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Hard cases of the multifacility location problem

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Abstract

Let μ be a rational-valued metric on a finite set T. We consider (a version of) the *multifacility location problem*: given a finite set $V \supseteq T$ and a function $c : \binom{V}{2} \to \mathbf{Z}_+$, attach each element $x \in V - T$ to an element $\gamma(x) \in T$ minimizing $\sum \left(c(xy)\mu(\gamma(x)\gamma(y)) : xy \in \binom{V}{2}\right)$, letting $\gamma(t) := t$ for each $t \in T$. Large classes of metrics μ have been known for which the problem is solvable in polynomial time. On the other hand, Dalhaus et al. (SIAM J. Comput. 23 (4) (1994) 864) showed that if $T = \{t_1, t_2, t_3\}$ and $\mu(t_i t_j) = 1$ for all $i \neq j$, then the problem (turning into the *minimum* 3-*terminal cut problem*) becomes strongly NP-hard. Extending that result and its generalization in (European J. Combin. 19 (1998) 71), we prove that for μ fixed, the problem is strongly NP-hard if the metric μ is nonmodular or if the underlying graph of μ is nonorientable (in a certain sense).

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1. Introduction

A *semimetric* on a set X is a function $d : X \times X \to \mathbf{R}_+$ that establishes *distances* on the pairs of elements (points) of X satisfying $d(x,x) = 0$, $d(x,y) = d(y,x)$ and $d(x,y) + d(y,z) \geqslant d(x,z)$, for all $x,y,z \in X$. We denote an unordered pair $\{x,y\}$ by xy, write $d(xy)$ instead of $d(x,y)$, and denote by $\binom{X}{2}$ the set of pairs xy of distinct elements of X. When $d(xy) > 0$ for all $xy \in \binom{X}{2}$, d is called a *metric*. A special case is the *path metric* d^G of a connected graph $G = (X,E)$, where $d^G(xy)$ is the minimum number of edges of a path in G connecting nodes x and y.

Suppose one is given a metric μ on a finite set T, a larger finite set $V \supseteq T$, and a function $c : \binom{V}{2} \to \mathbf{Z}_+$. One is asked for *attaching* each element $x \in V - T$ to an element $\gamma(x) \in T$ so as to minimize the value $\sum \left(c(xy)\mu(\gamma(x)\gamma(y)) : xy \in \binom{V}{2}\right)$, letting by definition $\gamma(t) := t$ for each $t \in T$. Such a problem is known as a version of the *multifacility location problem*. (In an interpretation, T is thought of as the set of points, each containing one existing facility, $V - T$ as the set of new facilities that are required to be placed into T, and $c(xy)$ as a measure of mutual communication or supporting task between facilities x and y. See [9].)

This admits a reformulation in terms of metric extensions as follows (see [5,8]). A semimetric m on V is called an *extension* of μ to V if m coincides with μ within T, and a 0-*extension* if, in addition, for each $x \in V$, there is $t \in T$ such that $m(xt) = 0$. Then the above task is equivalent to the *minimum* 0-*extension problem*:

Find a 0-extension m of μ to V such that the value

$$cm := \sum \left(c(xy)m(xy) : xy \in \binom{V}{2} \right) \text{ is as small as possible.} \tag{1.1}$$

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This problem generalizes the well-known multiterminal, or multiway, cut problem. It also arises as a discrete strengthening of the dual of a version of the multiflow maximization problem where the values of partial flows are weighted by a given metric (cf. [5,6]). The complexity of (1.1) varies depending on the input metric μ. Large classes of metrics have been found for which the problem is solvable in polynomial time (see [2,3,5,7]). All those metrics have a common feature: they are modular and their underlying graphs are orientable.

