A new three-point approximating $C^2$ subdivision scheme

Shahid S. Siddiqi*, Nadeem Ahmad

Department of Mathematics, Punjab University, Lahore 54590, Pakistan

Received 5 May 2006; received in revised form 25 August 2006; accepted 30 August 2006

Abstract

A new three-point approximating subdivision scheme is presented which generates $C^2$ curves. Its limit function has a support on $[-3, 2]$. The smoothness of the new scheme is shown using the Laurent polynomial method, and a comparison with the Hassan et al. [M.F. Hassan, N.A. Dodgson, Ternary and three-point univariate subdivision schemes, in: A. Cohen, J.-L. Merrien, L.L. Schumaker (Eds.), Curve and Surface Fitting: Sant-Malo 2002, Nashboro Press, Brentwood, 2003, pp. 199–208] scheme is depicted through two examples. It can be observed that the method developed generates a curve more consistent with the control polygon.

Keywords: Approximating subdivision scheme; Mask; Convergence and smoothness; Laurent polynomial

1. Introduction

Developing new subdivision schemes for curve designing has its own importance. Subdivision is becoming an important subject with many applications in fields including computer graphics, computer aided geometric design and computer animation, due to its simplicity and efficiency. A subdivision scheme defines a curve from an initial control polygon or a surface from an initial control mesh by subdividing them according to some refining rules, recursively.

In 1956 de Rham [3], a French mathematician, published on recursively corner cutting a piecewise linear approximation to obtain a smooth curve. Later on, Chaikin [1] introduced another corner cutting method which was published in 1974 and became very popular. A four-point $C^1$ interpolatory subdivision scheme was introduced by Deslauriers and Dubuc [4]; with the special choice of $w = \frac{1}{16}$, the scheme behaved exactly for cubic polynomial. Dyn [2] proved that a four-point interpolatory subdivision scheme is $C^1$ by means of eigenanalysis. Tang et al. [10] used the Laurent polynomial to get the same result. Hassan et al. [6] presented a four-point ternary interpolatory subdivision scheme with a tension parameter. The scheme was proved to be $C^2$ for a certain range of tension parameter. Limei et al. [11] developed a four-point approximating subdivision scheme for a quadrilateral net. The limiting surface was claimed to be $C^3$. Dyn et al. [7] recently presented a four-point approximating subdivision scheme that generates $C^2$ curves and is close to the interpolatory case. Weissman [9] introduced a six-point subdivision scheme by taking a convex combination of the two Deslauriers and Dubuc [4] schemes. Hassan et al. [8] presented a three-point approximating subdivision scheme that generates a $C^3$ curve.

* Corresponding author.

E-mail addresses: shahidsiddiqiprof@yahoo.co.uk (S.S. Siddiqi), nadeemahmadap@yahoo.co.uk (N. Ahmad).

0893-9659/$ - see front matter © 2006 Elsevier Ltd. All rights reserved.
doi:10.1016/j.aml.2006.08.022
In this work a new three-point approximating subdivision scheme is defined as follows:

\[ f_{2i+1}^{k+1} = af_i^{k-1} + bf_i^{k} + cf_{i+1}^{k}, \]
\[ f_{2i+1}^{k+1} = cf_i^{k-1} + bf_i^{k} + af_{i+1}^{k} \]

where \( \{f_i^0\} \) is a set of initial control points with \( a = \frac{1}{2}w^2, b = \frac{1}{2}(1 + 2w - 2w^2) \) and \( c = \frac{1}{2}(w - 1)^2 \). It may be noted that the polynomials \( a, b \) and \( c \) are quadratic B-spline basis functions. It will be shown that the scheme gives \( C^2 \) limit function for \( w = \frac{1}{4} \) with support on \([-3, 2] \). A comparison of the proposed scheme to that introduced by Hassan et al. [8] is illustrated in Figs. 1 and 2.

2. Preliminaries

A binary univariate subdivision scheme is defined in terms of a mask consisting of a finite set of non-zero coefficients \( a = \{a_i : i \in \mathbb{Z}\} \). The scheme is given by

\[ f_{i+1}^{k+1} = \sum_{j \in \mathbb{Z}} a_{i-2j} f_j^k, \quad i \in \mathbb{Z}. \]

Approximating subdivision schemes does not retain the points of stage \( k \) as a subset of the points of stage \( k + 1 \). Thus the general form of an approximating subdivision scheme is

\[ f_{2i}^{k+1} = \sum_{j \in \mathbb{Z}} a_{2j} f_j^k, \]
\[ f_{2i+1}^{k+1} = \sum_{j \in \mathbb{Z}} a_{2j+1} f_{i-j}^k. \]
The formal definition and the notion of convergence of subdivision scheme are as follows:

