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Recognizing clique graphs of directed and rooted path graphs

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Abstract

We describe characterizations for the classes of clique graphs of directed and rooted path graphs. The characterizations relate these classes to those of clique-Helly and strongly chordal graphs, respectively, which properly contain them. The characterizations lead to polynomial time algorithms for recognizing graphs of these classes. © 1999 Elsevier Science B.V. All rights reserved.

1. Introduction

Characterizing and recognizing clique graphs of classes of graphs is of interest in the context of intersection graphs and specially in the study of clique graphs. Also, it provides valuable knowledge about the structure of the graphs of the considered classes. A comprehensive reference for clique graphs and other graph operators is the book [16].

We examine clique graphs of some classes of chordal graphs. The clique graphs of chordal graphs correspond to the class of expanded trees [18]. These graphs are also known as dually chordal graphs [3] and tree-clique graphs [13]. In particular, they can be recognized in linear time [3]. Undirected path graphs form a proper subclass of chordal graphs. However, the class of their clique graphs coincides with that of chordal graphs [18], i.e. with the expanded trees. The clique graphs of directed path graphs correspond to the ACI graphs [14], while those of the rooted path graphs are the rooted expanded trees [2]. We refer to ACI graphs and rooted expanded trees by the names of dually DV and dually RDV graphs, respectively.

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In the present work we describe characterizations for the classes of dually DV graphs and dually RDV graphs. The theorems relate these classes to those of clique-Helly and strongly chordal graphs, respectively, which properly contain them. The characterizations lead to polynomial time algorithms for recognizing graphs of these classes. These recognition problems have not been solved so far.

G denotes an undirected graph, having vertex set $V(G)$ and edge set $E(G)$. Without loss of generality, the graphs here considered are connected. A *clique* of G is a complete subgraph of it, while a *maximal clique* is one not properly contained in any other. When the set of maximal cliques of G satisfies the Helly property, then G is called *clique-Helly*. The *clique graph* of G, denoted by $K(G)$, is the intersection graph of the maximal cliques of G. Write $K^2(G) = K(K(G))$.

A *tree* is a (connected undirected) graph with no cycles. A *directed tree* is an orientation of a tree. A *rooted tree* is a directed tree having exactly one vertex of indegree zero. Chordal graphs have been characterized as the intersection graphs of subtrees of a tree [4,9]. The intersection graphs of a set of paths P in a tree, directed tree and rooted tree T are the *undirected path* (*UV*) graphs, *directed path* (*DV*) graphs and *rooted path* (*RDV*) graphs, respectively. When $|V(T)|$ is minimum then each vertex $v_i \in V(T)$ corresponds to a maximal clique C_i of G, formed exactly by the vertices of G corresponding to the paths of P passing through v_i. This property applies to all UV, DV and RDV graphs ([11,15,10]). In this case, the pair T, P is a *representation* of G, while T is called a (*undirected, directed* and *rooted*) *clique-tree* of G, respectively. A unified study of these classes appears in [15], where the names UV, DV and RDV were introduced.

A *dually chordal graph* G is one admitting a spanning tree T such that, for each $(v,w) \in E(G)$, the vertices of the $v-w$ path in T form a clique in G. In this case, T is called a *canonical tree* of G. A *dually DV* (*dually RDV*) *graph* is a graph G admitting a spanning directed (rooted) tree T such that, for each $(v,w) \in E(G)$, T contains a directed $v-w$ or $w-v$ path, whose vertices form a clique in G. Such a path is called the *span* of (v,w) in T. Similarly, T is a *directed* (*rooted*) *canonical tree* of G. An edge $(v,w) \in E(G)$ is *maximal* in T when there is no edge whose span in T properly contains the span of (v,w).

Since rooted path graphs are directed path graphs and the latter are undirected path graphs, it follows that the class of dually RDV graphs is contained in that of dually DV graphs, and the latter contained in the class of chordal graphs. The containments are proper. For example, the graph of Fig. 1(a) is a dually DV graph and not a dually RDV. On the other hand, the one of Fig. 1(b) is dually chordal and not a dually DV graph.

In Section 2, we describe the characterizations of dually DV and RDV graphs. The corresponding recognition algorithms for these classes are given in Section 3. The method leads also to constructing directed and rooted canonical trees of the graphs. The complexities of the algorithms are $O(|E(G)|^4)$ and $O(|V(G)|^{2.38})$, respectively, for recognizing if G is a dually DV and a dually RDV graph. In the affirmative case of the recognition process, the algorithms exhibit a directed or rooted canonical tree, respectively.

