$, where B is the Borel subgroup of $\text{GL}(n)$ which is generated by all upper triangular matrices.

PROPOSITION 4.2. *Let $d \in \mathbb{N}$ and let $I \subset S$ be a graded ideal with d-linear resolution. Then* $\text{Gin}(I)$ *is stable, independent of the characteristic of K, and* $\beta_{i,i+j}(I) = \beta_{i,i+j}(\text{Gin}(I))$ *for all $i, j \in \mathbb{N}$.*

Proof. It is well known that $\text{reg}(\text{Gin}(I)) = \text{reg}(I)$. Therefore $\text{reg}(\text{Gin}(I)) = d$ and $\text{Gin}(I)$ also has a d-linear resolution. Reference [9, Proposition 10] implies that a Borel-fixed monomial ideal, which is generated in degree d, has regularity d if and only if it is stable. Thus we get that $\text{Gin}(I)$ is a stable ideal, independent of the characteristic of K.

Since I has a linear resolution, we obtain by the main result in [1] that $\beta_{i,i+j}(I) = \beta_{i,i+j}(\text{Gin}(I))$ for all $i, j \in \mathbb{N}$. ∎

THEOREM 4.3. *Let $d \in \mathbb{N}$, let $0 \leq k \leq \binom{n+d-1}{d}$, and let $I \subset S$ be a graded ideal with d-linear resolution and k generators. Then*

$$\beta_{i,i+j}(I) \geq \beta_{i,i+j}(I(d,k))$$

for all $i, j \in \mathbb{N}$.

Proof. This follows from Propositions 4.1 and 4.2. ∎

Consider the lexicographic order $>_{\text{lex}}$. Recall that for monomials x^a, $x^b \in S$ we have $x^a >_{\text{lex}} x^b$ if either $|a| > |b|$ or $|a| = |b|$ and there exists an integer r such that $a_r > b_r$ and $a_s = b_s$ for $s < r$. Then $L(x^a) = \{x^b : |b| = d, x^b \geq_{\text{lex}} x^a\}$ is the lex-segment of x^a. For a given $d \in \mathbb{N}$ and $0 \leq k \leq \binom{n+d-1}{d}$ there exists a unique ideal $J(d, k)$ which is generated in degree d by a lex-segment $L(x^a)$ for some monomial x^a with $|L(x^a)| = k$. It is easy to see that $J(d, k)$ is a lex-ideal; i.e., if $x^b >_{\text{lex}} x^c$ and $x^c \in J(d, k)$, then $x^b \in J(d, k)$. In particular, $J(d, k)$ is stable.

PROPOSITION 4.4. *Let $d \in \mathbb{N}$, let $0 \leq k \leq \binom{n+d-1}{d}$, and let $I \subset S$ be a graded ideal with d-linear resolution and k generators. Then*

$$\beta_{i, i+j}(I) \leq \beta_{i, i+j}(J(d, k))$$

for all $i, j \in \mathbb{N}$.

Proof. By [15, Theorem 31] we find a lex-ideal L with the same Hilbert function as I and $\beta_{i, i+j}(I) \leq \beta_{i, i+j}(L)$ for all $i, j \in \mathbb{N}$. We see that $\beta_{0, d}(L) = \beta_{0, d}(I) = k$ because these ideals share the same Hilbert function. It follows that $J(d, k) = (L_d)$, and, in particular, $G(J(d, k)) = G(L)_d$. Therefore

$$\beta_{i, i+d}(I) \leq \beta_{i, i+d}(L) = \beta_{i, i+d}(J(d, k))$$

for all $i, j \in \mathbb{N}$, where the last equality follows from $(*)$. ∎

Fix $d \in \mathbb{N}$ and $0 \leq k \leq \binom{n+d-1}{d}$. Let $\mathscr{B}(d, k)$ be the set of Betti sequences $\{\beta_{i, j}(I)\}$ where I is a graded ideal with d-linear resolution and $\beta_{0, d}(I) = k$. On $\mathscr{B}(d, k)$ we consider a partial order: We set $\{\beta_{i, j}(I)\} \geq \{\beta_{i, j}(J)\}$ if $\beta_{i, j}(I) \geq \beta_{i, j}(J)$ for all $i, j \in \mathbb{N}$.

COROLLARY 4.5. *Let $d \in \mathbb{N}$ and let $0 \leq k \leq \binom{n+d-1}{d}$. Then $\{\beta_{i, j}(I(d, k))\}$ is the unique minimal element and $\{\beta_{i, j}(J(d, k))\}$ is the unique maximal element of $\mathscr{B}(d, k)$.*

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