**Definition 1.** A subdivision scheme $S$ is uniformly convergent if for any initial data $f^0 = \{f_i : i \in \mathbb{Z}\}$, there exists a continuous function $f$ such that for any closed interval $I \subset \mathbb{R}$ it satisfies

$$\lim_{k \to \infty} \sup_{i \in 2^k I} |f^k_i - f(2^{-k} i)| = 0.$$  

Obviously $f = S^\infty f^0$. For each scheme $S$ with mask $a(z)$, $a(z)$ is defined as

$$a(z) = \sum_{i \in \mathbb{Z}} a_i z^i.$$  

Since the schemes under consideration have masks of finite support, the corresponding symbols are Laurent polynomials, namely polynomials in positive and negative powers of the variables.

**Theorem 1 (Dyn [5]).** Let $S$ be a convergent subdivision scheme with a mask $a$. Then

$$\sum_{j \in \mathbb{Z}} a_{2j} = \sum_{j \in \mathbb{Z}} a_{2j+1} = 1. \quad (2.1)$$

It follows from Theorem 1 that the symbol of a convergent subdivision scheme satisfies

$$a(-1) = 0 \quad \text{and} \quad a(1) = 2.$$  

This condition guarantees the existence of a related subdivision scheme for the divided differences of the original control points and the existence of the Laurent polynomial

$$a_1(z) = \frac{2z}{(1+z)} a(z).$$  

The subdivision $S_1$ with symbol $a_1(z)$ is related to $S$ with symbol $a(z)$ by the following theorem.

**Theorem 2 (Dyn [5]).** Let $S$ denote a subdivision scheme with symbol $a(z)$ satisfying (2.1). Then there exists a subdivision scheme $S_1$ with the property

$$df^k = S_1 df^{k-1},$$

where $f^k = S^k f^0$ and $df^k = \{(df^k)_i = 2^k (f^k_{i+1} - f^k_i) : i \in \mathbb{Z}\}$.

The convergence of $S$ can be determined by analyzing the subdivision scheme $\frac{1}{2} S_1$.

**Theorem 3 (Dyn [5]).** $S$ is a uniformly convergent subdivision scheme if and only if $\frac{1}{2} S_1$ converges uniformly to the zero function for all initial data $f^0$, that is

$$\lim_{k \to \infty} \left( \frac{1}{2} S_1 \right)^k f^0 = 0. \quad (2.2)$$

A scheme $S_1$ satisfying (2.2) for all initial data $f^0$ is termed contractive. By Theorem 3, checking the convergence of $S$ is equivalent to checking whether $S_1$ is contractive, which is equivalent to checking whether $\|\left(\frac{1}{2} S_1\right)^L\|_\infty < 1$, for some $L \in \mathbb{Z}^+$. Since there are two rules for computing the values at the next refinement level, one with the even coefficients of the mask and one with odd coefficients of the mask, the norm is defined as

$$\|S\| = \max \left\{ \sum_i |a_{2i}|, \sum_i |a_{2i+1}| \right\},$$

and

$$\left\| \left( \frac{1}{2} S \right)^L \right\|_\infty = \max \left\{ \sum_{\gamma} |a_{\gamma+2L, \gamma}| : \gamma = 0, 1, \ldots, 2L - 1 \right\},$$
Theorem 3

with a choice of \( m \) then

and

above subdivision scheme can be written as

where

\[
\frac{1}{2} S_1
\]

Theorem 4 (Dyn [5]). Let \( a(z) = \frac{(1+z)^m}{m!} b(z) \). If \( S_b \) is convergent, then \( S_a^\infty f^0 \in C^m(R) \) for any initial data \( f^0 \).

3. The smoothness of the three-point approximating subdivision scheme

The new three-point approximating subdivision scheme is defined as follows:

\[
f_{2i}^{k+1} = a f_i^k + b f_i^{k+1} + c f_{i+1}^k,
\]

\[
f_{2i+1}^{k+1} = c f_i^k + b f_i^{k+1} + a f_{i+1}^k
\]

where \( \{ f_i^0 \} \) is a set of initial control points with \( a = \frac{1}{2} w^2, b = \frac{1}{2} (1 + 2w - 2w^2) \) and \( c = \frac{1}{2} (w - 1)^2 \). For \( w = \frac{1}{4} \), the above subdivision scheme can be written as

\[
f_{2i}^{k+1} = \frac{1}{32} f_i^k + \frac{22}{32} f_i^{k+1} + \frac{9}{32} f_{i+1}^k,
\]

\[
f_{2i+1}^{k+1} = \frac{9}{32} f_i^k + \frac{22}{32} f_i^{k+1} + \frac{1}{32} f_{i+1}^k.
\]