Here a metric μ on T is called *modular* if any three points $s_0, s_1, s_2 \in T$ have a *median*, a node $z \in T$ satisfying $\mu(s_i z) + \mu(z s_j) = \mu(s_i s_j)$ for all $0 \leqslant i < j \leqslant 2$. A graph H is called modular if d^H is modular; in particular, H is bipartite. The *underlying graph* of a metric μ on T is the least graph $H(\mu) = (T, U)$ that enables us to restore μ if we know the distances of its edges. Formally, nodes $x, y \in T$ are adjacent in $H(\mu)$ if and only if each $z \in V - \{x, y\}$ satisfies $\mu(xz) + \mu(zy) > \mu(xy)$ (in other words, no $z \neq x, y$ lies *between* x and y regarding μ). A graph is called *orientable* if its edges can be oriented so that for any 4-circuit $C = (v_0, e_1, v_1, \ldots, e_4, v_4 = v_0)$ and $i = 1, 2$, the edge e_i is oriented from v_{i-1} to v_i if and only if the opposite edge e_{i+2} is oriented from v_{i+2} to v_{i+1}.

The simplest nonmodular metric is $\mu = d^{K_3}$, where K_n is the complete graph with n nodes. In this case (1.1) turns into the *minimum* (*capacity*) 3-*terminal cut problem* (3-TERMINAL CUT). Dalhaus et al. [4] showed that the latter is strongly NP-hard. This was extended to the path metrics of more general graphs.

Theorem 1.1 (Karzanov [5, Section 6]). *For a fixed connected graph H, problem* (1.1) with $\mu = d^H$ *is strongly NP-hard if H is nonmodular or nonorientable.*

In this paper, decreasing the existing gap between the hard and polynomial cases in the problem, we extend Theorem 1.1 to more general metrics as follows.

Theorem 1.2. *Problem* (1.1) *with a fixed rational-valued metric μ is strongly NP-hard if either*

(i) *μ is modular and $H(\mu)$ is nonorientable, or*
(ii) *μ is nonmodular.*

The intractability of 3-TERMINAL CUT is proved in [4] by use of a reduction from the maximum cut problem (MAX CUT). The crucial point is the construction of a certain "gadget" not obeying the standard submodular relation for graphs with two terminals. Borrowing that idea, paper [5] proves Theorem 1.1 by constructing gadgets with a similar property when problem (1.1) with the path metric of a nonmodular or nonorientable graph H is considered. We show that in the hypotheses of Theorem 1.2 the desired gadgets can be constructed as well, thus proving this theorem.

Section 2 explains how to extend the idea of reduction in [4] to our problem. Using it, we prove part (i) of Theorem 1.2 (which is technically simpler) in Section 3, and prove part (ii) in Section 4.

In what follows the set of 0-extensions of μ to V is denoted by $\mathrm{Ext}^0(\mu, V)$, and the minimum value cm in (1.1) by $\tau(V, c, \mu)$. Without loss of generality, we can consider only *integer* metrics μ (since the metrics μ in Theorem 1.2 are rational, and multiplying μ by a positive integer factor does not affect the problem).

2. Approach

Given a metric μ on T, a set $V \supset T$, a function $\binom{V}{2} \to \mathbf{Z}_+$, and (not necessarily distinct) elements $s, t \in T$ and $x, y \in V - T$, let $\tau(s, x|t, y)$ denote the minimum cm among all $m \in \mathrm{Ext}^0(\mu, V)$ such that $m(xs) = m(yt) = 0$.

The idea of a possible reduction from MAX CUT to (1.1) (naturally generalizing that from MAX CUT to 3-TERMINAL CUT in [4]) is as follows. Suppose that for one or another μ we are able to devise a pair (*gadget*) (V, c) with specified s, t, x, y satisfying the following condition:

$$\begin{aligned} &\text{(i)} \quad \tau(s,x|t,y) = \tau(s,y|t,x) = \hat{\tau}, \\ &\text{(ii)} \quad \tau(s,x|s,y) = \tau(t,x|t,y) = \hat{\tau} + \delta \quad \text{for some } \delta > 0, \\ &\text{(iii)} \quad \tau(s',x|t',y) \geqslant \hat{\tau} + \delta \quad \text{for all other pairs } s', t' \text{ in } T, \end{aligned} \tag{2.1}$$

