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VERY WELL COVERED GRAPHS

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A graph is well-covered if it has no isolated vertices and all the maximal stable (independent) sets have the same cardinality. If furthermore this cardinality is equal to $\frac{1}{2}n$, where n is the order of the graph, the graph is called 'very well covered'. The class of very well-covered graphs contains in particular the bipartite well-covered graphs studied by Ravindra. In this article, we characterize the very well covered-graphs and give some of their properties.

Un graphe est dit bien couvert s'il est sans sommets isolés et si tous ses ensembles stables (indépendants) maximaux ont la même cardinalité. Si, de plus, cette cardinalité est $\frac{1}{2}n$, où n désigne le nombre de sommets du graphe, le graphe est appelé 'très bien couvert'. La classe des graphes très bien couverts contient en particulier les graphes bipartis bien couverts étudiés par Ravindra. Dans cet article, nous caractérisons les graphes très bien couverts et établissons certaines de leurs propriétés.

0. Introduction

In what follows, G will denote a simple undirected graph $G(V, E)$ of order $n = |V|$.

A *stable* (or *independent*) set S is a set of nonadjacent vertices. α' (resp. α) will denote the minimum (resp. maximum) cardinality of a maximal stable set.

D is a *dominating* set if and only if every point of $V - D$ is adjacent to a point of D. We will denote by γ (resp. Γ) the minimum (resp. maximum) cardinality of a minimal dominating set.

Let us denote by $\Gamma(x)$ the set of vertices adjacent to x and, more generally, $\Gamma(A) = \bigcup_{x \in A} \Gamma(x)$ for $A \subset V$. A vertex x of A is said to be *redundant* in A if $x \cup \Gamma(x) \subset \Gamma(A - x) \cup \{A - x\}$. A set I of vertices containing no redundant vertex is called *irredundant*. Equivalently, I is irredundant if every point of I either is adjacent to no other point of I, or is adjacent to a point of $V - I$ itself adjacent to no other point of I. We will denote by ir (resp. IR) the minimum (resp. maximum) cardinality of a maximal irredundant set.

The different parameters $\alpha, \alpha', \gamma, \Gamma$, ir, IR have already been studied in many articles (see [2] for example). In particular we have the following relations:

maximal stable set		dominant & irredundant	$\Rightarrow$	irredundant maximal
$\Updownarrow$		$\Updownarrow$		
stable & dominant	$\Rightarrow$	dominant minimal		

Therefore

$$ir \leq \gamma \leq \alpha' \leq \alpha \leq \Gamma \leq IR.$$

A simple graph is said to be *well covered* if it has no isolated vertex and if $\alpha' = \alpha$, that is every maximal stable set is maximum. This notion has been introduced by Plummer in 1970 [5] and studied for bipartite graphs by Ravindra in 1977 [6] and also by Berge [1]. In particular it has been shown in [1] that well covered graphs are quasiregularizable. A graph is quasiregularizable if one can get a regular multigraph of non zero degree by eventually deleting some of its edges and replacing the other ones by several parallel edges. A graph is quasiregularizable if and only if $|\Gamma(S)| \geq |S|$ for every stable S of G; this is equivalent to say that for every stable set S of G there exists a matching of all of S into $V-S$. We will show that a similar characterization can be obtained by replacing 'stable set' by 'irredundant set'.

0.1. Proposition. *A simple graph $G(V, E)$ is quasiregularizable if and only if, for every irredundant set I of G, there exists a matching of all of I into $V-I$.*

Proof. Let G be a quasiregularizable graph, I an irredundant set of G and S the set of isolated points of I (possibly empty). S is a stable set and there exists a matching between S and $V-S$, and therefore between S and $V-I$. Furthermore every point x of $I-S$ is adjacent to a point x' of $V-I$ where $x' \notin \Gamma(I-x)$. In particular $x' \in V-I-\Gamma(S)$ and then, altogether, we get a matching of all of I into $V-I$. The converse is clear as every stable set is irredundant.

