Global existence and blow up of solutions for the inhomogeneous nonlinear Schrödinger equation in $\mathbb{R}^2$

Yanjin Wang

Graduate School of Mathematical Sciences, University of Tokyo, 3-8-1 Komaba, Meguro, Tokyo 153-8914, Japan

Received 20 December 2006
Available online 5 June 2007
Submitted by H.W. Broer

Abstract

This paper discusses a class of inhomogeneous nonlinear Schrödinger equation

$$\begin{cases}
    i\partial_t u(t, x) = -\Delta u(t, x) - V(x)|u(t, x)|^{p-1}u(t, x), \\
    u(0, x) = u_0(x),
\end{cases}$$

where $(t, x) \in \mathbb{R} \times \mathbb{R}^2$, $V(x)$ satisfies some assumptions.

By a constrained variational problem, we firstly define some cross-constrained invariant sets for the inhomogeneous nonlinear Schrödinger equation, then we obtain some sharp conditions for global existence and blow up of solutions. As a consequence it is shown that the solution is globally well-posed in $H^1_0(\mathbb{R}^2)$ with the $H^1$-norm of the initial data $u_0$ which is dominated by the minimal value of the constrained variational problem.

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Keywords: Schrödinger equation; Inhomogeneous nonlinearity; Global existence; Blow up

1. Introduction

In this paper we study a class of inhomogeneous nonlinear Schrödinger equation as follows,

$$\begin{cases}
    i\partial_t u(t, x) = -\Delta u(t, x) - g(x, |u|^2)u(t, x), \\
    u(0, x) = u_0(x),
\end{cases}$$

(1.1)

where $\Delta$ is the Laplacian operator on $\mathbb{R}^2$, $u : [0, T) \times \mathbb{R}^2 \to \mathbb{C}$ and $g(x, |u|^2)$ is some inhomogeneous nonlinearity, which means that the nonlinear term is $x$-dependence.

Models of this type of Schrödinger equation (1.1) are of interesting in various physical contexts, as well as in nonlinear optic and plasma physics, where for example, when $g(x, |u|^2) = V(x)|u|^{p-1}$ Eq. (1.1) can model light beam propagation in an inhomogeneous medium where $V(x)$ is proportional to the electron density [13].

There exist several papers devoted to the existence and stability of solitary waves of Eq. (1.1) for some cases $V(x)$ (see, e.g., [2,6,9,10]). Also, global existence and blow up of solutions of Schrödinger equation have been studied by
Merle for some type of inhomogeneous nonlinearity \( g(x, |u|^2) = V(x)|u|^\frac{4}{n} \) with \( n \geq 2 \) in [15], where it is shown that the solution of (1.1) is globally well-posed in \( H^1 \) with the \( L^2 \)-norm of the initial data \( u_0 \) bounded by some ground state. In other words, Merle found the sufficient conditions for the existence and nonexistence of the solution in [15]. And the critical case with some type of \( V(x) \) was also studied by Fibich and Wang [7]. Furthermore, recently the instability of standing waves was consider by Liu et al. [14], Fukuizumi and Ohta [8]. Moreover, more general inhomogeneous nonlinear Schrödinger equation was also considered in [16].

In this paper we will concentrate on \( g(x, |u|^2) = V(x)|u|^{p-1} \) with \( p \geq 3 \) in \( \mathbb{R}^2 \), here \( V(x) \) satisfies the following condition:

**Assumption 1.** \( V(x) \in C^1(\mathbb{R}^2) \) satisfies that there exists \( k > 0 \) such that

\[
0 \leq V(x) \leq k, \quad \forall x \in \mathbb{R}^2
\]

and

\[
\int V(x) \, dx \neq 0.
\]

And let \( \rho(x) = (p - 1)V(x) - x \cdot \nabla V(x) \geq 0 \) satisfies that \( \rho(x) \) is decreasing with respect to \( x \), that is to say, if \( |x'| \geq |x| \) then \( \rho(x') \leq \rho(x) \).

Without loss of generality, in the paper we also suppose that the initial data \( u_0 \) satisfies that

\[
\int V(x)|u_0(x)|^2 \, dx \neq 0.
\]

Indeed, if the above inequality is not satisfied, then it is obvious that \( V(x)u_0(x) \equiv 0 \) on \( \mathbb{R}^2 \), thus Eq. (1.1) is equivalent with linear Schrödinger equation.

