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Generalized Bergman kernels on symplectic manifolds

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Abstract

We study the near diagonal asymptotic expansion of the generalized Bergman kernel of the renormalized Bochner-Laplacian on high tensor powers of a positive line bundle over a compact symplectic manifold. We show how to compute the coefficients of the expansion by recurrence and give a closed formula for the first two of them. As a consequence, we calculate the density of states function of the Bochner-Laplacian and establish a symplectic version of the convergence of the induced Fubini–Study metric. We also discuss generalizations of the asymptotic expansion for non-compact or singular manifolds as well as their applications. Our approach is inspired by the analytic localization techniques of Bismut and Lebeau.

Keywords: Generalized Bergman kernel; Symplectic manifold

0. Introduction

The Bergman kernel for complex projective manifolds is the smooth kernel of the orthogonal projection from the space of smooth sections of a positive line bundle L on the space of holomorphic sections of L, or, equivalently, on the kernel of the Kodaira-Laplacian $\square^L = \bar\partial^L \bar\partial^{L*} + \bar\partial^{L*} \bar\partial^L$

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on L. It was studied in various generalities in [6,17,30–32,42,48,50,51], where the diagonal asymptotic expansion for high powers of L was established. Moreover, the coefficients in the diagonal asymptotic expansion encode geometric information about the underlying complex projective manifolds. The diagonal asymptotic expansion plays a crucial role in the recent work of Donaldson [26] where the existence of Kähler metrics with constant scalar curvature is shown to be closely related to Chow–Mumford stability.

Dai, Liu and Ma [20] studied the asymptotic expansion of the Bergman kernel of the spin^c Dirac operator associated to a positive line bundle on a compact symplectic manifold by relating it to that of the corresponding heat kernel. As a by product, they gave a new proof of the above results. Their approach is inspired by Local Index Theory, especially by the analytic localization techniques of Bismut and Lebeau [4, §11].

Another natural generalization of the operator $\square^L$ in symplectic geometry was initiated by Guillemin and Uribe [28]. In this very interesting short paper, they introduce a renormalized Bochner-Laplacian (cf. (0.4)) which is exactly $2\square^L$ in the Kähler case. The asymptotic of the spectrum of the renormalized Bochner-Laplacian on L^p when $p \to \infty$ is studied in various generalities in [9,15,28] by applying the analysis of Toeplitz structures (generalized Szegő projections) by Boutet de Monvel and Guillemin [13], and in [33] as a direct application of Lichnerowicz formula.

A large and important body of work about the Bergman kernel (to quote just a few [7,10,43]) uses yet another replacement of the $\overline{\partial}$-operator and of the notion of holomorphic section. It is based on a construction by Boutet de Monvel and Guillemin [13] of a first-order pseudodifferential operator D_b on the circle bundle associated to L, which imitates the $\overline{\partial}_b$ operator. However, D_b is neither canonically defined nor unique.

In this paper we will study the asymptotic expansion of the generalized Bergman kernel of the renormalized Bochner-Laplacian, namely the smooth kernel of the projection on its bound states as $p \to \infty$. Our motivation is to deal with a concrete, geometrically and canonically defined operator which allows detailed calculations of the expansion coefficients. Our method is different from the one using the parametrix construction of Boutet de Monvel and Guillemin and continues the line of thought of [20], having origins in the works of Demailly [22], Bismut [2] and Bismut and Vasserot [5]. We use the spectral gap of the renormalized Bochner-Laplacian, finite propagation speed for wave equations and rescaling of the renormalized Bochner-Laplacian near the diagonal. We can work directly on the base manifold and the passage to the associated circle bundle is not necessary.

We now explain our results in more detail. We work on a compact symplectic manifold (X, ω) of real dimension $2n$. Let (L, h^L) and (E, h^E) be two Hermitian vector bundles on X, endowed with Hermitian connections ∇^L and ∇^E. The curvatures of these connections are given by $R^L = (\nabla^L)^2$ and $R^E = (\nabla^E)^2$. We will assume throughout the paper that L is a line bundle satisfying the ***pre-quantization condition***:

$$\frac{\sqrt{-1}}{2\pi} R^L = \omega. \tag{0.1}$$

We choose an arbitrary[1] Riemannian metric g^{TX} on X. Let $\boldsymbol{J} : TX \to TX$ be the skew-adjoint linear map which satisfies the relation

[1] Usually one takes as primary data the symplectic form ω and an almost complex structure J with $\omega(Ju, Jv) = \omega(u, v)$ for each $u, v \in TX$ and $\omega(\cdot, J\cdot) > 0$, then defines $g^{TX}(u, v) := \omega(u, Jv)$. In this case $\boldsymbol{J} = J$. We prefer however to work with an arbitrary Riemannian metric in view of the applications, e.g., Theorem 3.11.

$$\omega(u,v) = g^{TX}(\boldsymbol{J}u, v) \quad \text{for } u, v \in TX. \tag{0.2}$$

Since $\boldsymbol{J}_x \in \mathrm{End}(T_x X)$ we can define the determinant function $\det \boldsymbol{J}$ on TX by $(\det \boldsymbol{J})(x) := \det \boldsymbol{J}_x$ for each $x \in X$.

There exists an almost complex structure $J : TX \to TX$ such that $g^{TX}(Ju, Jv) = g^{TX}(u,v)$, $\omega(Ju, Jv) = \omega(u,v)$ for every $u, v \in TX$ and $\omega(\cdot, J\cdot)$ defines a metric on TX. Indeed, if J satisfies these conditions, then J commutes with $\boldsymbol{J}$ and $-J\boldsymbol{J} \in \mathrm{End}(TX)$ is positive, so necessarily $J = \boldsymbol{J}(-\boldsymbol{J}^2)^{-1/2}$.

We introduce the Levi-Civita connection ∇^{TX} on (TX, g^{TX}) and let R^{TX} denote its curvature and r^X its scalar curvature (cf. (2.22)). By $\nabla^X J \in T^*X \otimes \mathrm{End}(TX)$ we mean the covariant derivative of J induced by ∇^{TX}.

Pursuant to the above choices of connections, we consider the induced Bochner-Laplacian $\Delta^{L^p \otimes E}$ acting on $\mathscr{C}^\infty(X, L^p \otimes E)$ (cf. (1.2)), where $L^p := L^{\otimes p}$. Further, we fix a smooth Hermitian section Φ of $\mathrm{End}(E)$ on X and define:

$$\tau(x) = -\pi \,\mathrm{Tr}|_{TX}[J\boldsymbol{J}],$$

$$\mu_0 = \inf_{u \in T_x X,\ x \in X} \sqrt{-1} R^L_x(u, Ju)/|u|^2_{g^{TX}} > 0, \tag{0.3}$$

$$\Delta_{p,\Phi} = \Delta^{L^p \otimes E} - p\tau + \Phi. \tag{0.4}$$

Note that for a local orthonormal frame $\{e_i\}_i$ of (TX, g^{TX}) near $x \in X$, we have $\tau(x) = \frac{\sqrt{-1}}{2} \sum_j R^L(e_j, Je_j)$. By (0.1) and since $\omega(\cdot, J\cdot)$ is a metric, we obtain $\tau(x) > 0$ for every $x \in X$.

Let $\mathrm{Spec}(A)$ denote the spectrum of an operator A. By [33, Cor. 1.2] (cf. also [5,9,15,28]), there exists $C_L > 0$ independent of p such that

$$\mathrm{Spec}(\Delta_{p,\Phi}) \subset [-C_L, C_L] \cup [2p\mu_0 - C_L, +\infty[. \tag{0.5}$$

The constant C_L can be estimated precisely by using the $\mathscr{C}^0$-norms of R^{TX}, R^E, R^L, $\nabla^X J$ and Φ, cf. [33, pp. 656–658].

Since $\Delta_{p,\Phi}$ is an elliptic operator on a compact manifold, it has discrete spectrum and its eigensections are smooth. Let $\mathcal{H}_p \subset \mathscr{C}^\infty(X, L^p \otimes E)$ be the direct sum of eigenspaces of $\Delta_{p,\Phi}$ corresponding to the eigenvalues belonging to $[-C_L, C_L]$. By [33, Cor. 1.2] (also cf. [9,28] for the case E trivial and $\boldsymbol{J} = J$) we have the following formula for p large enough:

$$\begin{aligned} \dim \mathcal{H}_p = d_p &= \int_X \mathrm{Td}(TX)\,\mathrm{ch}\big(L^p \otimes E\big) \\ &= \mathrm{rk}(E) \int_X \frac{c_1(L)^n}{n!} p^n + \int_X \left(c_1(E) + \frac{\mathrm{rk}(E)}{2} c_1(TX)\right) \frac{c_1(L)^{n-1}}{(n-1)!} p^{n-1} \\ &\quad + \mathscr{O}\big(p^{n-2}\big). \end{aligned} \tag{0.6}$$

As usual, $\mathrm{ch}(\cdot)$, $c_1(\cdot)$, $\mathrm{Td}(\cdot)$ are the Chern character, the first Chern class and the Todd class of the corresponding complex vector bundles (we consider here TX as a complex vector bundle with complex structure J).

The restriction to the diagonal of the generalized Bergman kernels (Definition 1.1, (1.3)) can be introduced as follows. We consider an arbitrary orthonormal basis $\{S_i^p\}_{i=1}^{d_p}$ of $\mathcal{H}_p$ with respect to the inner product (1.1) such that $\Delta_{p,\Phi} S_i^p = \lambda_{i,p} S_i^p$. We adhere to the convention that $\lambda^0 = 1$ for each $\lambda \in \mathbb{R}$. For $q \in \mathbb{N}$, we define $B_{q,p} \in \mathscr{C}^\infty(X, \operatorname{End}(E))$ by

$$B_{q,p}(x) = \sum_{i=1}^{d_p} \lambda_{i,p}^q S_i^p(x) \otimes \big(S_i^p(x)\big)^*. \tag{0.7}$$

Clearly, $B_{q,p}(x)$ does not depend on the choice of $\{S_i^p\}$ but is by construction dependent on the data g^{TX}, h^L, ∇^L, h^E, ∇^E, J and Φ.

In general, on any given manifold we fix a Riemannian metric and a covariant derivative. Pursuant to this choices, we form the pointwise norms, covariant derivative of order $l \in \mathbb{N}$ and the $\mathscr{C}^l$-norm of tensors.

Let $\mathscr{G}_X$ denote the set of Riemannian metrics on X. We say that a subset $G \subset \mathscr{G}_X$ is bounded below, if there exists $g_0^{TX} \in \mathscr{G}_X$ such that $g^{TX} \geqslant g_0^{TX}$ for all $g^{TX} \in G$.

A corollary of Theorem 1.19 is one of our main results:

Theorem 0.1. *There exist smooth coefficients $b_{q,r}(x) \in \operatorname{End}(E)_x$ such that*

$$b_{0,0} = (\det \boldsymbol{J})^{1/2} \operatorname{Id}_E, \tag{0.8}$$

and for every $k, l \in \mathbb{N}$ there exists $C_{k,l} > 0$ such that for every $x \in X$, $p \in \mathbb{N}$,

$$\left| \frac{1}{p^n} B_{q,p}(x) - \sum_{r=0}^{k} b_{q,r}(x) p^{-r} \right|_{\mathscr{C}^l} \leqslant C_{k,l} p^{-k-1}. \tag{0.9}$$

The coefficients $b_{q,r}(x)$ are polynomials in R^{TX}, R^E, Φ (and R^L), their derivatives of order $\leqslant 2(r+q)-2$ (resp. $2(r+q)$), and reciprocals of linear combinations of eigenvalues of $\boldsymbol{J}$ at x.

The expansion is uniform in the following sense. For a subset $\mathcal{M}$ of $\mathscr{D}$, the infinite dimensional manifold of all compatible tuples $(g^{TX}, h^L, \nabla^L, h^E, \nabla^E, J, \Phi)$, assume that:

(i) *for each fixed $k, l \in \mathbb{N}$ the covariant derivatives in the direction X up to order $2n+2k+2q+l+5$ of elements of $\mathcal{M}$ form a set of tensors on $X \times \mathcal{M}$ which is bounded in the $\mathscr{C}^l$-norm calculated in the direction of $\mathcal{M}$;*
(ii) *the projection of $\mathcal{M}$ on the space of Riemannian metrics $\mathscr{G}_X$ is bounded below.*

Then there exists $C_{k,l} = C_{k,l}(\mathcal{M})$ such that (0.9) *holds for all tuples of data from $\mathcal{M}$. Moreover, the $\mathscr{C}^l$-norm in* (0.9) *can be taken to be the $\mathscr{C}^l$ norm on $X \times \mathcal{M}$.*

We calculate further the coefficients $b_{0,1}$ and $b_{q,0}$, $q \geqslant 1$ as follows.[2]

[2] If $\{e_j\}$ is a local orthonormal frame of (TX, g^{TX}), then $|\nabla^X J|^2 = \sum_{ij} |(\nabla^X_{e_i} J)e_j|^2$; this is two times the corresponding $|\nabla^X J|^2$ from [34].

Theorem 0.2. *If $J = \boldsymbol{J}$, then for $q \geqslant 1$,*

$$b_{0,1} = \frac{1}{8\pi}\left[r^X + \frac{1}{4}\left|\nabla^X J\right|^2 + 2\sqrt{-1}R^E(e_j, Je_j)\right], \tag{0.10}$$

$$b_{q,0} = \left(\frac{1}{24}\left|\nabla^X J\right|^2 + \frac{\sqrt{-1}}{2}R^E(e_j, Je_j) + \Phi\right)^q. \tag{0.11}$$

Let us verify the compatibility of (0.10) with the Atiyah–Singer formula (0.6). Let $T^{(1,0)}X = \{v \in TX \otimes_{\mathbb{R}} \mathbb{C};\ Jv = \sqrt{-1}v\}$ be the almost complex tangent bundle on X and let $P^{1,0} = \frac{1}{2}(1 - \sqrt{-1}J)$ be the natural projection from $TX \otimes_{\mathbb{R}} \mathbb{C}$ onto $T^{(1,0)}X$. Then $\nabla^{1,0} = P^{1,0}\nabla^{TX}P^{1,0}$ is a Hermitian connection on $T^{(1,0)}X$, and the Chern–Weil representative of $c_1(TX)$ is $c_1(T^{(1,0)}X, \nabla^{1,0}) = \frac{\sqrt{-1}}{2\pi}\operatorname{Tr}|_{T^{(1,0)}X}(\nabla^{1,0})^2$. By (1.95),

$$\left(\nabla^{1,0}\right)^2 = P^{1,0}\left[R^{TX} - \frac{1}{4}\left(\nabla^X J\right) \wedge \left(\nabla^X J\right)\right]P^{1,0}. \tag{0.12}$$

Thus if $J = \boldsymbol{J}$, (0.12), (2.13), (2.15), (2.21) and (2.22) imply

$$\left\langle c_1\left(T^{(1,0)}X, \nabla^{1,0}\right), \omega\right\rangle = \frac{1}{4\pi}\left(r^X + \frac{1}{4}\left|\nabla^X J\right|^2\right). \tag{0.13}$$

Therefore, by integrating over X the expansion (0.9) for $k = 1$ we obtain (0.6), so (0.10) is compatible with (0.6).

Theorem 0.1 for $q = 0$ and (0.10) generalize the results of [17,31,51] and [50] to the symplectic case. The term $r^X + \frac{1}{4}|\nabla^X J|^2$ in (0.10) is called the Hermitian scalar curvature in the literature [27, Chap. 10] and is a natural substitute for the Riemannian scalar curvature in the almost-Kähler case. It was used by Donaldson [25] to define the moment map on the space of compatible almost-complex structures. We can view (0.11) as an extension and refinement of the results of [11], [28, §5] about the density of states function of $\Delta_{p,\Phi}$ (cf. Remark 3.2 for the details).

To clarify the relation between the renormalized Bochner-Laplacian and the pseudodifferential operator D_b^2 introduced by Boutet de Monvel and Guillemin [13], let us notice that (0.11) for $E = \mathbb{C}$ and $\boldsymbol{J} = J$ shows that these two operators could be equal only if $\Phi = -\frac{1}{24}|\nabla^X J|^2$.

Let us explain the strategy we apply in this paper. In the case considered in Dai, Liu and Ma [20] there is a spectral gap in the sense that the eigenvalues of the Laplacian are either 0 or tend to $+\infty$. This allows to obtain the key equation [20, (4.89)] and to prove the ***full off-diagonal expansion*** (cf. [20, Theorem 4.18]), which is needed to study the Bergman kernel on orbifolds.

However, in the current situation we have possibly different bounded eigenvalues (cf. (0.5) and (0.6)) so we proceed as follows. The first step is to use the spectral gap (0.5) and the finite propagation speed of solutions of hyperbolic equations which permit to localize the asymptotics near the diagonal. Then we rescale the renormalized Bochner–Laplace operator and obtain a formal expansion of the operator as $p \to \infty$. Finally we combine the Sobolev norm estimates contained in [20] and a formal power series technique to show that the formal expansion is indeed the real expansion.

In the course of the proof we also develop a method to compute the coefficients (cf. (1.110), (1.114)) which is new with respect to [31] and [20]. The final result is Theorem 1.19 where we obtain the ***near diagonal expansion*** of the generalized Bergman kernels. This result is enough for most applications.

We treat several applications of the asymptotic expansion of the renormalized Bochner-Laplacian. First we calculate the density of bounded eigenvalues of $\Delta_{p,\Phi}$. We show then how our method can be employed to study the Bergman kernel of the operator $\overline{\partial}+\overline{\partial}^*$ when X is Kähler but $\boldsymbol{J} \neq J$. This discussion applies also to the first-order pseudodifferential operator D_b of Boutet de Monvel and Guillemin [13], which was studied extensively by Shiffman and Zelditch [43]. We give further a symplectic version of the convergence of the induced Fubini–Study metric [48].

We include also generalizations for non-compact or singular manifolds. We have thus a unified treatment of the convergence of the induced Fubini–Study metric [10,12,43,48], the holomorphic Morse inequalities [2,8,22] and the characterization of Moishezon spaces [8,22,29,45] from the point of view of Bergman kernels.

Let us provide a short road-map of the paper. In Section 1, we prove Theorem 0.1. In Section 2, we compute the coefficients $b_{q,r}$, and thus establish Theorem 0.2. In Section 3, we explain the applications.

Some results of this paper have been announced in [34]. In [35] we shall study the Berezin–Toeplitz quantization on symplectic manifolds as an application of the asymptotic expansion of the Bergman kernel. We refer also the readers to our forthcoming book [36] for a comprehensive study of the Bergman kernels along the lines of the present paper.

1. Generalized Bergman kernels

As pointed out in Introduction, we will apply the same strategy as in [20]. However, we have to deal with the following problem. In the situation of [20] the operators D_p^2 have only one bounded eigenvalue as $p \to \infty$, namely 0, whereas in the present paper, we could have different eigenvalues of $\Delta_{p,\Phi}$ in the interval $[-C_L, C_L]$ as $p \to \infty$ (cf. (0.5) and (0.6); it is in principle possible to have d_p different eigenvalues of multiplicity one in $[-C_L, C_L]$). This prevents us to use directly the key equation [20, (4.89)] in order to get a full off-diagonal asymptotic expansion of the generalized Bergman kernels.

To overcome this difficulty, we first localize the asymptotics near the diagonal and by rescaling arguments we obtain a formal expansion of the considered operators as $p \to \infty$. In order to show that the formal expansion is indeed the real expansion we need to prove the vanishing of the coefficients $F_{q,r}$ $(r < 2q)$ in the expansion (1.77). We will introduce a formal power series technique which permits to show the vanishing of the latter coefficients and allows us also to give a method to compute the coefficients from (0.9).

The ideas used here are inspired by the technique of Local Index Theory, especially by [4, §10, 11].

This section is organized as follows. In Section 1.1, we explain that the asymptotic expansion of the generalized Bergman kernel $P_{q,p}(x, x')$ is local on X by using the spectral gap (0.5) and the finite propagation speed of solutions of hyperbolic equations. In Section 1.2, we obtain an asymptotic expansion of $\Delta_{p,\Phi}$ in normal coordinates. In Section 1.3, we adapt to our problem the Sobolev norm estimates developed in [20] and we study the uniform estimate of the generalized Bergman kernels of the renormalized Bochner-Laplacian $\mathscr{L}_t$. In Section 1.4, we study the Bergman kernel of the limit operator $\mathscr{L}_0$. In Section 1.5, we compute some coefficients $F_{q,r}$

($r \leqslant 2q$) of the asymptotic expansion from Theorem 1.13. Finally, we prove Theorem 0.1 in Section 1.6.

1.1. Localization of the problem

Let a^X be the injectivity radius of (X, g^{TX}). We fix $\varepsilon \in]0, a^X/4[$. We denote by $B^X(x,\epsilon)$ and $B^{T_xX}(0,\epsilon)$ the open balls in X and T_xX with center x and radius ϵ, respectively. Then the exponential map $T_xX \ni Z \to \exp_x^X(Z) \in X$ is a diffeomorphism from $B^{T_xX}(0,\epsilon)$ on $B^X(x,\epsilon)$ for $\epsilon \leqslant a^X$. From now on, we identify $B^{T_xX}(0,\epsilon)$ with $B^X(x,\epsilon)$ for $\epsilon \leqslant a^X$.

Let $\langle\cdot,\cdot\rangle_{L^p\otimes E}$ be the metric on $L^p \otimes E$ induced by h^L and h^E and dv_X be the Riemannian volume form of (TX, g^{TX}). The L^2-Hermitian product on $\mathscr{C}^\infty(X, L^p \otimes E)$, the space of smooth sections of $L^p \otimes E$, is given by

$$\langle s_1, s_2\rangle = \int_X \big\langle s_1(x), s_2(x)\big\rangle_{L^p\otimes E}\, dv_X(x). \tag{1.1}$$

We denote the corresponding norm with $\|\cdot\|_{L^2}$.

Let ∇^{TX} be the Levi-Civita connection of the metric g^{TX} and $\nabla^{L^p\otimes E}$ be the connection on $L^p \otimes E$ induced by ∇^L and ∇^E. Let $\{e_i\}_i$ be an orthonormal frame of TX. Then the Bochner-Laplacian on $L^p \otimes E$ is given by

$$\Delta^{L^p\otimes E} = -\sum_i \big[\big(\nabla^{L^p\otimes E}_{e_i}\big)^2 - \nabla^{L^p\otimes E}_{\nabla^{TX}_{e_i}e_i}\big]. \tag{1.2}$$

We consider the vector subspace $\mathcal{H}_p$ spanned by the eigensections of $\Delta_{p,\Phi} = \Delta^{L^p\otimes E} - p\tau + \Phi$ corresponding to eigenvalues in $[-C_L, C_L]$. Let $P_{\mathcal{H}_p}$ be the orthonormal projection from $\mathscr{C}^\infty(X, L^p \otimes E)$ onto $\mathcal{H}_p$.

Definition 1.1. The smooth kernel of $(\Delta_{p,\Phi})^q P_{\mathcal{H}_p}$, $q \geqslant 0$ (where $(\Delta_{p,\Phi})^0 = 1$), with respect to $dv_X(x')$ is denoted $P_{q,p}(x,x')$ and is called a ***generalized Bergman kernel*** of $\Delta_{p,\Phi}$.

The kernel $P_{q,p}(x,x')$ is a section of $\pi_1^*(L^p \otimes E) \otimes \pi_2^*(L^p \otimes E)^*$ over $X \times X$, where π_1 and π_2 are the projections of $X \times X$ on the first and second factor. Using the notations of (0.7) we can write

$$P_{q,p}(x,x') = \sum_{i=1}^{d_p} \lambda_{i,p}^q S_i^p(x) \otimes \big(S_i^p(x')\big)^* \in \big(L^p \otimes E\big)_x \otimes \big(L^p \otimes E\big)^*_{x'}. \tag{1.3}$$

Since $L_x^p \otimes (L_x^p)^*$ is canonically isomorphic to $\mathbb{C}$, the restriction of $P_{q,p}$ to the diagonal $\{(x,x)\colon x \in X\}$ can be identified to $B_{q,p} \in \mathscr{C}^\infty(X, E \otimes E^*) = \mathscr{C}^\infty(X, \mathrm{End}(E))$.

Let $f : \mathbb{R} \to [0,1]$ be a smooth even function such that $f(v) = 1$ for $|v| \leqslant \varepsilon/2$, and $f(v) = 0$ for $|v| \geqslant \varepsilon$. Set $F : \mathbb{R} \to \mathbb{R}$

$$F(a) = \Bigg(\int_{-\infty}^{+\infty} f(v)\,dv\Bigg)^{-1} \int_{-\infty}^{+\infty} e^{iva} f(v)\,dv. \tag{1.4}$$

Then $F(a)$ is an even function and lies in the Schwartz space $\mathcal{S}(\mathbb{R})$ and $F(0)=1$. Let $\widetilde{F}$ be the holomorphic function on $\mathbb{C}$ such that $\widetilde{F}(a^2)=F(a)$, for $a\in\mathbb{R}$. The restriction of $\widetilde{F}$ to $\mathbb{R}$ lies in the Schwartz space $\mathcal{S}(\mathbb{R})$.

We define by recurrence the functions $\widetilde{F}_k:\mathbb{R}\to\mathbb{R}$, for $k\in\mathbb{N}$, and the constants c_k, for $k\in\mathbb{N}^*$, as follows. We set $\widetilde{F}_0(a):=\widetilde{F}(a)$ and $c_1:=\frac{1}{1!}\widetilde{F}_0'(0)$. If $\widetilde{F}_0,\dots,\widetilde{F}_{k-1}$ and $c_1,\dots,c_k$ are already defined, set

$$\widetilde{F}_k(a)=\widetilde{F}(a)-\sum_{j=1}^{k}c_j a^j\widetilde{F}(a),\qquad c_{k+1}=\frac{1}{(k+1)!}\widetilde{F}_k^{(k+1)}(0). \tag{1.5}$$

Then $\widetilde{F}_k$ verifies

$$\widetilde{F}_k^{(i)}(0)=0\quad\text{for every } 0<i\leqslant k. \tag{1.6}$$

Proposition 1.2. *For every $k,m\in\mathbb{N}$, there exists $C_{k,m}>0$ such that for all $p\geqslant 1$ we have*

$$\left|\widetilde{F}_k\left(\frac{1}{\sqrt{p}}\Delta_{p,\Phi}\right)(x,x')-P_{0,p}(x,x')\right|_{\mathscr{C}^m(X\times X)}\leqslant C_{k,m}\,p^{-\frac{k}{2}+4(m+n+1)}. \tag{1.7}$$

Here the $\mathscr{C}^m$ norm is induced by ∇^L, ∇^E, h^L, h^E and g^{TX}.

Proof. By (1.4), for each $m\in\mathbb{N}$, there exists $C'_{k,m}>0$ such that

$$\sup_{a\in\mathbb{R}}|a|^m\left|\widetilde{F}_k(a)\right|\leqslant C'_{k,m}. \tag{1.8}$$

Set

$$G_{k,p}(a)=1_{[\sqrt{p}\mu_0,+\infty[}(a)\widetilde{F}_k(a),\qquad H_{k,p}(a)=1_{[0,\frac{C_L}{\sqrt{p}}]}\big(|a|\big)\widetilde{F}_k(a). \tag{1.9}$$

By (0.5), for p big enough,

$$\widetilde{F}_k\left(\frac{1}{\sqrt{p}}\Delta_{p,\Phi}\right)=G_{k,p}\left(\frac{1}{\sqrt{p}}\Delta_{p,\Phi}\right)+H_{k,p}\left(\frac{1}{\sqrt{p}}\Delta_{p,\Phi}\right). \tag{1.10}$$

Since X is compact, there exist $\{x_i\}_{i=1}^r$ such that $\{U_i=B^X(x_i,\varepsilon)\}_{i=1}^r$ is a covering of X. We identify $B^{T_{x_i}X}(0,\varepsilon)$ with $B^X(x_i,\varepsilon)$ by the exponential map as above. We identify $(L^p\otimes E)_Z$ for $Z\in B^{T_{x_i}X}(0,\varepsilon)$ to $(L^p\otimes E)_{x_i}$ by parallel transport with respect to the connection $\nabla^{L^p\otimes E}$ along the curve $\gamma_Z:[0,1]\ni u\to\exp^X_{x_i}(uZ)$. Let us take an orthonormal basis $\{e_i\}_i$ of $T_{x_i}X$ and let $\widetilde{e}_i(Z)$ be the parallel transport of e_i with respect to ∇^{TX} along the above curve γ_Z. We denote by Γ^E, Γ^L the corresponding connection forms of ∇^E, ∇^L with respect to some fixed frame for E, L which is parallel along the curve γ_Z under the trivialization on U_i. Denote by ∇_U is the ordinary differentiation operator on $T_{x_i}X$ in the direction U. Then

$$\nabla^{L^p\otimes E}_{e_j}=\nabla_{e_j}+p\Gamma^L(e_j)+\Gamma^E(e_j). \tag{1.11}$$

Let φ_i be a partition of unity associated to $\{U_i\}$. We define a Sobolev norm on the lth Sobolev space $\boldsymbol{H}^l(X, L^p \otimes E)$ by

$$\|s\|^2_{H^l_p} = \sum_i \sum_{k=0}^{l} \sum_{i_1,\dots,i_k=1}^{2n} \big\|\nabla_{e_{i_1}} \cdots \nabla_{e_{i_k}} (\varphi_i s)\big\|^2_{L^2}. \tag{1.12}$$

Then by (0.4), (1.2), (1.11), there exists $C > 0$ such that for any $p \geqslant 1$, $s \in \boldsymbol{H}^2(X, L^p \otimes E)$,

$$\|s\|_{H^2_p} \leqslant C\big(\|\Delta_{p,\Phi} s\|_{L^2} + p^2 \|s\|_{L^2}\big). \tag{1.13}$$

Let Q be a differential operator of order $m \in \mathbb{N}$ with scalar principal symbol and with compact support in U_i. Since $[\Delta_{p,\Phi}, Q]$ is a differential operator of order $m+1$ in which the coefficient of p^2 is a differential operator of order $m-1$, (1.13) implies

$$\begin{aligned}\|Qs\|_{H^2_p} &\leqslant C\big(\|\Delta_{p,\Phi} Qs\|_{L^2} + p^2 \|Qs\|_{L^2}\big)\\ &\leqslant C\big(\|Q\Delta_{p,\Phi} s\|_{L^2} + p\|s\|_{H^{m+1}_p} + p^2\|s\|_{H^{m-1}_p} + p^2\|Qs\|_{L^2}\big).\end{aligned} \tag{1.14}$$

Hence for every $m \in \mathbb{N}$ there exists $C_m > 0$ such that for all p we have

$$\|s\|_{H^{2m+2}_p} \leqslant C_m p^{4m+2} \sum_{j=0}^{m+1} \big\|\Delta^j_{p,\Phi} s\big\|_{L^2}. \tag{1.15}$$

Moreover, if $\mathbf{G}_{k,p}$ is one of the operators $G_{k,p}$ or $H_{k,p}$, then $\langle \Delta^{m'}_{p,\Phi} \mathbf{G}_{k,p}(\frac{1}{\sqrt{p}}\Delta_{p,\Phi}) Qs, s'\rangle = \langle s, Q^* \mathbf{G}_{k,p}(\frac{1}{\sqrt{p}}\Delta_{p,\Phi}) \Delta^{m'}_{p,\Phi} s'\rangle$. Hence from (1.6), (1.8), we infer that for $l, m' \in \mathbb{N}$, there exist $C, C' > 0$ such that for $p > 1$,

$$\begin{aligned}\bigg\|\Delta^{m'}_{p,\Phi} G_{k,p}\bigg(\frac{1}{\sqrt{p}}\Delta_{p,\Phi}\bigg) Qs\bigg\|_{L^2} &\leqslant Cp^{-l}\|s\|_{L^2},\\ \bigg\|\Delta^{m'}_{p,\Phi}\bigg(H_{k,p}\bigg(\frac{1}{\sqrt{p}}\Delta_{p,\Phi}\bigg) - P_{\mathcal{H}_p}\bigg) Qs\bigg\|_{L^2} &\leqslant C' p^{2m-\frac{k}{2}}\|s\|_{L^2}.\end{aligned} \tag{1.16}$$

We deduce from (1.15) and (1.16) that if P, Q are differential operators with compact support in U_i, U_j respectively, then for each $l \in \mathbb{N}$, there exists $C > 0$ such that for $p \geqslant 1$,

$$\begin{aligned}\bigg\|PG_{k,p}\bigg(\frac{1}{\sqrt{p}}\Delta_{p,\Phi}\bigg) Qs\bigg\|_{L^2} &\leqslant Cp^{-l}\|s\|_{L^2},\\ \bigg\|P\bigg(H_{k,p}\bigg(\frac{1}{\sqrt{p}}\Delta_{p,\Phi}\bigg) - P_{\mathcal{H}_p}\bigg) Qs\bigg\|_{L^2} &\leqslant Cp^{2(m+m')-\frac{k}{2}}\|s\|_{L^2}.\end{aligned} \tag{1.17}$$

Using the Sobolev inequality on $U_i \times U_j$ we see for every $l, m \in \mathbb{N}$, there exist $C_{l,m} > 0$ and $C_m > 0$ such that for all $p \geqslant 1$ the following estimates hold:

$$\left|G_{k,p}\left(\frac{1}{\sqrt{p}}\Delta_{p,\Phi}\right)(x,x')\right|_{\mathscr{C}^m} \leqslant C_{l,m}p^{-l},$$
$$\left|\left(H_{k,p}\left(\frac{1}{\sqrt{p}}\Delta_{p,\Phi}\right) - P_{0,p}\right)(x,x')\right|_{\mathscr{C}^m} \leqslant C_m p^{4(m+n+1)-\frac{k}{2}}. \tag{1.18}$$

By (1.10) and (1.18), we get our Proposition 1.2. □

Using (1.4), (1.5) and the finite propagation speed [19, §7.8], [47, §4.4], it is clear that for $x, x' \in X$, $\widetilde{F}_k(\frac{1}{\sqrt{p}}\Delta_{p,\Phi})(x,\cdot)$ only depends on the restriction of $\Delta_{p,\Phi}$ to $B^X(x, \varepsilon p^{-\frac{1}{4}})$, and $\widetilde{F}_k(\frac{1}{\sqrt{p}}\Delta_{p,\Phi})(x,x') = 0$, if $d(x,x') \geqslant \varepsilon p^{-\frac{1}{4}}$. This means that the asymptotic of $P_{q,p}(x,\cdot)$ when $p \to +\infty$, modulo $\mathscr{O}(p^{-\infty})$ (i.e. terms whose $\mathscr{C}^m$ norm is $\mathscr{O}(p^{-l})$ for every $l, m \in \mathbb{N}$), only depends on the restriction of $\Delta_{p,\Phi}$ to $B^X(x, \varepsilon p^{-\frac{1}{4}})$.