0.2. Corollary. *For every quasiregularizable graph $G(V, E)$ of order n, $IR \leq \frac{1}{2}n$. If $IR = \frac{1}{2}n$, then $\Gamma = IR$.*

Proof. The existence of a matching between a maximum irredundant set I and $V-I$ shows that $IR \leq \frac{1}{2}n$. If we have equality, I is irredundant and dominant and therefore minimal dominant and so $|I| \leq \Gamma$. But $\Gamma \leq IR$ for every graph and then $\Gamma = IR$.

We will consider here well covered graphs of even order n which moreover satisfy $\alpha' = \alpha = \frac{1}{2}n$, and which we will call '*very well covered*'. This class of graphs includes in particular bipartite well covered graphs [6], but also other ones (see example Fig. 1). In such graphs we have $\alpha' = \alpha = \Gamma = IR = \frac{1}{2}n$. The aim of this article is to characterize those graphs and give some of their properties. As every connected component G_i of a very well covered graph G is also very well covered (indeed the maximal stable sets of G are the unions of the maximal stable sets of the G_i), we will only deal with connected very well covered graphs; one can easily generalize this.

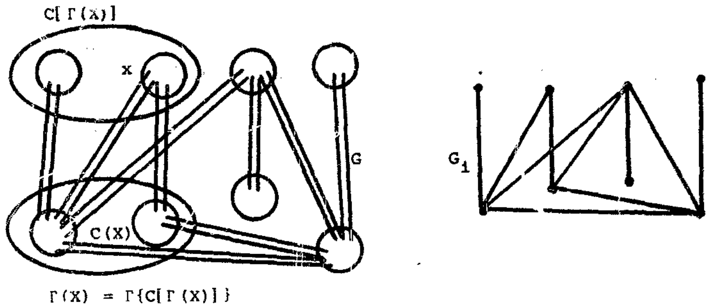

Fig. 1.

1. Characterization of very well covered graphs

1.1. Property (P). Let G be a graph with a perfect matching C. We will denote by $C(x)$ the point adjacent to x in C and $C(A)=\bigcup_{x\in A} C(x)$ for every $A\subset V$ (therefore we have $C[C(A)]=A$). We will say that C satisfies property (P) if, for every point x of V,

$$(y\in\Gamma(x),\, y\neq C(x)) \Rightarrow (y\notin\Gamma[C(x)] \quad\text{and}\quad y\in\Gamma(z)\,\forall z\in\Gamma[C(x)]).$$

(P) signifies that every neighbour of a point x, other than its matched point $C(x)$, is not adjacent to $C(x)$ but is adjacent to all neighbours of $C(x)$.

1.2. Theorem. *For a simple graph $G(V, E)$, the following properties are equivalent:*

(i) *G is very well covered.*

(ii) *There exists a perfect matching in G which satisfies the property* (P).

(iii) *There exists at least one perfect matching in G, and every perfect matching of G satisfies* (P).

Proof. (i)$\Rightarrow$(iii). Let $G(V, E)$ be a very well covered graph of order n and S be a maximal stable set of G. As $|S|=\frac{1}{2}n$, a matching between S and $V-S$ will be a perfect matching of G. Consider any perfect matching C of G. Every maximal stable set, of n elements, contains exactly one vertex of every pair $(x, C(x))$. Let $y\in\Gamma(x)$, $y\neq C(x)$ and $z\in\Gamma[C(x)]$, $z\neq x$. Every maximal set containing y also contains $C(x)$ since it doesn't contain x and then $y\notin\Gamma[C(x)]$ and $y\in\Gamma(z)$. Therefore C satisfies (P).

(iii)$\Rightarrow$(ii). Immediate.