Before going any further, we introduce some notations, which will be used throughout the paper. We denote \( \int_{\mathbb{R}^2} f(x) \, dx \) simply by \( \int f(x) \, dx \). The Lebesgue space is denoted as \( L^q = L^q(\mathbb{R}^2) = \{ u(x) \in \mathbb{C} : \|u\|_q = (\int |u|^q \, dx)^{\frac{1}{q}} < \infty \} \). The Sobolev space is defined by \( H^1 = H^1(\mathbb{R}^2) = \{ u \in S'(\mathbb{R}^2) : \|u\|_{H^1}^2 := \int |u(x)|^2 \, dx + \int |
abla u(x)|^2 \, dx < \infty \} \), and \( H^1_r = \{ u \in H^1 : u(x) = u(r), \ r = |x| \} \) with the norm \( \|u\|_{H^1} \). \( |x|u \in L^2 \) is equivalent with \( \int |x|^2|u(x)|^2 \, dx < \infty \). The notation \( t \to T^- \) means that \( t \to T \) and \( t < T \).

As the proof of Theorem 4.4.6 in [5], by Kato’s method (see [11,12]) we can easily prove that the local-time well-posedness, conservation of \( L^2 \)-mass and energy (1.4), (1.6), and identity (1.8) hold for Eq. (1.1) in \( C([0, T), H^1(\mathbb{R}^2)) \). We here state the following corresponding proposition without proof.

**Proposition 1.1.** Assume that \( 3 \leq p < \infty \). For \( u_0 \in H^1(\mathbb{R}^2) \), then there exists a unique local solution \( u \in C([0, T), H^1(\mathbb{R}^2)) \) of the Schrödinger equation (1.1) for some \( T \in [0, \infty) \) (maximal existence time), which satisfies the following conservation laws of the mass

\[
Q(t) = Q(0)
\]

where

\[
Q(t) := \int |u(t,x)|^2 \, dx,
\]

and energy

\[
E(t) = E(0)
\]

where

\[
E(t) := \frac{1}{2} \int |\nabla u(t,x)|^2 \, dx - \frac{1}{p+1} \int V(x)|u(t,x)|^{p+1} \, dx
\]

for every \( t \in [0, T) \). And here \( \nabla u(t,x) = \left( \frac{d}{dx^1} u(t,x), \ldots, \frac{d}{dx^n} u(t,x) \right)^T \).

Furthermore, we have the following alternatives: \( T = \infty \) or else \( T < \infty \) and \( \lim_{t \to T^-} \|u(t, \cdot)\|_{H^1(\mathbb{R}^2)} = \infty. \)
Moreover, if \( u_0 \in H^1(\mathbb{R}^2) \) also satisfies \( |x|u_0 \in L^2(\mathbb{R}^2) \), then the following identity holds
\[
\frac{d^2}{dt^2} \int |x|^2 |u(t,x)|^2 \, dx = 8P(u,t)
\] (1.8)
for every \( t \in [0, T) \), where
\[
P(v,t) := \int |\nabla v(t,x)|^2 \, dx - \frac{p-1}{p+1} \int V(x)|v(t,x)|^{p+1} \, dx + \frac{1}{p+1} \int x \cdot \nabla V(x)|v(t,x)|^{p+1} \, dx.
\] (1.9)

In the above proposition we take \( p \) from \([3, \infty)\). Indeed, when \( p \in (1, 3) \) Proposition 1.1 is still valid.

In a general way, if \( T < \infty \), then the solution is said to blow up in a finite time \( T \). And in the case \( T = \infty \) and \( \lim_{t \to T^-} \|\nabla u(t, \cdot)\|_2^2 = \infty \), we still call it as a blow-up solution of Eq. (1.1) in the paper. Otherwise, \( T = \infty \) and \( \lim_{t \to T^-} \|\nabla u(t, \cdot)\|_2^2 \) is bounded, then the solution exists globally in time on \([0, \infty)\). In fact, here the global existence of solution implies that \( \|\nabla u(t, \cdot)\|_2^2 \) is uniformly bounded on \([0, \infty)\) by the continuity of \( u(t,x) \) as a function of \( t \).