1.2. Rescaling and a Taylor expansion of the operator $\Delta_{p,\Phi}$

We fix $x_0 \in X$. From now on, we identify $B^{T_{x_0}X}(0, 4\varepsilon)$ with $B^X(x_0, 4\varepsilon)$. For $Z \in B^{T_{x_0}X}(0, 4\varepsilon)$ we identify L_Z, E_Z and $(L^p \otimes E)_Z$ to L_{x_0}, E_{x_0} and $(L^p \otimes E)_{x_0}$ by parallel transport with respect to the connections ∇^L, ∇^E and $\nabla^{L^p\otimes E}$ along the curve $\gamma_Z : [0,1] \ni u \to \exp^X_{x_0}(uZ)$. Let $\{e_i\}_i$ be an oriented orthonormal basis of $T_{x_0}X$, and let $\{e^i\}_i$ be its dual basis.

Let us identify $\mathbb{R}^{2n} \simeq T_{x_0}X$ by

$$\mathbb{R}^{2n} \ni (Z_1, \dots, Z_{2n}) \longmapsto \sum_i Z_i e_i \in T_{x_0}X. \tag{1.19}$$

For $\varepsilon > 0$ small enough, we will extend the geometric objects from $B^{T_{x_0}X}(0,\varepsilon)$ to $\mathbb{R}^{2n} \simeq T_{x_0}X$ such that $\Delta_{p,\Phi}$ becomes the restriction of a renormalized Bochner-Laplacian on $\mathbb{R}^{2n}$ associated to a Hermitian line bundle with positive curvature. In this way, we are able to replace X by $\mathbb{R}^{2n}$.

We denote in the sequel $X_0 = \mathbb{R}^{2n} \simeq T_{x_0}X$. We consider the trivial bundles L_0, E_0 with fibers L_{x_0}, E_{x_0} on X_0. We still denote by ∇^L, ∇^E, h^L, etc. the connections and metrics on L_0, E_0 on $B^{T_{x_0}X}(0, 4\varepsilon)$ induced by the above identification. Then h^L, h^E get identified to the constant metrics $h^{L_0} = h^{L_{x_0}}$, $h^{E_0} = h^{E_{x_0}}$.

Let $\rho : \mathbb{R} \to [0,1]$ be a smooth even function such that

$$\rho(v) = \begin{cases} 1 & \text{if } |v| < 2, \\ 0 & \text{if } |v| > 4. \end{cases} \tag{1.20}$$

Let $\varphi_\varepsilon : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is the map defined by $\varphi_\varepsilon(Z) = \rho(|Z|/\varepsilon)Z$. Then $\Phi_0 = \Phi \circ \varphi_\varepsilon$ is a smooth self-adjoint section of $\mathrm{End}(E_0)$ on X_0. We equip X_0 with the metric $g^{TX_0}(Z) = g^{TX}(\varphi_\varepsilon(Z))$ and with the complex structure $J_0(Z) = J(\varphi_\varepsilon(Z))$. Set $\nabla^{E_0} = \varphi_\varepsilon^*\nabla^E$. Then ∇^{E_0} is the extension of ∇^E on $B^{T_{x_0}X}(0,\varepsilon)$. If $\mathcal{R} = \sum_i Z_i e_i = Z$ denotes the radial vector field on $\mathbb{R}^{2n}$, then we define the Hermitian connection ∇^{L_0} on (L_0, h^{L_0}) by

$$\nabla^{L_0}\big|_Z = \varphi_\varepsilon^*\nabla^L + \frac{1}{2}\big(1 - \rho^2\big(|Z|/\varepsilon\big)\big) R^L_{x_0}(\mathcal{R}, \cdot). \tag{1.21}$$

Then we calculate easily that its curvature $R^{L_0} = (\nabla^{L_0})^2$ is

$$\begin{aligned} R^{L_0}(Z) &= \varphi_\varepsilon^* R^L + \frac{1}{2} d\big(\big(1-\rho^2\big(|Z|/\varepsilon\big)\big) R^L_{x_0}(\mathcal{R},\cdot)\big) \\ &= \big(1-\rho^2\big(|Z|/\varepsilon\big)\big) R^L_{x_0} + \rho^2\big(|Z|/\varepsilon\big) R^L_{\varphi_\varepsilon(Z)} \\ &\quad - (\rho\rho')\big(|Z|/\varepsilon\big) \frac{Z_i e^i}{\varepsilon|Z|} \wedge \big[R^L_{x_0}(\mathcal{R},\cdot) - R^L_{\varphi_\varepsilon(Z)}(\mathcal{R},\cdot)\big]. \end{aligned} \tag{1.22}$$

Recall that μ_0 was defined in (0.3) as the infimum on X of the smallest eigenvalue of $\sqrt{-1}R^L(\cdot, J\cdot)$ with respect to g^{TX}. Formula (1.22) shows that for $\varepsilon > 0$ small enough R^{L_0} is positive, i.e., $\sqrt{-1}\sum_j R^{L_0}(e_j, J_0 e_j) > 0$ for every local orthonormal frame $\{e_j\}$ of TX_0 and satisfies the following estimate for any $x_0 \in X$,

$$\inf\big\{\sqrt{-1}R^{L_0}_Z(u, J_0 u)/|u|^2_{g^{TX_0}}\colon u \in T_Z X_0,\ Z \in X_0\big\} \geqslant \frac{4}{5}\mu_0.$$

From now on we fix such an $\varepsilon > 0$.

Let $\Delta^{X_0}_{p,\Phi_0} = \Delta^{L_0^p \otimes E_0} - p\tau_0 + \Phi_0$ be the renormalized Bochner-Laplacian on $X_0 = \mathbb{R}^{2n} \simeq T_{x_0}X$ associated to the above data, as in (0.4). Observe that by the previous estimate R^{L_0} is uniformly positive on $\mathbb{R}^{2n}$, so by [33, (3.2), (3.11) and (3.12), pp. 656–658] the operator $\Delta^{X_0}_{p,\Phi_0}$ admits a spectral gap analogous to (0.5). Specifically, there exists $C_{L_0} > 0$ such that

$$\operatorname{Spec}\big(\Delta^{X_0}_{p,\Phi_0}\big) \subset [-C_{L_0}, C_{L_0}] \cup \left[\frac{8}{5}p\mu_0 - C_{L_0}, +\infty\right[. \tag{1.23}$$

Let S_L be a unit vector of L_{x_0}. Using S_L and the above discussion, we get an isometry $E_0 \otimes L_0^p \simeq E_{x_0}$. Let $P_{0,\mathcal{H}_p}$ be the spectral projection of $\Delta^{X_0}_{p,\Phi_0}$ from $\mathscr{C}^\infty(X_0, L_0^p \otimes E_0) \simeq \mathscr{C}^\infty(X_0, E_{x_0})$ corresponding to the interval $[-C_{L_0}, C_{L_0}]$, and let $P_{0,q,p}(x,x')$ ($q \geqslant 0$) be the smooth kernels of $P_{0,q,p} = (\Delta^{X_0}_{p,\Phi_0})^q P_{0,\mathcal{H}_p}$ (we set $(\Delta^{X_0}_{p,\Phi_0})^0 = 1$) with respect to the volume form $dv_{X_0}(x')$. The following proposition shows that $P_{q,p}$ and $P_{0,q,p}$ are asymptotically close on $B^{T_{x_0}X}(0,\varepsilon)$ in the $\mathscr{C}^\infty$-topology, as $p \to \infty$.

The following result is an analogue of [20, Proposition 4.4].

Proposition 1.3. *For every $l, m \in \mathbb{N}$, there exists $C_{l,m} > 0$ such that for $x, x' \in B^{T_{x_0}X}(0,\varepsilon)$, $x_0 \in X$,*

$$\big|(P_{0,q,p} - P_{q,p})(x,x')\big|_{\mathscr{C}^m} \leqslant C_{l,m} p^{-l}. \tag{1.24}$$

Proof. Using (1.4) and (1.23), we know that for $x, x' \in B^{T_{x_0}X}(0,\varepsilon)$,

$$\left|\widetilde{F}_k\left(\frac{1}{\sqrt{p}}\Delta_{p,\Phi}\right)(x,x') - P_{0,0,p}(x,x')\right|_{\mathscr{C}^m} \leqslant C_{k,m} p^{-\frac{k}{2}+4(m+n+1)}. \tag{1.25}$$

Thus from (1.7) and (1.25) for k big enough, we infer (1.24) in the particular case $q = 0$. Taking into account the definition of $P_{0,q,p}$ and $P_{q,p}$, (1.11) and the $q = 0$ case of (1.24) entail (1.24) in general. □

It suffices therefore to study the kernel $P_{0,q,p}$ and for this purpose we rescale the operator $\Delta^{X_0}_{p,\Phi_0}$. Let dv_{TX} be the Riemannian volume form of $(T_{x_0}X, g^{T_{x_0}X})$. Let $\kappa(Z)$ be the smooth positive function defined by the equation

$$dv_{X_0}(Z) = \kappa(Z)\, dv_{TX}(Z), \tag{1.26}$$

with $\kappa(0) = 1$. Denote by ∇_U the ordinary differentiation operator on $T_{x_0}X$ in the direction U, and set $\partial_i = \nabla_{e_i}$. If $\alpha = (\alpha_1, \dots, \alpha_{2n})$ is a multi-index, then set $Z^\alpha = Z_1^{\alpha_1} \cdots Z_{2n}^{\alpha_{2n}}$. We also denote by $(\partial^\alpha R^L)_{x_0}$ the tensor $(\partial^\alpha R^L)_{x_0}(e_i, e_j) = \partial^\alpha (R^L(e_i, e_j))_{x_0}$. Denote by $t = \frac{1}{\sqrt{p}}$. For $s \in \mathscr{C}^\infty(\mathbb{R}^{2n}, E_{x_0})$ and $Z \in \mathbb{R}^{2n}$, set

$$\begin{aligned}(S_t s)(Z) = s(Z/t), \qquad &\nabla_t = t S_t^{-1} \kappa^{\frac{1}{2}} \nabla^{L_0^p \otimes E_0} \kappa^{-\frac{1}{2}} S_t, \\ &\mathscr{L}_t = S_t^{-1} \frac{1}{p} \kappa^{\frac{1}{2}} \Delta^{X_0}_{p,\Phi_0} \kappa^{-\frac{1}{2}} S_t.\end{aligned} \tag{1.27}$$

The operator $\mathscr{L}_t$ is the rescaled operator, which we now develop in Taylor series. In the following result we draw on [20, Theorem 4.6] and calculate two more terms of the asymptotic expansion of $\mathscr{L}_t$.

Theorem 1.4. *There exist polynomials $\mathcal{A}_{i,j,r}$ (resp. $\mathcal{B}_{i,r}$, $\mathcal{C}_r$) ($r \in \mathbb{N}$, $i, j \in \{1, \dots, 2n\}$) in Z with the following properties*:

(i) *their coefficients are polynomials in R^{TX} (resp. R^{TX}, R^E, Φ, R^L) and their derivatives at x_0 up to order $r-2$ (resp. $r-2$, $r-2$, $r-2$, r),*
(ii) *$\mathcal{A}_{i,j,r}$ is a homogeneous polynomial in Z of degree r, the degree in Z of $\mathcal{B}_{i,r}$ is $\leqslant r+1$ (resp. $\mathcal{C}_r$ is $\leqslant r+2$), and has the same parity with $r-1$ (resp. r),*
(iii) *if we denote by*

$$\mathcal{O}_r = \mathcal{A}_{i,j,r} \nabla_{e_i} \nabla_{e_j} + \mathcal{B}_{i,r} \nabla_{e_i} + \mathcal{C}_r, \tag{1.28}$$

then

$$\mathscr{L}_t = \mathscr{L}_0 + \sum_{r=1}^{m} t^r \mathcal{O}_r + \mathscr{O}\big(t^{m+1}\big), \tag{1.29}$$

and there exists $m' \in \mathbb{N}$ so that for every $k \in \mathbb{N}$, $t \leqslant 1$, the derivatives up to order k of the coefficients of the operator $\mathscr{O}(t^{m+1})$ are dominated by $Ct^{m+1}(1+|Z|)^{m'}$. Moreover

$$\begin{aligned}\mathscr{L}_0 &= -\sum_j \left(\nabla_{e_j} + \frac{1}{2} R^L_{x_0}(Z, e_j)\right)^2 - \tau_{x_0}, \\ \mathcal{O}_1(Z) &= -\frac{2}{3}\big(\partial_j R^L\big)_{x_0}(\mathcal{R}, e_i) Z_j \left(\nabla_{e_i} + \frac{1}{2} R^L_{x_0}(\mathcal{R}, e_i)\right) - \frac{1}{3}\big(\partial_i R^L\big)_{x_0}(\mathcal{R}, e_i) - (\nabla_{\mathcal{R}} \tau)_{x_0},\end{aligned}$$

$$\begin{aligned}
\mathcal{O}_2(Z) = {} & \frac{1}{3}\big\langle R^{TX}_{x_0}(\mathcal{R}, e_i)\mathcal{R}, e_j\big\rangle_{x_0}\bigg(\nabla_{e_i} + \frac{1}{2}R^L_{x_0}(\mathcal{R}, e_i)\bigg)\bigg(\nabla_{e_j} + \frac{1}{2}R^L_{x_0}(\mathcal{R}, e_j)\bigg) \\
& + \bigg[\frac{2}{3}\big\langle R^{TX}_{x_0}(\mathcal{R}, e_j)e_j, e_i\big\rangle_{x_0} - \bigg(\frac{1}{2}\sum_{|\alpha|=2}\big(\partial^\alpha R^L\big)_{x_0}\frac{Z^\alpha}{\alpha!} + R^E_{x_0}\bigg)(\mathcal{R}, e_i)\bigg] \\
& \times\bigg(\nabla_{e_i} + \frac{1}{2}R^L_{x_0}(\mathcal{R}, e_i)\bigg) \\
& - \frac{1}{4}\nabla_{e_i}\bigg(\sum_{|\alpha|=2}\big(\partial^\alpha R^L\big)_{x_0}\frac{Z^\alpha}{\alpha!}(\mathcal{R}, e_i)\bigg) - \frac{1}{9}\sum_i\bigg[\sum_j\big(\partial_j R^L\big)_{x_0}(\mathcal{R}, e_i)Z_j\bigg]^2 \\
& - \frac{1}{12}\big[\mathscr{L}_0, \big\langle R^{TX}_{x_0}(\mathcal{R}, e_i)\mathcal{R}, e_i\big\rangle_{x_0}\big] - \sum_{|\alpha|=2}\big(\partial^\alpha\tau\big)_{x_0}\frac{Z^\alpha}{\alpha!} + \Phi_{x_0}. \qquad (1.30)
\end{aligned}$$

Proof. Set $g_{ij}(Z) = g^{TX}(e_i, e_j)(Z) = \langle e_i, e_j\rangle_Z$ and let $(g^{ij}(Z))$ be the inverse of the matrix $(g_{ij}(Z))$. By [1, Proposition 1.28], the Taylor expansion of $g_{ij}(Z)$ with respect to the basis $\{e_i\}$ up to order r is a polynomial of the Taylor expansion of R^{TX} up to order $r-2$, moreover

$$\begin{aligned}
g_{ij}(Z) &= \delta_{ij} + \frac{1}{3}\big\langle R^{TX}_{x_0}(\mathcal{R}, e_i)\mathcal{R}, e_j\big\rangle_{x_0} + \mathscr{O}\big(|Z|^3\big), \\
\kappa(Z) = \big|\det\big(g_{ij}(Z)\big)\big|^{1/2} &= 1 + \frac{1}{6}\big\langle R^{TX}_{x_0}(\mathcal{R}, e_i)\mathcal{R}, e_i\big\rangle_{x_0} + \mathscr{O}\big(|Z|^3\big). \qquad (1.31)
\end{aligned}$$

If Γ^l_{ij} is the connection form of ∇^{TX} with respect to the basis $\{e_i\}$, then we have $(\nabla^{TX}_{e_i}e_j)(Z) = \Gamma^l_{ij}(Z)e_l$. Owing to (1.31),

$$\begin{aligned}
\Gamma^l_{ij}(Z) &= \frac{1}{2}g^{lk}(\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij})(Z) \\
&= \frac{1}{3}\big[\big\langle R^{TX}_{x_0}(\mathcal{R}, e_j)e_i, e_l\big\rangle_{x_0} + \big\langle R^{TX}_{x_0}(\mathcal{R}, e_i)e_j, e_l\big\rangle_{x_0}\big] + \mathscr{O}\big(|Z|^2\big). \qquad (1.32)
\end{aligned}$$

Now by (1.2),

$$\Delta_{p,\Phi} = -g^{ij}\big(\nabla^{L^p\otimes E}_{e_i}\nabla^{L^p\otimes E}_{e_j} - \nabla^{L^p\otimes E}_{\nabla^{TX}_{e_i}e_j}\big) - p\tau + \Phi, \qquad (1.33)$$

so from (1.27) and (1.33) we infer the expression

$$\mathscr{L}_t = -g^{ij}(tZ)\big[\nabla_{t,e_i}\nabla_{t,e_j} - t\Gamma^l_{ij}(tZ)\nabla_{t,e_l}\big] - \tau(tZ) + t^2\Phi(tZ). \qquad (1.34)$$

Let Γ^E, Γ^L be the connection forms of ∇^E and ∇^L with respect to some fixed frames for E, L which are parallel along the curve γ_Z under our trivializations on $B^{T_{x_0}X}(0,\varepsilon)$. (1.27) yields on $B^{T_{x_0}X}(0,\varepsilon/t)$

$$\nabla_{t,e_i}|_Z = \kappa^{\frac{1}{2}}(tZ)\bigg(\nabla_{e_i} + \frac{1}{t}\Gamma^L(e_i)(tZ) + t\Gamma^E(e_i)(tZ)\bigg)\kappa^{-\frac{1}{2}}(tZ). \qquad (1.35)$$

Let $\Gamma^\bullet = \Gamma^E, \Gamma^L$ and $R^\bullet = R^E, R^L$, respectively. By [1, Proposition 1.18] the Taylor coefficients of $\Gamma^\bullet(e_j)(Z)$ at x_0 up to order r are only determined by those of $R^\bullet$ up to order $r-1$, and

$$\sum_{|\alpha|=r} \left(\partial^\alpha \Gamma^\bullet\right)_{x_0}(e_j)\frac{Z^\alpha}{\alpha!} = \frac{1}{r+1}\sum_{|\alpha|=r-1} \left(\partial^\alpha R^\bullet\right)_{x_0}(\mathcal{R}, e_j)\frac{Z^\alpha}{\alpha!}. \tag{1.36}$$

Owing to (1.31), (1.36)

$$\begin{aligned}
\mathscr{L}_t = {} & -\left(\delta_{ij} - \frac{t^2}{3}\left\langle R^{TX}_{x_0}(\mathcal{R}, e_i)\mathcal{R}, e_j\right\rangle + \mathscr{O}\left(t^3\right)\right)\kappa^{\frac{1}{2}}(tZ) \\
& \times \Bigg\{\left[\nabla_{e_i} + \left(\frac{1}{2}R^L_{x_0} + \frac{t}{3}\left(\partial_k R^L\right)_{x_0} Z_k + \frac{t^2}{4}\sum_{|\alpha|=2}\left(\partial^\alpha R^L\right)_{x_0}\frac{Z^\alpha}{\alpha!} + \frac{t^2}{2}R^E_{x_0}\right)(\mathcal{R}, e_i) + \mathscr{O}\left(t^3\right)\right] \\
& \times \left[\nabla_{e_j} + \left(\frac{1}{2}R^L_{x_0} + \frac{t}{3}\left(\partial_k R^L\right)_{x_0} Z_k + \frac{t^2}{4}\sum_{|\alpha|=2}\left(\partial^\alpha R^L\right)_{x_0}\frac{Z^\alpha}{\alpha!} + \frac{t^2}{2}R^E_{x_0}\right)(\mathcal{R}, e_j) + \mathscr{O}\left(t^3\right)\right] \\
& - t\Gamma^l_{ij}(tZ)\left(\nabla_{e_l} + \frac{1}{2}R^L_{x_0}(\mathcal{R}, e_l) + \mathscr{O}(t)\right)\Bigg\}\kappa^{-\frac{1}{2}}(tZ) \\
& - \tau_{x_0} - t(\nabla_{\mathcal{R}}\tau)_{x_0} - t^2\sum_{|\alpha|=2}\left(\partial^\alpha\tau\right)_{x_0}\frac{Z^\alpha}{\alpha!} + t^2\Phi_{x_0} + \mathscr{O}\left(t^3\right).
\end{aligned} \tag{1.37}$$

Relations (1.31) and (1.34)–(1.37) settle our theorem. □

1.3. Uniform estimate of the generalized Bergman kernels

We shall estimate the Sobolev norm of the resolvent of $\mathscr{L}_t$ so we introduce the following norms. We denote by $\langle\cdot,\cdot\rangle_{0,L^2}$ and $\|\cdot\|_{0,L^2}$ the inner product and the L^2 norm on $\mathscr{C}^\infty(X_0, E_{x_0})$ induced by g^{TX_0}, h^{E_0} as in (1.1). For $s \in \mathscr{C}^\infty_0(X_0, E_{x_0})$ set

$$\begin{aligned}
\|s\|^2_{t,0} &= \|s\|^2_0 = \int_{\mathbb{R}^{2n}} \left|s(Z)\right|^2_{h^{E_{x_0}}} dv_{TX}(Z), \\
\|s\|^2_{t,m} &= \sum_{l=0}^{m}\sum_{i_1,\dots,i_l=1}^{2n} \|\nabla_{t,e_{i_1}}\cdots\nabla_{t,e_{i_l}} s\|^2_{t,0}.
\end{aligned} \tag{1.38}$$

We denote by $\langle s', s\rangle_{t,0}$ the inner product on $\mathscr{C}^\infty(X_0, E_{x_0})$ corresponding to $\|\cdot\|^2_{t,0}$. Let $\boldsymbol{H}^m_t$ be the Sobolev space of order m with norm $\|\cdot\|_{t,m}$. Let $\boldsymbol{H}^{-1}_t$ be the Sobolev space of order -1 and let $\|\cdot\|_{t,-1}$ be the norm on $\boldsymbol{H}^{-1}_t$ defined by $\|s\|_{t,-1} = \sup_{0\neq s'\in \boldsymbol{H}^1_t} |\langle s, s'\rangle_{t,0}|/\|s'\|_{t,1}$. If $A \in \mathscr{L}(\boldsymbol{H}^m_t, \boldsymbol{H}^{m'}_t)$ for $m, m' \in \mathbb{Z}$, then we denote by $\|A\|^{m,m'}_t$ the norm of A with respect to the norms $\|\cdot\|_{t,m}$ and $\|\cdot\|_{t,m'}$.

Remark 1.5. Note that $\Delta^{X_0}_{p,\Phi_0}$ is self-adjoint with respect to $\|\cdot\|_0$, thus by (1.26), (1.27) and (1.38), $\mathscr{L}_t$ is a formally self-adjoint elliptic operator with respect to $\|\cdot\|_0$, and is a smooth family of operators with the parameter $x_0 \in X$. Thus $\mathscr{L}_0$ and $\mathcal{O}_r$ are also formally self-adjoint with respect to $\|\cdot\|_0$. This will simplify the computation of the coefficients $b_{0,1}$ in (0.11) (cf. Section 2.3) and explains why we prefer to conjugate with $\kappa^{1/2}$ comparing to [20, (3.38)].

Theorems 1.6–1.9 are the analogues of [20, Theorems 4.7–4.10]. We include the proofs for the sake of completeness.

Theorem 1.6. *There exist constants $C_1, C_2, C_3 > 0$ such that for $t \in]0,1]$ and every $s, s' \in \mathscr{C}_0^\infty(\mathbb{R}^{2n}, E_{x_0})$,*

$$\begin{aligned} \langle \mathscr{L}_t s, s\rangle_{t,0} &\geqslant C_1 \|s\|_{t,1}^2 - C_2 \|s\|_{t,0}^2, \\ \left|\langle \mathscr{L}_t s, s'\rangle_{t,0}\right| &\leqslant C_3 \|s\|_{t,1}\|s'\|_{t,1}. \end{aligned} \tag{1.39}$$

Proof. Relations (0.4) and (1.2) yield

$$\langle \Delta_{p,\Phi} s, s\rangle_{0,L^2} = \left\|\nabla^{L_0^p\otimes E_0} s\right\|_{0,L^2}^2 - \left\langle (p\tau - \Phi)s, s\right\rangle_{0,L^2}. \tag{1.40}$$

Thus (1.27), (1.38) and (1.40) applied to $\kappa^{-1/2} S_t s$ instead of s, yield

$$\langle \mathscr{L}_t s, s\rangle_{t,0} = \|\nabla_t s\|_{t,0}^2 - \left\langle \left(S_t^{-1}\left(\tau - t^2\Phi\right)\right)s, s\right\rangle_{t,0}, \tag{1.41}$$

which implies (1.39). □

Let δ be the counterclockwise oriented circle in $\mathbb{C}$ of center 0 and radius $\mu_0/4$.

Theorem 1.7. *There exists $t_0 > 0$ such that the resolvent $(\lambda - \mathscr{L}_t)^{-1}$ exists for all $\lambda \in \delta$, $t \in]0, t_0]$. There exists $C > 0$ such that for all $t \in]0, t_0]$, $\lambda \in \delta$, and all $x_0 \in X$ we have*

$$\left\|(\lambda - \mathscr{L}_t)^{-1}\right\|_t^{0,0} \leqslant C, \qquad \left\|(\lambda - \mathscr{L}_t)^{-1}\right\|_t^{-1,1} \leqslant C. \tag{1.42}$$

Proof. By (1.23), (1.27), for t small enough,

$$\operatorname{Spec}(\mathscr{L}_t) \subset \left[-C_{L_0}t^2, C_{L_0}t^2\right] \cup [\mu_0, +\infty[. \tag{1.43}$$

Thus the resolvent $(\lambda - \mathscr{L}_t)^{-1}$ exists for $\lambda \in \delta$ and t small enough, and we get the first inequality of (1.42). By (1.39), $(\lambda_0 - \mathscr{L}_t)^{-1}$ exists for $\lambda_0 \in \mathbb{R}$, $\lambda_0 \leqslant -2C_2$, and $\|(\lambda_0 - \mathscr{L}_t)^{-1}\|_t^{-1,1} \leqslant \frac{1}{C_1}$. Now,

$$(\lambda - \mathscr{L}_t)^{-1} = (\lambda_0 - \mathscr{L}_t)^{-1} - (\lambda - \lambda_0)(\lambda - \mathscr{L}_t)^{-1}(\lambda_0 - \mathscr{L}_t)^{-1}. \tag{1.44}$$

Thus for $\lambda \in \delta$, from (1.44), we get

$$\left\|(\lambda - \mathscr{L}_t)^{-1}\right\|_t^{-1,0} \leqslant \frac{1}{C_1}\left(1 + \frac{4}{\mu_0}|\lambda - \lambda_0|\right). \tag{1.45}$$

Changing the last two factors in (1.44) and applying (1.45) we get

$$\left\|(\lambda-\mathscr{L}_t)^{-1}\right\|_t^{-1,1}\leqslant\frac{1}{C_1}+\frac{|\lambda-\lambda_0|}{{C_1}^2}\left(1+\frac{4}{\mu_0}|\lambda-\lambda_0|\right)\leqslant C. \tag{1.46}$$

The proof of our theorem is complete. □

Proposition 1.8. *Take $m\in\mathbb{N}^*$. There exists $C_m>0$ such that for $t\in\,]0,1]$, $Q_1,\dots,Q_m\in\{\nabla_{t,e_i},Z_i\}_{i=1}^{2n}$ and $s,s'\in\mathscr{C}_0^\infty(X_0,E_{x_0})$,*

$$\left|\left\langle\left[Q_1,\left[Q_2,\dots,[Q_m,\mathscr{L}_t]\dots\right]\right]s,s'\right\rangle_{t,0}\right|\leqslant C_m\|s\|_{t,1}\|s'\|_{t,1}. \tag{1.47}$$

Proof. Note that $[\nabla_{t,e_i},Z_j]=\delta_{ij}$, hence (1.34) implies that $[Z_j,\mathscr{L}_t]$ verifies (1.47). On the other hand, we obtain from (1.27)

$$[\nabla_{t,e_i},\nabla_{t,e_j}]=\left(R^{L_0}(tZ)+t^2R^{E_0}(tZ)\right)(e_i,e_j). \tag{1.48}$$

Thus from (1.34) and (1.48), we know that $[\nabla_{t,e_k},\mathscr{L}_t]$ has the same structure as $\mathscr{L}_t$ for $t\in\,]0,1]$, i.e. $[\nabla_{t,e_k},\mathscr{L}_t]$ has the same type as

$$\sum_{ij}a_{ij}(t,tZ)\nabla_{t,e_i}\nabla_{t,e_j}+\sum_i b_i(t,tZ)\nabla_{t,e_i}+c(t,tZ), \tag{1.49}$$

and $a_{ij}(t,Z)$, $b_i(t,Z)$, $c(t,Z)$ and their derivatives in Z are uniformly bounded for $Z\in\mathbb{R}^{2n}$, $t\in[0,1]$. Moreover they are polynomials in t.

If $(\nabla_{t,e_i})^*$ is the adjoint of ∇_{t,e_i} with respect to $\langle\cdot,\cdot\rangle_{t,0}$, then (1.38) yields

$$(\nabla_{t,e_i})^*=-\nabla_{t,e_i}-t\left(\kappa^{-1}(e_i\kappa)\right)(tZ). \tag{1.50}$$

Thus by (1.49) and (1.50), (1.47) is verified for $m=1$.