(ii)$\Rightarrow$(i). A graph G containing a perfect matching C is of even order, has no isolated vertex, and every stable set has at most $\frac{1}{2}n$ elements. Let us suppose that C satisfies (P) and that there exists a maximal stable set S and a pair $(x, C(x))$ with no point in S. As S is maximal, $x\notin S\Rightarrow\exists y$, $y\in S\cap\Gamma(x)$ and $C(x)\notin S\Rightarrow\exists z$,

$z \in S \cap \Gamma[C(x)]$. From (P), $y \neq z$ since $y \notin \Gamma[C(x)]$, and $y \in \Gamma(z)$ which is impossible since S is a stable set. Therefore every maximal stable set contains a point of every pair $(x, C(x))$ and has $\frac{1}{2}n$ elements. So G is very well covered.

1.3. Corollary. *In every well covered graph $G(V, E)$, we have $\Gamma\{C[\Gamma(A)]\} = \Gamma(A)$ for every perfect matching C and every $A \subset V$.*

Proof. As $C(A) \subset \Gamma(A)$ then $A \subset C[\Gamma(A)]$ and

$$\Gamma(A) \subset \Gamma\{C[\Gamma(A)]\}.$$

Conversely, let $z \in \Gamma\{C[\Gamma(A)]\}$. z is adjacent to a point y of $C[\Gamma(A)]$ and $C(y) \in \Gamma(A)$. If $z = C(y)$, then $z \in \Gamma(A)$. If $z \neq C(y)$, then, by property (P), z is adjacent to all the neighbours of $C(y)$ and therefore to a point of A; we have again $z \in \Gamma(A)$.

1.4. Corollary. *In a very well covered graph, for every perfect matching and every point x, $\Gamma(x) \cap C[\Gamma(x)] = \emptyset$ and $C[\Gamma(x)]$ is a stable set.*

Proof. If $y \in \Gamma(x) \cap C[\Gamma(x)]$, then y and $C(y)$ are adjacent to x which is impossible. From Corollary 1.3, $\Gamma(x) \cap C[\Gamma(x)] = \emptyset$ can be written as $\Gamma\{C[\Gamma(x)]\} \cap C[\Gamma(x)] = \emptyset$ which shows that $C[\Gamma(x)]$ is stable.

1.5. Corollary (Ravindra [6]). *A bipartite graph G is well covered if and only if there exists in G a perfect matching C such that, for every pair $(x, C(x))$, the subgraph induced by $\Gamma(x) \cup \Gamma[C(x)]$ is complete bipartite.*

Proof. If G is bipartite well covered, it is very well covered and in this case the property (P) can be reduced to

$$\forall x \in V, \quad (y \in \Gamma(x), y \neq C(x)) \Rightarrow (y \in \Gamma(z), \forall z \in \Gamma[C(x)]).$$

2. Reduction of very well covered graphs

2.1. Equivalence relation. Let C be a perfect matching of a very well covered graph G. We will say that x and y are equivalent if either $x = y$ or if $x \in \Gamma[C(y)]$ and $y \in \Gamma[C(x)]$ (or, equivalently, $C(y) \in \Gamma(x)$ and $C(x) \in \Gamma(y)$).

This relation is an equivalence relation. Indeed if $y \neq x$ is equivalent to x and $x \neq z$ is equivalent to z, then $y \in \Gamma[C(x)]$ and $C(z) \in \Gamma(x)$; therefore, from (P), $y \in \Gamma[C(z)]$. Also, similarly, $z \in \Gamma[C(x)]$ and $C(y) \in \Gamma(x)$; therefore $z \in \Gamma[C(y)]$. Then y and z are equivalent. We will denote by X the class of equivalence of x. The equivalence classes satisfy the following properties (which are consequences of the definitions and of property (P)):

P_1: The equivalence classes form a partition of V into stable sets.

P_2: The class of $C(x)$ is $C(X)$; the subgraph induced by $X \cup C(X)$ is complete bipartite with $|X| = |C(X)|$.

Let us suppose now that there exists an edge between X and Y with $Y \neq C(X)$, then:

P_3: The subgraph induced by $X \cup Y$ is complete bipartite.

P_4: There is no edge between X and $C(Y)$.

P_5: There is no edge between $C(X)$ and $C(Y)$.