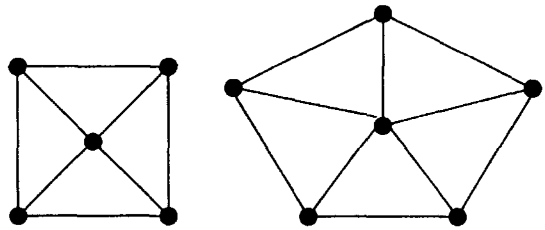

Fig. 1.

2. The characterizations

In this section, we describe characterizations of dually DV and dually RDV graphs. The following are known results on these classes of graphs.

Theorem 1 (Gutierrez and Zucchiello [14]). *The clique graphs of the DV graphs are exactly the dually DV graphs, and the clique graphs of the dually DV graphs are exactly the DV graphs.*

Theorem 2 (Bornstein and Szwarcfiter [2]). *The clique graphs of the RDV graphs are exactly the dually RDV graphs, and the clique graphs of the dually RDV graphs are exactly the RDV graphs.*

The following notation is usefull for our purposes. Let G be a graph. By G' we denote the graph obtained from G by adding to each $v \in V(G)$ a new vertex v' and an edge (v, v'). Call v' the *image* of v in G'.

The theorem below characterizes dually DV graphs.

Theorem 3. *G is a dually DV graph if and only if G is clique-Helly and $K(G')$ is a DV graph.*

Proof. The hypothesis is that G is a dually DV graph. Dually chordal graphs are necessarily clique-Helly [1]. Therefore G is clique-Helly. Let T be a directed canonical tree of G. For each $v \in V(T)$, add to T a new vertex v' and an edge (v, v'), directed arbitrarily. The resulting tree is a directed canonical tree of G'. Consequently, G' is also a dually DV graph. By Theorem 1, $K(G')$ is a DV graph.

Conversely, by hypothesis G is clique-Helly and $K(G')$ a DV graph. Assume that G has at least two vertices, otherwise the theorem is trivial. First, we show that G' is clique-Helly too. For each $v \in V(G)$, let v' be the image of v in G'. Let S be a set of pairwise intersecting maximal cliques of G'. Either S is formed solely by maximal cliques of G, or S has exactly one maximal clique containing the image v' of some $v \in V(G)$, besides cliques of G. In the first case, the maximal cliques of S contain a common vertex, because G is clique-Helly. In the second, all of them contain v. Therefore G' is clique-Helly. On the other hand, no two vertices of G' have the same neighbourhood and the only dominated vertices in this graph are the images. In this situation, we can apply Escalante's Theorem [7] and conclude that $K^2(G') = G$. Consequently, G is the clique graph of a DV graph. By Theorem 1, it follows that G is a dually DV graph. □

Theorem 4. *The following conditions are equivalent for a graph G:*
(1) *G is a dually RDV graph,*
(2) *G is clique-Helly and $K(G')$ is a RDV graph,*
(3) *G is strongly chordal and $K(G')$ is a RDV graph.*

Proof. "(1) ⇒ (3)" Assume that G is a dually RDV graph. By Theorem 2, G is the clique graph of some RDV graph. On the other hand, RDV graphs are strongly chordal. In addition, clique graphs of strongly chordal graphs are strongly chordal [1]. Therefore G is strongly chordal. For the second condition, let T be a rooted canonical tree of G. Similarly as before, let T' be the tree obtained from T by adding a new vertex v' and an edge (v, v'), this time directed from v to v', for each $v \in V(T)$. Then T' is a rooted canonical tree of G'. Consequently, G' is a dually RDV graph. By Theorem 2, $K(G')$ is a RDV graph.

"(3) ⇒ (2)" is obvious.

"(2) ⇒ (1)" Assume that G is clique-Helly and $K(G')$ a RDV graph. As in the proof of Theorem 3, G' is clique-Helly. Again, similarly as above, we conclude by [7] that $K^2(G') = G$. Consequently, using the hypothesis, it follows that G is a dually RDV graph. □

3. The algorithms

We describe below algorithms for recognizing dually DV and dually RDV graphs, as well as for constructing directed canonical trees and rooted canonical trees, respectively. The algorithms are applications of Theorems 3 and 4.

The recognition algorithm for dually DV graphs is as follows. Given a graph G, verify if it is clique-Helly. If the answer is negative, stop, as G is not a dually DV graph. If G is indeed clique-Helly, construct G' and $K(G')$. Finally, verify if $K(G')$ is a DV graph. G is a dually DV graph in the affirmative case, and otherwise is not.