By recurrence, it follows that $[Q_1,[Q_2,\dots,[Q_m,\mathscr{L}_t]\dots]]$ has the same structure (1.49) as $\mathscr{L}_t$, so from (1.50) we get the required assertion. □

Theorem 1.9. *For every $t\in\,]0,t_0]$, $\lambda\in\delta$, $m\in\mathbb{N}$, the resolvent $(\lambda-\mathscr{L}_t)^{-1}$ maps $\boldsymbol{H}_t^m$ into $\boldsymbol{H}_t^{m+1}$. Moreover, for every $\alpha\in\mathbb{N}^{2n}$, there exists $C_{\alpha,m}>0$ such that for $t\in\,]0,t_0]$, $\lambda\in\delta$, $s\in\mathscr{C}^\infty(X_0,E_{x_0})$,*

$$\left\|Z^\alpha(\lambda-\mathscr{L}_t)^{-1}s\right\|_{t,m+1}\leqslant C_{\alpha,m}\sum_{\alpha'\leqslant\alpha}\left\|Z^{\alpha'}s\right\|_{t,m}. \tag{1.51}$$

Proof. For $Q_1,\dots,Q_m\in\{\nabla_{t,e_i}\}_{i=1}^{2n}$, $Q_{m+1},\dots,Q_{m+|\alpha|}\in\{Z_i\}_{i=1}^{2n}$, we can express $Q_1\cdots Q_{m+|\alpha|}(\lambda-\mathscr{L}_t)^{-1}$ as a linear combination of operators of the type

$$\left[Q_1,\left[Q_2,\dots,\left[Q_{m'},(\lambda-\mathscr{L}_t)^{-1}\right]\dots\right]\right]Q_{m'+1}\cdots Q_{m+|\alpha|},\quad m'\leqslant m+|\alpha|. \tag{1.52}$$

Let $\mathscr{R}_t$ be the family of operators $\mathscr{R}_t = \{[Q_{j_1}, [Q_{j_2}, \dots, [Q_{j_l}, \mathscr{L}_t] \dots]]\}$. Clearly, every commutator $[Q_1, [Q_2, \dots, [Q_{m'}, (\lambda - \mathscr{L}_t)^{-1}] \dots]]$ is a linear combination of operators of the form

$$(\lambda - \mathscr{L}_t)^{-1} R_1 (\lambda - \mathscr{L}_t)^{-1} R_2 \cdots R_{m'} (\lambda - \mathscr{L}_t)^{-1} \tag{1.53}$$

with $R_1, \dots, R_{m'} \in \mathscr{R}_t$.

From Proposition 1.8 we deduce that the norm $\|\cdot\|_t^{1,-1}$ of the operators $R_j \in \mathscr{R}_t$ is uniformly bounded from above by a constant. Hence by Theorem 1.7 there exists $C > 0$, such that the norm $\|\cdot\|_t^{0,1}$ of operators (1.53) is dominated by C. □

The next step is to convert the estimates for the resolvent into estimates for the spectral projection $\mathcal{P}_{0,t} : (\mathscr{C}^\infty(X_0, E_{x_0}), \|\cdot\|_0) \to (\mathscr{C}^\infty(X_0, E_{x_0}), \|\cdot\|_0)$ of $\mathscr{L}_t$ corresponding to the interval $[-C_{L_0} t^2, C_{L_0} t^2]$. Let $\mathcal{P}_{q,t}(Z, Z') = \mathcal{P}_{q,t,x_0}(Z, Z')$ (with $Z, Z' \in X_0$, $q \geqslant 0$) be the smooth kernel of $\mathcal{P}_{q,t} = (\mathscr{L}_t)^q \mathcal{P}_{0,t}$ (we set $(\mathscr{L}_t)^0 = 1$) with respect to $dv_{TX}(Z')$. Note that $\mathscr{L}_t$ is a family of differential operators on $T_{x_0}X$ with coefficients in $\mathrm{End}(E)_{x_0}$. Let $\pi : TX \times_X TX \to X$ be the natural projection from the fiberwise product of TX on X. Then we can view $\mathcal{P}_{q,t}(Z, Z')$ as a smooth section of $\pi^*(\mathrm{End}(E))$ over $TX \times_X TX$ by identifying a section $S \in \mathscr{C}^\infty(TX \times_X TX, \pi^* \mathrm{End}(E))$ with the family $(S_x)_{x \in X}$, where $S_x = S|_{\pi^{-1}(x)}$. Let $\nabla^{\mathrm{End}(E)}$ be the connection on $\mathrm{End}(E)$ induced by ∇^E. Then $\nabla^{\pi^* \mathrm{End}(E)}$ induces naturally a $\mathscr{C}^m$-norm of S for the parameter $x_0 \in X$.

In the following result we adapt [20, Theorem 4.11] to the present situation.

Theorem 1.10. *For every $m, m', r \in \mathbb{N}$, $\sigma > 0$, there exists $C > 0$, such that for $t \in \,]0, t_0]$, $Z, Z' \in T_{x_0}X$, $|Z|, |Z'| \leqslant \sigma$,*

$$\sup_{|\alpha|+|\alpha'| \leqslant m} \left| \frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^\alpha \partial Z'^{\alpha'}} \frac{\partial^r}{\partial t^r} \mathcal{P}_{q,t}(Z, Z') \right|_{\mathscr{C}^{m'}(X)} \leqslant C. \tag{1.54}$$

Here $\mathscr{C}^{m'}(X)$ is the $\mathscr{C}^{m'}$ norm for the parameter $x_0 \in X$.

Proof. By (1.43), for every $k \in \mathbb{N}^*$, $q \geqslant 0$,

$$\mathcal{P}_{q,t} = (\mathscr{L}_t)^q \mathcal{P}_{0,t} = \frac{1}{2\pi i} \begin{pmatrix} q+k-1 \\ k-1 \end{pmatrix}^{-1} \int_\delta \lambda^{q+k-1} (\lambda - \mathscr{L}_t)^{-k} \, d\lambda. \tag{1.55}$$

For $m \in \mathbb{N}$, let $\mathcal{Q}^m$ be the set of operators $\{\nabla_{t,e_{i_1}} \cdots \nabla_{t,e_{i_j}}\}_{j \leqslant m}$. From Theorem 1.9, we deduce that if $Q \in \mathcal{Q}^m$, then there is $C_m > 0$ such that

$$\left\| Q(\lambda - \mathscr{L}_t)^{-m} \right\|_t^{0,0} \leqslant C_m, \quad \text{for all } \lambda \in \delta. \tag{1.56}$$

Observe that $\mathscr{L}_t$ is formally self-adjoint with respect to $\|\cdot\|_{t,0}$, so after taking the adjoint of (1.56), we have

$$\left\| (\lambda - \mathscr{L}_t)^{-m} Q \right\|_t^{0,0} \leqslant C_m. \tag{1.57}$$

From (1.55), (1.56) and (1.57), we obtain

$$\|Q\mathcal{P}_{q,t}Q'\|_t^{0,0} \leqslant C_m, \quad \text{for } Q, Q' \in \mathcal{Q}^m. \tag{1.58}$$

Let $|\cdot|_{(\sigma),m}$ be the usual Sobolev norm on $\mathscr{C}^\infty(B^{T_{x_0}X}(0,\sigma+1), E_{x_0})$ induced by $h^{E_{x_0}}$ and the volume form $dv_{TX}(Z)$ as in (1.38). Let $\|A\|_{(\sigma)}$ be the operator norm of $A \in \mathscr{L}(L^2(B^{T_{x_0}X}(0,\sigma+1), E_{x_0}))$ with respect to $|\cdot|_{(\sigma),0}$. Observe that by (1.35), (1.38), for $m>0$, there exists $C_\sigma > 0$ such that for $s \in \mathscr{C}^\infty(X_0, E_{x_0})$, $\operatorname{supp}(s) \subset B^{T_{x_0}X}(0,\sigma+1)$,

$$\frac{1}{C_\sigma}\|s\|_{t,m} \leqslant |s|_{(\sigma),m} \leqslant C_\sigma \|s\|_{t,m}. \tag{1.59}$$

Now (1.58) and (1.59) together with Sobolev's inequalities imply

$$\sup_{|Z|,|Z'|\leqslant\sigma} \left|Q_Z Q'_{Z'}\mathcal{P}_{q,t}(Z,Z')\right| \leqslant C, \quad \text{for } Q, Q' \in \mathcal{Q}^m. \tag{1.60}$$

Thanks to $\Gamma^L(e_i)(0)=0$, (1.35) and (1.60), estimate (1.54) holds for $r=m'=0$. To obtain (1.54) for $r \geqslant 1$ and $m'=0$, note that from (1.55),

$$\frac{\partial^r}{\partial t^r}\mathcal{P}_{q,t} = \frac{1}{2\pi i}\begin{pmatrix} q+k-1 \\ k-1\end{pmatrix}^{-1} \int_\delta \lambda^{q+k-1}\frac{\partial^r}{\partial t^r}(\lambda-\mathscr{L}_t)^{-k}\,d\lambda, \quad \text{for } k \geqslant 1. \tag{1.61}$$

Set

$$I_{k,r} = \left\{(\mathbf{k},\mathbf{r}) = (k_i,r_i) \,\Big|\, \sum_{i=0}^j k_i = k+j,\ \sum_{i=1}^j r_i = r,\ k_i, r_i \in \mathbb{N}^*\right\}. \tag{1.62}$$

Then there exist $a_{\mathbf{r}}^{\mathbf{k}} \in \mathbb{R}$ such that

$$\begin{aligned} A_{\mathbf{r}}^{\mathbf{k}}(\lambda,t) &= (\lambda-\mathscr{L}_t)^{-k_0}\frac{\partial^{r_1}\mathscr{L}_t}{\partial t^{r_1}}(\lambda-\mathscr{L}_t)^{-k_1}\cdots\frac{\partial^{r_j}\mathscr{L}_t}{\partial t^{r_j}}(\lambda-\mathscr{L}_t)^{-k_j}, \\ \frac{\partial^r}{\partial t^r}(\lambda-\mathscr{L}_t)^{-k} &= \sum_{(\mathbf{k},\mathbf{r})\in I_{k,r}} a_{\mathbf{r}}^{\mathbf{k}} A_{\mathbf{r}}^{\mathbf{k}}(\lambda,t). \end{aligned} \tag{1.63}$$

We claim that $A_{\mathbf{r}}^{\mathbf{k}}(\lambda,t)$ is well defined and for every $m \in \mathbb{N}$, $k > 2(m+r+1)$, $Q, Q' \in \mathcal{Q}^m$, there exists $C>0$ such that for $\lambda \in \delta$, $t \in\,]0,t_0]$,

$$\left\|QA_{\mathbf{r}}^{\mathbf{k}}(\lambda,t)Q's\right\|_{t,0} \leqslant C\sum_{|\beta|\leqslant 2r}\left\|Z^\beta s\right\|_{t,0}. \tag{1.64}$$

In fact, by (1.34), $\frac{\partial^r}{\partial t^r}\mathscr{L}_t$ is a combination of

$$\frac{\partial^{r_1}}{\partial t^{r_1}}\big(g^{ij}(tZ)\big)\cdot\left(\frac{\partial^{r_2}}{\partial t^{r_2}}\nabla_{t,e_i}\right)\cdot\left(\frac{\partial^{r_3}}{\partial t^{r_3}}\nabla_{t,e_j}\right),$$
$$\frac{\partial^{r_1}}{\partial t^{r_1}}\big(d(tZ)\big),\qquad \frac{\partial^{r_1}}{\partial t^{r_1}}\big(d_i(tZ)\big)\cdot\left(\frac{\partial^{r_2}}{\partial t^{r_2}}\nabla_{t,e_i}\right).$$

If $r_1\geqslant 1$, then $\frac{\partial^{r_1}}{\partial t^{r_1}}(d(tZ))$ (resp. $\frac{\partial^{r_1}}{\partial t^{r_1}}\nabla_{t,e_i}$), are functions of the type $d'(tZ)Z^\beta$, where $|\beta|\leqslant r_1$ (resp. $|\beta|\leqslant r_1+1$) and $d'(Z)$ and its derivatives with respect to Z are bounded smooth functions in Z.

Let $\mathscr{R}'_t$ be the family of operators of the type

$$\mathscr{R}'_t=\big\{\big[f_{j_1}Q_{j_1},\big[f_{j_2}Q_{j_2},\dots,[f_{j_l}Q_{j_l},\mathscr{L}_t]\dots\big]\big]\big\}$$

where f_{j_i} are smooth functions with bounded derivatives, and $Q_{j_i}\in\{\nabla_{t,e_l},Z_l\}_{l=1}^{2n}$.

To handle the operator $A^{\mathbf{k}}_{\mathbf{r}}(\lambda,t)Q'$, we will, as above, move all the terms Z^β in $d'(tZ)Z^\beta$ to the right-hand side of this operator. To do so, we always use the commutator relations, in the sense that each time we consider only the commutator only for Z_i, and not for Z^β with $|\beta|>1$. Then $A^{\mathbf{k}}_{\mathbf{r}}(\lambda,t)Q'$ turns out to be of the form $\sum_{|\beta|\leqslant 2r}L^t_\beta Q''_\beta Z^\beta$, and Q''_β is obtained from Q' and its commutation with Z^β. Next we move all the terms ∇_{t,e_i} in $\frac{\partial^{r_j}\mathscr{L}_t}{\partial t^{r_j}}$ to the right-hand side of the operator L^t_β. Then as in the proof of Theorem 1.9, we get finally that $QA^{\mathbf{k}}_{\mathbf{r}}(\lambda,t)Q'$ is of the form $\sum_{|\beta|\leqslant 2r}\mathscr{L}^t_\beta Z^\beta$, where $\mathscr{L}^t_\beta$ is a linear combination of operators of the form

$$Q(\lambda-\mathscr{L}_t)^{-k'_0}R_1(\lambda-\mathscr{L}_t)^{-k'_1}R_2\cdots R_{l'}(\lambda-\mathscr{L}_t)^{-k'_{l'}}Q'''Q'',$$

with $R_1,\dots,R_{l'}\in\mathscr{R}'_t$, $Q'''\in\mathcal{Q}^{2r}$, $Q''\in\mathcal{Q}^m$, $|\beta|\leqslant 2r$, and Q'' is obtained from Q' and its commutation with Z^β. Since $k>2(m+r+1)$, we can use the same argument as in (1.56) and (1.57) to split the above operator in the following two parts

$$Q(\lambda-\mathscr{L}_t)^{-k'_0}R_1(\lambda-\mathscr{L}_t)^{-k'_1}R_2\cdots R_i(\lambda-\mathscr{L}_t)^{-k''_i},$$
$$(\lambda-\mathscr{L}_t)^{-(k'_i-k''_i)}\cdots R_{l'}(\lambda-\mathscr{L}_t)^{-k'_{l'}}Q'''Q'',$$

such that the $\|\cdot\|_t^{0,0}$-norm of each part is bounded by C for $\lambda\in\delta$. Thus the proof of (1.64) is complete.

By (1.61), (1.63) and the above argument, we get the estimate (1.54) with $m'=0$. Finally, for every vector U on X we have

$$\nabla_U^{\pi^*\mathrm{End}(E)}\mathcal{P}_{q,t}=\frac{1}{2\pi i}\binom{q+k-1}{k-1}^{-1}\int_\delta\lambda^{q+k-1}\nabla_U^{\pi^*\mathrm{End}(E)}(\lambda-\mathscr{L}_t)^{-k}\,d\lambda.\tag{1.65}$$

Now we use a similar formula as (1.63) for $\nabla_U^{\pi^*\mathrm{End}(E)}(\lambda-\mathscr{L}_t)^{-k}$ by replacing $\frac{\partial^{r_1}\mathscr{L}_t}{\partial t^{r_1}}$ by $\nabla_U^{\pi^*\mathrm{End}(E)}\mathscr{L}_t$, and remark that $\nabla_U^{\pi^*\mathrm{End}(E)}\mathscr{L}_t$ is a differential operator on $T_{x_0}X$ with the same structure as $\mathscr{L}_t$. Then by the above argument, we conclude that (1.54) holds for $m'\geqslant 1$. □

For k big enough, set

$$F_{q,r} := \frac{1}{2\pi i r!}\binom{q+k-1}{k-1}^{-1}\int_{\delta}\lambda^{q+k-1}\sum_{(\mathbf{k},\mathbf{r})\in I_{k,r}} a^{\mathbf{k}}_{\mathbf{r}} A^{\mathbf{k}}_{\mathbf{r}}(\lambda,0)\,d\lambda,$$
$$F_{q,r,t} := \frac{1}{r!}\frac{\partial^r}{\partial t^r}\mathcal{P}_{q,t} - F_{q,r}. \tag{1.66}$$

Let $F_{q,r}(\cdot,\cdot)\in\mathscr{C}^\infty(TX\times_X TX,\pi^*\operatorname{End}(E))$ be the smooth kernel of $F_{q,r}$ with respect to $dv_{TX}(Z')$. In what follows we need the following observation: the limit of $\|\cdot\|_{t,m}$ for $t\to 0$ exists, and we denote it by $\|\cdot\|_{0,m}$.

Theorems 1.11, 1.12 are the analogues of [20, Theorems 4.14, 4.15]. We include the proofs for the sake of completeness.

Theorem 1.11. *For every $r\geqslant 0$, $k>0$, there exists $C>0$ such that for $t\in[0,t_0]$, $\lambda\in\delta$,*

$$\left\|\left(\frac{\partial^r\mathscr{L}_t}{\partial t^r}-\frac{\partial^r\mathscr{L}_t}{\partial t^r}\bigg|_{t=0}\right)s\right\|_{t,-1}\leqslant Ct\sum_{|\alpha|\leqslant r+3}\left\|Z^\alpha s\right\|_{0,1},$$
$$\left\|\left(\frac{\partial^r}{\partial t^r}(\lambda-\mathscr{L}_t)^{-k}-\sum_{(\mathbf{k},\mathbf{r})\in I_{k,r}} a^{\mathbf{k}}_{\mathbf{r}} A^{\mathbf{k}}_{\mathbf{r}}(\lambda,0)\right)s\right\|_{0,0}\leqslant Ct\sum_{|\alpha|\leqslant 4r+3}\left\|Z^\alpha s\right\|_{0,0}. \tag{1.67}$$

Proof. Note that by (1.35), (1.38), for $t\in[0,1]$, $k\geqslant 1$, $s\in\mathscr{C}^\infty(X_0,E_{x_0})$,

$$\|s\|_{t,0}=\|s\|_{0,0},\qquad \|s\|_{t,k}\leqslant C\sum_{|\alpha|\leqslant k}\left\|Z^\alpha s\right\|_{0,k}. \tag{1.68}$$

Using the Taylor expansion in the variable t in (1.34), we are lead to the following estimate for compactly supported s, s':

$$\left|\left\langle\left(\frac{\partial^r\mathscr{L}_t}{\partial t^r}-\frac{\partial^r\mathscr{L}_t}{\partial t^r}\bigg|_{t=0}\right)s,s'\right\rangle_{0,0}\right|\leqslant Ct\|s'\|_{t,1}\sum_{|\alpha|\leqslant r+3}\left\|Z^\alpha s\right\|_{0,1}. \tag{1.69}$$

Thus we get the first inequality of (1.67). Note that

$$(\lambda-\mathscr{L}_t)^{-1}-(\lambda-\mathscr{L}_0)^{-1}=(\lambda-\mathscr{L}_t)^{-1}(\mathscr{L}_t-\mathscr{L}_0)(\lambda-\mathscr{L}_0)^{-1}. \tag{1.70}$$

After taking the limit, we know that Theorems 1.7–1.9 still hold for $t=0$. Now from Theorem 1.9 for $\mathscr{L}_0$, (1.69) and (1.70),

$$\left\|\left((\lambda-\mathscr{L}_t)^{-1}-(\lambda-\mathscr{L}_0)^{-1}\right)s\right\|_{0,0}\leqslant Ct\sum_{|\alpha|\leqslant 3}\left\|Z^\alpha s\right\|_{0,0}. \tag{1.71}$$

Note that $\nabla_{0,e_j}=\nabla_{e_j}+\frac{1}{2}R^L_{x_0}(\mathcal{R},e_j)$ by (1.35). If we denote by $\mathscr{L}_{\lambda,t}=\lambda-\mathscr{L}_t$, then

$$A^{\mathbf{k}}_{\mathbf{r}}(\lambda,t)-A^{\mathbf{k}}_{\mathbf{r}}(\lambda,0)=\sum_{i=1}^{j}\mathscr{L}^{-k_0}_{\lambda,t}\cdots\left(\frac{\partial^{r_i}\mathscr{L}_t}{\partial t^{r_i}}-\frac{\partial^{r_i}\mathscr{L}_t}{\partial t^{r_i}}\bigg|_{t=0}\right)\mathscr{L}^{-k_i}_{\lambda,0}\cdots\mathscr{L}^{-k_j}_{\lambda,0}$$

$$+\sum_{i=0}^{j}\mathscr{L}^{-k_0}_{\lambda,t}\cdots\left(\mathscr{L}^{-k_i}_{\lambda,t}-\mathscr{L}^{-k_i}_{\lambda,0}\right)\left(\frac{\partial^{r_{i+1}}\mathscr{L}_t}{\partial t^{r_{i+1}}}\bigg|_{t=0}\right)\cdots\mathscr{L}^{-k_j}_{\lambda,0}. \tag{1.72}$$

From the discussion after (1.64), formulas (1.42), (1.63) and (1.71), we get the second inequality of (1.67). □

Theorem 1.12. *For $\sigma>0$, there exists $C>0$ such that for $t\in\,]0,t_0]$, $Z,Z'\in T_{x_0}X$, $|Z|,|Z'|\leqslant\sigma$,*

$$\left|F_{q,r,t}(Z,Z')\right|\leqslant Ct^{1/(2n+1)}. \tag{1.73}$$

Proof. By (1.61), (1.66) and (1.67), there exists $C>0$ such that for $t\in\,]0,t_0]$,

$$\|F_{q,r,t}\|_{(\sigma)}\leqslant Ct. \tag{1.74}$$

Let $\phi:\mathbb{R}^{2n}\to[0,1]$ be a smooth function with compact support, which equals 1 near 0 and such that $\int_{T_{x_0}X}\phi(Z)\,dv_{TX}(Z)=1$. Take $\nu\in\,]0,1]$. By the proof of Theorem 1.10 and (1.66), $F_{q,r}$ verifies the similar inequality as in (1.54) with $r=0$. Thus by (1.54), there exists $C>0$ such that

$$\bigg|\left\langle F_{q,r,t}(Z,Z')U,U'\right\rangle-\int\limits_{T_{x_0}X\times T_{x_0}X}\left\langle F_{q,r,t}(Z-W,Z'-W')U,U'\right\rangle$$
$$\times\frac{1}{\nu^{4n}}\phi(W/\nu)\phi(W'/\nu)\,dv_{TX}(W)\,dv_{TX}(W')\bigg|\leqslant C\nu|U||U'|, \tag{1.75}$$

for all $|Z|,|Z'|\leqslant\sigma$ and $U,U'\in E_{x_0}$. On the other hand, by (1.74),

$$\bigg|\int\limits_{T_{x_0}X\times T_{x_0}X}\left\langle F_{q,r,t}(Z-W,Z'-W')U,U'\right\rangle$$
$$\times\frac{1}{\nu^{4n}}\phi(W/\nu)\phi(W'/\nu)\,dv_{TX}(W)\,dv_{TX}(W')\bigg|\leqslant Ct\frac{1}{\nu^{2n}}|U||U'|. \tag{1.76}$$

By combining (1.75) with (1.76) and taking $\nu=t^{1/(2n+1)}$ we obtain (1.73). □

Finally, we prove the following off-diagonal estimate for the kernel of $\mathcal{P}_{q,t}$.

Theorem 1.13. *For every $j, m, m' \in \mathbb{N}$, $\sigma > 0$, there exists $C > 0$ such that*

$$\sup_{|\alpha|+|\alpha'|\leqslant m} \left| \frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^\alpha \partial Z'^{\alpha'}} \left(\mathcal{P}_{q,t} - \sum_{r=0}^{j} F_{q,r} t^r \right)(Z, Z') \right|_{\mathscr{C}^{m'}(X)} \leqslant C t^{j+1} \tag{1.77}$$

for all $t \in]0,1]$ and all $Z, Z' \in T_{x_0}X$ satisfying $|Z|, |Z'| \leqslant \sigma$.

Proof. By (1.66) and (1.73) we have

$$\frac{1}{r!} \frac{\partial^r}{\partial t^r} \mathcal{P}_{q,t} \bigg|_{t=0} = F_{q,r}. \tag{1.78}$$

Recall that the Taylor expansion with integral rest of some $G \in \mathscr{C}^{j+1}([0,1])$ is

$$G(t) - \sum_{r=0}^{j} \frac{1}{r!} \frac{\partial^r G}{\partial t^r}(0) t^r = \frac{1}{j!} \int_0^t (t - t_0)^j \frac{\partial^{j+1} G}{\partial t^{j+1}}(t_0)\, dt_0, \quad t \in [0,1]. \tag{1.79}$$

Theorem 1.10 and (1.66) show that the estimate (1.54) holds if we replace $\frac{1}{r!}\frac{\partial^r}{\partial t^r}\mathcal{P}_{q,t}$ with $F_{q,r}$. Using this new estimate together with (1.54), (1.66), (1.79), we obtain (1.77). □

1.4. Bergman kernel of $\mathscr{L}_0$

The almost complex structure J induces a splitting $TX \otimes_{\mathbb{R}} \mathbb{C} = T^{(1,0)}X \oplus T^{(0,1)}X$, where $T^{(1,0)}X$ and $T^{(0,1)}X$ are the eigenbundles of J corresponding to the eigenvalues $\sqrt{-1}$ and $-\sqrt{-1}$ respectively. We denote by $\det_{\mathbb{C}}$ the determinant function on the complex bundle $T^{(1,0)}X$. Set

$$\mathcal{J} = -2\pi\sqrt{-1}\boldsymbol{J}. \tag{1.80}$$

By (0.2), $\mathcal{J} \in \operatorname{End}(T^{(1,0)}X)$ is positive, and $\mathcal{J}$ acting on TX is skew-adjoint. For each tensor ψ on X, we denote by $\nabla^X \psi$ the covariant derivative of ψ induced by ∇^{TX}. Thus $\nabla^X \mathcal{J}, \nabla^X J \in T^*X \otimes \operatorname{End}(TX)$, $\nabla^X \nabla^X \mathcal{J} \in T^*X \otimes T^*X \otimes \operatorname{End}(TX)$.

We also adopt the convention that all tensors will be evaluated at the base point $x_0 \in X$, and most of the time, we will omit the subscript x_0.

Let P^N be the orthogonal projection from $(L^2(\mathbb{R}^{2n}, E_{x_0}), \|\cdot\|_0 = \|\cdot\|_{t,0})$ onto $N = \operatorname{Ker}(\mathscr{L}_0)$, and let $P^N(Z, Z')$ be the smooth kernel of P^N with respect to $dv_{TX}(Z)$. Then $P^N(Z, Z')$ is the Bergman kernel of $\mathscr{L}_0$. For $Z, Z' \in T_{x_0}X$, we have

$$P^N(Z, Z') = \frac{\det_{\mathbb{C}} \mathcal{J}_{x_0}}{(2\pi)^n} \exp\left(-\frac{1}{4} \big\langle \big(\mathcal{J}_{x_0}^2\big)^{1/2} (Z - Z'), (Z - Z') \big\rangle + \frac{1}{2} \langle \mathcal{J}_{x_0} Z, Z' \rangle \right). \tag{1.81}$$

Now we discuss the eigenvalues and eigenfunctions of $\mathscr{L}_0$ in detail. We choose an orthonormal basis $\{w_i\}_{i=1}^n$ of $T_{x_0}^{(1,0)}X$, such that

$$\mathcal{J}_{x_0} = \operatorname{diag}(a_1, \ldots, a_n) \in \operatorname{End}\big(T_{x_0}^{(1,0)}X\big), \tag{1.82}$$

with $0 < a_1 \leqslant a_2 \leqslant \cdots \leqslant a_n$, and let $\{w^j\}_{j=1}^n$ be its dual basis. Then $e_{2j-1} = \frac{1}{\sqrt{2}}(w_j + \overline{w}_j)$ and $e_{2j} = \frac{\sqrt{-1}}{\sqrt{2}}(w_j - \overline{w}_j)$, $j = 1, \dots, n$, form an orthonormal basis of $T_{x_0}X$. We use the coordinates on $T_{x_0}X \simeq \mathbb{R}^{2n}$ induced by $\{e_i\}$ as in (1.19) and in what follows we also introduce the complex coordinates $z = (z_1, \dots, z_n)$ on $\mathbb{C}^n \simeq \mathbb{R}^{2n}$. Thus $Z = z + \bar{z}$, and $w_i = \sqrt{2}\frac{\partial}{\partial z_i}$, $\overline{w}_i = \sqrt{2}\frac{\partial}{\partial \bar{z}_i}$. We will also identify z to $\sum_i z_i \frac{\partial}{\partial z_i}$ and $\bar{z}$ to $\sum_i \bar{z}_i \frac{\partial}{\partial \bar{z}_i}$ when we consider z and $\bar{z}$ as vector fields. Remark that

$$\left|\frac{\partial}{\partial z_i}\right|^2 = \left|\frac{\partial}{\partial \bar{z}_i}\right|^2 = \frac{1}{2}, \quad \text{so that} \quad |z|^2 = |\bar{z}|^2 = \frac{1}{2}|Z|^2. \tag{1.83}$$

It is very useful to rewrite $\mathscr{L}_0$ by using the creation and annihilation operators. Set

$$\nabla_{0,\cdot} = \nabla_\cdot + \frac{1}{2} R^L_{x_0}(\mathcal{R}, \cdot), \quad b_i = -2\nabla_{0,\frac{\partial}{\partial z_i}}, \quad b_i^+ = 2\nabla_{0,\frac{\partial}{\partial \bar{z}_i}}, \quad b = (b_1, \dots, b_n). \tag{1.84}$$

Then by (1.80) and (1.82), we have

$$b_i = -2\frac{\partial}{\partial z_i} + \frac{1}{2} a_i \bar{z}_i, \qquad b_i^+ = 2\frac{\partial}{\partial \bar{z}_i} + \frac{1}{2} a_i z_i, \tag{1.85}$$

and for every polynomial $g(z, \bar{z})$ on z and $\bar{z}$,

$$\begin{gathered}
\big[b_i, b_j^+\big] = b_i b_j^+ - b_j^+ b_i = -2a_i\delta_{ij}, \\
[b_i, b_j] = \big[b_i^+, b_j^+\big] = 0, \\
\big[g(z,\bar{z}), b_j\big] = 2\frac{\partial}{\partial z_j} g(z,\bar{z}), \qquad \big[g(z,\bar{z}), b_j^+\big] = -2\frac{\partial}{\partial \bar{z}_j} g(z,\bar{z}).
\end{gathered} \tag{1.86}$$

By (0.3) and (1.82), $\tau_{x_0} = \sum_i a_i$. Thus from (1.30), (1.82), (1.84)–(1.86), we deduce

$$\mathscr{L}_0 = \sum_i b_i b_i^+. \tag{1.87}$$

Remark 1.14. Let $L = \mathbb{C}$ be the trivial holomorphic line bundle on $\mathbb{C}^n$ with the canonical Section 1. Let h^L be the metric on L defined by $|1|_{h^L}(z) := e^{-\frac{1}{4}\sum_{j=1}^n a_j |z_j|^2} =: h(z)$ for $z \in \mathbb{C}^n$. Let $g^{T\mathbb{C}^n}$ be the Euclidean metric on $\mathbb{C}^n$. Then $\mathscr{L}_0$ is twice the corresponding Kodaira-Laplacian $\bar{\partial}^{L*}\bar{\partial}^L$ under the trivialization of L by using the unit section $e^{\frac{1}{4}\sum_{j=1}^n a_j |z_j|^2} 1$. Let $\bar{\partial}^*$ be the adjoint of the Dolbeault operator $\bar{\partial}$ associated to L with the trivial metric on L. In fact, under the canonical trivialization by 1,

$$\bar{\partial}^L = \bar{\partial}, \qquad \bar{\partial}^{L*} = h^{-2}\bar{\partial}^* h^2.$$

Set

$$\bar{\partial}_h = h\bar{\partial}h^{-1}, \qquad \bar{\partial}_h^* = h^{-1}\bar{\partial}^* h.$$

Then

$$b_j^+ = 2[i_{\frac{\partial}{\partial \bar z_j}}, \bar\partial_h], \qquad b_j = \left[\bar\partial_h^*, d\bar z_j \wedge\right],$$
$$\bar\partial_h = \frac{1}{2}\sum_j d\bar z_j \wedge b_j^+, \qquad \bar\partial_h^* = \sum_j i_{\frac{\partial}{\partial \bar z_j}} b_j.$$

Under the trivialization by $h^{-1}\cdot 1$, we know the Kodaira-Laplacian $\bar\partial^{L*}\bar\partial^L + \bar\partial^L\bar\partial^{L*}$ is $h(\bar\partial^{L*}\bar\partial^L + \bar\partial^L\bar\partial^{L*})h^{-1} = \bar\partial_h^*\bar\partial_h + \bar\partial_h\bar\partial_h^*$, and its restriction on functions is $\frac{1}{2}\mathscr{L}_0$.