Now let us consider the identity (1.8) in the case \( V(x) = c \neq 0 \). If \( u_0 \in H^1(\mathbb{R}^2) \) satisfies \( |x|u_0 \in L^2(\mathbb{R}^2) \), which means that \( \int |x|^2 |u_0(x)|^2 \, dx < \infty \), then for all \( t \in [0, T) \) we have
\[
\frac{d^2}{dt^2} \int |x|^2 |u(t,x)|^2 \, dx = 16E(t) - 8c \frac{p-3}{p+1} \int |u(t,x)|^{p+1} \, dx,
\] (1.10)
where \( E(t) \) is defined by (1.7) with \( V(x) = c \). By variational calculus, Cazenave [5] and Zhang [18,19] have studied the global existence and blow up of solutions for the homogeneous nonlinear Schrödinger equation.

But for Eq. (1.1), the term \( \int x \cdot \nabla V(x)|u(t,x)|^{p+1} \, dx \) of (1.9) will increase difficulty in analysis to obtain the blow-up results. Thus we cannot directly use the method of homogeneous nonlinear Schrödinger equation to Eq. (1.1). We have to construct some cross-constrained invariant sets for (1.1). Thus a constrained variational problem is firstly introduced. And then constructing three cross-constrained invariant sets we obtain some sharp conditions for the global existence and blow up of solutions for Cauchy problem of Eq. (1.1). Especially, in this paper we find an interesting phenomenon that maybe the solution of the inhomogeneous nonlinear Schrödinger equation (1.1) blows up when \( t \to \infty \), which does not appear in the homogeneous case of Eq. (1.1) in [5,18,19]. At last using the sharp condition we give out a sufficient condition for the global well-posedness of Eq. (1.1).

The plan of the paper is as follows: In Section 2 we prove the existence of solutions of a variational problem by the method as [3]. In Section 3, we claim some sharp conditions for global existence and blow up of solutions of Eq. (1.1). And then we give out some sufficient condition of the initial data for the global existence. In the last section, we make some concluding remarks.

### 2. Constrained variational problem

We begin with the following definitions
\[
S(\phi) := \frac{1}{2} \int |\nabla \phi(x)|^2 \, dx,
\] (2.1)
\[
I(\phi) := \frac{1}{2} \int \left( |\phi(x)|^2 - \frac{2}{p+1} V(x)|\phi(x)|^{p+1} \right) \, dx,
\] (2.2)
and
\[
M := \{ \phi \in H^1_t(\mathbb{R}^2) : I(\phi) = 0, \phi \neq 0 \}.
\] (2.3)

To construct invariant sets, we next introduce a constrained variational problem
\[
s := \inf_{\phi \in M} S(\phi).
\] (2.4)

Now we are in a position to state a theorem about the constrained variational problem:
Theorem 2.1. There exists $\Phi \in M$ such that

$$S(\Phi) = \inf_{\phi \in M} S(\phi) = s > 0.$$  \hfill (2.5)

Proof. Firstly we claim that $M \neq \emptyset$. Indeed, if $I(\phi) \neq 0$ then we set $\phi^\lambda = \lambda \phi(x)$. And it follows that

$$I(\phi^\lambda) = \frac{1}{2} \lambda^2 \int \left( |\phi(x)|^2 - \frac{2}{p+1} \lambda^{p-1} V(x)|\phi(x)|^{p+1} \right) dx. \hfill (2.6)$$

Since $\lambda^{-2} I(\phi^\lambda) > 0$ as $\lambda \to 0$ and $\lambda^{-2} I(\phi^\lambda) \to -\infty$ as $\lambda \to \infty$, by the continuity of $I(\phi^\lambda)$ as a function of $\lambda$ there exists $\lambda_1 > 0$ such that $I(\phi^{\lambda_1}) = 0$.

It is obvious that $S(\phi) > 0$, which implies $S$ is bounded below on $M$. Thus we take a sequence $\{\phi_n\}_{n=1}^\infty$ from $M$, which is a minimizing sequence of $S$ on $M$, that is, $\lim_{n \to \infty} S(\phi_n) = s$.

Since $S$ is bounded below on $M$, we see that $\phi_n$ is bounded in $H^1(R^2)$. Then it follows that there exists a subsequence, denoted still by $\{\phi_n\}_{n \in \mathbb{N}}$, such that

$$\phi_n \rightharpoonup \phi_\infty \text{ weakly in } H^1(R^2). \hfill (2.7)$$

We here recall the following compactness lemma [17]:

$$H^1(R^2) \hookrightarrow L^\nu \text{ is compact,} \hfill (2.8)$$

where $2 < \nu < \infty$.