where $\hat{\tau}$ stands for $\tau(V, c, \mu)$ (an analog of the "violated submodularity" in [4]). Take an instance of (a 2-terminal version of) MAX CUT whose input consists of a graph $\Gamma = (W, Z)$ and specified nodes p, q. It is required to find a subset $X \subset W$ such that $p \in X \not\ni q$ and the number of edges of Γ connecting X and $W - X$ is maximum. Similar to the construction in [4], replace each edge $uv \in Z$ by a copy of the gadget, identifying the element x as above with one node among u, v, the element y with the other node, s with p, and t with q. Also identify the corresponding elements of $T-\{s,t\}$ in these copies. This results in a set $\mathscr{V}$ and a function ζ on the pairs of elements of $\mathscr{V}$ (determined in a natural way by the copies of c when the involved copies of the gadget are glued together). Using (2.1), it is not difficult to conclude that if m is a minimum 0-extension for $\mu,\mathscr{V},\zeta$, then the set $X:=\{v\in W: m(pv)=0\}$ induces a maximum cardinality cut for Γ,p,q. Therefore, (1.1) for the given μ is NP-hard (as this is so for MAX CUT); moreover, it is strongly NP-hard because the gadget size is a constant depending only on μ.

Thus, our aim is to construct gadgets satisfying (2.1) for the metrics μ as in Theorem 1.2.

3. Modular metrics with nonorientable underlying graphs

In this section we prove Theorem 1.2 in case (i). We use the following result of Bandelt [1]:

if μ is a modular metric, then the underlying graph $H(\mu)$ is modular,

$\mu(e)=\mu(e')$ holds for any two nonajacent edges e,e' occurring in a

4-circuit of $H(\mu)$, and a path P in $H(\mu)$ is shortest

if and only if P is a shortest path for μ. (3.1)

(For a shorter proof and relevant facts, see [6, Section 2].)

Let μ be an integer modular metric on T whose underlying graph $H=H(\mu)$ is nonorientable. The construction of a gadget satisfying (2.1) is close to that for the path metric of a nonorientable bipartite graph in [5].

Since H is nonorientable, there exist edges $e_0,e_1,\dots,e_{k-1},e_k=e_0$ of H along with 4-circuits $C_0,\dots,C_{k-1}$, which yield the "twist" (or constitute an *orientation-reversing dual cycle*). More precisely,

for $i=0,\dots,k-1$, $e_i=s_it_i$ and $e_{i+1}=s_{i+1}t_{i+1}$ are opposite edges

in the 4-circuit $C_i=s_it_it_{i+1}s_{i+1}s_i$ of H, and $t_k=s_0$ (and $s_k=t_0$). (3.2)

(One can choose such a sequence with all edges different, but this is not important for us.) Since μ is modular, we have by (3.1) that

for $i=0,\dots,k-1, \mu(e_i)$ is a constant h, and $\mu(s_is_{i+1})=\mu(t_it_{i+1})=: f_i$. (3.3)

We denote t_i by s_{i+k} and take indices modulo $2k$. The desired gadget is represented by the graph $G=(V,E)$ with edge weights c, where $V=T\cup\{z_0,\dots,z_{2k-1}\}$ and for $i=0,\dots,2k-1$,

(i) z_i is adjacent to both s_i and s_{i+k}, and $c(z_is_i)=c(z_is_{i+k})=N$ for a positive integer N (specified below);
(ii) z_i and z_{i+1} are adjacent, and $c(z_iz_{i+1})=1$.

Fig. 1 illustrates G for $k=4$. We assign $s:=s_0$, $t:=t_0$, $x:=z_0$ and $y:=z_k$, and formally extend c by zero to $\binom{V}{2}-E$. We claim validity of (2.1).