Theorem 1.15. *The spectrum of the restriction of $\mathscr{L}_0$ on $L^2(\mathbb{R}^{2n})$ is given by*

$$\operatorname{Spec}(\mathscr{L}_0|_{L^2(\mathbb{R}^{2n})}) = \left\{ 2\sum_{i=1}^n \alpha_i a_i \colon \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n \right\} \tag{1.88}$$

and an orthogonal basis of the eigenspace of $2\sum_{i=1}^n \alpha_i a_i$ is given by

$$b^\alpha\left(z^\beta \exp\left(-\frac{1}{4}\sum_i a_i |z_i|^2\right)\right), \quad \text{with } \beta \in \mathbb{N}^n. \tag{1.89}$$

Proof. First observe that for all $\beta \in \mathbb{N}^n$ the functions $z^\beta \exp(-\frac{1}{4}\sum_i a_i|z_i|^2)$ are annihilated by the operators b_j^+, $j = 1, \ldots, n$, thus they are in the kernel of $\mathscr{L}_0|_{L^2(\mathbb{R}^{2n})}$. Using (1.86) we see that the functions (1.89) are eigenfunctions of $\mathscr{L}_0|_{L^2(\mathbb{R}^{2n})}$ with eigenvalue $2\sum_{i=1}^n \alpha_i a_i$. But the space spanned by (1.89) includes all the rescaled Hermite polynomials multiplied by $\exp(-\frac{1}{4}\sum_i a_i|z_i|^2)$, which is an orthogonal basis of $L^2(\mathbb{R}^{2n})$ by [46, §8.6]. Thus the eigenfunctions in (1.89) are all the eigenfunctions of $\mathscr{L}_0|_{L^2(\mathbb{R}^{2n})}$. The proof of Theorem 1.15 is complete. □

We deduce from Theorem 1.15 that the following functions build an orthonormal basis of $\operatorname{Ker}(\mathscr{L}_0|_{L^2(\mathbb{R}^{2n})})$:

$$\left(\frac{a^\beta}{(2\pi)^n 2^{|\beta|}\beta!}\prod_{i=1}^n a_i\right)^{1/2} z^\beta \exp\left(-\frac{1}{4}\sum_{j=1}^n a_j|z_j|^2\right), \quad \beta \in \mathbb{N}^n. \tag{1.90}$$

Calculating the Schwartz kernel of P^N using the basis (1.90), we recover (1.81):

$$P^N(Z, Z') = \frac{1}{(2\pi)^n}\prod_{i=1}^n a_i \exp\left(-\frac{1}{4}\sum_i a_i\left(|z_i|^2 + |z_i'|^2 - 2z_i\bar z_i'\right)\right). \tag{1.91}$$

Recall that the operators $\mathcal{O}_1, \mathcal{O}_2$ were defined in (1.30). Theorem 1.16 below is crucial in proving the vanishing result of $F_{q,r}$ (cf. Theorem 1.18).

We denote by $\langle\cdot,\cdot\rangle$ the $\mathbb{C}$-bilinear form on $TX \otimes_\mathbb{R} \mathbb{C}$ induced by the metric g^{TX}.

Theorem 1.16. *We have the relation*

$$P^N \mathcal{O}_1 P^N = 0. \tag{1.92}$$

Proof. From (0.2), for $U, V, W \in TX$, $\langle (\nabla^X_U \boldsymbol{J})V, W\rangle = (\nabla^X_U \omega)(V, W)$, thus

$$\big\langle \big(\nabla^X_U \boldsymbol{J}\big)V, W\big\rangle + \big\langle \big(\nabla^X_V \boldsymbol{J}\big)W, U\big\rangle + \big\langle \big(\nabla^X_W \boldsymbol{J}\big)U, V\big\rangle = d\omega(U, V, W) = 0. \tag{1.93}$$

By (0.2) and (0.3),

$$\begin{aligned} R^L(U, V) &= \langle \mathcal{J}U, V\rangle, \\ \big(\nabla^X_U R^L\big)(V, W) &= \big\langle \big(\nabla^X_U \mathcal{J}\big)V, W\big\rangle, \\ \nabla_U \tau &= -\frac{\sqrt{-1}}{2} \operatorname{Tr}\Big|_{TX} \big[\nabla^X_U (J\mathcal{J})\big]. \end{aligned} \tag{1.94}$$

Since J and $\mathcal{J} \in \operatorname{End}(TX)$ are skew-adjoint and commute, $\nabla^X_U J$, $\nabla^X_U \mathcal{J}$ are skew-adjoint and $\nabla^X_U (J\mathcal{J})$ is symmetric. From $J^2 = -\operatorname{Id}$, we know that

$$J\big(\nabla^X J\big) + \big(\nabla^X J\big)J = 0, \tag{1.95}$$

thus $\nabla^X_U J$ exchanges $T^{(1,0)}X$ and $T^{(0,1)}X$. From (1.82), (1.93) and (1.94), we have

$$\begin{aligned} (\nabla_{\mathcal{R}} \tau)_{x_0} &= -2\sqrt{-1}\Big\langle \big(\nabla^X_{\mathcal{R}} (J\mathcal{J})\big) \frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_i}\Big\rangle = 2\Big\langle \big(\nabla^X_{\mathcal{R}} \mathcal{J}\big) \frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_i}\Big\rangle, \\ \big(\partial_i R^L\big)_{x_0}(\mathcal{R}, e_i) &= 2\Big\langle \big(\nabla^X_{\frac{\partial}{\partial z_i}} \mathcal{J}\big)\mathcal{R}, \frac{\partial}{\partial \bar{z}_i}\Big\rangle + 2\Big\langle \big(\nabla^X_{\frac{\partial}{\partial \bar{z}_i}} \mathcal{J}\big)\mathcal{R}, \frac{\partial}{\partial z_i}\Big\rangle \\ &= 4\Big\langle \big(\nabla^X_{\frac{\partial}{\partial z_i}} \mathcal{J}\big)\mathcal{R}, \frac{\partial}{\partial \bar{z}_i}\Big\rangle - 2\Big\langle \big(\nabla^X_{\mathcal{R}} \mathcal{J}\big) \frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_i}\Big\rangle. \end{aligned} \tag{1.96}$$

From (1.30), (1.86), (1.94) and (1.96), we infer

$$\begin{aligned} \mathcal{O}_1 = &-\frac{2}{3}\Big[\Big\langle \big(\nabla^X_{\mathcal{R}} \mathcal{J}\big)\mathcal{R}, \frac{\partial}{\partial z_i}\Big\rangle b_i^+ - \Big\langle \big(\nabla^X_{\mathcal{R}} \mathcal{J}\big)\mathcal{R}, \frac{\partial}{\partial \bar{z}_i}\Big\rangle b_i \\ &+ 2\Big\langle \big(\nabla^X_{\frac{\partial}{\partial z_i}} \mathcal{J}\big)\mathcal{R}, \frac{\partial}{\partial \bar{z}_i}\Big\rangle + 2\Big\langle \big(\nabla^X_{\mathcal{R}} \mathcal{J}\big) \frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_i}\Big\rangle\Big] \\ = &-\frac{2}{3}\Big[\Big\langle \big(\nabla^X_{\mathcal{R}} \mathcal{J}\big)\mathcal{R}, \frac{\partial}{\partial z_i}\Big\rangle b_i^+ - b_i\Big\langle \big(\nabla^X_{\mathcal{R}} \mathcal{J}\big)\mathcal{R}, \frac{\partial}{\partial \bar{z}_i}\Big\rangle\Big]. \end{aligned} \tag{1.97}$$

Note that by (1.85) and (1.91),

$$\big(b_i^+ P^N\big)(Z, Z') = 0, \qquad \big(b_i P^N\big)(Z, Z') = a_i(\bar{z}_i - \bar{z}'_i) P^N(Z, Z'). \tag{1.98}$$

We learn from (1.98) that for every polynomial $g(z,\bar z)$ in $z,\bar z$ we can write $g(z,\bar z)P^N(Z,Z')$ as sums of $b^\beta g_\beta(z,\bar z')P^N(Z,Z')$ with $g_\beta(z,\bar z')$ polynomials in $z,\bar z'$. By Theorem 1.15,

$$P^N b^\alpha g(z,\bar z)P^N=0, \quad \text{for } |\alpha|>0, \tag{1.99}$$

and relations (1.97)–(1.99) yield the desired relation (1.92). □

1.5. Evaluation of $F_{q,r}$

For $s\in\mathbb{R}$, let $\lfloor s\rfloor$ denote the greatest integer which is less than or equal to s. Let $f(\lambda,t)$ be a formal power series with values in $\mathrm{End}(L^2(\mathbb{R}^{2n},E_{x_0}))$

$$f(\lambda,t)=\sum_{r=0}^{\infty}t^r f_r(\lambda), \quad f_r(\lambda)\in \mathrm{End}\big(L^2\big(\mathbb{R}^{2n},E_{x_0}\big)\big). \tag{1.100}$$

By (1.29), consider the equation of formal power series for $\lambda\in\delta$,

$$\left(-\mathscr{L}_0+\lambda-\sum_{r=1}^{\infty}t^r\mathcal{O}_r\right)f(\lambda,t)=\mathrm{Id}_{L^2(\mathbb{R}^{2n},E_{x_0})}. \tag{1.101}$$

Let $N^\perp$ be the orthogonal space of N in $L^2(\mathbb{R}^{2n},E_{x_0})$, and $P^{N^\perp}$ be the orthogonal projection from $L^2(\mathbb{R}^{2n},E_{x_0})$ to $N^\perp$. We decompose $f(\lambda,t)$ according the splitting $L^2(\mathbb{R}^{2n},E_{x_0})=N\oplus N^\perp$,

$$g_r(\lambda)=P^N f_r(\lambda), \qquad f_r^\perp(\lambda)=P^{N^\perp}f_r(\lambda). \tag{1.102}$$

Using (1.102) and identifying the powers of t in (1.101), we find that

$$\begin{aligned} g_0(\lambda)&=\frac{1}{\lambda}P^N, \qquad f_0^\perp(\lambda)=(\lambda-\mathscr{L}_0)^{-1}P^{N^\perp},\\ f_r^\perp(\lambda)&=(\lambda-\mathscr{L}_0)^{-1}\sum_{j=1}^{r}P^{N^\perp}\mathcal{O}_j f_{r-j}(\lambda),\\ g_r(\lambda)&=\frac{1}{\lambda}\sum_{j=1}^{r}P^N\mathcal{O}_j f_{r-j}(\lambda). \end{aligned} \tag{1.103}$$

Lemma 1.17. *For $r\in\mathbb{N}$, $\lambda^{\lfloor\frac{r}{2}\rfloor+1}g_r(\lambda)$, $\lambda^{\lfloor\frac{r+1}{2}\rfloor}f_r^\perp(\lambda)$ are holomorphic functions for $|\lambda|\leqslant\mu_0/4$ and*

$$\big(\lambda^{r+1}g_{2r}\big)(0)=\big(P^N\mathcal{O}_2P^N-P^N\mathcal{O}_1\mathscr{L}_0^{-1}P^{N^\perp}\mathcal{O}_1P^N\big)^rP^N. \tag{1.104}$$

Proof. By (1.103) we know that Lemma 1.17 is true for $r = 0$. Assume that Lemma 1.17 is true for $r \leqslant m$. Now, by Theorem 1.15, (1.103) and the recurrence assumption, it follows that $\lambda^{\lfloor \frac{m}{2} \rfloor + 1} f_{m+1}^{\perp}(\lambda)$ is holomorphic for $|\lambda| \leqslant \mu_0/4$, and

$$\lambda^{\lfloor \frac{m+1}{2} \rfloor + 1} g_{m+1}(\lambda) = \lambda^{\lfloor \frac{m+1}{2} \rfloor} \sum_{i=1}^{m+1} P^N \mathcal{O}_i \big[g_{m+1-i}(\lambda) + f_{m+1-i}^{\perp}(\lambda) \big]. \tag{1.105}$$

By our recurrence assumption, $\lambda^{\lfloor \frac{m}{2} \rfloor} g_{m-2}(\lambda)$, $\lambda^{\lfloor \frac{m+1}{2} \rfloor - 1} g_{m-j-1}(\lambda)$, $\lambda^{\lfloor \frac{m+1}{2} \rfloor - 1} f_{m-j}^{\perp}(\lambda)$, $\lambda^{\lfloor \frac{m+1}{2} \rfloor} f_m^{\perp}(\lambda)$, $\lambda^{\lfloor \frac{m}{2} \rfloor} f_{m-1}^{\perp}(\lambda)$ are holomorphic for $|\lambda| \leqslant \mu_0/4$, $j \geqslant 2$. Thus by Theorem 1.16, (1.103) and (1.105), $\lambda^{\lfloor \frac{m+1}{2} \rfloor + 1} g_{m+1}(\lambda)$ is also holomorphic for $|\lambda| \leqslant \mu_0/4$, and

$$\begin{aligned}
\big(\lambda^{\lfloor \frac{m+1}{2} \rfloor + 1} g_{m+1}\big)(0) &= \big(\lambda^{\lfloor \frac{m+1}{2} \rfloor} \big(P^N \mathcal{O}_1 f_m^{\perp} + P^N \mathcal{O}_2 \big(g_{m-1} + f_{m-1}^{\perp}\big) + P^N \mathcal{O}_3 g_{m-2}\big)\big)(0) \\
&= \big(-P^N \mathcal{O}_1 \mathscr{L}_0^{-1} P^{N^{\perp}} \mathcal{O}_1 + P^N \mathcal{O}_2\big) \big(\lambda^{\lfloor \frac{m+1}{2} \rfloor} \big(g_{m-1} + f_{m-1}^{\perp}\big)\big)(0) \\
&\quad + \big(-P^N \mathcal{O}_1 \mathscr{L}_0^{-1} P^{N^{\perp}} \mathcal{O}_2 + P^N \mathcal{O}_3\big) \big(\lambda^{\lfloor \frac{m+1}{2} \rfloor} g_{m-2}\big)(0).
\end{aligned} \tag{1.106}$$

If m is odd, then $\lfloor \frac{m+1}{2} \rfloor = \lfloor \frac{m}{2} \rfloor + 1$, so by (1.106) and the recurrence assumption,

$$\begin{aligned}
\big(\lambda^{\lfloor \frac{m+1}{2} \rfloor + 1} g_{m+1}\big)(0) &= P^N \big(-\mathcal{O}_1 \mathscr{L}_0^{-1} P^{N^{\perp}} \mathcal{O}_1 + \mathcal{O}_2\big) P^N \big(\lambda^{\lfloor \frac{m-1}{2} \rfloor + 1} g_{m-1}\big)(0) \\
&= \big(P^N \mathcal{O}_2 P^N - P^N \mathcal{O}_1 \mathscr{L}_0^{-1} P^{N^{\perp}} \mathcal{O}_1 P^N\big)^{\lfloor \frac{m+1}{2} \rfloor} P^N.
\end{aligned} \tag{1.107}$$

The proof of Lemma 1.17 is complete. □

Theorem 1.18. *There exist polynomials $J_{q,r}(Z, Z')$ in Z, Z' with the same parity as r and* $\deg J_{q,r}(Z, Z') \leqslant 3r$, *whose coefficients are polynomials in R^{TX}, R^E, Φ (and R^L) and their derivatives of order $\leqslant r - 2$ (resp. $\leqslant r$), and reciprocals of linear combinations of eigenvalues of $\boldsymbol{J}$ at x_0, such that*

$$F_{q,r}(Z, Z') = J_{q,r}(Z, Z') P^N(Z, Z'). \tag{1.108}$$

Moreover,

$$\begin{aligned}
F_{0,0} &= P^N, \\
F_{q,r} &= 0, \quad \text{for } q > 0,\ r < 2q, \\
F_{q,2q} &= \big(P^N \mathcal{O}_2 P^N - P^N \mathcal{O}_1 \mathscr{L}_0^{-1} P^{N^{\perp}} \mathcal{O}_1 P^N\big)^q P^N \quad \text{for } q > 0.
\end{aligned} \tag{1.109}$$

Proof. Recall that $\mathcal{P}_{q,t} = (\mathscr{L}_t)^q \mathcal{P}_{0,t}$. By (1.55), $\mathcal{P}_{q,t} = \frac{1}{2\pi i} \int_\delta \lambda^q (\lambda - \mathscr{L}_t)^{-1} d\lambda$. Thus by (1.61), (1.66), (1.78) and (1.102),

$$F_{q,r} = \frac{1}{2\pi i} \int_\delta \lambda^q g_r(\lambda)\, d\lambda + \frac{1}{2\pi i} \int_\delta \lambda^q f_r^{\perp}(\lambda)\, d\lambda. \tag{1.110}$$

From Lemma 1.17 and (1.110), we get (1.109). Generally, from Theorems 1.4, 1.15, Remark 1.5, (1.91), (1.103), (1.110) and the residue formula, we conclude that $F_{q,r}$ has the form (1.108). □

From Theorems 1.15, 1.16, (1.103), (1.110) and the residue formula, we can get $F_{q,r}$ by using the operators $\mathscr{L}_0^{-1}$, P^N, $P^{N^\perp}$, $\mathcal{O}_k$ (with $k \leqslant r$). This gives us a direct method to compute $F_{q,r}$ in view of Theorem 1.15. In particular, we get [3]

$$\begin{aligned} F_{0,1} &= -P^N \mathcal{O}_1 \mathscr{L}_0^{-1} P^{N^\perp} - P^{N^\perp} \mathscr{L}_0^{-1} \mathcal{O}_1 P^N, \\ F_{0,2} &= \frac{1}{2\pi i} \int_\delta \left[(\lambda - \mathscr{L}_0)^{-1} P^{N^\perp} (\mathcal{O}_1 f_1 + \mathcal{O}_2 f_0)(\lambda) + \frac{1}{\lambda} P^N (\mathcal{O}_1 f_1 + \mathcal{O}_2 f_0)(\lambda) \right] d\lambda \\ &= \mathscr{L}_0^{-1} P^{N^\perp} \mathcal{O}_1 \mathscr{L}_0^{-1} P^{N^\perp} \mathcal{O}_1 P^N - \mathscr{L}_0^{-1} P^{N^\perp} \mathcal{O}_2 P^N \\ &\quad + P^N \mathcal{O}_1 \mathscr{L}_0^{-1} P^{N^\perp} \mathcal{O}_1 \mathscr{L}_0^{-1} P^{N^\perp} - P^N \mathcal{O}_2 \mathscr{L}_0^{-1} P^{N^\perp} \\ &\quad + P^{N^\perp} \mathscr{L}_0^{-1} \mathcal{O}_1 P^N \mathcal{O}_1 \mathscr{L}_0^{-1} P^{N^\perp} - P^N \mathcal{O}_1 \mathscr{L}_0^{-2} P^{N^\perp} \mathcal{O}_1 P^N. \end{aligned} \tag{1.111}$$

1.6. Proof of Theorem 0.1

Recall that $P_{0,q,p} = (\Delta_{p,\Phi_0}^{X_0})^q P_{0,\mathcal{H}_p}$. Owing to (1.26), (1.27) we have

$$P_{0,q,p}(Z, Z') = t^{-2n-2q} \kappa^{-\frac{1}{2}}(Z) \mathcal{P}_{q,t}(Z/t, Z'/t) \kappa^{-\frac{1}{2}}(Z'), \quad \text{for all } Z, Z' \in \mathbb{R}^{2n}. \tag{1.112}$$

By (1.24), (1.112), Proposition 1.3, Theorems 1.13 and 1.18, we get the following main technical result of this paper, called ***the near off-diagonal expansion*** of the generalized Bergman kernels:

Theorem 1.19. *For every $j, m, m' \in \mathbb{N}$, $j \geqslant 2q$ and $\sigma > 0$, there exists $C > 0$ such that the estimate*

$$\begin{aligned} \sup_{|\alpha|+|\alpha'| \leqslant m} \Bigg| \frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^\alpha \partial Z'^{\alpha'}} \Bigg(\frac{1}{p^n} P_{q,p}(Z, Z') \\ - \sum_{r=2q}^{j} F_{q,r}(\sqrt{p} Z, \sqrt{p} Z') \kappa^{-\frac{1}{2}}(Z) \kappa^{-\frac{1}{2}}(Z') p^{-\frac{r}{2}+q} \Bigg) \Bigg|_{\mathscr{C}^{m'}(X)} \leqslant C p^{-\frac{j-m+1}{2}+q} \end{aligned} \tag{1.113}$$

holds for all $p \geqslant 1$ and all $Z, Z' \in T_{x_0} X$ with $|Z|, |Z'| \leqslant \sigma/\sqrt{p}$.

Set now $Z = Z' = 0$ in (1.113). By Theorem 1.18, we obtain (0.9) and

$$b_{q,r}(x_0) = F_{q,2r+2q}(0,0). \tag{1.114}$$

Hence (0.8) follows from (1.81) and (1.114). The statement about the structure of $b_{q,r}$ follows from Theorems 1.15 and 1.18.

[3] The formula $F_{0,2}$ in [34, (20)] missed the last two terms here which are zero at $(0,0)$ if $\boldsymbol{J} = J$, cf. (2.32).

To prove the uniformity statement of Theorem 0.1, we notice that in the proof of Theorem 1.10, we only use the derivatives up to order $2n+m+m'+r+2$ of the coefficients of $\mathscr{L}_t$. Therefore, in view of (1.79), the constants in Theorems 1.10, 1.12 (resp. Theorem 1.13) are uniformly bounded, if the $\mathscr{C}^{2n+m+m'+r+3}$-norms (resp. $\mathscr{C}^{2n+m+m'+j+4}$-norms) on X (with respect to a fixed metric g_0^{TX}) of a family $\mathcal{M}$ of $(g^{TX}, h^L, \nabla^L, h^E, \nabla^E, J, \Phi)$ are bounded and the family of components g^{TX} of $\mathcal{M}$ is bounded below. (Note that $\Delta^{L^p\otimes E}$ includes one derivative on ∇^L, ∇^E; that is why we have to add one derivative to the orders $2n+m+m'+r+2$ and $2n+m+m'+j+3$ respectively.)

Moreover, taking derivatives with respect to the data $(g^{TX}, h^L, \nabla^L, h^E, \nabla^E, J, \Phi)$ we obtain an equation similar to (1.65), where $x_0 \in X$ plays now the role of a parameter. Thus the $\mathscr{C}^{m'}$-norm in (1.113) can also include the data $(g^{TX}, h^L, \nabla^L, h^E, \nabla^E, J, \Phi)$ if the $\mathscr{C}^{m'}$-norms (with respect to the parameter $x_0 \in X$) of the derivatives of the above data up to order $2n+m+j+4$ are bounded. Hence we can determine a constant $C_{k,l}$ such that (0.9) holds uniformly for all data in a set $\mathcal{M}$ satisfying conditions (i) and (ii) of Theorem 0.1. To obtain (0.9) we apply (1.113) with $j=2k+2q+1$. This completes the proof of Theorem 0.1.

2. Computing the coefficients $b_{q,r}$

In principle, Theorem 1.15, Eqs. (1.103), (1.110) and the residue formula give us a direct method to calculate $b_{q,r}$ by recurrence. Actually, it is computable for the first few terms $b_{q,r}$ in (0.9) in this way.

This section is organized as follows. In Section 2.1, we will give a simplified formula for $\mathcal{O}_2 P^N$ without the assumption $\boldsymbol{J}=J$. In Sections 2.2, 2.3, we will compute $b_{q,0}$ and $b_{0,1}$ under the assumption $\boldsymbol{J}=J$, thus proving Theorem 0.2.

In this section, we use the notation in Section 1.4, and all tensors will be evaluated at the base point $x_0 \in X$. Recall that the operators $\mathcal{O}_1, \mathcal{O}_2$ were defined in (1.30). We denote by $\langle\cdot,\cdot\rangle$ the $\mathbb{C}$-bilinear form on $TX\otimes_{\mathbb{R}}\mathbb{C}$ induced by the metric g^{TX}.

2.1. *A formula for $\mathcal{O}_2 P^N$*

We will use the following lemma to evaluate $b_{q,r}$ in (0.9).

Lemma 2.1. *The following relation holds*:

$$
\begin{aligned}
\mathcal{O}_2 P^N = \bigg\{ & \frac{1}{3} b_i b_j \Big\langle R^{TX}\Big(\mathcal{R}, \frac{\partial}{\partial \bar z_i}\Big)\mathcal{R}, \frac{\partial}{\partial \bar z_j}\Big\rangle + \frac{1}{2} b_i \sum_{|\alpha|=2} \big(\partial^\alpha R^L\big)_{x_0}\Big(\mathcal{R}, \frac{\partial}{\partial \bar z_i}\Big)\frac{Z^\alpha}{\alpha!} \\
& + \frac{4}{3} b_j \Big[\Big\langle R^{TX}\Big(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar z_i}\Big)\mathcal{R}, \frac{\partial}{\partial \bar z_j}\Big\rangle - \Big\langle R^{TX}\Big(\mathcal{R}, \frac{\partial}{\partial z_i}\Big)\frac{\partial}{\partial \bar z_i}, \frac{\partial}{\partial \bar z_j}\Big\rangle\Big] + R^E\Big(\mathcal{R}, \frac{\partial}{\partial \bar z_i}\Big) b_i \\
& + \Big\langle (\nabla^X \nabla^X \mathcal{J})_{(\mathcal{R},\mathcal{R})} \frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar z_i}\Big\rangle + 4\Big\langle R^{TX}\Big(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j}\Big)\frac{\partial}{\partial \bar z_i}, \frac{\partial}{\partial \bar z_j}\Big\rangle\bigg\} P^N \\
& + \Big(-\frac{1}{3}\mathscr{L}_0 \Big\langle R^{TX}\Big(\mathcal{R}, \frac{\partial}{\partial z_j}\Big)\mathcal{R}, \frac{\partial}{\partial \bar z_j}\Big\rangle + \frac{1}{9}\big|(\nabla^X_{\mathcal{R}} \mathcal{J})\mathcal{R}\big|^2 - \sum_{|\alpha|=2} \big(\partial^\alpha \tau\big)_{x_0} \frac{Z^\alpha}{\alpha!} + \Phi\Big) P^N.
\end{aligned}
\tag{2.1}
$$

Proof. The definition of $\nabla^X\nabla^X\mathcal{J}$, R^{TX} and (1.93) imply, for $U,V,W,Y\in TX$,

$$\begin{gathered}\big\langle R^{TX}(U,V)W,Y\big\rangle=\big\langle R^{TX}(W,Y)U,V\big\rangle,\\ R^{TX}(U,V)W+R^{TX}(V,W)U+R^{TX}(W,U)V=0,\\ \big(\nabla^X\nabla^X\mathcal{J}\big)_{(U,V)}-\big(\nabla^X\nabla^X\mathcal{J}\big)_{(V,U)}=\big[R^{TX}(U,V),\mathcal{J}\big],\\ \big\langle\big(\nabla^X\nabla^X\mathcal{J}\big)_{(Y,U)}V,W\big\rangle+\big\langle\big(\nabla^X\nabla^X\mathcal{J}\big)_{(Y,V)}W,U\big\rangle+\big\langle\big(\nabla^X\nabla^X\mathcal{J}\big)_{(Y,W)}U,V\big\rangle=0.\end{gathered}\tag{2.2}$$

Set

$$\begin{aligned}I_1&=\frac12\sum_{|\alpha|=2}\big(\partial^\alpha R^L\big)_{x_0}\Big(\mathcal{R},\frac{\partial}{\partial\bar z_i}\Big)\frac{Z^\alpha}{\alpha!}b_i\\&\quad-\frac12\frac{\partial}{\partial\bar z_i}\Big(\sum_{|\alpha|=2}\big(\partial^\alpha R^L\big)_{x_0}\Big(\mathcal{R},\frac{\partial}{\partial z_i}\Big)\frac{Z^\alpha}{\alpha!}\Big)-\frac12\frac{\partial}{\partial z_i}\Big(\sum_{|\alpha|=2}\big(\partial^\alpha R^L\big)_{x_0}\Big(\mathcal{R},\frac{\partial}{\partial\bar z_i}\Big)\frac{Z^\alpha}{\alpha!}\Big),\\ I_2&=\frac13\Big\langle R^{TX}_{x_0}\Big(\mathcal{R},\frac{\partial}{\partial\bar z_i}\Big)\mathcal{R},\frac{\partial}{\partial\bar z_j}\Big\rangle b_ib_j\\&\quad-\frac43\Big[\Big\langle R^{TX}_{x_0}\Big(\mathcal{R},\frac{\partial}{\partial\bar z_i}\Big)\frac{\partial}{\partial z_i},\frac{\partial}{\partial\bar z_j}\Big\rangle+\Big\langle R^{TX}_{x_0}\Big(\mathcal{R},\frac{\partial}{\partial z_i}\Big)\frac{\partial}{\partial\bar z_i},\frac{\partial}{\partial\bar z_j}\Big\rangle\Big]b_j.\end{aligned}\tag{2.3}$$

Note that for every 1-form ψ we have $\psi(e_i)\nabla_{0,e_i}=-\psi(\frac{\partial}{\partial\bar z_i})b_i+\psi(\frac{\partial}{\partial z_i})b_i^+$. Due to (1.30), (1.80), (1.86), (1.94), (2.3), the first formula of (2.2), and since $\mathcal{J}$ is purely imaginary, we obtain

$$\begin{aligned}\mathcal{O}_2&=I_1+I_2-\frac13\Big[\mathscr{L}_0,\Big\langle R^{TX}\Big(\mathcal{R},\frac{\partial}{\partial z_j}\Big)\mathcal{R},\frac{\partial}{\partial\bar z_j}\Big\rangle\Big]+R^E\Big(\mathcal{R},\frac{\partial}{\partial\bar z_i}\Big)b_i\\&\quad+\frac13\Big[\Big\langle R^{TX}\Big(\mathcal{R},\frac{\partial}{\partial z_i}\Big)\mathcal{R},\frac{\partial}{\partial\bar z_j}\Big\rangle\big(-2b_jb_i^+-2a_i\delta_{ij}\big)+\Big\langle R^{TX}\Big(\mathcal{R},\frac{\partial}{\partial z_i}\Big)\mathcal{R},\frac{\partial}{\partial z_j}\Big\rangle b_i^+b_j^+\Big]\\&\quad+\Big(\frac23\Big\langle R^{TX}(\mathcal{R},e_j)e_j,\frac{\partial}{\partial z_i}\Big\rangle-\Big(\frac12\sum_{|\alpha|=2}\big(\partial^\alpha R^L\big)_{x_0}\frac{Z^\alpha}{\alpha!}+R^E\Big)\Big(\mathcal{R},\frac{\partial}{\partial z_i}\Big)\Big)b_i^+\\&\quad+\frac19\big|\big(\nabla^X_{\mathcal{R}}\mathcal{J}\big)\mathcal{R}\big|^2-\sum_{|\alpha|=2}\big(\partial^\alpha\tau\big)_{x_0}\frac{Z^\alpha}{\alpha!}+\Phi.\end{aligned}\tag{2.4}$$

In normal coordinates we have $(\nabla^{TX}_{e_i}e_j)_{x_0}=0$. So we deduce from (1.32) that the following relation holds at x_0:

$$\begin{aligned}\nabla_{e_j}\nabla_{e_i}\langle\mathcal{J}e_k,e_l\rangle&=\big\langle\big(\nabla^X_{e_j}\nabla^X_{e_i}\mathcal{J}\big)e_k+\mathcal{J}\big(\nabla^{TX}_{e_j}\nabla^{TX}_{e_i}e_k\big),e_l\big\rangle+\big\langle\mathcal{J}e_k,\nabla^{TX}_{e_j}\nabla^{TX}_{e_i}e_l\big\rangle\\&=\big\langle\big(\nabla^X_{e_j}\nabla^X_{e_i}\mathcal{J}\big)e_k,e_l\big\rangle-\frac13\big\langle R^{TX}(e_j,e_i)e_k+R^{TX}(e_j,e_k)e_i,\mathcal{J}e_l\big\rangle\\&\quad+\frac13\big\langle R^{TX}(e_j,e_i)e_l+R^{TX}(e_j,e_l)e_i,\mathcal{J}e_k\big\rangle.\end{aligned}\tag{2.5}$$