P_6: If furthermore there exists one edge between $C(X)$ and Z, then there is an edge between Y and Z.

From properties P_1, P_2, P_3, one can associate to the graph G the quotient graph G_i obtained by replacing each class X of G by one vertex called X; two vertices are joined in G_i if there exists an edge between the two classes X and Y in G, which is equivalent by P_3 to say that the subgraph of G induced by $X \cup Y$ is complete bipartite.

2.2. Proposition. *G_i is very well covered.*

That follows from P_4 and P_6.

2.3. Theorem. *G_i is the same for every choice of a perfect matching in G.*

Proof. If C is a perfect matching in G and X a class, it is sufficient to show that every other perfect matching in G matches X with $C(X)$ (not necessarily in the same way as C). Let x be a vertex of G, X the class of x for a perfect matching C. As $\Gamma\{C[\Gamma(X)]\} = \Gamma(X)$ (Corollary 1.3) and $|\Gamma(X)| = |C[\Gamma(X)]|$, every other perfect matching of G matches also the points of $\Gamma(X)$ with those of $C[\Gamma(X)]$, and in particular the points of $C(X)$, included in $\Gamma(X)$, with points of $C[\Gamma(X)]$ (Fig. 1). But, as no point of $C(X)$ is joined to $C[\Gamma(X)] - X$ (property P_5), every perfect matching of G matches X with $C(X)$.

Theorem 2.3 shows that to a very well covered graph one can associate, in a unique way, an irreducible very well covered graph. Conversely, let us consider the following operation: in a graph G, which has a perfect matching C, replace a vertex x by a stable set S and $C(x)$ by a stable set S' with $|S| = |S'|$. Then join every point of S to every point of S' and to every vertex y adjacent to x in G. Also join every point of S' to every point z adjacent to $C(x)$ in G. This operation is a particular case of the join of graphs and one can denote the graph obtained by $G^{S,S'}_{x,C(x)}$ (see [4] for example). Then we get immediately the following proposition:

2.4. Proposition. *If G is very well covered, $G^{S,S'}_{x,C(x)}$ is also very well covered.*

Moreover one can get all very well covered graphs from irreducible very well covered graphs by applying the above operation to the different matched points $(x, C(x))$.

In summary, to study very well covered graphs (in particular to study their properties) one can only study the irreducible ones. Those are characterized by the following theorem:

2.5. Theorem. *The following properties are equivalent:*

(i) *G is a very well covered irreducible.*

(ii) *G has a perfect matching C which satisfies the properties* (P) *and* P_5: $x \in \Gamma(y) \Rightarrow C(x) \notin \Gamma[C(y)]$.

(iii) *G has a unique perfect matching and this matching satisfies* (P).

Remark. P_5 means that G does not contain a cycle C_4 with two edges of the matching C.

Proof of 2.5. (i)$\Rightarrow$(ii). This is a consequence of the properties P_i.

(ii)$\Rightarrow$(iii). If C satisfies P_5, each class consists of a unique vertex. From the proof of 2.3, every perfect matching matches X with $C(X)$ and then G has a unique perfect matching.

(iii)$\Rightarrow$(i). Theorem 1.2 shows that G is very well covered. If G has a unique perfect matching, each class consists of a unique point and then G is irreducible.

3. Degrees in an irreducible very well covered graph

We will denote $\bar{x}$ the only vertex matched to x in an irreducible very well covered graph, $d(x)$ the degree of a vertex x, and δ (resp. Δ) the minimum (resp. maximum) degree of G.