Indeed, associate each 0-extension $m\in \mathrm{Ext}^0(\mu,V)$ with the attaching map $\gamma:\{z_0,\dots,z_{2k-1}\}\to T$ such that $\gamma(z_i)=s_j$ if $m(z_is_j)=0$; we denote m by m^γ. If $\gamma(z_i)=v$, then, letting $\varepsilon:=\mu(s_iv)+\mu(vs_{i+k})-\mu(s_is_{i+k})$, the contribution to the *volume* cm^γ due to the edges $e=z_is_i$ and $e'=z_is_{i+k}$ is equal to

$$c(e)m^\gamma(e)+c(e')m^\gamma(e')=N(m^\gamma(e)+m^\gamma(e'))=Nh+N\varepsilon$$

(h is defined in (3.3)). We have $\varepsilon=0$ if $v\in\{s_i,s_{i+k}\}$, and $\varepsilon\geqslant 1$ otherwise (since s_it_i is an edge of $H(\mu)$). Hence, every map γ pretending to be optimal or nearly optimal must attach each z_i to either s_i or s_{i+k} when N is chosen sufficiently large (e.g., $N=1+2k\max\{\mu(st): s,t\in T\}$).

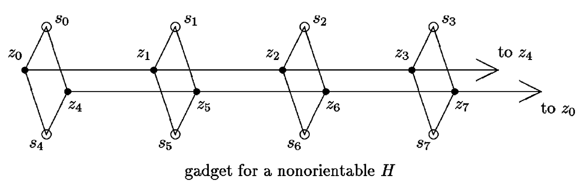

Fig. 1. Gadget for a nonorientable H.

Next, if z_i is attached to s_i (resp. s_{i+k}) and z_{i+1} is attached to s_{i+1} (resp. s_{i+1+k}), then the edge $u=z_iz_{i+1}$ contributes $c(u)m^\gamma(u)=f_i$ (cf. (3.3)), letting $f_j=f_{j+k}$. On the other hand, if z_i is attached to s_i (resp. s_{i+k}) while z_{i+1} to s_{i+1+k} (resp. s_{i+1}), then the contribution becomes $h+f_i=\mu(s_it_{i+1})$ (since the shortest paths $s_it_it_{i+1}$ and $t_is_is_{i+1}$ of $H(\mu)$ are shortest for μ, by (3.1)).

So we can conclude that $\hat{\tau}=2khN+2(f_1+\cdots+f_k)$, and there are precisely two optimal 0-extensions, namely, m^{γ_1} and m^{γ_2}, where $\gamma_1(z_i)=s_i$ and $\gamma_2(z_i)=s_{i+k}$ for $i=0,\dots,2k-1$. This gives (i) in (2.1). Furthermore, one can see that if m^γ is the least-volume 0-extension induced by a map γ that brings both x,y either to s or to t, then $m^\gamma(z_jz_{j+1})=h+f_j$ for precisely two numbers $j\in\{0,\dots,2k-1\}$ such that $f_j=\min\{f_1,\dots,f_k\}$. So $cm^\gamma=\hat{\tau}+2h$, yielding (2.1) (ii). Finally, (iii) is guaranteed by the choice of N.

Thus, (1.1) with μ modular and $H(\mu)$ nonorientable is strongly NP-hard.

4. Nonmodular metrics

Next we prove Theorem 1.2 in case (ii). Let μ be an integer nonmodular metric on T.

For $x,y,z\in T$, let $\Delta(x,y,z)$ denote the value (*perimeter*) $\mu(xy)+\mu(yz)+\mu(zx)$. We fix a medianless triplet $\{s_0,s_1,s_2\}$ such that $\Delta(s_0,s_1,s_2)$ is minimum, denoted by $\bar{\Delta}$. By technical reasons, we put $s_{i+3}:=s_i$ for $i=0,1,2$, and take indices modulo 6. In the gadget (G,c) (whose structure is more involved compared with the corresponding unweighted case in [5]) the graph $G=(V,E)$ is such that

$$V=T\cup Z,\quad Z=\{z_0,\dots,z_5\}\quad\text{and}\quad E=E_1\cup E_2\cup E_3.$$