From (1.94) and (2.5) we obtain

$$\sum_{|\alpha|=2} \big(\partial^\alpha R^L\big)_{x_0}(e_k, e_l)\frac{Z^\alpha}{\alpha!} = \frac{1}{2}\big(\nabla_{e_j}\nabla_{e_i}\langle \mathcal{J} e_k, e_l\rangle\big)_{x_0} Z_i Z_j$$
$$= \frac{1}{2}\big\langle\big(\nabla^X\nabla^X\mathcal{J}\big)_{(\mathcal{R},\mathcal{R})} e_k, e_l\big\rangle + \frac{1}{6}\big[\big\langle R^{TX}(\mathcal{R}, e_l)\mathcal{R}, \mathcal{J} e_k\big\rangle - \big\langle R^{TX}(\mathcal{R}, e_k)\mathcal{R}, \mathcal{J} e_l\big\rangle\big]. \tag{2.6}$$

Thus

$$\sum_{|\alpha|=2} \big(\partial^\alpha R^L\big)_{x_0}(\mathcal{R}, e_l)\frac{Z^\alpha}{\alpha!} = \frac{1}{2}\big\langle\big(\nabla^X\nabla^X\mathcal{J}\big)_{(\mathcal{R},\mathcal{R})}\mathcal{R}, e_l\big\rangle + \frac{1}{6}\big\langle R^{TX}(\mathcal{R}, \mathcal{J}\mathcal{R})\mathcal{R}, e_l\big\rangle. \tag{2.7}$$

Using (1.86), (2.3) and (2.7) we compute

$$I_1 = \frac{1}{2} b_i \sum_{|\alpha|=2} \big(\partial^\alpha R^L\big)_{x_0}\Big(\mathcal{R}, \frac{\partial}{\partial \bar{z}_i}\Big)\frac{Z^\alpha}{\alpha!}$$
$$+ \frac{1}{12}\Big[\frac{\partial}{\partial z_i}\Big\langle R^{TX}_{x_0}(\mathcal{R}, \mathcal{J}\mathcal{R})\mathcal{R}, \frac{\partial}{\partial \bar{z}_i}\Big\rangle - \frac{\partial}{\partial \bar{z}_i}\Big\langle R^{TX}_{x_0}(\mathcal{R}, \mathcal{J}\mathcal{R})\mathcal{R}, \frac{\partial}{\partial z_i}\Big\rangle\Big]$$
$$+ \frac{1}{4}\Big[\frac{\partial}{\partial z_i}\Big\langle\big(\nabla^X\nabla^X\mathcal{J}\big)_{(\mathcal{R},\mathcal{R})}\mathcal{R}, \frac{\partial}{\partial \bar{z}_i}\Big\rangle - \frac{\partial}{\partial \bar{z}_i}\Big\langle\big(\nabla^X\nabla^X\mathcal{J}\big)_{(\mathcal{R},\mathcal{R})}\mathcal{R}, \frac{\partial}{\partial z_i}\Big\rangle\Big]. \tag{2.8}$$

Note that $\mathcal{J}$ and $(\nabla^X\nabla^X\mathcal{J})_{(Y,U)}$ are skew-adjoint, by (1.82) and (2.2). Hence

$$\frac{\partial}{\partial z_i}\Big\langle\big(\nabla^X\nabla^X\mathcal{J}\big)_{(\mathcal{R},\mathcal{R})}\mathcal{R}, \frac{\partial}{\partial \bar{z}_i}\Big\rangle - \frac{\partial}{\partial \bar{z}_i}\Big\langle\big(\nabla^X\nabla^X\mathcal{J}\big)_{(\mathcal{R},\mathcal{R})}\mathcal{R}, \frac{\partial}{\partial z_i}\Big\rangle$$
$$= \Big\langle 2\big(\nabla^X\nabla^X\mathcal{J}\big)_{(\mathcal{R},\mathcal{R})}\frac{\partial}{\partial z_i} + 2\big(\nabla^X\nabla^X\mathcal{J}\big)_{(\mathcal{R},\frac{\partial}{\partial z_i})}\mathcal{R} + \Big[R^{TX}\Big(\frac{\partial}{\partial z_i}, \mathcal{R}\Big), \mathcal{J}\Big]\mathcal{R}, \frac{\partial}{\partial \bar{z}_i}\Big\rangle$$
$$- \Big\langle 2\big(\nabla^X\nabla^X\mathcal{J}\big)_{(\mathcal{R},\frac{\partial}{\partial \bar{z}_i})}\mathcal{R} + \Big[R^{TX}\Big(\frac{\partial}{\partial \bar{z}_i}, \mathcal{R}\Big), \mathcal{J}\Big]\mathcal{R}, \frac{\partial}{\partial z_i}\Big\rangle$$
$$= 4\Big\langle\big(\nabla^X\nabla^X\mathcal{J}\big)_{(\mathcal{R},\mathcal{R})}\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_i}\Big\rangle + \Big\langle 2a_i R^{TX}\Big(\mathcal{R}, \frac{\partial}{\partial z_i}\Big)\mathcal{R} - R^{TX}(\mathcal{R}, \mathcal{J}\mathcal{R})\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_i}\Big\rangle, \tag{2.9}$$

and

$$\frac{\partial}{\partial z_i}\Big\langle R^{TX}(\mathcal{R}, \mathcal{J}\mathcal{R})\mathcal{R}, \frac{\partial}{\partial \bar{z}_i}\Big\rangle - \frac{\partial}{\partial \bar{z}_i}\Big\langle R^{TX}(\mathcal{R}, \mathcal{J}\mathcal{R})\mathcal{R}, \frac{\partial}{\partial z_i}\Big\rangle$$
$$= 2\Big\langle R^{TX}(\mathcal{R}, \mathcal{J}\mathcal{R})\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_i}\Big\rangle + 2a_i\Big\langle R^{TX}\Big(\mathcal{R}, \frac{\partial}{\partial \bar{z}_i}\Big)\mathcal{R}, \frac{\partial}{\partial z_i}\Big\rangle$$
$$+ \Big\langle R^{TX}\Big(\frac{\partial}{\partial z_i}, \mathcal{J}\mathcal{R}\Big)\mathcal{R}, \frac{\partial}{\partial \bar{z}_i}\Big\rangle - \Big\langle R^{TX}\Big(\frac{\partial}{\partial \bar{z}_i}, \mathcal{J}\mathcal{R}\Big)\mathcal{R}, \frac{\partial}{\partial z_i}\Big\rangle$$
$$= 3\Big\langle R^{TX}(\mathcal{R}, \mathcal{J}\mathcal{R})\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_i}\Big\rangle + 2a_i\Big\langle R^{TX}\Big(\mathcal{R}, \frac{\partial}{\partial z_i}\Big)\mathcal{R}, \frac{\partial}{\partial \bar{z}_i}\Big\rangle.$$

Thus by (2.8)–(2.9),

$$I_1 = \frac{1}{2} b_i \sum_{|\alpha|=2} \big(\partial^\alpha R^L\big)_{x_0}\left(\mathcal{R}, \frac{\partial}{\partial \bar z_i}\right)\frac{Z^\alpha}{\alpha!} + \left\langle \big(\nabla^X\nabla^X \mathcal{J}\big)_{(\mathcal{R},\mathcal{R})}\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar z_i}\right\rangle$$
$$+ \frac{2}{3} a_i \left\langle R^{TX}\left(\mathcal{R}, \frac{\partial}{\partial z_i}\right)\mathcal{R}, \frac{\partial}{\partial \bar z_i}\right\rangle. \tag{2.10}$$

Now by (1.86), (2.2) and (2.3) we calculate:

$$\begin{aligned} I_2 = {}& \frac{4}{3} b_j \left[\left\langle R^{TX}\left(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar z_i}\right)\mathcal{R}, \frac{\partial}{\partial \bar z_j}\right\rangle - \left\langle R^{TX}\left(\mathcal{R}, \frac{\partial}{\partial z_i}\right)\frac{\partial}{\partial \bar z_i}, \frac{\partial}{\partial \bar z_j}\right\rangle\right] \\ & + \frac{1}{3} b_i b_j \left\langle R^{TX}\left(\mathcal{R}, \frac{\partial}{\partial \bar z_i}\right)\mathcal{R}, \frac{\partial}{\partial \bar z_j}\right\rangle - \frac{8}{3}\left\langle R^{TX}\left(\frac{\partial}{\partial z_j}, \frac{\partial}{\partial z_i}\right)\frac{\partial}{\partial \bar z_i}, \frac{\partial}{\partial \bar z_j}\right\rangle \\ & + \frac{4}{3}\left[\left\langle R^{TX}\left(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar z_i}\right)\frac{\partial}{\partial z_j}, \frac{\partial}{\partial \bar z_j}\right\rangle - \left\langle R^{TX}\left(\frac{\partial}{\partial z_j}, \frac{\partial}{\partial \bar z_i}\right)\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar z_j}\right\rangle\right] \\ = {}& \frac{4}{3} b_j \left[\left\langle R^{TX}\left(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar z_i}\right)\mathcal{R}, \frac{\partial}{\partial \bar z_j}\right\rangle - \left\langle R^{TX}\left(\mathcal{R}, \frac{\partial}{\partial z_i}\right)\frac{\partial}{\partial \bar z_i}, \frac{\partial}{\partial \bar z_j}\right\rangle\right] \\ & + \frac{1}{3} b_i b_j \left\langle R^{TX}\left(\mathcal{R}, \frac{\partial}{\partial \bar z_i}\right)\mathcal{R}, \frac{\partial}{\partial \bar z_j}\right\rangle + 4\left\langle R^{TX}\left(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j}\right)\frac{\partial}{\partial \bar z_i}, \frac{\partial}{\partial \bar z_j}\right\rangle. \end{aligned} \tag{2.11}$$

Finally (1.87), (1.98), (2.4), (2.10) and (2.11) yield (2.1). □

Now (1.94), (1.97), (1.99), (1.108), (1.109) and (2.1) entail

$$\begin{aligned} F_{1,2} = {}& J_{1,2} P^N \\ = {}& \left(2R^E\left(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar z_i}\right) + 4\left\langle R^{TX}\left(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j}\right)\frac{\partial}{\partial \bar z_i}, \frac{\partial}{\partial \bar z_j}\right\rangle + \Phi\right)P^N \\ & + P^N\left(\left\langle \big(\nabla^X\nabla^X \mathcal{J}\big)_{(\mathcal{R},\mathcal{R})}\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar z_i}\right\rangle + \frac{\sqrt{-1}}{4}\operatorname{Tr}\Big|_{TX}\big(\nabla^X\nabla^X(J\mathcal{J})\big)_{(\mathcal{R},\mathcal{R})}\right. \\ & \left. + \frac{1}{9}\big|(\nabla^X_{\mathcal{R}}\mathcal{J})\mathcal{R}\big|^2 + \frac{4}{9}\left\langle (\nabla^X_{\mathcal{R}}\mathcal{J})\mathcal{R}, \frac{\partial}{\partial z_i}\right\rangle b_i^+ \mathscr{L}_0^{-1} b_i \left\langle (\nabla^X_{\mathcal{R}}\mathcal{J})\mathcal{R}, \frac{\partial}{\partial \bar z_i}\right\rangle\right)P^N. \end{aligned} \tag{2.12}$$

2.2. *The coefficients* $b_{q,0}$

In the rest of this section we assume that $\boldsymbol{J} = J$. A very useful observation is that (1.93) and (1.95) imply

$$\begin{aligned} & \mathcal{J} = -2\pi\sqrt{-1}J \text{ and } a_i = 2\pi \text{ in (1.82)}, \ \tau = 2\pi n. \\ & \nabla^X_U J \text{ is skew-adjoint and the tensor } \langle (\nabla^X_{\cdot} J)\cdot, \cdot\rangle \text{ is of the type} \\ & \big(T^{*(1,0)}X\big)^{\otimes 3} \oplus \big(T^{*(0,1)}X\big)^{\otimes 3}. \end{aligned} \tag{2.13}$$

Before computing $b_{q,0}$, we establish the relation between the scalar curvature r^X and $|\nabla^X J|^2$.

Lemma 2.2.

$$r^X = 8\left\langle R^{TX}\left(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_j}\right)\frac{\partial}{\partial z_j}, \frac{\partial}{\partial \bar{z}_i}\right\rangle - \frac{1}{4}\left|\nabla^X J\right|^2. \tag{2.14}$$

Proof. By (2.13),

$$\left|\nabla^X J\right|^2 = 4\langle(\nabla^X_{\frac{\partial}{\partial z_i}} J)e_j, (\nabla^X_{\frac{\partial}{\partial \bar{z}_i}} J)e_j\rangle = 8\left\langle(\nabla^X_{\frac{\partial}{\partial z_i}} J)\frac{\partial}{\partial z_j}, (\nabla^X_{\frac{\partial}{\partial \bar{z}_i}} J)\frac{\partial}{\partial \bar{z}_j}\right\rangle. \tag{2.15}$$

Using (1.83), (1.93) and (2.13) we get

$$\begin{aligned}
&\left\langle(\nabla^X_{\frac{\partial}{\partial z_j}} J)\frac{\partial}{\partial z_i}, (\nabla^X_{\frac{\partial}{\partial \bar{z}_i}} J)\frac{\partial}{\partial \bar{z}_j}\right\rangle \\
&\quad = 2\left\langle(\nabla^X_{\frac{\partial}{\partial z_j}} J)\frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_k}\right\rangle\left\langle(\nabla^X_{\frac{\partial}{\partial \bar{z}_i}} J)\frac{\partial}{\partial \bar{z}_j}, \frac{\partial}{\partial \bar{z}_k}\right\rangle \\
&\quad = 2\left\langle(\nabla^X_{\frac{\partial}{\partial z_i}} J)\frac{\partial}{\partial z_k} - (\nabla^X_{\frac{\partial}{\partial z_k}} J)\frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j}\right\rangle\left\langle(\nabla^X_{\frac{\partial}{\partial \bar{z}_i}} J)\frac{\partial}{\partial \bar{z}_k}, \frac{\partial}{\partial \bar{z}_j}\right\rangle \\
&\quad = \left\langle(\nabla^X_{\frac{\partial}{\partial z_i}} J)\frac{\partial}{\partial z_k}, (\nabla^X_{\frac{\partial}{\partial \bar{z}_i}} J)\frac{\partial}{\partial \bar{z}_k}\right\rangle - \left\langle(\nabla^X_{\frac{\partial}{\partial z_k}} J)\frac{\partial}{\partial z_i}, (\nabla^X_{\frac{\partial}{\partial \bar{z}_i}} J)\frac{\partial}{\partial \bar{z}_k}\right\rangle.
\end{aligned} \tag{2.16}$$

By (2.15) and (2.16),

$$\left\langle(\nabla^X_{\frac{\partial}{\partial z_j}} J)\frac{\partial}{\partial z_i}, (\nabla^X_{\frac{\partial}{\partial \bar{z}_i}} J)\frac{\partial}{\partial \bar{z}_j}\right\rangle = \frac{1}{16}\left|\nabla^X J\right|^2. \tag{2.17}$$

Now, from (1.95), we get

$$\left(\nabla^X\nabla^X J\right)_{(U,V)} J + \left(\nabla^X_U J\right)\circ\left(\nabla^X_V J\right) + \left(\nabla^X_V J\right)\circ\left(\nabla^X_U J\right) + J\left(\nabla^X\nabla^X J\right)_{(U,V)} = 0. \tag{2.18}$$

We infer from (2.2), (2.13) and (2.18) that for all $u_1, u_2, u_3 \in T^{(1,0)}X$, $\bar{v}_1, \bar{v}_2 \in T^{(0,1)}X$ the following holds:

$$\begin{gathered}
\left(\nabla^X\nabla^X J\right)_{(u_1,u_2)} u_3, \left(\nabla^X\nabla^X J\right)_{(\bar{v}_1,\bar{v}_2)} u_3 \in T^{(0,1)}X, \\
\left(\nabla^X\nabla^X J\right)_{(u_1,\bar{v}_2)} u_3 \in T^{(1,0)}X, \\
2\sqrt{-1}\langle\left(\nabla^X\nabla^X J\right)_{(u_1,\bar{v}_1)} u_2, \bar{v}_2\rangle = \langle\left(\nabla^X_{u_1} J\right)u_2, \left(\nabla^X_{\bar{v}_1} J\right)\bar{v}_2\rangle.
\end{gathered} \tag{2.19}$$

(The second equation of (2.19) follows from the first line and (2.2)). Formulas (2.2) and (2.19) yield

$$\begin{aligned}\big\langle\big(\nabla^X\nabla^X J\big)_{(u_1,u_2)}\bar v_1,\bar v_2\big\rangle &= -\big\langle\big(\nabla^X\nabla^X J\big)_{(u_1,\bar v_1)}\bar v_2,u_2\big\rangle-\big\langle\big(\nabla^X\nabla^X J\big)_{(u_1,\bar v_2)}u_2,\bar v_1\big\rangle\\ &= \frac{1}{2\sqrt{-1}}\big\langle\big(\nabla^X_{u_1}J\big)u_2,\big(\nabla^X_{\bar v_1}J\big)\bar v_2-\big(\nabla^X_{\bar v_2}J\big)\bar v_1\big\rangle.\end{aligned}\tag{2.20}$$

From (2.2), (2.16), (2.17) and (2.20), we deduce

$$\begin{aligned}&\left\langle R^{TX}\left(\frac{\partial}{\partial z_i},\frac{\partial}{\partial z_j}\right)\frac{\partial}{\partial \bar z_i},\frac{\partial}{\partial \bar z_j}\right\rangle\\ &\quad=\frac{\sqrt{-1}}{2}\left\langle\left[R^{TX}\left(\frac{\partial}{\partial z_i},\frac{\partial}{\partial z_j}\right),J\right]\frac{\partial}{\partial \bar z_i},\frac{\partial}{\partial \bar z_j}\right\rangle\\ &\quad=\frac{\sqrt{-1}}{2}\left\langle\left(\big(\nabla^X\nabla^X J\big)_{(\frac{\partial}{\partial z_i},\frac{\partial}{\partial z_j})}-\big(\nabla^X\nabla^X J\big)_{(\frac{\partial}{\partial z_j},\frac{\partial}{\partial z_i})}\right)\frac{\partial}{\partial \bar z_i},\frac{\partial}{\partial \bar z_j}\right\rangle\\ &\quad=\frac14\left\langle\big(\nabla^X_{\frac{\partial}{\partial z_i}}J\big)\frac{\partial}{\partial z_j},\big(\nabla^X_{\frac{\partial}{\partial \bar z_i}}J\big)\frac{\partial}{\partial \bar z_j}\right\rangle\\ &\quad=\frac{1}{32}\big|\nabla^X J\big|^2.\end{aligned}\tag{2.21}$$

The scalar curvature r^X of (X,g^{TX}) is given by

$$\begin{aligned}r^X&=-\big\langle R^{TX}(e_i,e_j)e_i,e_j\big\rangle=-4\left\langle R^{TX}\left(\frac{\partial}{\partial z_i},e_j\right)\frac{\partial}{\partial \bar z_i},e_j\right\rangle\\ &=-8\left\langle R^{TX}\left(\frac{\partial}{\partial z_i},\frac{\partial}{\partial z_j}\right)\frac{\partial}{\partial \bar z_i},\frac{\partial}{\partial \bar z_j}\right\rangle-8\left\langle R^{TX}\left(\frac{\partial}{\partial z_i},\frac{\partial}{\partial \bar z_j}\right)\frac{\partial}{\partial \bar z_i},\frac{\partial}{\partial z_j}\right\rangle.\end{aligned}\tag{2.22}$$

In conclusion, relations (2.21) and (2.22) imply (2.14). □

From (1.97) and (2.13) we know

$$\mathcal{O}_1=\frac23 b_i\left\langle\big(\nabla^X_{\bar z}\mathcal{J}\big)\bar z,\frac{\partial}{\partial \bar z_i}\right\rangle-\frac23\left\langle\big(\nabla^X_{z}\mathcal{J}\big)z,\frac{\partial}{\partial z_i}\right\rangle b_i^+.\tag{2.23}$$

Hence by (1.86), (1.98), (2.13) and (2.23),

$$\begin{aligned}\big(\mathcal{O}_1P^N\big)(Z,Z')&=\frac23\left(b_i\left\langle\big(\nabla^X_{\bar z}\mathcal{J}\big)\bar z,\frac{\partial}{\partial \bar z_i}\right\rangle P^N\right)(Z,Z')\\ &=\frac23\left\{\left(\frac{b_ib_j}{2\pi}\left\langle\big(\nabla^X_{\frac{\partial}{\partial \bar z_j}}\mathcal{J}\big)\bar z',\frac{\partial}{\partial \bar z_i}\right\rangle+b_i\left\langle\big(\nabla^X_{\bar z'}\mathcal{J}\big)\bar z',\frac{\partial}{\partial \bar z_i}\right\rangle\right)P^N\right\}(Z,Z').\end{aligned}\tag{2.24}$$

By Theorem 1.15, (1.99) and (2.24), we have

$$\begin{aligned}&\big(\mathscr{L}_0^{-1}P^{N^\perp}\mathcal{O}_1P^N\big)(Z,Z')\\ &\quad=\frac23\left\{\left(\frac{b_ib_j}{16\pi^2}\left\langle\big(\nabla^X_{\frac{\partial}{\partial \bar z_j}}\mathcal{J}\big)\bar z',\frac{\partial}{\partial \bar z_i}\right\rangle+\frac{b_i}{4\pi}\left\langle\big(\nabla^X_{\bar z'}\mathcal{J}\big)\bar z',\frac{\partial}{\partial \bar z_i}\right\rangle\right)P^N\right\}(Z,Z'),\end{aligned}$$

$$P^N\mathcal{O}_1\mathscr{L}_0^{-1}P^{N^\perp}\mathcal{O}_1P^N = -\frac{2}{3}P^N\Big\langle(\nabla^X_z\mathcal{J})z, \frac{\partial}{\partial z_k}\Big\rangle b_k^+\mathscr{L}_0^{-1}P^{N^\perp}\mathcal{O}_1P^N. \tag{2.25}$$

Now, (1.83), (1.86), (1.98), (2.23) and (2.25) imply

$$\begin{aligned}
&\frac{2}{3}\Big(\Big\langle(\nabla^X_z\mathcal{J})z, \frac{\partial}{\partial z_k}\Big\rangle b_k^+\mathscr{L}_0^{-1}P^{N^\perp}\mathcal{O}_1P^N\Big)(Z,Z')\\
&\quad= \frac{4}{9}\Big\{\Big\langle(\nabla^X_z\mathcal{J})z, \frac{\partial}{\partial z_k}\Big\rangle\Big(-\frac{b_i}{4\pi}\Big\langle(\nabla^X_{\frac{\partial}{\partial\bar z_i}}\mathcal{J})\frac{\partial}{\partial\bar z_k}+(\nabla^X_{\frac{\partial}{\partial\bar z_k}}\mathcal{J})\frac{\partial}{\partial\bar z_i}, \bar z'\Big\rangle\\
&\qquad+\Big\langle(\nabla^X_{\bar z'}\mathcal{J})\bar z', \frac{\partial}{\partial\bar z_k}\Big\rangle\Big)P^N\Big\}(Z,Z')\\
&\quad= \Big\{\Big[-\frac{b_i}{9\pi}\Big\langle(\nabla^X_z\mathcal{J})z, \frac{\partial}{\partial z_k}\Big\rangle\Big\langle(\nabla^X_{\frac{\partial}{\partial\bar z_i}}\mathcal{J})\frac{\partial}{\partial\bar z_k}+(\nabla^X_{\frac{\partial}{\partial\bar z_k}}\mathcal{J})\frac{\partial}{\partial\bar z_i}, \bar z'\Big\rangle\\
&\qquad-\frac{2}{9\pi}\Big\langle(\nabla^X_z\mathcal{J})\frac{\partial}{\partial z_i}+(\nabla^X_{\frac{\partial}{\partial z_i}}\mathcal{J})z, \frac{\partial}{\partial z_k}\Big\rangle\Big\langle(\nabla^X_{\frac{\partial}{\partial\bar z_i}}\mathcal{J})\frac{\partial}{\partial\bar z_k}+(\nabla^X_{\frac{\partial}{\partial\bar z_k}}\mathcal{J})\frac{\partial}{\partial\bar z_i}, \bar z'\Big\rangle\\
&\qquad+\frac{2}{9}\big\langle(\nabla^X_z\mathcal{J})z, (\nabla^X_{\bar z'}\mathcal{J})\bar z'\big\rangle\Big]P^N\Big\}(Z,Z').
\end{aligned} \tag{2.26}$$

Thanks to (1.98), (2.13), (2.15) and (2.16) we obtain

$$\begin{aligned}
&\frac{1}{9}\big|(\nabla^X_{\mathcal{R}}\mathcal{J})\mathcal{R}\big|^2P^N(Z,Z')\\
&\quad= \frac{8\pi^2}{9}\big\langle(\nabla^X_z J)z, (\nabla^X_{\bar z}J)\bar z\big\rangle P^N(Z,Z')\\
&\quad= \frac{8\pi^2}{9}\Big\{\Big\langle(\nabla^X_z J)z, \frac{b_ib_j}{4\pi^2}(\nabla^X_{\frac{\partial}{\partial\bar z_i}}J)\frac{\partial}{\partial\bar z_j}+\frac{b_i}{2\pi}\Big[(\nabla^X_{\frac{\partial}{\partial\bar z_i}}J)\bar z'+(\nabla^X_{\bar z'}J)\frac{\partial}{\partial\bar z_i}\Big]\\
&\qquad+(\nabla^X_{\bar z'}J)\bar z'\Big\rangle P^N\Big\}(Z,Z')\\
&\quad= \frac{8\pi^2}{9}\Big\{\Big[\Big\langle\frac{b_ib_j}{4\pi^2}(\nabla^X_{\frac{\partial}{\partial\bar z_i}}J)\frac{\partial}{\partial\bar z_j}+\frac{b_i}{2\pi}\Big((\nabla^X_{\frac{\partial}{\partial\bar z_i}}J)\bar z'+(\nabla^X_{\bar z'}J)\frac{\partial}{\partial\bar z_i}\Big), (\nabla^X_z J)z\Big\rangle\\
&\qquad+\frac{b_i}{2\pi^2}\Big\langle(\nabla^X_{\frac{\partial}{\partial z_j}}J)z+(\nabla^X_z J)\frac{\partial}{\partial z_j}, (\nabla^X_{\frac{\partial}{\partial\bar z_i}}J)\frac{\partial}{\partial\bar z_j}+(\nabla^X_{\frac{\partial}{\partial\bar z_j}}J)\frac{\partial}{\partial\bar z_i}\Big\rangle\\
&\qquad+\frac{1}{\pi}\Big\langle(\nabla^X_{\frac{\partial}{\partial z_i}}J)z+(\nabla^X_z J)\frac{\partial}{\partial z_i}, (\nabla^X_{\frac{\partial}{\partial\bar z_i}}J)\bar z'+(\nabla^X_{\bar z'}J)\frac{\partial}{\partial\bar z_i}\Big\rangle\\
&\qquad+\big\langle(\nabla^X_z J)z, (\nabla^X_{\bar z'}J)\bar z'\big\rangle+\frac{3}{16\pi^2}\big|\nabla^X J\big|^2\Big]P^N\Big\}(Z,Z').
\end{aligned} \tag{2.27}$$

Taking into account (2.2), (2.19) and the equality $\langle[R^{TX}(\bar z, z), J]\frac{\partial}{\partial z_i}, \frac{\partial}{\partial\bar z_i}\rangle = 0$, we get

$$\Big\langle \big(\nabla^X\nabla^X J\big)_{(\mathcal{R},\mathcal{R})}\frac{\partial}{\partial z_i},\frac{\partial}{\partial \bar z_i}\Big\rangle = \Big\langle \big(\nabla^X\nabla^X J\big)_{(z,\bar z)}\frac{\partial}{\partial z_i}+\big(\nabla^X\nabla^X J\big)_{(\bar z,z)}\frac{\partial}{\partial z_i},\frac{\partial}{\partial \bar z_i}\Big\rangle$$
$$= -\sqrt{-1}\Big\langle \big(\nabla^X_z J\big)\frac{\partial}{\partial z_i},\big(\nabla^X_{\bar z} J\big)\frac{\partial}{\partial \bar z_i}\Big\rangle. \tag{2.28}$$

From (1.98), (2.15) and (2.28),

$$\begin{aligned}
&\Big\langle \big(\nabla^X\nabla^X \mathcal{J}\big)_{(\mathcal{R},\mathcal{R})}\frac{\partial}{\partial z_i},\frac{\partial}{\partial \bar z_i}\Big\rangle P^N(Z,Z')\\
&\quad = -2\pi\Big\langle \big(\nabla^X_z J\big)\frac{\partial}{\partial z_i},\big(\nabla^X_{\bar z} J\big)\frac{\partial}{\partial \bar z_i}\Big\rangle P^N(Z,Z')\\
&\quad = -2\pi\Big\{\Big\langle \big(\nabla^X_z J\big)\frac{\partial}{\partial z_i},\frac{b_j}{2\pi}\big(\nabla^X_{\frac{\partial}{\partial \bar z_j}} J\big)\frac{\partial}{\partial \bar z_i}+\big(\nabla^X_{\bar z'} J\big)\frac{\partial}{\partial \bar z_i}\Big\rangle P^N\Big\}(Z,Z')\\
&\quad = -\Big\{\Big[\Big\langle b_j\big(\nabla^X_{\frac{\partial}{\partial \bar z_j}} J\big)\frac{\partial}{\partial \bar z_i}+2\pi\big(\nabla^X_{\bar z'} J\big)\frac{\partial}{\partial \bar z_i},\big(\nabla^X_z J\big)\frac{\partial}{\partial z_i}\Big\rangle+\frac14\big|\nabla^X J\big|^2\Big]P^N\Big\}(Z,Z').
\end{aligned} \tag{2.29}$$

Recall that the polynomial $J_{q,2q}(Z,Z')$ was defined in (1.108) and (1.109). The equality $J\mathcal{J} = 2\pi\sqrt{-1}$, and (1.99), (2.12), (2.21), (2.23)–(2.29) show that $J_{1,2}(Z,Z')$ is a polynomial in $z,\bar z'$, and each monomial of $J_{1,2}$ has the same degree in z and $\bar z'$; moreover

$$J_{1,2}(0,0) = \frac{1}{24}\big|\nabla^X J\big|^2_{x_0} + 2R^E_{x_0}\Big(\frac{\partial}{\partial z_i},\frac{\partial}{\partial \bar z_i}\Big) + \Phi_{x_0}. \tag{2.30}$$

Using (1.91) with $a_i = 2\pi$, (1.109) and the recurrence, we infer that each monomial of $J_{q,2q}$ has the same degree in z and $\bar z'$, and

$$J_{q,2q}(0,0) = \big(J_{1,2}(0,0)\big)^q. \tag{2.31}$$

In view of (1.91), (1.108), (1.114), (2.30) and (2.31) we obtain (0.11).

2.3. *The coefficient* $b_{0,1}$

By (1.114), we need to compute $F_{0,2}(0,0)$. By (1.98), (2.24) and (2.25), we know that

$$\big(\mathcal{O}_1 P^N\big)(Z,0) = 0, \qquad \big(\mathscr{L}_0^{-1}P^{N^\perp}\mathcal{O}_1 P^N\big)(0,Z') = 0. \tag{2.32}$$

Thus the first and last two terms in (1.111) are zero at $(0,0)$. Thus we only need to compute $-(\mathscr{L}_0^{-1}P^{N^\perp}\mathcal{O}_2 P^N)(0,0)$, since the third and fourth terms in (1.111) are adjoint of the first two terms by Remark 1.5.