3.1. Proposition. *An irreducible very well covered connected graph of order n has the following properties:*

(1) *For each z in V, $(y \in \Gamma(z), y \neq \bar{z}) \Rightarrow d(\bar{y}) < d(z)$.*

(2) *Every vertex has at least a neighbour matched to a vertex of degree 1. Therefore $\delta = 1$.*

(3) *If $d(z) = \Delta$, $d(\bar{z}) = 1$.*

(4) *$\Delta \leq \frac{1}{2}n$ and $\Delta > 2$ if $n > 4$.*

(5) *If G has q vertices of degree 1, $q \geq 2$ and $q \leq \frac{1}{2}n$ if $G \neq K_2$.*

(6) *If $q = 2$, then $n = \Delta$ and G is bipartite. If $q > 2$, then $n \leq q(\Delta - 1)$.*

Proof. (1) Let z be a vertex of G and $y \in \Gamma(z)$, $y \neq \bar{z}$. Every neighbour x of $\bar{y}$ is also neighbour of z and therefore $\Gamma(\bar{y}) \subset \Gamma(z)$. The inclusion is strict because $\bar{z} \notin \Gamma(\bar{y})$ (property P_5) and therefore $d(\bar{y}) < d(z)$. The properties (2) to (5) can be proved in the same manner (see [3]).

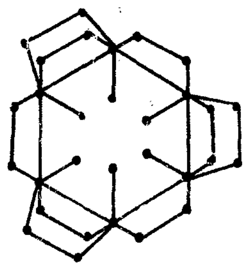

Fig. 2.

(6) Let z be a point of G with $d(z)=\Delta$. Then $d(\bar{z})=1$ (3) and there exists $y \in \Gamma(z)$ such that $d(\bar{y})=1$ (2). Let us suppose that there exists a $z \in G$ with $d(z)=\Delta$ such that z has an unique neighbour $y \neq z$ with $d(\bar{y})=1$. For each $x \in \Gamma(z)$, $\bar{x}$ has all its neighbours in $\Gamma(z)$ and has a neighbour matched with a vertex of degree 1. This neighbour of x is necessarily y. Therefore $\Gamma(y)=\overline{\Gamma(z)}$. Then $\Gamma(z)$ and $\overline{\Gamma(z)}$ are stable sets (Corollary 1.4); no point of $\overline{\Gamma(y)}=\Gamma(z)$ is joined to $V-\Gamma(y)$ (Corollary 1.3); and, as G is connected, $V=\Gamma(z)\cup\overline{\Gamma(z)}$, G is bipartite, $q=2$ and $\Delta=n$.

If $q=2$, z satisfies this hypothesis and so $\Delta=n$ and G is bipartite. If $q>2$, every vertex of degree Δ has at least two neighbours matched to a vertex of degree 1. Then, q and Δ given, let us determine the maximum order of G. Let $\bar{y}_1$, $\bar{y}_2, \ldots, \bar{y}_q$ be the q vertices of degree 1. Each vertex y_i of degree Δ is adjacent to $\bar{y}_i$, at least two other vertices y_j, and at most $\Delta-3$ other vertices of G. Each vertex y_i with $d(y_i)\leq\Delta-1$ is adjacent to $\bar{y}_i$, at least an other vertex y_j and at most $(\Delta-1)-2=\Delta-3$ other vertices of G. As every vertex of G is adjacent to a y_i, G will have $2q$ vertices y_i and $\bar{y}_i$ and at most $(\Delta-3)q$ other vertices. Therefore $n\leq(\Delta-1)q$.

This proof can be used also to show that the bound is the best possible and to construct all graphs for which $\frac{1}{2}n=\lfloor\frac{1}{2}(\Delta-1)q\rfloor$ (see [3]). The graph given in Fig. 2 is an example of such a graph with $\Delta=6$, $q=6$, $n=30$.

We will now give again a characterization of well covered trees due to Ravindra [6].

3.2. Proposition. *A tree T ($T\neq K_2$) is well covered if and only if it has an even order and $\frac{1}{2}n$ vertices of degree* 1.

Proof. A tree T is bipartite and then, if it is well covered, it is very well covered and moreover irreducible since it has no cycle. So its order is even and it has at most $\frac{1}{2}n$ vertices of degree 1. If it has less than $\frac{1}{2}n$ such vertices, there exists a pair $(x,\bar{x})$ with $d(x)>1$ and $d(\bar{x})>1$.