For $i=1,2,3$, we assign certain numbers $c_i(e)$ (specified below) to the edges $e\in E_i$, and define the weight $c(e)$ of e to be $N_ic_i(e)$. The factors N_1,N_2,N_3 are chosen so that $N_1=1$, N_2 is sufficiently large, and N_3 is sufficiently large with respect to N_2. Informally speaking, the "heavy" edges of E_3 provide that (at optimality or almost optimality) each point z_j must come in the interval $I_j:=\{v\in T:\mu(s_{j-1}v)+\mu(vs_{j+1})=\mu(s_{j-1}s_{j+1})\}$, then the "medium" edges of E_2 force z_j to choose only between the endpoints s_{j-1},s_{j+1} of I_j, and finally the "light" edges of E_1 provide the desired property (2.1).

As before, m^γ denotes the 0-extension of μ to V induced by $\gamma:Z\to T$. A sequence $P=(v_1,\dots,v_k)$ of elements of T is called a *path on* T, and P is called *shortest* if $\mu(v_1v_2)+\cdots+\mu(v_{k-1}v_k)=\mu(v_1v_k)$. Define

$$d_i:=d_{i+3}:=\mu(s_{i-1}s_{i+1}).$$

The set E_3 consists of edges $e_j=z_js_{j-1}$ and $e'_j=z_js_{j+1}$ with $c_3(e_j)=c_3(e'_j)=1$ for $j=0,\dots,5$. Then the contribution to cm^γ due to e_j and e'_j is N_3d_j if $\gamma(z_j)\in I_j$, and at least $N_3d_j+N_3$ otherwise, implying that z_j should be mapped into I_j, by the choice of N_3. The minimality of $\bar{\Delta}$ provides the following useful property.

Statement 4.1. *For any $v\in I_j$, at least one of the paths $P=(s_j,s_{j-1},v)$ and $P'=(s_j,s_{j+1},v)$ is shortest.*

Proof. Let for definiteness $j=1$. Suppose P' is not shortest. Then $\mu(s_1v)<|P'|=\mu(s_1s_2)+\mu(s_2v)$ and $\mu(s_0v)=\mu(s_0s_2)-\mu(s_2v)$ imply $\Delta(s_1,v,s_0)<\bar{\Delta}$. So s_1,v,s_0 have a median w. If $w=s_0$, then P is shortest, as required. Otherwise we have $\Delta(s_1,w,s_2)<\bar{\Delta}$ (since $\mu(s_1w)<\mu(s_1s_0)$ and the path (s_2,v,w,s_0) is, obviously, shortest). Then s_1,w,s_2 have a median q. It is easy to see that q is a median for s_0,s_1,s_2; a contradiction. □

Now we explain the construction of E_2 and c_2. Each $z=z_j$ ($j=0,\dots,5$) is connected to each s_i ($i=0,1,2$) by edge $u_i=zs_i$ on which the value of c_2 is defined by

$$c_2(u_i):=a_i:=(d_{i-1}+d_{i+1}-d_i)/(d_{i-1}d_{i+1}) \tag{4.1}$$

(a_i is positive and does not depend on j). Suppose z is mapped by γ to some s_i, say $\gamma(z)=s_1$. Then, up to a factor of N_2, the contribution to cm^γ from the edges u_0,u_1,u_2 (concerning z) is

$$\begin{aligned}d_2a_0+d_0a_2&=d_2(d_1+d_2-d_0)/(d_1d_2)+d_0(d_1+d_0-d_2)/(d_0d_1)\\&=(d_1+d_2-d_0)/d_1+(d_1+d_0-d_2)/d_1=2.\end{aligned} \tag{4.2}$$

On the other hand, the contribution grows when z_j occurs in the interior of any interval I_i.

Statement 4.2. *Let* $v \in I_i - \{s_{i-1}, s_{i+1}\}$. *Then* $\sigma := \sum(a_k\mu(s_kv) : k = 0,1,2) > 2$.