Let $h_i(z)$ and $f_{ij}(z)$ $(i,j=1,\dots,n)$ be arbitrary polynomials in z. By Theorem 1.15, (1.86), (1.91) and (1.98) with $a_i = 2\pi$, we have

$$\big(b_i h_i P^N\big)(0,0) = -2\frac{\partial h_i}{\partial z_i}(0), \qquad \big(b_i b_j f_{ij} P^N\big)(0,0) = 4\frac{\partial^2 f_{ij}}{\partial z_i\partial z_j}(0),$$
$$\big(\mathscr{L}_0^{-1} b_i f_{ij} b_j P^N\big)(0,0) = -\frac{1}{2\pi}\frac{\partial^2 f_{ij}}{\partial z_i\partial z_j}(0). \tag{2.33}$$

Owing to Theorem 1.15, (2.15), (2.17), (2.27) and (2.33),

$$
\begin{aligned}
&-\frac{1}{9}\big(\mathscr{L}_0^{-1}P^{N^\perp}\big|(\nabla^X_{\mathcal{R}}\mathcal{J})\mathcal{R}\big|^2P^N\big)(0,0)\\
&=-\frac{8}{9}\bigg\{\bigg[\frac{b_ib_j}{32\pi}\bigg\langle(\nabla^X_z J)z,(\nabla^X_{\frac{\partial}{\partial\bar z_i}}J)\frac{\partial}{\partial\bar z_j}\bigg\rangle\\
&\quad+\frac{b_i}{8\pi}\bigg\langle(\nabla^X_{\frac{\partial}{\partial z_j}}J)z+(\nabla^X_z J)\frac{\partial}{\partial z_j},(\nabla^X_{\frac{\partial}{\partial\bar z_i}}J)\frac{\partial}{\partial\bar z_j}+(\nabla^X_{\frac{\partial}{\partial\bar z_j}}J)\frac{\partial}{\partial\bar z_i}\bigg\rangle\bigg]P^N\bigg\}(0,0)\\
&=\frac{1}{9\pi}\bigg\langle(\nabla^X_{\frac{\partial}{\partial z_i}}J)\frac{\partial}{\partial z_j}+(\nabla^X_{\frac{\partial}{\partial z_j}}J)\frac{\partial}{\partial z_i},(\nabla^X_{\frac{\partial}{\partial\bar z_i}}J)\frac{\partial}{\partial\bar z_j}+2(\nabla^X_{\frac{\partial}{\partial\bar z_j}}J)\frac{\partial}{\partial\bar z_i}\bigg\rangle\\
&=\frac{1}{16\pi}\big|\nabla^X J\big|^2,
\end{aligned}
\tag{2.34}
$$

and by Theorem 1.15, (2.15), (2.29) and (2.33),

$$
\begin{aligned}
&-\bigg(\mathscr{L}_0^{-1}P^{N^\perp}\bigg\langle\big(\nabla^X\nabla^X\mathcal{J}\big)_{(\mathcal{R},\mathcal{R})}\frac{\partial}{\partial z_i},\frac{\partial}{\partial\bar z_i}\bigg\rangle P^N\bigg)(0,0)\\
&=\bigg(\frac{b_j}{4\pi}\bigg\langle(\nabla^X_z J)\frac{\partial}{\partial z_i},(\nabla^X_{\frac{\partial}{\partial\bar z_j}}J)\frac{\partial}{\partial\bar z_i}\bigg\rangle P^N\bigg)(0,0)\\
&=-\frac{1}{16\pi}\big|\nabla^X J\big|^2.
\end{aligned}
\tag{2.35}
$$

Observe that (1.98) shows that for every polynomial $g(z)$ in z, the constant term of $\frac{1}{P^N}\frac{b^\alpha}{2^{|\alpha|}}g(z)P^N$ is the constant term of $(\frac{\partial}{\partial z})^\alpha g$. Thus, in view of (1.98), when calculating $-\mathscr{L}_0^{-1}P^{N^\perp}\mathcal{O}_2P^N$, the contribution of $\frac{1}{2}b_i\sum_{|\alpha|=2}(\partial^\alpha R^L)_{x_0}(\mathcal{R},\frac{\partial}{\partial\bar z_i})\frac{Z^\alpha}{\alpha!}$ in $\mathcal{O}_2$ consists of the terms whose total degree of b_i and $\bar z_j$ is the same as the degree of z. Hence we only need to consider the contribution from the terms where the degree of z is 2. Using (2.2), (2.7), (2.13), (2.19), (2.20) and the equality $\langle[R^{TX}(\bar z,z),\mathcal{J}]z,\frac{\partial}{\partial\bar z_i}\rangle=0$, this contribution is

$$
\begin{aligned}
I_3&=\frac{1}{4}b_i\bigg[\bigg\langle\big(\nabla^X\nabla^X\mathcal{J}\big)_{(z,z)}\bar z+\big(\nabla^X\nabla^X\mathcal{J}\big)_{(z,\bar z)}z+\big(\nabla^X\nabla^X\mathcal{J}\big)_{(\bar z,z)}z,\frac{\partial}{\partial\bar z_i}\bigg\rangle\\
&\quad+\frac{1}{3}\bigg\langle R^{TX}(\bar z,\mathcal{J}z)z+R^{TX}(z,\mathcal{J}\bar z)z,\frac{\partial}{\partial\bar z_i}\bigg\rangle\bigg]\\
&=-\frac{\pi}{4}b_i\bigg[\bigg\langle(\nabla^X_z J)z,3(\nabla^X_{\bar z}J)\frac{\partial}{\partial\bar z_i}-(\nabla^X_{\frac{\partial}{\partial\bar z_i}}J)\bar z\bigg\rangle+\frac{4}{3}\bigg\langle R^{TX}(z,\bar z)z,\frac{\partial}{\partial\bar z_i}\bigg\rangle\bigg].
\end{aligned}
\tag{2.36}
$$

Therefore, from (1.98), (2.2), (2.17), (2.21), (2.33) and (2.36), we get

$$
\begin{aligned}
&-\big(\mathscr{L}_0^{-1}P^{N^\perp}I_3P^N\big)(0,0)\\
&=\frac{\pi}{4}\bigg\{\mathscr{L}_0^{-1}b_i\bigg[\frac{4}{3}\bigg\langle R^{TX}\bigg(z,\frac{\partial}{\partial\bar z_j}\bigg)z,\frac{\partial}{\partial\bar z_i}\bigg\rangle
\end{aligned}
$$

$$
\begin{aligned}
&+\Big\langle \big(\nabla^X_z J\big)z, 3\big(\nabla^X_{\frac{\partial}{\partial \bar z_j}} J\big)\frac{\partial}{\partial \bar z_i} - \big(\nabla^X_{\frac{\partial}{\partial \bar z_i}} J\big)\frac{\partial}{\partial \bar z_j}\Big\rangle\Big]\frac{b_j}{2\pi}P^N\Big\}(0,0)\\
&= -\frac{1}{16\pi}\Big[\frac{4}{3}\Big\langle R^{TX}\Big(\frac{\partial}{\partial z_j},\frac{\partial}{\partial \bar z_j}\Big)\frac{\partial}{\partial z_i} + R^{TX}\Big(\frac{\partial}{\partial z_i},\frac{\partial}{\partial \bar z_j}\Big)\frac{\partial}{\partial z_j},\frac{\partial}{\partial \bar z_i}\Big\rangle\\
&\quad+\Big\langle \big(\nabla^X_{\frac{\partial}{\partial z_i}} J\big)\frac{\partial}{\partial z_j} + \big(\nabla^X_{\frac{\partial}{\partial z_j}} J\big)\frac{\partial}{\partial z_i}, 3\big(\nabla^X_{\frac{\partial}{\partial \bar z_i}} J\big)\frac{\partial}{\partial \bar z_j} - \big(\nabla^X_{\frac{\partial}{\partial \bar z_j}} J\big)\frac{\partial}{\partial \bar z_i}\Big\rangle\Big]\\
&= -\frac{5}{192\pi}\big|\nabla^X J\big|^2 - \frac{1}{6\pi}\Big\langle R^{TX}\Big(\frac{\partial}{\partial z_i},\frac{\partial}{\partial \bar z_j}\Big)\frac{\partial}{\partial z_j},\frac{\partial}{\partial \bar z_i}\Big\rangle. \qquad (2.37)
\end{aligned}
$$

Thanks to (1.98), (2.21) and (2.33) we have

$$
\begin{aligned}
&\frac{1}{3}\Big(P^{N^\perp}\Big\langle R^{TX}\Big(\mathcal{R},\frac{\partial}{\partial z_i}\Big)\mathcal{R},\frac{\partial}{\partial \bar z_i}\Big\rangle P^N\Big)(0,0)\\
&= \frac{1}{3}\Big(P^{N^\perp}\Big\langle R^{TX}\Big(z,\frac{\partial}{\partial z_i}\Big)\frac{\partial}{\partial \bar z_j} + R^{TX}\Big(\frac{\partial}{\partial \bar z_j},\frac{\partial}{\partial z_i}\Big)z,\frac{\partial}{\partial \bar z_i}\Big\rangle\frac{b_j}{2\pi}P^N\Big)(0,0)\\
&= -\frac{1}{3\pi}\Big[\Big\langle R^{TX}\Big(\frac{\partial}{\partial z_j},\frac{\partial}{\partial z_i}\Big)\frac{\partial}{\partial \bar z_j} + R^{TX}\Big(\frac{\partial}{\partial \bar z_j},\frac{\partial}{\partial z_i}\Big)\frac{\partial}{\partial z_j},\frac{\partial}{\partial \bar z_i}\Big\rangle\Big]\\
&= -\frac{1}{96\pi}\big|\nabla^X J\big|^2 + \frac{1}{3\pi}\Big\langle R^{TX}\Big(\frac{\partial}{\partial z_i},\frac{\partial}{\partial \bar z_j}\Big)\frac{\partial}{\partial z_j},\frac{\partial}{\partial \bar z_i}\Big\rangle. \qquad (2.38)
\end{aligned}
$$

By (2.1), (2.13), (2.33), (2.34), (2.35), (2.37), (2.38) and the discussion above (2.36), we have

$$
\begin{aligned}
&-\big(\mathscr{L}_0^{-1}P^{N^\perp}\mathcal{O}_2P^N\big)(0,0)\\
&= -\Big\{\Big[\frac{b_ib_j}{24\pi}\Big\langle R^{TX}\Big(z,\frac{\partial}{\partial \bar z_i}\Big)z,\frac{\partial}{\partial \bar z_j}\Big\rangle + \frac{b_i}{4\pi}R^E\Big(z,\frac{\partial}{\partial \bar z_i}\Big)\\
&\quad+\frac{b_j}{3\pi}\Big(\Big\langle R^{TX}\Big(\frac{\partial}{\partial z_i},\frac{\partial}{\partial \bar z_i}\Big)z,\frac{\partial}{\partial \bar z_j}\Big\rangle - \Big\langle R^{TX}\Big(z,\frac{\partial}{\partial z_i}\Big)\frac{\partial}{\partial \bar z_i},\frac{\partial}{\partial \bar z_j}\Big\rangle\Big)\Big]P^N\Big\}(0,0)\\
&\quad+\frac{1}{3}\Big(P^{N^\perp}\Big\langle R^{TX}\Big(\mathcal{R},\frac{\partial}{\partial z_i}\Big)\mathcal{R},\frac{\partial}{\partial \bar z_i}\Big\rangle P^N\Big)(0,0) - \big(\mathscr{L}_0^{-1}P^{N^\perp}I_3P^N\big)(0,0)\\
&= -\frac{1}{6\pi}\Big\langle R^{TX}\Big(\frac{\partial}{\partial z_i},\frac{\partial}{\partial \bar z_i}\Big)\frac{\partial}{\partial z_j} + R^{TX}\Big(\frac{\partial}{\partial z_j},\frac{\partial}{\partial \bar z_i}\Big)\frac{\partial}{\partial z_i},\frac{\partial}{\partial \bar z_j}\Big\rangle + \frac{1}{2\pi}R^E\Big(\frac{\partial}{\partial z_i},\frac{\partial}{\partial \bar z_i}\Big)\\
&\quad+\frac{2}{3\pi}\Big[\Big\langle R^{TX}\Big(\frac{\partial}{\partial z_i},\frac{\partial}{\partial \bar z_i}\Big)\frac{\partial}{\partial z_j},\frac{\partial}{\partial \bar z_j}\Big\rangle - \Big\langle R^{TX}\Big(\frac{\partial}{\partial z_j},\frac{\partial}{\partial z_i}\Big)\frac{\partial}{\partial \bar z_i},\frac{\partial}{\partial \bar z_j}\Big\rangle\Big]\\
&\quad-\frac{7}{192\pi}\big|\nabla^X J\big|^2 + \frac{1}{6\pi}\Big\langle R^{TX}\Big(\frac{\partial}{\partial z_i},\frac{\partial}{\partial \bar z_j}\Big)\frac{\partial}{\partial z_j},\frac{\partial}{\partial \bar z_i}\Big\rangle\\
&= \frac{1}{2\pi}\Big\langle R^{TX}\Big(\frac{\partial}{\partial z_i},\frac{\partial}{\partial \bar z_j}\Big)\frac{\partial}{\partial z_j},\frac{\partial}{\partial \bar z_i}\Big\rangle + \frac{1}{2\pi}R^E\Big(\frac{\partial}{\partial z_i},\frac{\partial}{\partial \bar z_i}\Big). \qquad (2.39)
\end{aligned}
$$

Formulas (2.14), (2.39) and the discussion at the beginning of Section 2.3 yield finally

$$b_{0,1}(x_0) = F_{0,2}(0,0) = -\big(\mathscr{L}_0^{-1} P^{N^\perp} \mathcal{O}_2 P^N\big)(0,0) - \big(\big(\mathscr{L}_0^{-1} P^{N^\perp} \mathcal{O}_2 P^N\big)(0,0)\big)^*$$
$$= \frac{1}{8\pi}\bigg[r_{x_0}^X + \frac{1}{4}\big|\nabla^X J\big|_{x_0}^2 + 2\sqrt{-1} R_{x_0}^E(e_j, Je_j)\bigg]. \tag{2.40}$$

The proof of Theorem 0.2 is complete.

Remark 2.3. In the Kähler case, i.e., if J is integrable and L, E are holomorphic, then $\mathcal{O}_1 = 0$, and the above computation simplifies a lot.

3. Applications

In this section, we discuss various applications of our results. In Section 3.1, we study the density of states function of $\Delta_{p,\Phi}$. In Section 3.2, we explain how to handle the first-order pseudodifferential operator D_b of Boutet de Monvel and Guillemin [13] which was studied extensively by Shiffman and Zelditch [43]. In Section 3.3, we prove a symplectic version of the convergence of the Fubini–Study metric of an ample line bundle [48]. In Section 3.4, we show how to handle the operator $\bar\partial + \bar\partial^*$ when X is Kähler but $\boldsymbol{J} \neq J$. Finally, in Sections 3.5, 3.6, we establish some generalizations for non-compact or singular manifolds.

3.1. Density of states function

Let (X,ω) be a compact symplectic manifold of real dimension $2n$ and (L, ∇^L, h^L) is a prequantum line bundle as in (0.1). Assume that E is the trivial bundle $\mathbb{C}$, $\Phi = 0$ and $\boldsymbol{J} = J$. The latter means, by (0.2), that g^{TX} is the Riemannian metric associated to ω and J. We denote by $\operatorname{vol}(X) = \int_X \frac{\omega^n}{n!}$ the Riemannian volume of (X, g^{TX}). Recall that d_p is defined in (0.6).

Our aim is to describe the asymptotic distribution of the energies of the bound states as p tends to infinity. We define the spectrum counting function of $\Delta_p := \Delta_{p,0}$ by $N_p(\lambda) = \#\{i\colon \lambda_{i,p} \leqslant \lambda\}$ with $\lambda_{i,p}$ the eigenvalues of Δ_p as in (0.7), and the spectral density measure on $[-C_L, C_L]$ by

$$\nu_p = \frac{1}{d_p}\frac{d}{d\lambda}N_p(\lambda), \quad \lambda \in [-C_L, C_L]. \tag{3.1}$$

Clearly, ν_p is a sum of Dirac measures supported on $\operatorname{Spec}(\Delta_p) \cap [-C_L, C_L]$. Set

$$\varrho : X \longrightarrow \mathbb{R}, \qquad \varrho(x) = \frac{1}{24}\big|\nabla^X J\big|^2. \tag{3.2}$$

Theorem 3.1. *The weak limit of the sequence $\{\nu_p\}_{p\geqslant 1}$ is the direct image measure $\varrho_*(\frac{1}{\operatorname{vol}(X)}\frac{\omega^n}{n!})$, that is, for every continuous function $f \in \mathscr{C}^0([-C_L, C_L])$, we have*

$$\lim_{p\to\infty} \int_{-C_L}^{C_L} f\, d\nu_p = \frac{1}{\operatorname{vol}(X)} \int_X (f\circ\varrho)\frac{\omega^n}{n!}. \tag{3.3}$$

Proof. By (0.7), we have for $q \geqslant 1$ (now E is trivial): $B_{q,p}(x) = \sum_{i=1}^{d_p} \lambda_{i,p}^q |S_i^p(x)|^2$, which yields by integration over X,

$$\frac{1}{d_p} \int_X B_{q,p}\, dv_X = \frac{1}{d_p} \sum_{i=1}^{d_p} \lambda_{i,p}^q = \int_{-C_L}^{C_L} \lambda^q\, d\nu_p(\lambda), \tag{3.4}$$

since S_i^p have unit L^2 norm. On the other hand, (0.6), (0.9) and (0.11) entail for $p \to \infty$,

$$\begin{aligned} \frac{1}{d_p} \int_X B_{q,p}\, dv_X &= \frac{p^n}{d_p} \int_X b_{q,0}\, dv_X + \frac{\mathscr{O}(p^{n-1})}{d_p} \\ &= \frac{1}{\operatorname{vol}(X)} \int_X \varrho^q\, dv_X + \mathscr{O}\big(p^{-1}\big). \end{aligned} \tag{3.5}$$

We infer from (3.4) and (3.5) that (3.3) holds for $f(\lambda) = \lambda^q$, $q \geqslant 1$. Since this is obviously true for $f(\lambda) \equiv 1$, too, we deduce it holds for all polynomials. Upon invoking the Weierstrass approximation theorem, we get (3.3) for all continuous functions on $[-C_L, C_L]$. This completes the proof. □

Remark 3.2. A function ϱ satisfying (3.3) is called spectral density function. Its existence and uniqueness were demonstrated by Guillemin and Uribe [28]. What concerns the explicit formula of ϱ, the paper [11] is dedicated to its computation. Our formula (3.2) is different from [11, Theorem 1.2].[4]

An interesting corollary of (3.2) and (3.3) is the following result which was first stated in [11, Cor. 1.3].

Corollary 3.3. *The spectral density function is identically zero if and only if* (X, J, ω) *is Kähler.*

Remark 3.4. Theorem 3.1 can be slightly generalized. Assume namely that $\boldsymbol{J} = J$ and E is a Hermitian vector bundle as in the Introduction such that $R^E = \eta \otimes \mathrm{Id}_E$ where η is a 2-form. Suppose that $\Phi = \varphi\, \mathrm{Id}_E$ where φ a real function on X. Then there exists a spectrum density function satisfying (3.3) given by

$$\varrho : X \longrightarrow \mathbb{R}, \qquad \varrho(x) = \frac{1}{24}\big|\nabla^X J\big|^2 + \frac{\sqrt{-1}}{2}\eta(e_j, Je_j) + \varphi. \tag{3.6}$$

The proof is similar to the previous one, since $\operatorname{Tr}_{E_x}[B_{q,p}(x)] = \sum_{i=1}^{d_p} \lambda_{i,p}^q |S_i^p(x)|^2$.

[4] Indeed, [11, Theorem 1.2] gives $\varrho(x) = -\frac{5}{24}|\nabla^X J|^2$. Note that [11, equation after (3.11)] shows that the principal terms of $\frac{\partial}{\partial s}$, $\frac{\partial}{\partial y^j}$ are ∂_0, $T_j^l \partial_l$, respectively. Hence the leading term of G_{0j} in [11, (3.7)] should be $\kappa^{-1/2} b_j^{(1)}$ (but this was missed therein). Now, from [11, (3.5)], $b_j^{(1)}$ is $\frac{1}{2}\langle Jz, T_j^l \partial_l\rangle$. Thus the expression of $\mathcal{L}_0$ in [11, (3.8)] is incorrect.

3.2. Almost-holomorphic Szegő kernels

We use the notations and assumptions from Section 3.1, especially, $\boldsymbol{J} = J$. Then $\tau = 2\pi n$.

Let $Y = \{u \in L^*,\ |u|_{h^{L^*}} = 1\}$ be the unit circle bundle in L^*. Then the smooth sections of L^p can be identified to the smooth functions

$$\mathscr{C}^\infty(Y)_p = \{ f \in \mathscr{C}^\infty(Y, \mathbb{C});\ f(ye^{i\theta}) = e^{ip\theta} f(y) \text{ for } e^{i\theta} \in S^1,\ y \in Y \},$$

here $ye^{i\theta}$ is the S^1 action on Y.

The connection ∇^L on L induces a connection on the S^1-principal bundle $\pi : Y \to X$ and this induces a corresponding horizontal bundle $T^HY \subset TY$. We denote by $d\theta^2$ the standard metric on $S^1 = \mathbb{R}/2\pi\mathbb{Z}$. We introduce the metric $g^{TY} = \pi^* g^{TX} \oplus d\theta^2$ on TY corresponding to the direct sum $TY = T^HY \oplus TS^1$. Associated to (Y, g^{TY}) there is the Bochner-Laplacian Δ_Y acting on functions on Y. By construction, Δ_Y commutes with the generator ∂_θ of the circle action, and so it commutes with the horizontal Laplacian

$$\Delta_h = \Delta_Y + \partial_\theta^2. \tag{3.7}$$

Δ_h acting on $\mathscr{C}^\infty(Y)_p$ is identical to Δ^{L^p} on $\mathscr{C}^\infty(X, L^p)$ (cf. [10, §2.1]).

By the construction of [13, Lemma 14.11, Theorem A 5.9], [14], [28, (3.13)], there exists a self-adjoint second-order pseudodifferential operator Q on Y such that

$$V = \Delta_h + \sqrt{-1}\tau\partial_\theta - Q \tag{3.8}$$

is a self-adjoint pseudodifferential operator of order zero on Y, and V and Q commute with the S^1-action. The orthogonal projection Π onto the kernel of Q is called the *Szegő projector* associated with the almost CR manifold Y. In fact, the Szegő projector is not unique or canonically defined, but the above construction defines a canonical choice of Π modulo smoothing operators. In the complex case, the construction produces the usual Szegő projector Π.

We denote the operators on $\mathscr{C}^\infty(X, L^p)$ corresponding to Q, V, Π by Q_p, V_p, Π_p, especially, $V_p(x, y) = \frac{1}{2\pi}\int_0^{2\pi} e^{-ip\theta} V(xe^{i\theta}, y)\, d\theta$. Then by (3.8),

$$Q_p = \Delta^{L^p} - p\tau - V_p. \tag{3.9}$$

By [28, §4], there exists $\mu_1 > 0$ such that for p large,

$$\operatorname{Spec}(Q_p) \subset \{0\} \cup [\mu_1 p, +\infty[. \tag{3.10}$$

Since the operator V_p is uniformly bounded in p, (0.5), (0.6) imply that for p large we have

$$\dim \operatorname{Ker}(Q_p) = d_p = \int_X \operatorname{Td}(TX) \operatorname{ch}(L^p). \tag{3.11}$$

Formula (3.11) was first obtained by Borthwick and Uribe [9].

Now we explain how to study the Szegő projector Π_p[5] using the methods of the present paper. Recall that $\widetilde{F}$ is the function defined after (1.4). Let $\Pi_p(x,x')$, $\widetilde{F}(Q_p)(x,x')$ be the smooth kernels of Π_p, $\widetilde{F}(Q_p)$ with respect to the Riemannian volume form $dv_X(x')$.

Note that V_p is a 0-order pseudodifferential operator on X induced from a 0-order pseudodifferential operator on Y. Thus (3.9) and (3.10) entail the analogue of [20, Proposition 3.1] (cf. Proposition 1.2): for every $l,m\in\mathbb{N}$, there exists $C_{l,m}>0$ such that for $p\geqslant 1$,

$$\left|\widetilde{F}(Q_p)(x,x')-\Pi_p(x,x')\right|_{\mathscr{C}^m(X\times X)}\leqslant C_{l,m}\,p^{-l}. \tag{3.12}$$

By finite propagation speed [47, §4.4], we know that $\widetilde{F}(Q_p)(x,x')$ only depends on the restriction of Q_p to $B^X(x,\varepsilon)$, and is zero if $d(x,x')\geqslant\varepsilon$. It follows that the asymptotic of $\Pi_p(x,x')$ as $p\to\infty$ is localized on a neighborhood of x. Thus we can translate our analysis from X to the manifold $\mathbb{R}^{2n}\simeq T_{x_0}X=:X_0$ as in Section 1.2. Proceeding as in Section 1.2 we extend ∇^L to a Hermitian connection ∇^{L_0} on $(L_0,h^{L_0})=(X_0\times L_{x_0},h^{L_{x_0}})$ on $T_{x_0}X$ such that the curvature R^{L_0} is positive and $R^{L_0}=R^L_{x_0}$ outside a compact set.

Now, by using a micro-local partition of unity, one can still construct the operator Q^{X_0} as in [13, Lemma 14.11, Theorem A 5.9], [14], [28, (3.13)], such that V^{X_0} differs from V by a smooth operator in a neighborhood of 0 in X_0, and Q^{X_0} still verifies (3.10). Thus we can work on $\mathscr{C}^\infty(X_0,\mathbb{C})$ as in Section 1.3. Similar to (1.27) we rescale then the coordinates and use the norm (1.38). Then $V_p^{X_0}$ is a 0th order pseudodifferential operator on X_0 induced from a 0th order pseudodifferential operator on Y_0. This guarantees that the operator obtained by rescaling $V_p^{X_0}$ has an expansion similar to (1.29) with leading term t^2R_2, in the sense of pseudodifferential operators.

Using (3.10), [20, (3.89)] and similar arguments to those from [20, Theorem 4.18], we can also get the following full off-diagonal expansion (3.13) of the Szegő kernel Π_p. More precisely, recalling that $P^N(Z,Z')$ is the Bergman kernel of $\mathscr{L}_0$ as in (1.81) and (1.91) with $a_j=2\pi$, we have:

Theorem 3.5. *For every $r\geqslant 0$ there exist a polynomial $\boldsymbol{j}_r(Z,Z')$ in Z,Z' with the same parity as r, such that $\boldsymbol{j}_0=1$, and a constant $C''>0$ with the property that for every $k,m,m'\in\mathbb{N}$ and $\varepsilon>0$ there exist $N\in\mathbb{N}$, $C>0$ so that the following estimate holds*

$$\begin{aligned}&\left|\frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^\alpha\partial Z'^{\alpha'}}\left(\frac{1}{p^n}\Pi_p(Z,Z')-\sum_{r=0}^k(\boldsymbol{j}_rP^N)(\sqrt{p}Z,\sqrt{p}Z')\kappa^{-\frac12}(Z)\kappa^{-\frac12}(Z')p^{-r/2}\right)\right|_{\mathscr{C}^{m'}(X)}\\&\quad\leqslant Cp^{-(k+1-m)/2}\left(1+|\sqrt{p}Z|+|\sqrt{p}Z'|\right)^N\exp\left(-\sqrt{C''\mu_1}\sqrt{p}|Z-Z'|\right)+\mathscr{O}\left(p^{-\infty}\right),\end{aligned} \tag{3.13}$$

for all $x_0\in X$ and $\alpha,\alpha'\in\mathbb{Z}^{2n}$ with $|\alpha|+|\alpha'|\leqslant m$, $Z,Z'\in T_{x_0}X$ with $|Z|,|Z'|\leqslant\varepsilon$, and all $p\geqslant 1$.

In (3.13) we use the trivializations from Section 1.2; $\mathscr{C}^{m'}(X)$ is the $\mathscr{C}^{m'}$-norm for the parameter $x_0\in X$. A function is said to be $\mathscr{O}(p^{-\infty})$ if for every $l,l_1\in\mathbb{N}$, there exists $C_{l,l_1}>0$ such that

[5] As Professor Sjőstrand pointed out to us, in general, $\Pi_p-P_{0,p}$ is not $\mathscr{O}(p^{-\infty})$ as $p\to\infty$, where $P_{0,p}$ is the smooth kernel of the operator $\Delta_{0,p}$ (Definition 1.1). This can also be seen from the presence of a contribution coming from Φ in the expression (0.9) of the coefficient $b_{0,2}$.

its $\mathscr{C}^{l_1}$-norm is dominated by $C_{l,l_1} p^{-l}$. The term $\kappa^{-\frac{1}{2}}$ in (3.13) comes from the conjugation of the operators as in (1.113). We leave the details and the generalization in the case of the presence of a non-trivial twisting vector bundle E to the interested reader.

Theorem 3.5 is closely related to [10,30,43]. More precisely, Shiffman and Zelditch [43, Theorem 1] prove a similar result for two cases: either for $|Z|, |Z'| \leqslant C/\sqrt{p}$ or for $Z = 0, |Z'| \leqslant Cp^{-1/3}$, with $C > 0$ fixed. This is explained in detail in the recent [44, Theorem 2.4] of the same authors.

3.3. Symplectic version of Kodaira embedding theorem

Let (X, ω) be a compact symplectic manifold of real dimension $2n$ and let (L, ∇^L, h^L) be a pre-quantum line bundle and let g^{TX} be a Riemannian metric on X as in Introduction.

Recall that $\mathcal{H}_p \subset \mathscr{C}^\infty(X, L^p)$ is the vector subspace spanned of those eigensections of $\Delta_p = \Delta^{L^p} - \tau p$ corresponding to eigenvalues from $[-C_L, C_L]$. We denote by $\mathbb{P}\mathcal{H}_p^*$ the projective space associated to the dual of $\mathcal{H}_p$ and we identify $\mathbb{P}\mathcal{H}_p^*$ with the Grassmannian of hyperplanes in $\mathcal{H}_p$. The *base locus* of $\mathcal{H}_p$ is the set $\mathrm{Bl}(\mathcal{H}_p) = \{x \in X\colon\ s(x) = 0 \text{ for all } s \in \mathcal{H}_p\}$. As in algebraic geometry, we define the Kodaira map

$$\begin{aligned} &\Phi_p : X \setminus \mathrm{Bl}(\mathcal{H}_p) \longrightarrow \mathbb{P}\mathcal{H}_p^*, \\ &\Phi_p(x) = \big\{s \in \mathcal{H}_p\colon\ s(x) = 0\big\} \end{aligned} \tag{3.14}$$

which sends $x \in X \setminus \mathrm{Bl}(\mathcal{H}_p)$ to the hyperplane of sections vanishing at x. Note that $\mathcal{H}_p$ is endowed with the induced L^2 Hermitian product (1.1) so there is a well-defined Fubini–Study metric g_{FS} on $\mathbb{P}\mathcal{H}_p^*$ with the associated form ω_{FS}.

Theorem 3.6. *Let (L, ∇^L) be a pre-quantum line bundle over a compact symplectic manifold (X, ω). The following assertions hold true*:

(i) *For large p, the Kodaira maps $\Phi_p : X \to \mathbb{P}\mathcal{H}_p^*$ are well defined.*
(ii) *The induced Fubini–Study metric $\frac{1}{p}\Phi_p^*(\omega_{FS})$ converges in the $\mathscr{C}^\infty$ topology to ω; for each $l \geqslant 0$ there exists $C_l > 0$ such that*

$$\left|\frac{1}{p}\Phi_p^*(\omega_{FS}) - \omega\right|_{\mathscr{C}^l} \leqslant \frac{C_l}{p}. \tag{3.15}$$

(iii) *For large p the Kodaira maps Φ_p are embeddings.*

Remark 3.7. (1) Assume that X is Kähler and L is a holomorphic bundle. If $\square^{L^p}$ denotes the Kodaira-Laplacian on L^p, then $\Delta_p = 2\square^{L^p}$, so $\mathcal{H}_p$ coincides with the space $H^0(X, L^p)$ of holomorphic sections of L^p. Then (i) and (iii) are simply the Kodaira embedding theorem. Assertion (ii) is due to Tian [48, Theorem A] as an answer to a conjecture of Yau. In [48] the case $l = 2$ is considered and the left-hand side of (3.15) is estimated by $C_l/\sqrt{p}$. Ruan [42] proved the $\mathscr{C}^\infty$ convergence and improved the bound to C_l/p. Both papers use the peak section method, based on L^2-estimates for $\bar\partial$. Finally, Catlin and Zelditch, independently, deduced (ii) from the asymptotic expansion of the Szegő kernel [17,51]. Bouche [12] proved that the induced Fubini–Study metric $(\Phi_p^* h^{\mathscr{O}(1)})^{1/p}$ on L converges in the $\mathscr{C}^0$ topology to the initial metric h^L.