The algorithm for recognizing dually RDV graphs is next described. Given a graph G, check whether G is strongly chordal. If not, stop, and conclude that G is not a dually RDV graph. In the affirmative case, construct G' and $K(G')$. Verify if $K(G')$ is a RDV graph. If it is, G is a dually RDV graph, otherwise it is not.

In order to evaluate the complexities of the above algorithms, we need to obtain bounds on the number of maximal cliques of the graphs. The lemma below describes the maximal cliques of dually DV graphs.

Lemma 1. *Let G be a dually DV graph and T a directed canonical tree of it. Then there exists a one-to-one correspondence between maximal cliques of G and maximal edges of G in T.*

Proof. First we show that every maximal clique C induces a directed path in T. With this purpose, observe that the following weaker statement must be true. T contains a directed path p passing through all vertices of C. Because otherwise, there exists a pair of vertices $x, y \in V(C)$ such that T has neither an $x - y$ nor a $y - x$ path. However $x, y \in V(C)$ implies $(x, y) \in E(G)$. The latter, together with the definition of directed canonical tree imply that T contains either an $x - y$ or a $y - x$ path. This contradiction assures that p indeed exists. Let, this time, x and y be the first and the last vertices of $V(C)$ in the path p, respectively. Examine the directed subpath $x - y$ of p. Clearly, $x - y$ contains all vertices of C. Since T is a directed canonical tree and $(x, y) \in E(G)$ it follows that the vertices of the $x - y$ path in T induce a clique C' in G. It follows that C' and C must coincide, otherwise C is not a maximal clique. Consequently, $x - y$ is the path induced in T by the vertices of C.

Let now C be a maximal clique of G. Then $V(C)$ induces a $v - w$ path in T. Clearly, $(v, w) \in E(G)$, because $v, w \in V(C)$. Then this path $v - w$ is precisely the span of (v, w). In addition, (v, w) is a maximal edge, otherwise C cannot be a maximal clique. Choose (v, w) to be the edge corresponding to C.

Conversely, let $(v, w) \in E(G)$ be a maximal edge. Let C be a maximal clique containing (v, w). We have seen above that $V(C)$ induces a path p in T. Clearly, p contains the span of (v, w) in T. This containment cannot be proper, otherwise (v, w) would not be a maximal edge. Hence p and the span of (v, w) coincide. Since p contains exactly the vertices of C it follows that there exists precisely one maximal clique of G containing (v, w). That is, select C to correspond to the edge (v, w). □

Dually chordal graphs may contain an exponential number of maximal cliques, since all graphs obtained from complete graphs on $2p+1$ vertices by deleting p independent edges are expanded trees. In contrast, Lemma 1 assures that if G is a dually DV graph then it contains at most $|E(G)|$ maximal cliques. Finally, when G is a dually RDV graph then it has no more than $|V(G)|$ maximal cliques, as dually RDV graphs are necessarily chordal.

Let the dually DV graph recognition algorithm be applied to the graph G, where $|V(G)| = n$ and $|E(G)| = m$. Its complexity can be evaluated as follows. Recognizing if G is a clique-Helly graph can be done in $O(n^3 m)$ steps [17]. Constructing G' from G requires $O(n)$ steps. The following determines an upper bound for the number of steps involved in the construction of $K(G')$. Generate the set of maximal cliques of G', using the algorithm [19]. If G is a dually DV graph then G' has no more than $n + m$ maximal cliques, by Lemma 1. Hence the execution of [19] can be stopped after the $(n + m + 1)$-st maximal clique is enumerated. In this case, the recognition process is also stopped with a negative answer. The complexity of [19] is the product of the number of vertices and edges per maximal clique effectively generated. Hence the entire execution of [19] is bounded by $O(nm^2)$. The same bound applies to the actual construction of $K(G')$, Clearly, $K(G')$ has $O(m)$ vertices and $O(m^2)$ edges. By a similar argument as above, we may assume that the latter graph has $O(m)$ maximal cliques. Because if $K(G')$ has more maximal cliques than vertices, it follows that G is not a dually DV graph and the recognition process can again be stopped. Finally, to recognize if $K(G')$ is a DV graph would take $O(m^4)$ steps [15]. Therefore the latter expression is the complexity of the algorithm.

The dually RDV graph recognition algorithm is faster. First, observe that the graph G' must be chordal in this case. Otherwise, G is not strongly chordal and therefore not a dually RDV graph. In this case, the construction of the clique graph $K(G')$ can be performed as indicated by the lemma below.