Proof. Assume $i = 0$ and let $\mu(s_1v) = \varepsilon$ and $\mu(s_0v) = d_2 + \varepsilon$ (cf. Statement 4.1). Then

$$\sigma = (d_2 + \varepsilon)a_0 + \varepsilon a_1 + (d_0 - \varepsilon)a_2 = d_2a_0 + d_0a_2$$

$$+ \varepsilon(a_0 + a_1 - a_2) = 2 + \varepsilon(a_0 + a_1 - a_2),$$

in view of (4.2). We observe that $a_0 + a_1 - a_2 > 0$. Indeed,

$$d_0d_1d_2(a_0 + a_1 - a_2)$$

$$=(d_0d_1 + d_0d_2 - d_0^2) + (d_1d_0 + d_1d_2 - d_1^2) - (d_2d_0 + d_2d_1 - d_2^2)$$

$$=2d_0d_1 - d_0^2 - d_1^2 + d_2^2 = d_2^2 - (d_0 - d_1)^2 > 0$$

since $d_2 > d_0 - d_1$. So $\sigma > 2$. □

Thus, by an appropriate choice of constants N_2 and N_3, each point z_j must be mapped to either s_{j-1} or s_{j+1}. Such a map γ is called *feasible*. We now construct the crucial set E_1 and function c_1. The set E_1 consists of six edges $g_j = z_jz_{j+1}$, $j = 0, \ldots, 5$, forming the 6-circuit C (this is similar to the construction in [5] motivated by Dalhaus et al. [4]). The core is how to assign c_1. For $i = 0,1,2$, let $h_i := h_{i+3} := (d_{i-1} + d_{i+1} - d_i)/2$. These numbers would be just the distances from s_0, s_1, s_2 to their median if it existed, i.e.,

$$d_i = h_{i-1} + h_{i+1}. \tag{4.3}$$

We define

$$c_1(z_jz_{j+1}) := c_1(z_{j+3}z_{j+4}) := h_{j-1} \quad \text{for } j = 0,1,2. \tag{4.4}$$

For $\gamma : Z \to T$, let ζ^γ denote $\sum(c_1(g_j)m^\gamma(g_j) : j = 0, \ldots, 5)$, i.e., ζ^γ is the contribution to cm^γ from the edges of C. The analysis below will depend on the numbers

$$\rho = 2(h_0h_1 + h_1h_2 + h_2h_0) \quad \text{and} \quad \alpha = 2\min\{h_0^2, h_1^2, h_2^2\}. \tag{4.5}$$

W.l.o.g., assume $h_0 \leqslant h_1, h_2$, i.e., $\alpha = 2h_0^2$. Our aim is to show that (2.1) holds if we take as s, t, x, y the elements s_0, s_2, z_1, z_4, respectively.

To show this, consider the map γ_1 as drawn in Fig. 2(a), i.e., $\gamma_1(z_j)$ is s_{j+1} for $j = 0,2,4$ and s_{j-1} for $j = 1,3,5$. This γ_1 attaches x to s and y to t. In view of (4.3)–(4.5), we have

$$\zeta^{\gamma_1} = c_1(g_0)\mu(\gamma_1(z_0)\gamma_1(z_1)) + \cdots + c_1(g_5)\mu(\gamma_1(z_5)\gamma_1(z_0))$$

$$= h_2d_2 + h_0 \times 0 + h_1d_1 + h_2 \times 0 + h_0d_0 + h_1 \times 0$$

$$= h_2(h_0 + h_1) + h_1(h_0 + h_2) + h_0(h_1 + h_2) = \rho.$$

Similarly, $\zeta^{\gamma_2} = \rho$ for the symmetric map γ_2 which is defined by $\gamma_2(z_j) = \gamma_1(z_{j+3})$, attaching x to t and y to s. We shall see later that γ_1 and γ_2 are just optimal maps for our gadget.