(2) Borthwick and Uribe [10, Theorem 1.1], Shiffman and Zelditch [43, Theorems 2, 3] prove a different symplectic version of [48, Theorem A] when $\boldsymbol{J} = J$. Instead of $\mathcal{H}_p$, they use the space $H^0_J(X, L^p) := \mathrm{Im}(\Pi_p)$ (cf. [10, p. 601], [43, §2.3] and Section 3.2 of the present paper) of 'almost holomorphic sections' proposed by Boutet de Monvel and Guillemin [13,14].

Proof. Let us first give an alternate description of the map Φ_p which relates it to the Bergman kernel. Let $\{S_i^p\}_{i=1}^{d_p}$ be an arbitrary orthonormal basis of $\mathcal{H}_p$ with respect to the Hermitian product (1.1). Once we have fixed a basis, we obtain an identification $\mathcal{H}_p \cong \mathcal{H}_p^* \cong \mathbb{C}^{d_p}$ and $\mathbb{P}\mathcal{H}_p^* \cong \mathbb{CP}^{d_p-1}$. Consider the commutative diagram:

$$\begin{array}{ccc} X \setminus \mathrm{Bl}(\mathcal{H}_p) & \xrightarrow{\Phi_p} & \mathbb{P}\mathcal{H}_p^* \\ \downarrow \mathrm{Id} & & \downarrow \cong \\ X \setminus \mathrm{Bl}(\mathcal{H}_p) & \xrightarrow{\widetilde{\Phi}_p} & \mathbb{CP}^{d_p-1}. \end{array} \tag{3.16}$$

Then

$$\Phi_p^*(\omega_{FS}) = \widetilde{\Phi}_p^*\left(\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log\sum_{j=1}^{d_p}|w_j|^2\right), \tag{3.17}$$

where $[w_1, \dots, w_{d_p}]$ are homogeneous coordinates in $\mathbb{CP}^{d_p-1}$. To describe $\widetilde{\Phi}_p$ in a neighborhood of a point $x_0 \in X \setminus \mathrm{Bl}(\mathcal{H}_p)$, we choose a local unity frame e_L of L and write $S_i^p = f_i^p e_L^{\otimes p}$ for some smooth functions f_i^p. Then

$$\widetilde{\Phi}_p(x) = \left[f_1^p(x), \dots, f_{d_p}^p(x)\right], \tag{3.18}$$

and this does not depend on the choice of the frame e_L.

(i) Let us choose an unit frame e_L of L. Then $|S_i^p|^2 = |f_i^p|^2|e_L|^{2p} = |f_i^p|^2$, hence

$$B_{0,p} = \sum_{i=1}^{d_p}|S_i^p|^2 = \sum_{i=1}^{d_p}|f_i^p|^2.$$

Since $b_{0,0} > 0$, the asymptotic expansion (0.9) shows that $B_{0,p}$ does not vanish on X for p large enough, so the sections $\{S_i^p\}_{i=1}^{d_p}$ have no common zeroes. Therefore Φ_p and $\widetilde{\Phi}_p$ are defined on all X.

(ii) Let us fix $x_0 \in X$. We identify a small geodesic ball $B^X(x_0, \varepsilon)$ to $B^{T_{x_0}X}(0, \varepsilon)$ by means of the exponential map and consider a trivialization of L as in Section 1.2, i.e. we trivialize L by using a unit frame $e_L(Z)$ which is parallel with respect to ∇^L along $[0, 1] \ni u \to uZ$ for $Z \in B^{T_{x_0}X}(0, \varepsilon)$. Let $\|w\|^2 = \sum_{j=1}^{d_p}|w_j|^2$. We can express the Fubini–Study metric as

$$\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log\left(\|w\|^2\right) = \frac{\sqrt{-1}}{2\pi}\left[\frac{1}{\|w\|^2}\sum_{j=1}^{d_p}dw_j \wedge d\overline{w}_j - \frac{1}{\|w\|^4}\sum_{j,k=1}^{d_p}\overline{w}_j w_k\, dw_j \wedge d\overline{w}_k\right],$$

and therefore, from (3.18),

$$\begin{aligned}\Phi_p^*(\omega_{FS})(x_0) &= \frac{\sqrt{-1}}{2\pi}\left[\frac{1}{|f^p|^2}\sum_{j=1}^{d_p} df_j^p \wedge d\overline{f_j^p} - \frac{1}{|f^p|^4}\sum_{j,k=1}^{d_p}\overline{f_j^p}\, f_k^p\, df_j^p \wedge d\overline{f_k^p}\right](x_0)\\ &= \frac{\sqrt{-1}}{2\pi}\big[f^p(x_0,x_0)^{-1}d_x d_y f^p(x,y) - f^p(x_0,x_0)^{-2}d_x f^p(x,y)\\ &\quad \wedge d_y f^p(x,y)\big]\big|_{x=y=x_0},\end{aligned} \tag{3.19}$$

where $f^p(x,y)=\sum_{i=1}^{d_p} f_i^p(x)\overline{f_i^p}(y)$ and $|f^p(x)|^2=f^p(x,x)$. Since

$$P_{0,p}(x,y)=f^p(x,y)e_L^p(x)\otimes e_L^p(y)^*, \tag{3.20}$$

thus $P_{0,p}(x,y)$ is $f^p(x,y)$ under our trivialization of L. By (1.31), Theorem 1.18, and (1.113), we obtain

$$\begin{aligned}\frac{1}{p}\Phi_p^*(\omega_{FS})(x_0) &= \frac{\sqrt{-1}}{2\pi}\left[\frac{1}{F_{0,0}}d_x d_y F_{0,0} - \frac{1}{F_{0,0}^2}d_x F_{0,0}\wedge d_y F_{0,0}\right](0,0)\\ &\quad - \frac{\sqrt{-1}}{2\pi}\frac{1}{\sqrt{p}}\left[\frac{1}{F_{0,0}^2}(d_x F_{0,1}\wedge d_y F_{0,0} + d_x F_{0,0}\wedge d_y F_{0,1})\right](0,0)\\ &\quad + \mathscr{O}(1/p).\end{aligned} \tag{3.21}$$

Using again (1.91) and (1.109), we obtain

$$\frac{1}{p}\Phi_p^*(\omega_{FS})(x_0) = \frac{\sqrt{-1}}{4\pi}\sum_{j=1}^{n} a_j dz_j\wedge d\bar z_j|_{x_0} + \mathscr{O}\left(\frac{1}{p}\right) = \omega(x_0)+\mathscr{O}\left(\frac{1}{p}\right), \tag{3.22}$$

and the convergence takes place in the $\mathscr{C}^\infty$ topology with respect to $x_0\in X$.

(iii) Since X is compact, we have to prove two things for p sufficiently large: (a) Φ_p are immersions and (b) Φ_p are injective. We note that (a) follows immediately from (3.15).

To prove (b) let us assume the contrary, namely that there exists a sequence of distinct points $x_p\neq y_p$ such that $\Phi_p(x_p)=\Phi_p(y_p)$. Relation (3.16) implies that $\widetilde{\Phi}_p(x_p)=\widetilde{\Phi}_p(y_p)$, where $\widetilde{\Phi}_p$ is defined by an arbitrary choice of basis.

The key observation is that Theorem 1.19 ensures the existence of a sequence of *peak sections* at each point of X. The construction goes like follows. Let $x_0\in X$ be fixed. Since Φ_p is base point free for large p, we can consider the hyperplane $\Phi_p(x_0)$ of all sections of $\mathcal{H}_p$ vanishing at x_0. We construct then an orthonormal basis $\{S_i^p\}_{i=1}^{d_p}$ of $\mathcal{H}_p$ such that the first d_p-1 elements belong to $\Phi_p(x_0)$. Then $S_{d_p}^p$ is a unit norm generator of the orthogonal complement of $\Phi_p(x_0)$, and will be denoted by $S_{x_0}^p$. This is a peak section at x_0. We note first that $|S_{x_0}^p(x_0)|^2=B_{0,p}(x_0)$ and $P_{0,p}(x,x_0)=S_{x_0}^p(x)\otimes S_{x_0}^p(x_0)^*$ and therefore

$$S_{x_0}^p(x)=\frac{1}{B_{0,p}(x_0)}P_{0,p}(x,x_0)\cdot S_{x_0}^p(x_0). \tag{3.23}$$

From (1.113) we deduce that for a sequence $\{r_p\}$ with $r_p \to 0$ and $r_p\sqrt{p} \to \infty$,

$$\int\limits_{B(x_0,r_p)} \left|S^p_{x_0}(x)\right|^2 dv_X(x) = 1 - \mathscr{O}(1/p), \quad \text{for } p \to \infty. \tag{3.24}$$

Relation (3.24) explains the term 'peak section': when p grows, the mass of $S^p_{x_0}$ concentrates near x_0. Since $\Phi_p(x_p) = \Phi_p(y_p)$ we can construct as before the peak section $S^p_{x_p} = S^p_{y_p}$ as the unit norm generator of the orthogonal complement of $\Phi_p(x_p) = \Phi_p(y_p)$. We fix in the sequel such a section which peaks at both x_p and y_p.

We consider the distance $d(x_p, y_p)$ between the two points x_p and y_p. By passing to a subsequence we have two possibilities: either $\sqrt{p}\, d(x_p, y_p) \to \infty$ as $p \to \infty$ or there exists a constant $C > 0$ such that $d(x_p, y_p) \leqslant C/\sqrt{p}$ for all p.

Assume that the first possibility is true. For large p, we learn from relation (3.24) that the mass of $S^p_{x_p} = S^p_{y_p}$ (which is 1) concentrates both in neighborhoods $B(x_p, r_p)$ and $B(y_p, r_p)$ with $r_p = d(x_p, y_p)/2$ and approaches therefore 2 if $p \to \infty$. This is a contradiction which rules out the first possibility.

To exclude the second possibility we follow [43]. We identify as usual $B^X(x_p, \varepsilon)$ to $B^{T_{x_p}X}(0, \varepsilon)$ so the point y_p gets identified to $Z_p/\sqrt{p}$ where $Z_p \in B^{T_{x_p}X}(0, C)$. We define then

$$f_p : [0,1] \longrightarrow \mathbb{R}, \qquad f_p(t) = \frac{|S^p_{x_p}(tZ_p/\sqrt{p})|^2}{B_{0,p}(tZ_p/\sqrt{p})}. \tag{3.25}$$

We have $f_p(0) = f_p(1) = 1$ (again because $S^p_{x_p} = S^p_{y_p}$) and $f_p(t) \leqslant 1$ by the definition of the generalized Bergman kernel. We deduce the existence of a point $t_p \in]0,1[$ such that $f''_p(t_p) = 0$. The expansion (1.113) and formulas (3.23), (3.25) imply the estimate

$$f_p(t) = e^{-\frac{t^2}{4}\sum_j a_j |z_{p,j}|^2}\bigl(1 + g_p(tZ_p)/\sqrt{p}\bigr) \tag{3.26}$$

where the $\mathscr{C}^2$ norm of g_p over $B^{T_{x_p}X}(0, C)$ is uniformly bounded in p. We infer from (3.26) that $|Z_p|_0^2 := \frac{1}{4}\sum_j a_j |z_{p,j}|^2 = \mathscr{O}(1/\sqrt{p})$. Using the limited expansion $e^x = 1 + x + x^2\varphi(x)$ for $x = t^2|Z_p|_0^2$ in (3.26) and taking derivatives, we obtain

$$f''_p(t) = -2|Z_p|_0^2 + \mathscr{O}\bigl(|Z_p|_0^4\bigr) + \mathscr{O}\bigl(|Z_p|_0^2/\sqrt{p}\bigr) = \bigl(-2 + \mathscr{O}(1/\sqrt{p})\bigr)|Z_p|_0^2.$$

Evaluating the latter expression at t_p we get $0 = f''_p(t_p) = (-2 + \mathscr{O}(1/\sqrt{p}))|Z_p|_0^2$, which is a contradiction since by assumption $Z_p \neq 0$. This finishes the proof of (iii). □

Remark 3.8. Let us point out complementary results which are analogues of [10, (1.3)–(1.5)] for the spaces $\mathcal{H}_p$. Computing as in (3.19) the pull-back $\Phi_p^* h_{FS}$ of the Hermitian metric $h_{FS} = g_{FS} - \sqrt{-1}\,\omega_{FS}$ on $\mathbb{P}\mathcal{H}_p^*$, we get the similar inequality to (3.15) for g_{FS} and $\omega(\cdot, J\cdot)$. Thus, Φ_p are asymptotically symplectic and isometric. Moreover, arguing as in [10, Proposition 4.4] we can show that Φ_p are 'nearly holomorphic':

$$\frac{1}{p}\|\partial\Phi_p\| \geqslant C, \quad \frac{1}{p}\|\bar{\partial}\Phi_p\| = \mathscr{O}(1/p), \quad \text{for some } C > 0, \tag{3.27}$$

uniformly on X, where $\|\cdot\|$ is the pointwise operator norm.

3.4. Holomorphic case revisited

In this section we assume that (X, J, ω) is Kähler, the vector bundles E, L are holomorphic on X, and ∇^E, ∇^L are the holomorphic Hermitian connections on (E, h^E), (L, h^L). As usual, $\frac{\sqrt{-1}}{2\pi}R^L = \omega$.

However, we will work with an arbitrary (*non-Kähler*) Riemannian metric g^{TX} on TX compatible with J. That is, in general $\boldsymbol{J} \neq J$, where $\boldsymbol{J}$ is defined in (0.2). The use of non-Kähler metrics is useful in Section 3.6, for example. Set[6]

$$\Theta(X, Y) = g^{TX}(JX, Y). \tag{3.28}$$

Then the 2-form Θ need not to be closed. We denote by $T^{(1,0)}X$, $T^{(0,1)}X$ the holomorphic and anti-holomorphic tangent bundles as in Section 1.4. Let $\{e_i\}$ be an orthonormal frame of (TX, g^{TX}).

Let $g^{TX}_\omega(\cdot,\cdot) := \omega(\cdot, J\cdot)$ be the metric on TX induced by ω, J. We will use a subscript ω to indicate the objects corresponding to g^{TX}_ω, especially r^X_ω is the scalar curvature of (TX, g^{TX}_ω), and Δ_ω is the Bochner-Laplacian as in (1.2) associated to g^{TX}_ω.

Let $\overline{\partial}^{L^p\otimes E,*}$ be the formal adjoint of the Dolbeault operator $\overline{\partial}^{L^p\otimes E}$ on the Dolbeault complex $\Omega^{0,\bullet}(X, L^p\otimes E)$ with the Hermitian product induced by g^{TX}, h^L, h^E as in (1.1). Set

$$D_p = \sqrt{2}\big(\overline{\partial}^{L^p\otimes E} + \overline{\partial}^{L^p\otimes E,*}\big).$$

Then

$$D_p^2 = 2\big(\overline{\partial}^{L^p\otimes E}\,\overline{\partial}^{L^p\otimes E,*} + \overline{\partial}^{L^p\otimes E,*}\,\overline{\partial}^{L^p\otimes E}\big)$$

preserves the $\mathbb{Z}$-grading of $\Omega^{0,\bullet}(X, L^p\otimes E)$. Then for p large enough,

$$\operatorname{Ker}(D_p) = \operatorname{Ker}\big(D_p^2\big) = H^0\big(X, L^p\otimes E\big). \tag{3.29}$$

Here D_p is not a spin^c Dirac operator on $\Omega^{0,\bullet}(X, L^p\otimes E)$, and D_p^2 is not a renormalized Bochner-Laplacian as in (0.4).

Let $P_p(x, x')$ $(x, x' \in X)$ be the smooth kernel of the orthogonal projection P_p from $\mathscr{C}^\infty(X, L^p\otimes E)$ on $\operatorname{Ker}(D_p^2)$ with respect to the Riemannian volume form $dv_X(x')$ for p large enough. Recall that we denote by $\det_{\mathbb{C}}$ the determinant function on the complex bundle $T^{(1,0)}X$. We denote by $|\boldsymbol{J}| = (-\boldsymbol{J}^2)^{-1/2}$, then $\det_{\mathbb{C}}|\boldsymbol{J}| = (2\pi)^{-n}\Pi_i a_i$ under the notation in (1.82). Now we explain how to put it in the frame of our work.

[6] The convention here differs from [3, (2.1)] by a factor -1.

Theorem 3.9. *The smooth kernel $P_p(x,x')$ has a full off-diagonal asymptotic expansion analogous to* (3.13) *with $\boldsymbol{j}_0 = 1$ as $p \to \infty$. The corresponding term $b_{0,1}$ in the expansion* (0.9) *of $B_{0,p}(x) := P_p(x,x)$ is given by*

$$b_{0,1} = \frac{\det_{\mathbb{C}} |\boldsymbol{J}|}{8\pi}\big[r_\omega^X - 2\Delta_\omega\big(\log\big(\det_{\mathbb{C}}|\boldsymbol{J}|\big)\big) + 4R^E(w_{\omega,j}, \overline{w}_{\omega,j})\big]. \tag{3.30}$$

here $\{w_{\omega,j}\}$ is an orthonormal basis of $(T^{(1,0)}X, g_\omega^{TX})$.

Proof. As pointed out in [33, Remark 3.1], [5, Theorem 1] implies that there exist $\mu_0, C_L > 0$ such that for every $p \in \mathbb{N}$ and $s \in \Omega^{>0}(X, L^p \otimes E) := \bigoplus_{q\geqslant 1} \Omega^{0,q}(X, L^p \otimes E)$,

$$\|D_p s\|_{L^2}^2 \geqslant (2p\mu_0 - C_L)\|s\|_{L^2}^2. \tag{3.31}$$

Moreover $\operatorname{Spec}(D_p^2) \subset \{0\} \cup [2p\mu_0 - C_L, +\infty[$.

Let S^{-B} denote the 1-form with values in antisymmetric elements of $\operatorname{End}(TX)$ which satisfies

$$\big\langle S^{-B}(U)V, W\big\rangle = \frac{\sqrt{-1}}{2}\big((\partial - \bar\partial)\Theta\big)(U,V,W) \tag{3.32}$$

for all $U, V, W \in TX$. The Bismut connection ∇^{-B} on TX is defined by

$$\nabla^{-B} = \nabla^{TX} + S^{-B}. \tag{3.33}$$

Then by [3, Prop. 2.5], ∇^{-B} preserves the metric g^{TX} and the complex structure of TX. Let $\nabla^{\det}$ be the holomorphic Hermitian connection on $\det(T^{(1,0)}X)$ with its curvature $R^{\det}$. Then these two connections induce naturally a unique connection on $\Lambda(T^{*(0,1)}X)$ which preserves its $\mathbb{Z}$-grading, and with the connections ∇^L, ∇^E, we get a connection ∇^{-B,E_p} on $\Lambda(T^{*(0,1)}X) \otimes L^p \otimes E$. Let Δ^{-B,E_p} be the Laplacian on $\Lambda(T^{*(0,1)}X) \otimes L^p \otimes E$ induced by ∇^{-B,E_p} as in (1.2). For each $v \in TX$ with decomposition $v = v_{1,0} + v_{0,1} \in T^{(1,0)}X \oplus T^{(0,1)}X$, let $\bar v_{1,0}^* \in T^{*(0,1)}X$ be the metric dual of $v_{1,0}$. Then

$$c(v) = \sqrt{2}\big(\bar v_{1,0}^* \wedge - i_{v_{0,1}}\big)$$

defines the Clifford action of v on $\Lambda(T^{*(0,1)}X)$, where $\wedge$ and i denote the exterior and interior product respectively. We define a map ${}^c : \Lambda(T^*X) \to C(TX)$, the Clifford bundle of TX, by sending $e^{i_1} \wedge \cdots \wedge e^{i_j}$ to $c(e_{i_1}) \cdots c(e_{i_j})$ for $i_1 < \cdots < i_j$. For $B \in \Lambda^3(T^*X)$, set $|B|^2 = \sum_{i<j<k} |B(e_i, e_j, e_k)|^2$. Then we can formulate [3, Theorem 2.3] as follows:

$$D_p^2 = \Delta^{-B,E_p} + \frac{r^X}{4} + {}^c\bigg(R^E + pR^L + \frac{1}{2}R^{\det}\bigg) + \frac{\sqrt{-1}}{2}{}^c(\bar\partial\partial\Theta) - \frac{1}{8}\big|(\partial - \bar\partial)\Theta\big|^2. \tag{3.34}$$

We use now the connection ∇^{-B,E_p} instead of ∇^{E_p} in [20, §2]. Then by (3.31), (3.34), everything goes through perfectly well and as in [20, Theorem 4.18], so we can directly apply the result in [20] to get the *full off-diagonal* asymptotic expansion of the Bergman kernel. Since the above construction preserves the $\mathbb{Z}$-grading on $\Omega^{0,\bullet}(X, L^p \otimes E)$, we can also directly work on $\mathscr{C}^\infty(X, L^p \otimes E)$.

Now, we need to compute the corresponding $b_{0,1}$. We endow E with the metric $h^E_\omega := (\det_{\mathbb{C}}|\boldsymbol{J}|)^{-1}h^E$ and let R^E_ω be the curvature associated to the holomorphic Hermitian connection of (E, h^E_ω). Then

$$R^E_\omega = R^E - \bar{\partial}\partial\log\bigl(\det_{\mathbb{C}}|\boldsymbol{J}|\bigr). \tag{3.35}$$

Thus

$$\sqrt{-1}R^E_\omega(e_{\omega,j}, Je_{\omega,j}) = 2R^E_\omega(w_{\omega,j}, \overline{w}_{\omega,j}) = \sqrt{-1}R^E(e_{\omega,j}, Je_{\omega,j}) - \Delta_\omega\log\bigl(\det_{\mathbb{C}}|\boldsymbol{J}|\bigr). \tag{3.36}$$

Let $\langle\cdot,\cdot\rangle_\omega$ be the L^2-Hermitian product on $\mathscr{C}^\infty(X, L^p\otimes E)$ induced by g^{TX}_ω, h^L, h^E_ω. Then

$$\bigl(\mathscr{C}^\infty\bigl(X, L^p\otimes E\bigr), \langle\cdot,\cdot\rangle_\omega\bigr) = \bigl(\mathscr{C}^\infty\bigl(X, L^p\otimes E\bigr), \langle\cdot,\cdot\rangle\bigr), \qquad dv_{X,\omega} = \bigl(\det_{\mathbb{C}}|\boldsymbol{J}|\bigr)\,dv_X. \tag{3.37}$$

Observe that $H^0(X, L^p\otimes E)$ does not depend on g^{TX}, h^L or h^E. If $P_{\omega,p}(x,x')$, $(x,x'\in X)$ denotes the smooth kernel of the orthogonal projection from $(\mathscr{C}^\infty(X, L^p\otimes E), \langle\cdot,\cdot\rangle_\omega)$ onto $H^0(X, L^p\otimes E)$ with respect to $dv_{X,\omega}(x)$, we have

$$P_p(x,x') = \bigl(\det_{\mathbb{C}}|\boldsymbol{J}|(x')\bigr)P_{\omega,p}(x,x'). \tag{3.38}$$

Now for the kernel $P_{\omega,p}(x,x')$, we can apply Theorem 0.1 (or [20, Theorem 1.3]) since $g^{TX}_\omega(\cdot,\cdot) = \omega(\cdot, J\cdot)$ is a Kähler metric on TX, and (3.30) follows from (0.8) and (3.36). □

Remark 3.10. The argument in this subsection goes through the orbifold case as in [20, Section 4.2].

3.5. Generalizations to non-compact manifolds

As in Section 3.4, we consider a complex Hermitian manifold (X, J, Θ) of dimension n, where J is the complex structure and Θ is the $(1,1)$ form associated to a Riemannian metric g^{TX} compatible with J as in (3.28). The Hermitian torsion of Θ is $\mathcal{T} = [i(\Theta), \partial\Theta]$, where $i(\Theta) = (\Theta\wedge\cdot)^*$ is the interior multiplication with Θ. Let (L, h^L) and (E, h^E) be holomorphic Hermitian vector bundles over X, with $\mathrm{rk}(L) = 1$. We denote by R^L, R^E and $R^{\det}$ the curvatures of the holomorphic Hermitian connections ∇^L, ∇^E and $\nabla^{\det}$ on L, E and $\det(T^{(1,0)}X)$. Let $J^L\in\mathrm{End}(TX)$ be the endomorphism satisfying $\frac{\sqrt{-1}}{2\pi}R^L(\cdot,\cdot) = \Theta(J^L\cdot,\cdot)$. The line bundle L is supposed to be positive and we set $\omega = \frac{\sqrt{-1}}{2\pi}R^L$. We also keep the notations g^{TX}_ω, Δ_ω and r^X_ω when we refer to Section 3.4.

The space of holomorphic sections of $L^p\otimes E$ which are L^2 with respect to the norm given by (1.1) is denoted by $H^0_{(2)}(X, L^p\otimes E)$. Let $P_p(x,x')$ $(x,x'\in X)$ be the Schwartz kernel of the orthogonal projection P_p from the L^2 section of $L^p\otimes E$ onto $H^0_{(2)}(X, L^p\otimes E)$ with respect to the Riemannian volume form $dv_X(x')$ associated to (X,Θ). Then by the ellipticity of the Kodaira-Laplacian and Schwartz kernel theorem, we know $P_p(x,x')$ is $\mathscr{C}^\infty$. We set $B_p(x) := P_p(x,x)\in\mathscr{C}^\infty(X, \mathrm{End}(E))$.

For a $(1,1)$-form Ω, we write $\Omega > 0$ (resp. $\geqslant 0$) if $\Omega(\cdot, J\cdot) > 0$ (resp. $\geqslant 0$). For two $(1,1)$-forms Ω and Ω' we write $\Omega > \Omega'$ (resp. $\Omega\geqslant\Omega'$) if $\Omega - \Omega' > 0$ (resp. $\Omega - \Omega'\geqslant 0$). We have the following generalization of Theorem 0.1.

Theorem 3.11. *Assume that (X,Θ) is a complete Hermitian manifold. Suppose that there exist $\varepsilon>0$, $C>0$ such that*:

$$\sqrt{-1}R^L \geqslant \varepsilon\Theta, \qquad \sqrt{-1}\big(R^{\det}+R^E\big) \geqslant -C\Theta\,\mathrm{Id}_E, \qquad |\partial\Theta|_{g^{TX}} < C. \tag{3.39}$$

Then for every compact $K\subset X$, the kernel $P_p(x,x')$ has a full off-diagonal asymptotic expansion analogous to (3.13) *with $\boldsymbol{j}_0=\mathrm{Id}_E$ as $p\to\infty$, uniformly for every $x,x'\in K$. Especially there exist coefficients $b_r\in\mathscr{C}^\infty(X,\mathrm{End}(E))$, $r\in\mathbb{N}$, such that for every compact set $K\subset X$ and every $k,l\in\mathbb{N}$, there exists $C_{k,l,K}>0$ with*

$$\left|\frac{1}{p^n}B_p(x)-\sum_{r=0}^{k} b_r(x)p^{-r}\right|_{\mathscr{C}^l(K)} \leqslant C_{k,l,K}\,p^{-k-1}, \quad \textit{for all } p\in\mathbb{N}^*. \tag{3.40}$$

Moreover, $b_0=\det_{\mathbb{C}}|J^L|$ and b_1 equals $b_{0,1}$ given in (3.30).

Let us remark that if $L=K_X:=\det(T^{*(1,0)}X)$ is the canonical line bundle on X, the first two conditions in (3.39) are to be replaced by

$$h^L \text{ is induced by } \Theta \text{ and } \sqrt{-1}R^{\det} < -\varepsilon\Theta, \qquad \sqrt{-1}R^E > -C\Theta\,\mathrm{Id}_E \tag{3.41}$$

and the conclusions are still valid. If (X,Θ) is Kähler then $\partial\Theta=0$, so the third condition in (3.39) is trivially satisfied.

Proof. By the argument in Section 1.1, if the Kodaira-Laplacian $\square^{L^p\otimes E}=\frac{1}{2}\Delta_p:=\frac{1}{2}\Delta_{p,0}$ acting on sections of $L^p\otimes E$ has a spectral gap as in (0.5), then we can localize the problem, and we get directly (3.40) from Section 1.3. Observe that $D_p^2|_{\Omega^{0,0}}=\Delta_p$. In general, on a non-compact manifold, we define a self-adjoint extension of D_p^2 by

$$\begin{gathered}\operatorname{Dom} D_p^2=\big\{u\in\operatorname{Dom}\bar\partial_p^E\cap\operatorname{Dom}\bar\partial_p^{E,*}\colon\ \bar\partial_p^E u\in\operatorname{Dom}\bar\partial_p^{E,*},\ \bar\partial_p^{E,*}u\in\operatorname{Dom}\bar\partial_p^E\big\},\\ D_p^2u=2\big(\bar\partial_p^E\bar\partial_p^{E,*}+\bar\partial_p^{E,*}\bar\partial_p^E\big)u, \quad \text{for } u\in\operatorname{Dom}D_p^2,\end{gathered}$$

where we set $\bar\partial_p^E:=\bar\partial^{L^p\otimes E}$. The quadratic form associated to D_p^2 is the form pQ_p given by

$$\begin{gathered}\operatorname{Dom}Q_p:=\operatorname{Dom}\bar\partial_p^E\cap\operatorname{Dom}\bar\partial^{E_p,*},\\ pQ_p(u,v)=2\big\langle\bar\partial_p^E u,\bar\partial_p^E v\big\rangle+2\big\langle\bar\partial_p^{E,*}u,\bar\partial_p^{E,*}v\big\rangle, \quad u,v\in\operatorname{Dom}Q_p.\end{gathered} \tag{3.42}$$

In the previous formulas $\bar\partial_p^E$ is the maximal extension of $\bar\partial_p^E$ to L^2 forms and $\bar\partial_p^{E,*}$ is its Hilbert space adjoint. We denote by $\Omega_0^{0,\bullet}(X,L^p\otimes E)$ the space of smooth compactly supported forms and by $L^2_{0,\bullet}(X,L^p\otimes E)$ the corresponding L^2-completion.

Under hypothesis (3.39) there exists $\mu>0$ such that for p large enough

$$pQ_p(u)\geqslant\mu p\|u\|^2, \quad u\in\operatorname{Dom}Q_p\cap L^2_{0,q}\big(X,L^p\otimes E\big) \text{ for } q>0. \tag{3.43}$$

Indeed, the estimate holds for $u \in \Omega_0^{0,q}(X, L^p \otimes E)$ since the Bochner–Kodaira–Nakano formula with torsion term of Demailly ([23, Th. 0.3], [41, Th. 1.5], [5, (8)], or [36, Cor. 1.4.17]) delivers

$$pQ_p(u) \geqslant \frac{4}{3}\big\langle \big(pR^L + R^E + R^{\det}\big)(w_i, \overline{w}_j)\overline{w}^j \wedge i_{\overline{w}_i} u, u\big\rangle \\ - \frac{2}{3}\big(\|\overline{\mathcal{T}}\tilde{u}\|^2 + \|\mathcal{T}^*\tilde{u}\|^2 + \|\overline{\mathcal{T}}^*\tilde{u}\|^2\big) \tag{3.44}$$

for $u \in \Omega_0^{0,q}(X, L^p \otimes E)$, where $\{w_i\}$ is an orthonormal frame of $T^{(1,0)}X$ and $\tilde{u} \in \Omega_0^{n,q}(X, L^p \otimes E \otimes K_X^*)$ is induced by u and the canonical identification $K_X \otimes K_X^* \simeq \mathbb{C}$. Relations (3.44) and (3.39) imply (3.43) for $u \in \Omega_0^{0,q}(X, L^p \otimes E)$. Since $\Omega_0^{0,\bullet}(X, L^p \otimes E)$ is dense in $\operatorname{Dom} Q_p$ with respect to the graph norm (due to the completeness of the metric g^{TX}), (3.43) holds in general.