Lemma 2. *The clique graph of a chordal graph G can be computed in* $O(|V(G)|^{2.38})$ *steps.*

Proof. The following is an algorithm for constructing $K(G)$ with the desired complexity. First, verify whether G is a chordal graph by constructing a perfect elimination ordering of it, in linear time. From such an ordering, obtain the set $\mathscr{C}$ of maximal cliques of G. This requires $O(|V(G)|^2)$ steps [12]. Define the bipartite graph B , with vertex set $V(G) \cup \mathscr{C}$, and such that $v \in V(G)$ is adjacent in B to $C \in \mathscr{C}$ precisely when C contains v in G. Clearly, $|V(B)| \leqslant 2|V(G)|$. It follows that $K(G)$ is exactly the subgraph induced in B^2 by $\mathscr{C}$. Therefore an adjacency matrix of $K(G)$ can be constructed as follows. Compute the square of an adjacency matrix of B. This can be performed in $O(|V(G)|^{2.38})$ steps [5]. Then replace by 1 any entry greater or equal to than 1. Finally, the submatrix defined by the lines and columns corresponding to $\mathscr{C}$ is an adjacency matrix of $K(G')$. □

We can now evaluate the complexity of the dually RDV graph algorithm. As before, let G be the input graph, $|V(G)| = n$ and $|E(G)| = m$. To recognize whether G is strongly chordal requires $O(n^2)$ steps [8]. In the present case, we can assume that G' has at most $2n$ maximal cliques. By Lemma 2, $K(G')$ can be computed in $O(n^{2.38})$ steps. Recognizing whether $K(G')$ is a RDV graph can be done in time $O(n^2)$ [6]. Consequently, the complexity is $O(n^{2.38})$.

It remains to describe the methods for obtaining directed and rooted canonical trees of dually DV and dually RDV graphs, respectively. The lemmas below indicate the constructions.

Lemma 3. *Let G be dually DV graph and T a directed tree. Then T is a directed canonical tree of G if and only if T is a directed clique-tree of* $K(G')$.

Proof. For each $v_i \in V(G)$, denote by v_i' its image in G' and by $C(v_i)$ the set of maximal cliques of G containing v_i. Consequently, $C(v_i)$ corresponds to a clique in $K(G')$. Such a clique, together with the vertex of $K(G')$ corresponding to the maximal clique formed by the edge (v_i, v_i') of G', define a maximal clique C_i' of $K(G')$. Since G is a dually DV graph, G is clique-Helly and so is G'. The latter implies that $K(G')$ has no maximal cliques besides C_i', $1 \leqslant i \leqslant |V(G)|$. Therefore the vertices of G are in one-to-one correpondence with the maximal cliques of $K(G')$.

The hypothesis is that T is directed canonical tree of G. We are going to define a set of paths P of T as follows. For each $v_i \in V(T)$, there is a path in P composed by the sole vertex v_i. In addition, for each maximal edge (v, w) of G with respect to T, the span of (v, w) is included in P. Each path of P corresponds to a vertex of $K(G')$. In addition, two vertices of $K(G')$ are adjacent precisely when their corresponding paths intersect. Hence T, P is a representation of $K(G')$. That is, T is a directed clique-tree.

Conversely, the hypothesis is that T is a directed clique-tree of $K(G')$. Let P be a set of paths of T, such that T, P is a representation of $K(G')$. Let $v_i, v_j \in V(G)$. We show that $(v_i, v_j) \in E(G)$ if and only if there is a path in P passing through both v_i, v_j. This fact implies that T is a directed canonical tree of G, as required. Suppose that v_i, v_j are adjacent in G. The corresponding elements in $K(G')$ are two intersecting maximal cliques C_i', C_j', since every clique of G containing the edge (v_i, v_j) belongs to both C_i' and C_j'. Consequently, there must be a path of P containing the vertices v_i, v_j of T. Finally, suppose that P contains a path from v_i to v_j in T. Then the maximal cliques C_i', C_j' of $K(G')$ correponding to v_i, v_j must intersect. Consequently, v_i, v_j are adjacent, completing the proof. □

The corresponding lemma for the rooted case is given below. Its proof is similar.

Lemma 4. *Let G be dually RDV graph and T a rooted tree. Then T is a rooted canonical tree of G if and only if T is a rooted clique-tree of $K(G')$.*

By Lemmas 4 and 5, to compute a directed or rooted canonical tree of a dually DV or dually RDV graph G, respectively, is equivalent to finding a directed or rooted clique-tree of $K(G')$. Such clique-trees are obtained in the recognition algorithms, as by-products of the steps of recognizing that $K(G')$ is a DV or RDV graph, respectively. Hence the algorithms also obtain the directed or rooted canonical trees.

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