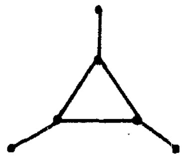

Fig. 3.

Let $y \in \Gamma(x)$, $y \neq \bar{x}$ and $z \in \Gamma(\bar{x})$, $z \neq x$. From property (P) y and z are adjacent which is impossible since $x\,y\,z\,\bar{x}\,x$ would be a cycle of T.

Conversely, the perfect matching consisting of the pendent edges satisfies (P).

However, an irreducible very well covered graph of order n which has $\frac{1}{2}n$ vertices of degree 1 is not necessarily a tree. For an example see Fig. 3.

4. Degrees in very well covered graphs

4.1. Proposition. *In a very well covered connected graph, if x is a vertex of minimum degree δ, then the vertex X which represents the class of x in the associated irreducible graph G_i is of degree* 1.

Proof. For every vertex x of G, $d(x) = \sum_{x_j \in \Gamma(x)} |X_j|$, where X_j is the class of x_j taken only once.

If G is complete bipartite, $G_i = K_2$ and every vertex in G_i has degree 1. If G is not complete bipartite, there exist vertices of degree at least 2 in G_i. Let us show that for every vertex x which class has a degree at least 2 in G_i, there exists a vertex of degree less than $d(x)$, which shows that $d(x) \neq \delta$. Indeed there exists $y \in \Gamma(x)$, $y \notin \bar{X}$ such that the class $\bar{Y}$ has degree 1 in G_i (Proposition 3.1). Then

$$d(x) \geq |X| + |Y| > |Y| = d(\bar{y}) \geq \delta.$$

We will give the two following corollaries without proof.

4.2. Corollary. *In a very well covered graph, $\delta = \inf_{d_i(X_j)=1} |X_j|$ where $d_i(X_j)$ is the degree of X_j in G_i.*

4.3. Corollary. *In a very well covered graph of order n, either $\delta = \frac{1}{2}n$ and then G is complete bipartite, or $\delta \leq n/2q$ where q is the number of vertices of degree* 1 *of the irreducible associated graph G_i.*

In the following proposition, we generalize a result established by Ravindra for well covered bipartite regular graphs [6] and by Berge for well covered bipartite regularizable graphs [4]. Recall that a graph is regularizable if one can get a regular multigraph of nonzero degree by multiplying each of its edges by an nonzero integer. Then a regular graph is regularizable. One possible characterization is that G is regularizable if and only if for every stable set S of G, $|S| \leqslant |\Gamma(S)|$, and $|S| = |\Gamma(S)| \Rightarrow \Gamma[\Gamma(S)] = S$. Regularizable graphs are studied in [4].

4.4. Proposition. *A very well covered connected graph is regularizable if and only if it is complete bipartite.*

Proof. If G is complete bipartite, it is very well covered and regular.

Conversely let G be a very well covered connected regularizable graph. A class X of degree 1 in the irreducible associated graph G_i is a stable set which satisfies $\Gamma(X) = \bar{X}$, that is to say $|\Gamma(X)| = |X|$ which implies $\Gamma[\Gamma(X)] = X$. Since G is connected, $V = X \cup \bar{X}$ and G is complete bipartite.

5. Dominating and irredundant sets in very well covered graphs

5.1. Proposition. *If G is a very well covered connected graph, with $G \neq K_2$, then* $\mathrm{ir}(G) \geqslant q$ *where q is the number of classes of degree 1 in the associated irreducible very well covered graph G_i.*

Proof. It suffices to show that every maximal irredundant set I contains at least q vertices. Remark that if I contains a vertex b, either b is isolated in I and I contains all the vertices of the class B of b; or b has a neighbour adjacent to no other vertex of I, and I does not contain any other vertex of B since these vertices have the same neighbours than b.

If G is complete bipartite with $G \neq K_2$, then $q = 2$. Either I is a stable set of $\frac{1}{2}n \geqslant 2$ vertices, or contains two adjacent vertices and $|I| \geqslant 2$.