Next, consider $f \in \operatorname{Dom}\Delta_p \cap L^2_{0,0}(X, L^p \otimes E)$ and set $u = \overline{\partial}_p^E f$. It follows from the definition of the Laplacian and (3.43) that

$$\|\Delta_p f\|^2 = 4\big\langle \overline{\partial}_p^{E,*}u, \overline{\partial}_p^{E,*}u\big\rangle = 2pQ_p(u) \geqslant 2\mu p\|u\|^2 = \mu p\langle \Delta_p f, f\rangle. \tag{3.45}$$

This clearly implies

$$\operatorname{Spec}(\Delta_p) \subset \{0\} \cup [p\mu, \infty[\quad \text{for large } p.$$

What concerns b_1, the argument leading to (3.35)–(3.38) still holds locally, thus we get b_1 from (3.30). $\square$

Theorem 3.11 permits an immediate generalization of Tian's convergence theorem. Tian [48, Theorem 4.1] already generalized the convergence in the $\mathscr{C}^2$ topology and convergence rate $1/\sqrt{p}$ to complete Kähler manifolds X with some conditions on their Ricci curvature. When X is a quasi-projective manifold the generalization is used to prove estimates involving the Ricci form and results about its extension to a smooth projective compactification of X.

Another easy consequence of Theorem 3.11 are holomorphic Morse inequalities for the space $H^0_{(2)}(X, L^p)$.

For simplicity we consider now $\operatorname{rk}(E) = 1$, with the important case $E = K_X = \det(T^{*(1,0)}X)$ in mind. Choose an orthonormal basis $(S_i^p)_{i\geqslant 1}$ of $H^0_{(2)}(X, L^p \otimes E)$. For each local holomorphic frames e_L and e_E of L and E we have

$$S_i^p = f_i^p e_L^{\otimes p} \otimes e_E \tag{3.46}$$

for some local holomorphic functions f_i^p. Then $B_p(x) = P_p(x,x) = \sum_{i\geqslant 1}|S_i^p(x)|^2 = \sum_{i\geqslant 1}|f_i^p(x)|^2 |e_L^{\otimes p}|^2_{h^{L^p}} |e_E|^2_{h^E}$ is a smooth function. Observe that the quantity $\sum_{i\geqslant 1}|f_i^p(x)|^2$ is not globally defined, but the current

$$\omega_p = \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log\Big(\sum_{i\geqslant 1}\big|f_i^p(x)\big|^2\Big) \tag{3.47}$$

is well defined globally on X. Indeed, since $R^L = -\partial\bar\partial \log |e_L|^2_{h^L}$ and $R^E = -\partial\bar\partial \log |e_E|^2_{h^E}$ we have

$$\frac{1}{p}\omega_p - \frac{\sqrt{-1}}{2\pi} R^L = \frac{\sqrt{-1}}{2\pi p}\partial\bar\partial \log B_p + \frac{\sqrt{-1}}{2\pi p} R^E. \tag{3.48}$$

If E is trivial of rank one and $\dim H^0_{(2)}(X, L^p) < \infty$, we have by (3.14) that $\omega_p = \Phi_p^*(\omega_{FS})$ where Φ_p is defined as in (3.14) with $\mathscr{H}_p$ replaced by $H^0_{(2)}(X, L^p)$.

We will call a connected complex manifold X *Andreotti pseudoconcave* if there exists a non-empty relatively compact open set $M \Subset X$ with smooth boundary ∂M such that the Levi form of M restricted to the analytic tangent space $T^{(1,0)}\partial M$ has at least one negative eigenvalue at each point of ∂M.

Corollary 3.12. *Assume that* $\mathrm{rk}(E) = 1$ *and* (3.39) *holds true. Then*:

(a) *for each compact set* $K \subset X$ *the restriction* $\omega_p|_K$ *is a smooth* $(1,1)$*-form for sufficiently large* p; *moreover, for every* $l \in \mathbb{N}$ *there exists a constant* $C_{l,K}$ *such that*

$$\left|\frac{1}{p}\omega_p - \frac{\sqrt{-1}}{2\pi} R^L\right|_{\mathscr{C}^l(K)} \leqslant \frac{C_{l,K}}{p};$$

(b) *the Morse inequalities hold in bidegree* $(0,0)$:

$$\liminf_{p\to\infty} p^{-n} \dim H^0_{(2)}\big(X, L^p \otimes E\big) \geqslant \frac{1}{n!}\int_X \left(\frac{\sqrt{-1}}{2\pi} R^L\right)^n; \tag{3.49}$$

(c) *if* X *is Andreotti pseudoconcave, then the manifold* (X, Θ) *has finite volume.*

Proof. Due to (3.40), B_p does not vanish on any given compact set K for p sufficiently large. Thus, (a) is a consequence of (3.40) and (3.48).

Part (b) follows from Fatou's lemma, applied on X with the measure $\Theta^n/n!$ to the sequence $p^{-n}B_p$ which converges pointwise to $(\det J^L)^{1/2} = \big(\frac{\sqrt{-1}}{2\pi} R^L\big)^n/\Theta^n$ on X.

If X is Andreotti pseudoconcave, then $\dim H^0(X, F) < \infty$ for every holomorphic line bundle F on X. Moreover, it is shown in [37] and [36, Theorem 3.4.5] that there exists a constant $C > 0$ such that for all $p \geqslant 1$ we have $\dim H^0(X, L^p) \leqslant Cp^n$. Assertion (c) follows immediately from the latter estimate and (3.49). □

Remark 3.13. Under the hypothesis (3.41), the inequality (3.49) (with E trivial) is [39, Theorem 1.1] of Nadel and Tsuji, where Demailly's holomorphic inequalities [22] on compact sets $K \subset X$ were used. The volume estimate is essential in their compactification theorem of complete Kähler manifolds with negative Ricci curvature (a generalization of the fact that arithmetic varieties can be complex-analytically compactified). The Morse inequalities (3.49) were also used by Napier and Ramachandran [40] to show that some quotients of the unit ball in $\mathbb{C}^n$ $(n > 2)$ having a strongly pseudoconvex end have finite topological type (for the compactification of such quotients see also [38]).

Another generalization is a version of Theorem 0.1 for covering manifolds. Let $\widetilde{X}$ be a paracompact smooth manifold, such that there is a discrete group Γ acting freely on $\widetilde{X}$ with a compact quotient $X = \widetilde{X}/\Gamma$. Let $\pi_\Gamma : \widetilde{X} \to X$ be the projection. Assume that there exists a Γ-invariant pre-quantum line bundle $\widetilde{L}$ on $\widetilde{X}$ and a Γ-invariant connection $\nabla^{\widetilde{L}}$ such that $\widetilde{\omega} = \frac{\sqrt{-1}}{2\pi}(\nabla^{\widetilde{L}})^2$ is non-degenerate. We endow $\widetilde{X}$ with a Γ-invariant Riemannian metric $g^{T\widetilde{X}}$. Let $\widetilde{J}$ be an Γ-invariant almost complex structure on $T\widetilde{X}$ which is separately compatible with $\widetilde{\omega}$ and $g^{T\widetilde{X}}$. Then $\widetilde{\mathbf{J}}$, $g^{T\widetilde{X}}$, $\widetilde{\omega}$, $\widetilde{J}$, $\widetilde{L}$, $\widetilde{E}$ are the pull-back of the corresponding objects in Introduction by the projection $\pi_\Gamma : \widetilde{X} \to X$. Let Φ be a smooth Hermitian section of $\mathrm{End}(E)$, and $\widetilde{\Phi} = \Phi \circ \pi_\Gamma$. Then the renormalized Bochner-Laplacian $\widetilde{\Delta}_{p,\widetilde{\Phi}}$ is

$$\widetilde{\Delta}_{p,\widetilde{\Phi}} = \Delta^{\widetilde{L}^p \otimes \widetilde{E}} - p(\tau \circ \pi_\Gamma) + \widetilde{\Phi}$$

which is an essentially self-adjoint operator. It is shown in [33, Corollary 4.7] that

$$\mathrm{Spec}(\widetilde{\Delta}_{p,\widetilde{\Phi}}) \subset [-C_L, C_L] \cup [2p\mu_0 - C_L, +\infty[, \tag{3.50}$$

where C_L is the same constant as in Introduction and μ_0 is introduced in (0.3). Let $\widetilde{\mathcal{H}}_p$ be the eigenspace of $\widetilde{\Delta}_{p,\widetilde{\Phi}}$ with the eigenvalues in $[-C_L, C_L]$:

$$\widetilde{\mathcal{H}}_p = \mathrm{Range}\, E\big([-C_L, C_L], \widetilde{\Delta}_{p,\widetilde{\Phi}}\big), \tag{3.51}$$

where $E(\cdot, \widetilde{\Delta}_{p,\widetilde{\Phi}})$ is the spectral measure of $\widetilde{\Delta}_{p,\widetilde{\Phi}}$. From [33, Corollary 4.7], the von Neumann dimension of $\widetilde{\mathcal{H}}_p$ equals $d_p = \dim \mathcal{H}_p$ for p large enough. Finally, we define the generalized Bergman kernel $\widetilde{P}_{q,p}$ of $\widetilde{\Delta}_{p,\widetilde{\Phi}}$ as in Definition 1.1. Unlike most of the objects on $\widetilde{X}$, $\widetilde{P}_{q,p}$ is not Γ-invariant.

Theorem 3.14. *We fix $0 < \varepsilon_0 < \inf_{x \in X}\{$injectivity radius of $x\}$. Then for every $k, l \in \mathbb{N}$, there exists $C_{k,l} > 0$ such that for all $x, x' \in \widetilde{X}$, $p \in \mathbb{N}^*$, the following estimates hold*:

$$\begin{aligned} \big|\widetilde{P}_{q,p}(x,x') - P_{q,p}\big(\pi_\Gamma(x), \pi_\Gamma(x')\big)\big|_{\mathscr{C}^l} &\leqslant C_{k,l}\, p^{-k-1}, \quad \text{if } d(x,x') < \varepsilon_0, \\ \big|\widetilde{P}_{q,p}(x,x')\big|_{\mathscr{C}^l} &\leqslant C_{k,l,}\, p^{-k-1}, \quad \text{if } d(x,x') \geqslant \varepsilon_0. \end{aligned} \tag{3.52}$$

Especially, $\widetilde{P}_{q,p}(x,x)$ has uniformly on $\widetilde{X}$ the same asymptotic expansion as that of $P_{q,p}(\pi_\Gamma(x), \pi_\Gamma(x))$ given in Theorem 0.1.

Proof. Let $\{\varphi_i\}$ be a partition of unity subordinate to $\{U_i = B^X(x_i, \varepsilon)\}$ as in Section 1.1. Then $\{\widetilde{\varphi}_{\gamma,i} = \varphi_i \circ \pi_\Gamma\}$ is a partition of unity subordinate to $\{\widetilde{U}_{\gamma,i}\}$ where $\pi_\Gamma^{-1}(U_i) = \bigcup_{\gamma \in \Gamma} \widetilde{U}_{\gamma,i}$ and $\widetilde{U}_{\gamma_1,i}$ and $\widetilde{U}_{\gamma_2,i}$ are disjoint for $\gamma_1 \neq \gamma_2$. The proof of Proposition 1.2 still holds for the pair $\{\widetilde{\varphi}_{\gamma,i}\}$, $\{\widetilde{U}_{\gamma,i}\}$, since we can apply the Sobolev embedding theorems with uniform constant on $\widetilde{U}_{\gamma,i}$. Thus, the analogue of (1.7) holds uniformly on $\widetilde{X}$. Using the finite propagation speed as at the end of Section 1.1, we conclude. □

Remark 3.15. Theorem 3.14 can be generalized for coverings of non-compact manifolds in the spirit of Theorem 3.11. Let (X, Θ) be a complete Kähler manifold, (L, h^L) be a holomorphic line bundle on X and let $\pi_\Gamma : \widetilde{X} \to X$ be a Galois covering of $X = \widetilde{X}/\Gamma$. Let $\widetilde{\Theta}$ and $(\widetilde{L}, h^{\widetilde{L}})$

be the inverse images of Θ and (L, h^L) through π_Γ. If (X, Θ) and (L, h^L) satisfy one of the conditions (3.39) or (3.41), $(\widetilde{X}, \widetilde{\Theta})$ and $(\widetilde{L}, h^{\widetilde{L}})$ have the same properties. We obtain therefore as in (3.49) (by integrating over a fundamental domain):

$$\liminf_{p\to\infty} p^{-n} \dim_\Gamma H^0_{(2)}\big(\widetilde{X}, \widetilde{L}^p\big) \geqslant \frac{1}{n!} \int_X \left(\frac{\sqrt{-1}}{2\pi} R^L\right)^n, \tag{3.53}$$

where $\dim_\Gamma$ is the von Neumann dimension of the Γ-module $H^0_{(2)}(X, L^p)$. Such type of inequalities was proved in [49] and they imply weak Lefschetz theorems à la Nori.

The example of non-compact manifolds emphasizes very well our approach to the existence of the asymptotic expansion of the Bergman kernel of Laplacian type operators when the power of the line bundle tends to infinity. In fact, the argument in Section 1.1 shows that the spectral gap property allows to localize our problem whether the manifold X is compact or not. Thus from the argument in Section 1.3 or [20, §4.4], it implies the existence of the asymptotic expansion. Moreover, the formal power series artifice in Section 1.5 gives a general way to compute the coefficients. As an example, we state the following result which is an extension of [20, Theorem 4.18] to non-compact case and we use the notation therein. Let (X, g^{TX}) be a Riemannian manifold with almost complex structure J which is compatible with g^{TX}, and let (L, h^L, ∇^L) and (E, h^E, ∇^E) be Hermitian bundles as in Introduction. We consider the associated spin^c Dirac operator D_p. Let $R^{T^{(1,0)}X}$ be the curvature of the connection on $T^{(1,0)}X$ induced by ∇^{TX} by projection. We denote by $I_{\mathbb{C}\otimes E}$ the projection from $\Lambda(T^{*(0,1)}X) \otimes E$ onto $\mathbb{C} \otimes E$ under the decomposition $\Lambda(T^{*(0,1)}X) = \mathbb{C} \oplus \Lambda^{>0}(T^{*(0,1)}X)$.

Theorem 3.16. *Suppose that (X, g^{TX}) is complete and the scalar curvature r^X of (X, g^{TX}), R^E and* $\mathrm{Tr}[R^{T^{(1,0)}X}]$ *are uniformly bounded on (X, g^{TX}). Assume also that there exists $\varepsilon > 0$ such that on X,*

$$\sqrt{-1} R^L(\cdot, J\cdot) > \varepsilon g^{TX}(\cdot, \cdot). \tag{3.54}$$

Then the smooth kernel $P_p(x, x')$ with respect to $dv_X(x')$ of the orthogonal projection P_p from $L^2_{0,\bullet}(X, L^p \otimes E)$ onto $\mathrm{Ker}(D_p)$ *has a full off-diagonal expansion as $p \to \infty$ uniformly on compact sets of X, analogous to Theorem* 3.11. *In the present case $\boldsymbol{j}_0 = I_{\mathbb{C}\otimes E}$.*

Proof. By the proof of [33, Theorem 2.5], we know that the spectral gap property $\mathrm{Spec}(D_p^2) \subset \{0\} \cup [2\mu_0 p - C_L, \infty[$ still holds under our condition. Then the arguments outlined above allow to conclude. □

3.6. Singular polarizations

Let (X, J) be a compact complex manifold. A *singular Kähler metric* on X is a closed, strictly positive $(1,1)$-current ω. This means there exist locally strictly plurisubharmonic functions $\varphi \in L^1_{\mathrm{loc}}$ such that $\sqrt{-1}\partial\overline{\partial}\varphi = \omega$.

If the cohomology class of ω in $H^2(X, \mathbb{R})$ is integral, there exists a holomorphic line bundle (L, h^L), endowed with a singular Hermitian metric, such that $\frac{\sqrt{-1}}{2\pi} R^L = \omega$ in the sense of currents. We call (L, h^L) a *singular polarization* of ω. If we change the metric h^L, the curvature of the new metric will be in the same cohomology class as ω. In this case we speak of a polarization of $[\omega] \in H^2(X, \mathbb{R})$. Our purpose is to define an appropriate notion of polarized section of L^p, possibly by changing the metric of L, and study the associated Bergman kernel.

First recall that a Hermitian metric h^L is called *singular* if it is given in local trivialization by functions $e^{-\varphi}$ with $\varphi \in L^1_{\mathrm{loc}}$. The curvature current R^L of h^L is well defined and given locally by the currents $\partial\bar{\partial}\varphi$.

By the approximation theorem of Demailly [24, Theorem 1.1], we can assume that h^L is smooth outside a proper analytic set $\Sigma \subset X$. Using this fundamental fact, we will introduce in the sequel the *generalized Poincaré metric* on $X \setminus \Sigma$. Let $\pi : \widetilde{X} \to X$ be a resolution of singularities such that $\pi : \widetilde{X} \setminus \pi^{-1}(\Sigma) \to X \setminus \Sigma$ is biholomorphic and $\pi^{-1}(\Sigma)$ is a divisor with only simple normal crossings. Let $g_0^{T\widetilde{X}}$ be an arbitrary smooth J-invariant metric on $\widetilde{X}$ and $\Theta'(\cdot,\cdot) = g_0^{T\widetilde{X}}(J\,\cdot,\cdot)$ the corresponding $(1,1)$-form. The generalized Poincaré metric on $X \setminus \Sigma = \widetilde{X} \setminus \pi^{-1}(\Sigma)$ is defined by (cf. [16, §2], [18, §6])

$$\Theta_{\varepsilon_0} = \Theta' - \varepsilon_0\sqrt{-1}\sum_i \partial\bar{\partial}\log\bigl(\bigl(-\log\|\sigma_i\|_i^2\bigr)^2\bigr), \quad 0 < \varepsilon_0 \ll 1 \text{ fixed}, \tag{3.55}$$

where $\pi^{-1}(\Sigma) = \bigcup_i \Sigma_i$ is the decomposition into irreducible components Σ_i of $\pi^{-1}(\Sigma)$ and each Σ_i is non-singular; σ_i are sections of the associated holomorphic line bundle $[\Sigma_i]$ which vanish to first order on Σ_i, and $\|\sigma_i\|_i$ is the norm for a smooth Hermitian metric on $[\Sigma_i]$ such that $\|\sigma_i\|_i < 1$. The first part of the following lemma generalizes previous work on the generalized Poincaré metric [16,18,52].

Lemma 3.17.

(i) *The generalized Poincaré metric* (3.55) *is a complete Hermitian metric of finite volume. Its Hermitian torsion $\mathcal{T}_{\varepsilon_0} = [i(\Theta_{\varepsilon_0}), \partial\Theta_{\varepsilon_0}]$ and the curvature $R^{\det}$ are bounded.*

(ii) *If (E, h^E) is a holomorphic vector bundle over X with smooth Hermitian metric h^E and*

$$H^0_{(2)}(X \setminus \Sigma, E) = \bigl\{u \in L^2_{0,0}\bigl(X \setminus \Sigma, E, \Theta_{\varepsilon_0}, h^E\bigr) \colon \bar{\partial}^E u = 0\bigr\}$$

then

$$H^0_{(2)}(X \setminus \Sigma, E) = H^0(X, E).$$

Proof. To describe the metric more precisely we denote by $\mathbb{D}$ the unit disc in $\mathbb{C}$ and $\mathbb{D}^* = \mathbb{D} \setminus \{0\}$. On the product $(\mathbb{D}^*)^l \times \mathbb{D}^{n-l}$ we introduce the metric

$$\omega_P = \frac{\sqrt{-1}}{2}\sum_{k=1}^{l} \frac{dz_k \wedge d\bar{z}_k}{|z_k|^2(\log|z_k|^2)^2} + \frac{\sqrt{-1}}{2}\sum_{k=l+1}^{n} dz_k \wedge d\bar{z}_k. \tag{3.56}$$

For each point $x \in \pi^{-1}(\Sigma)$ there exists a coordinate neighbourhood U of x isomorphic to $\mathbb{D}^n$ in which $(\widetilde{X} \setminus \pi^{-1}(\Sigma)) \cap U = \{z = (z_1, \ldots, z_n)\colon z_1 \neq 0, \ldots, z_l \neq 0\}$. Such coordinates are called special. We endow $(\widetilde{X} \setminus \pi^{-1}(\Sigma)) \cap U \cong (\mathbb{D}^*)^l \times \mathbb{D}^{n-l}$ with the metric (3.56). We have

$$-\sqrt{-1}\partial\bar{\partial}\log\bigl(\bigl(-\log\|\sigma_i\|_i^2\bigr)^2\bigr) = 2\sqrt{-1}\left(\frac{R^{[\Sigma_i]}}{\log\|\sigma_i\|_i^2} + \frac{\partial\log\|\sigma_i\|_i^2 \wedge \bar{\partial}\log\|\sigma_i\|_i^2}{(\log\|\sigma_i\|_i^2)^2}\right). \tag{3.57}$$

Since the terms $R^{[\Sigma_i]}/\log\|\sigma_i\|_i^2$ tend to zero as we approach Σ,

$$\Theta' + 2\sqrt{-1}\varepsilon_0 \sum_i \frac{R^{[\Sigma_i]}}{\log\|\sigma_i\|_i^2} > 0, \tag{3.58}$$

for ε_0 small enough. The last term in (3.57) is $\geqslant 0$, since $\sqrt{-1}\partial g \wedge \bar{\partial} g \geqslant 0$ for every real function g on $\widetilde{X}$. Thus Θ_{ε_0} is positive for ε_0 small enough.

We choose special coordinates in a neighborhood U of x_0 in which Σ_j has the equation $z_j = 0$ for $j = 1, \dots, k$ and Σ_j, $j > k$, do not meet U. Then for $1 \leqslant i \leqslant k$, $\|\sigma_i\|_i^2 = \varphi_i |z_i|^2$ for some positive smooth function φ_i on U and

$$\frac{\partial \log\|\sigma_i\|_i^2 \wedge \bar{\partial}\log\|\sigma_i\|_i^2}{(\log\|\sigma_i\|_i^2)^2} = \frac{dz_i \wedge d\bar{z}_i + \psi_i}{|z_i|^2(\log\|\sigma_i\|_i^2)^2} \tag{3.59}$$

where ψ_i is a smooth $(1,1)$-form on U such that $\psi_i|_{z_i=0} = 0$.

As in [52, Prop. 3.4], we show using (3.57) and (3.59) that the metrics (3.55) and (3.56) are equivalent for $|z_i|$ small. From this the first assertion of (i) follows.

Recall that $R^{\det}$ is the curvature of the holomorphic Hermitian connection on $\det(T^{(1,0)}X)$ with respect to the Hermitian metric induced by Θ_{ε_0}. We wish to show that there exist a constant $C > 0$ such that

$$-C\Theta_{\varepsilon_0} < \sqrt{-1}R^{\det} < C\Theta_{\varepsilon_0}, \qquad |\mathcal{T}_{\varepsilon_0}|_{\Theta_{\varepsilon_0}} < C, \tag{3.60}$$

where $\mathcal{T}_{\varepsilon_0} = [i(\Theta_{\varepsilon_0}), \partial\Theta_{\varepsilon_0}]$ is the Hermitian torsion operator of Θ_{ε_0} and $|\mathcal{T}_{\varepsilon_0}|_{\Theta_{\varepsilon_0}}$ is its norm with respect to Θ_{ε_0}. Since $\partial\Theta_{\varepsilon_0} = \partial\Theta'$ by (3.55), $\partial\Theta_{\varepsilon_0}$ extends smoothly over $\widetilde{X}$, and thus we get the second relation of (3.60).

We turn now to the first condition of (3.60). By (3.55), (3.57) and (3.59), we know that

$$\Theta_{\varepsilon_0}^n = \frac{2^k\varepsilon_0^k + \beta(z)}{\prod_{i=1}^k |z_i|^2(\log\|\sigma_i\|_i^2)^2} \prod_{j=1}^n (\sqrt{-1}dz_j \wedge d\bar{z}_j) =: \gamma(z) \prod_{j=1}^n (\sqrt{-1}dz_j \wedge d\bar{z}_j). \tag{3.61}$$

Here $\beta(z)$ is a polynomial in the functions $a_{i\alpha}(z)|z_i|^2(\log\|\sigma_i\|_i^2)^2$, $b_{i\alpha}(z)|z_i|^2\log\|\sigma_i\|_i^2$ and $c_{i\alpha}(z)$ $(1 \leqslant i \leqslant k)$, with $a_{i\alpha}, b_{i\alpha}$ smooth functions on U and $c_{i\alpha}$ smooth functions on U such that $c_{i\alpha}(z)|_{z_i=0} = 0$. Moreover, $2^k\varepsilon_0^k + \beta(z)$ is positive on U as Θ_{ε_0} is positive. Since

$$\left|\frac{\partial}{\partial z_1} \wedge \cdots \wedge \frac{\partial}{\partial z_n}\right|^2_{\Theta_{\varepsilon_0}} \prod_{j=1}^n (\sqrt{-1}dz_j \wedge d\bar{z}_j) = \Theta_{\varepsilon_0}^n, \tag{3.62}$$

we get from (3.61) and (3.62),

$$R^{\det} = -\partial\bar{\partial}\log\gamma(z) = -\partial\bar{\partial}\log\big(2^k\varepsilon_0^k + \beta(z)\big) + \sum_{i=1}^k \partial\bar{\partial}\log\big((\log\|\sigma_i\|_i^2)^2\big). \tag{3.63}$$

By (3.57), the last term of (3.63) is bounded with respect to Θ_{ε_0}. To examine the first term of the sum, we write

$$\partial\bar{\partial}\log\bigl(2^k\varepsilon_0^k+\beta(z)\bigr)=\frac{\partial\bar{\partial}\beta(z)}{2^k\varepsilon_0^k+\beta(z)}-\frac{\partial\beta(z)\wedge\bar{\partial}\beta(z)}{(2^k\varepsilon_0^k+\beta(z))^2}. \tag{3.64}$$

Now we observe that for $W_i(z)=|z_i|^2(\log\|\sigma_i\|_i^2)^2$ or $|z_i|^2\log\|\sigma_i\|_i^2$, the terms $\partial\bar{\partial}W_i(z)$, $\partial W_i(z)$, $\bar{\partial}W_i(z)$ are bounded with respect to the Poincaré metric (3.56), thus with respect to Θ_{ε_0}. Combining with the form of β given after (3.61), this completes the proof of (3.60).

Let us prove (ii). First observe that Θ_{ε_0} dominates the Euclidean metric in special coordinates near $\pi^{-1}(\Sigma)$, being equivalent with (3.56). Therefore it dominates some positive multiple of each smooth Hermitian metric on $\widetilde{X}$. We deduce that, given a smooth Hermitian metric Θ'' on X, there exists a constant $c>0$ such that $\Theta_{\varepsilon_0}\geqslant c\Theta''$ on $X\setminus\Sigma$. It follows that elements of $H^0_{(2)}(X\setminus\Sigma,E)$ are L^2 integrable with respect to the smooth metrics Θ'' and h^E over X, which entails they extend holomorphically to sections of $H^0(X,E)$ by [21, Lemme 6.9]. We have therefore $H^0_{(2)}(X\setminus\Sigma,E)\subset H^0(X,E)$. The reverse inclusion follows from the finiteness of the volume of $X\setminus\Sigma$ in the Poincaré metric. □

We can construct as in [45, §4], [36, Lemma 6.2.2] a singular Hermitian line bundle $(\widetilde{L},h^{\widetilde{L}})$ on $\widetilde{X}$ which is strictly positive and $\widetilde{L}|_{\widetilde{X}\setminus\pi^{-1}(\Sigma)}\cong\pi^*(L^{p_0})$, for some $p_0\in\mathbb{N}$. We introduce on $L|_{X\setminus\Sigma}$ the metric $(h^{\widetilde{L}})^{1/p_0}$ whose curvature extends to a strictly positive $(1,1)$-current on $\widetilde{X}$. Set

$$h^L_\varepsilon=\bigl(h^{\widetilde{L}}\bigr)^{1/p_0}\prod_i\bigl(-\log\|\sigma_i\|_i^2\bigr)^\varepsilon,\quad 0<\varepsilon\ll 1, \tag{3.65a}$$

$$H^0_{(2)}\bigl(X\setminus\Sigma,L^p\bigr)=\bigl\{u\in L^2_{0,0}\bigl(X\setminus\Sigma,L^p,\Theta_{\varepsilon_0},h^{L^p}_\varepsilon\bigr)\colon\ \bar{\partial}^{L^p}u=0\bigr\}. \tag{3.65b}$$

The space $H^0_{(2)}(X\setminus\Sigma,L^p)$ is the space of L^2-holomorphic sections relative to the metrics Θ_{ε_0} on $X\setminus\Sigma$ and h^L_ε on $L|_{X\setminus\Sigma}$. Since $(h^{\widetilde{L}})^{1/p_0}$ is bounded away from zero (having plurisubharmonic weights), the elements of this space are L^2 integrable with respect to the Poincaré metric and a smooth metric h^L_* of L over whole X. By Lemma 3.17(ii) we have $H^0_{(2)}(X\setminus\Sigma,L^p)\subset H^0(X,L^p)$. (Here we cannot infer the other inclusion since h^L_ε might blow up to infinity on Σ.) The space $H^0_{(2)}(X\setminus\Sigma,L^p)$ is our space of polarized sections of L^p.

Corollary 3.18. *Let (X,ω) be a compact complex manifold with a singular Kähler metric with integral cohomology class. Let (L,h^L) be a singular polarization of $[\omega]$ with strictly positive curvature current having singular support along a proper analytic set Σ. Let (E,h^E) be a holomorphic Hermitian vector bundle on X. Then the Bergman kernel associated to the orthogonal projection from the space of L^2-sections of $L^p\otimes E$ with respect to Θ_{ε_0}, $h^{L^p}_\varepsilon\otimes h^E$ on $X\setminus\Sigma$ onto the space of polarized sections*

$$H^0_{(2)}\bigl(X\setminus\Sigma,L^p\otimes E\bigr)=\bigl\{u\in L^2_{0,0}\bigl(X\setminus\Sigma,L^p\otimes E,\Theta_{\varepsilon_0},h^{L^p}_\varepsilon\otimes h^E\bigr)\colon\ \bar{\partial}^{L^p\otimes E}u=0\bigr\}$$

has the asymptotic expansion as in Theorem 3.11 *for $X\setminus\Sigma$.*

Proof. We will apply Theorem 3.11 to the non-Kähler Hermitian manifold $(X \setminus \Sigma, \Theta_{\varepsilon_0})$ equipped with the Hermitian bundle $(L|_{X\setminus\Sigma}, h^L_\varepsilon)$ and (E, h^E). Certainly, R^E is bounded. Thus we have to show that there exist constants $\eta > 0$, $C > 0$ such that

$$\sqrt{-1}R^{(L|_{X\setminus\Sigma},h^L_\varepsilon)} > \eta\Theta_{\varepsilon_0}, \quad \sqrt{-1}R^{\det} > -C\Theta_{\varepsilon_0}, \quad |\mathcal{T}_{\varepsilon_0}|_{\Theta_{\varepsilon_0}} < C. \tag{3.66}$$

The first relation results for all ε_0 small enough from (3.55), (3.65a) and the fact that the curvature of $(h^{\widetilde{L}})^{1/p_0}$ extends to a strictly positive (1, 1)-current on $\widetilde{X}$ (dominating a small positive multiple of Θ' on $\widetilde{X}$). The second and third relations were proved in (3.60). This completes the proof of Corollary 3.18. □

Remark 3.19. (a) Corollary 3.18 with $E = \mathbb{C}$ gives an alternative proof of the characterization of Moishezon manifolds given by Ji and Shiffman [29], Bonavero [8] and Takayama [45]. Indeed, each Moishezon manifold possesses a strictly positive singular polarization (L, h^L). Conversely, suppose X has such a polarization. Then as in (3.49), we have $\dim H^0_{(2)}(X \setminus \Sigma, L^p) \geqslant Cp^n$ for some $C > 0$ and p large enough. Since $H^0_{(2)}(X \setminus \Sigma, L^p) \subset H^0(X, L^p)$, it follows that L is big and X is Moishezon. A detailed account of the characterization of Moishezon manifolds, including the present method, can be found in [36, Chapter 2 and §6.2].

(b) Using Moishezon's fundamental result which states that a Moishezon manifold can be transformed into a projective manifold by a finite succession of blow-ups along smooth centers [36, Theorem 2.2.16], one can prove that every big line bundle L on a compact complex manifold carries a singular Hermitian metric having strictly positive curvature current with singularities along a proper analytic set (see e.g. [36, Lemma 2.3.6]).

(c) The results of this section hold also for reduced compact complex spaces X possessing a holomorphic line bundle L with singular Hermitian metric h^L having positive curvature current (see [45] for definitions). This is just a matter of desingularizing X. As space of polarized sections we obtain $H^0_{(2)}(X \setminus \Sigma, L^p)$ where Σ is an analytic set containing the singular set of X.

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