The maps pretending to provide (ii) in (2.1) are γ_3 and γ_4 illustrated in Fig. 2(b) and (c); here both x, y are mapped by γ_3 to s, and by γ_4 to t. We have

$$\zeta^{\gamma_3} = h_2d_2 + h_0d_2 + h_1 \times 0 + h_2d_2 + h_0d_2 + h_1 \times 0 = (2h_2 + 2h_0)(h_0 + h_1)$$

$$= 2h_2h_0 + 2h_2h_1 + 2h_0^2 + 2h_0h_1 = \rho + \alpha$$

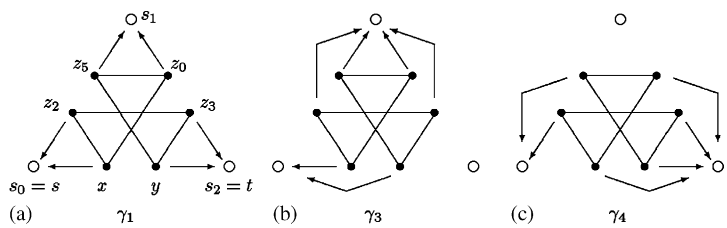

Fig. 2. (a) γ_1, (b) γ_3, (c) γ_4.

and

$$\zeta^{\gamma_4} = h_2 \times 0 + h_0 d_1 + h_1 d_1 + h_2 \times 0 + h_0 d_1 + h_1 d_1 = (2h_0 + 2h_1)(h_0 + h_2)$$
$$= 2h_0^2 + 2h_0 h_2 + 2h_1 h_0 + 2h_1 h_2 = \rho + \alpha.$$

Now (2.1) is obtained from the following.

Statement 4.3. *Let γ be a feasible map different from γ_1 and γ_2. Then $\zeta^\gamma \geqslant \rho + \alpha$.*

Proof. By (4.3), ζ^γ is representable as a nonnegative integer combination of products $h_i h_j$ for $0 \leqslant i, j \leqslant 5$ (including $i = j$). The contribution ζ_j to cm^γ from a single edge $g_j = z_j z_{j+1}$ is as follows:

$$\begin{aligned}
&\text{(i) if } \gamma(z_j) = \gamma(z_{j+1}) = s_{j-1}, \text{ then } \zeta_j = 0;\\
&\text{(ii) if } \gamma(z_j) = s_{j+1} \text{ and } \gamma(z_{j+1}) = s_j, \text{ then } \zeta_j = h_{j-1} d_{j-1} = h_{j-1} h_j + h_{j-1} h_{j+1};\\
&\text{(iii) if } \gamma(z_j) = s_{j+1} \text{ and } \gamma(z_{j+1}) = s_{j-1}, \text{ then } \zeta_j = h_{j-1} d_j = h_{j-1} h_{j+1} + h_{j-1}^2;\\
&\text{(iv) if } \gamma(z_j) = s_{j-1} \text{ and } \gamma(z_{j+1}) = s_j, \text{ then } \zeta_j = h_{j-1} d_{j+1} = h_{j-1} h_j + h_{j-1}^2.
\end{aligned} \tag{4.6}$$

We call g_j *slanting* if case (iii) or (iv) of (4.6) takes place. If no edge of C is slanting, then γ is either γ_1 or γ_2. Otherwise C contains at least two slanting edges. In this case we observe from (4.6) that the representation of ζ^γ includes $h_i^2 + h_j^2$ (or $2h_i^2$) for some i, j, which is at least α. So the result would follow from the fact that the representation includes the term $2h_i h_j$ for each $0 \leqslant i < j \leqslant 2$.

To see the latter for $i = 0$ and $j = 2$ say, consider the edges g_0 and g_1. By (4.3), g_0 contributes $h_0 h_2$ in cases (ii) and (iv), i.e., when $\gamma(z_1) = s_0$. And if $\gamma(z_1) = s_2$, then g_1 contributes $h_0 h_2$. Similarly, the pair g_3, g_4 contributes $h_0 h_2$.

This completes the proof of Theorem 1.2. □

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