If G is not complete bipartite, let Y_j, $1 \leqslant j \leqslant q$, be the q classes with $d_i(\bar{Y}_j) = 1$. We will find an injective application from the set of q pairs $(Y_j, \bar{Y}_j)$ into I.

Let us take first, if there exists someone, a pair $(Y, \bar{Y})$ with $d(\bar{Y}) = 1$ and $(Y \cup \bar{Y}) \cap I = \emptyset$. Since I is maximal, for each $\bar{y} \in \bar{Y}$, $I \cup \bar{y}$ is redundant. But $\bar{y}$, isolated in $I \cup \bar{y}$, is irredundant. Therefore there exists at least a vertex b in I redundant $I \cup \bar{y}$. b is not isolated in I otherwise it would be adjacent to $\bar{y}$ and then element of Y, contrary to the hypothesis $Y \cap I = \emptyset$. Then $I \cap B = \{b\}$ and b has neighbours in $V - I$, adjacent to no other point in I, but adjacent to $\bar{y}$. All these neighbours are the vertices of Y. The vertex b will be associated to the pair $(Y, \bar{Y})$.

$$d_i(B) \geqslant |\bar{B}| + |Y| > 1.$$

Let us also show that $d_i(\bar{B}) > 1$. If $|B| = |\bar{B}| = 1$, either $\bar{b} \notin I$, or $\bar{b} \in I$ and then $d_i(\bar{B}) > 1$ otherwise $\bar{b}$ would be redundant in I. If $|B| = |\bar{B}| > 1$, since all the vertices of $\bar{B}$ are adjacent to the vertex b of I, I contains at most one vertex in $\bar{B}$. In the two cases, either $d(\bar{B})>1$, or $\bar{B}$ contains a vertex c not in I. c, adjacent to b, but not to $\bar{y}$, is adjacent to another vertex of I, not in B, and therefore $d_i(\bar{B})>1$.

In the same way, to another pair $(Z,\bar{Z})$, with $d(\bar{Z})=1$ and $(Z\cup\bar{Z})\cap I=\emptyset$, is associated a vertex d of I. The vertices of Y, adjacent to b, are not adjacent to $\bar{z}$ since $d_i(\bar{z})=1$; then b is irredundant in $I\cup\bar{z}$ and d is different from b.

Let us take now the pairs $(T,\bar{T})$ with $d_i(\bar{T})=1$ and $(T\cup\bar{T})\cap I\neq\emptyset$. Such a pair is not a pair $(B,\bar{B})$ considered before, since $d_i(B)>1$ and $d_i(\bar{B})>1$. We will associate to $(T,\bar{T})$ one vertex of $(T\cup\bar{T})\cap I$. So the injective application is well defined; therefore $|I|\geq q$ and $\text{ir}(G)\geq q$.

5.2. Proposition. *Let G be a very well covered graph, and $\bar{Y}_j$, $1\leq j\leq q$, the vertices of degree 1 in the associated irreducible graph. Then every subgraph A containing one vertex of each class Y_j is a dominating set.*

Proof. Let z be a vertex of G, with class Z in the associated irreducible graph. From Proposition 3.1, Z has a neighbour Y_i. z is adjacent to all the vertices in the class Y_i and therefore to a point of A.

5.3. Theorem. *Let G be a very well covered connected graph, with $G\neq K_2$, then $\text{ir}=\gamma=q$ where q is the number of vertices of degree 1 in the associated irreducible graph.*

Proof. From Propositions 5.1 and 5.2, $\text{ir}\geq q$ and $\gamma\leq q$. In every graph $\text{ir}\leq\gamma$ and therefore $\text{ir}=\gamma=q$.

5.4. Remark. The reader can find other results in [3]. In particular it is proved that if G is a very well covered irreducible graph, then its number of edges is $m\leq\frac{1}{8}n(n+2)$ and that the diameter of a very well covered graph is less than or equal to $q+1$.

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