The Laurent polynomial \( a(z) \) for the mask of the scheme can be written as

\[
a(z) = \frac{9}{32} z^{-3} + \frac{1}{32} z^{-2} + \frac{22}{32} z^{-1} + \frac{22}{32} z + \frac{1}{32} z^1 + \frac{9}{32} z^2.
\]

In order to prove the smoothness of this scheme to be \( C^2 \) by the Laurent polynomial method, let

\[
b^{[m,L]}(z) = \frac{1}{2L} a_m^{[L]}(z), \quad m = 1, 2, \ldots, L
\]

where

\[
a_m(z) = \frac{2z}{1+z} a_{m-1} = \left( \frac{2z}{1+z} \right)^m a(z),
\]

and

\[
a_m^{[L]}(z) = \prod_{j=0}^{L-1} a_m(z^{2^j}).
\]

With a choice of \( m = 1 \) and \( L = 1 \), the above gives

\[
b^{[1,1]}(z) = \frac{1}{2} a_1(z) = \frac{1}{32} z^{-1} + \frac{7}{16} z + \frac{1}{4} z^2 + \frac{1}{32} z^3.
\]

Since the norm of subdivision \( \frac{1}{2} S_1 \)

\[
\left\| \frac{1}{2} S_1 \right\|_{\infty} = \max \left\{ \Sigma_{\beta} |b^{[1,1]}_{y+2\beta}| : y = 0, 1 \right\}
\]

\[
= \max \left\{ \frac{1}{2}, \frac{1}{2} \right\} = \frac{1}{2} < 1,
\]

then \( \frac{1}{2} S_1 \) is contractive, by Theorem 3, and so is convergent.

In order to prove the three-point scheme to be \( C^1 \), consider \( m = 2 \) and \( L = 1 \); the Laurent polynomial gives

\[
b^{[2,1]}(z) = \frac{1}{2} a_2(z) = \frac{1}{16} + \frac{7}{16} z + \frac{7}{16} z^2 + \frac{1}{16} z^3.
\]
Since the norm of subdivision $\frac{1}{2} S_2$
\[ \left\| \frac{1}{2} S_2 \right\|_{\infty} = \max \left\{ \sum_{\beta} |b^{[1,1]}_{\gamma+2\beta}| : \gamma = 0, 1 \right\} \]
\[ = \max \left\{ \frac{1}{2}, \frac{1}{2} \right\} = \frac{1}{2} < 1 \]
then subdivision scheme $\frac{1}{2} S_2$ is contractive. Consequently, by Theorem 4, $S_1$ is convergent and $S \in C^1$.

In order to prove the three-point scheme to be $C^2$, consider $m = 3$ and $L = 1$; the Laurent polynomial gives
\[ b^{[3,1]}(z) = \frac{1}{2} a_3(z) = \frac{1}{8} z^3 + \frac{3}{4} z^2 + \frac{1}{8} z. \]

Since the norm of subdivision $\frac{1}{2} S_3$
\[ \left\| \frac{1}{2} S_3 \right\|_{\infty} = \max \left\{ \sum_{\beta} |b^{[3,1]}_{\gamma+2\beta}| : \gamma = 0, 1 \right\} \]
\[ = \max \left\{ \frac{3}{4}, \frac{1}{4} \right\} = \frac{3}{4} < 1, \]
then subdivision scheme $\frac{1}{2} S_3$ is contractive. Consequently, by Theorem 4, $S_2$ is convergent and $S \in C^2$.

4. Examples

Two examples are given as depicted in Figs. 1 and 2. In the figures the continuous curve represents the scheme developed in this work, the dashed curve represents the scheme introduced by Hassan et al. [8], after four subdivisions. It can be observed that the scheme developed in this work gives better results as compared to that of Hassan et al. [8].

5. Conclusion

The new three-point approximating subdivision scheme is introduced. The scheme is analyzed, using the Laurent polynomial method. The scheme is proven to be $C^2$ for $w = \frac{1}{4}$ with limit function supported on $[-3, 2]$ and gives a better approximation than that introduced by Hassan et al. [8] as illustrated in Figs. 1 and 2. It can also be observed that the curve generated by the scheme is more consistent with the control polygon.

References