Thus by (2.7) and (2.8) we know

$$\phi_n \to \phi_\infty \text{ strongly in } L^{p+1}(R^2). \hfill (2.9)$$

Next we assert that $\phi_\infty \neq 0$. In fact, consider any $\phi \neq 0$ in $H^1_0(R^2)$ such that $I(\phi) = 0$, then by Assumption 1 we see

$$\int |\phi(x)|^2 dx = \frac{2}{p+1} \int V(x)|\phi(x)|^{p+1} dx \leq k \frac{2}{p+1} \int |\phi(x)|^{p+1} dx. \hfill (2.10)$$

On the other hand, using Gagliardo–Nirenberg inequality (see [1]) we have

$$\|\phi\|_{p+1} \leq c \|\nabla \phi\|_2 \|\phi\|_2 \hfill (2.11)$$

for some $c > 0$.

From (2.10) and (2.11) it follows that there exists a constant $v > 0$ such that

$$\|\nabla \phi\|_2 \geq v. \hfill (2.12)$$

Thus since $\{\phi_n\} \subset M$, then by (2.12) we see that $\|\nabla \phi_\infty\|_2 \geq v > 0$, which implies that $\phi_\infty \neq 0$.

In addition, by the weak-lower semi-continuity of $H^1$-norm,

$$2I(\phi_\infty) \leq \lim_{n \to \infty} \int |\phi_n(x)|^2 dx - \frac{1}{p+1} \int V(x)|\phi_n(x)|^{p+1} dx = 0, \hfill (2.13)$$

$$2s = \lim_{n \to \infty} \int |\nabla \phi_n(x)|^2 dx \geq \int |\nabla \phi_\infty(x)|^2 dx. \hfill (2.14)$$

If $I(\phi_\infty) < 0$, we let $\phi_0^\lambda(x) = \lambda \phi_\infty(x)$. Noting the fact that $\lambda^{-2} I(\phi_0^\lambda(x)) > 0$ when $\lambda \to 0$, by the continuity of $I(\phi_0^\lambda(x))$ as a function of $\lambda$, we can find some $\lambda_0 \in (0, 1)$ such that $I(\phi_0^{\lambda_0}) = 0$. Thus, we see that

$$S(\phi_0^{\lambda_0}) = \frac{\lambda_0^2}{2} \int |\nabla \phi_\infty(x)|^2 dx < s, \hfill (2.15)$$

which induces a contradiction with the definition of $s$. Consequently, with letting $\Phi = \phi_\infty$, it holds that $\Phi \in M$ and $s = S(\Phi) > 0$. \qed
Remark 2.1. In the above proof if we use the Schwarz spherical rearrangement $\tilde{\phi}_n$ (see A.III [4]) of the function $\phi_n$, then by the properties of the Schwarz spherical rearrangement we easily see that $S(\tilde{\phi}_n) \leq S(\phi_n)$, but we here cannot determine whether $I(\tilde{\phi}_n) = 0$. Thus, in the definition of $M$ we use $H^1_1$ so as to utilize the compactness lemma in the proof of Theorem 2.1.

3. Sharp conditions for global existence and blow-up

To give out sharp conditions for global existence and blow up of solutions of Eq. (1.1), we need to introduce some cross-constrained invariant sets for the Schrödinger equation (1.1) with inhomogeneous nonlinearity as follows:

$$
\Gamma_1 := \{ \phi \in H^1_1(\mathbb{R}^2): I(\phi) > 0, \ L(\phi) < s \},
$$

$$
\Gamma_2 := \{ \phi \in H^1_1(\mathbb{R}^2): I(\phi) < 0, \ L(\phi) < s, \ P(\phi) < 0 \},
$$

$$
\Gamma_3 := \{ \phi \in H^1_1(\mathbb{R}^2): I(\phi) < 0, \ L(\phi) < s, \ P(\phi) > 0 \},
$$

where

$$
P(\phi) := \int |\nabla \phi(x)|^2\,dx - \frac{p-1}{p+1} \int V(x)|\phi(x)|^{p+1}\,dx + \frac{1}{p+1} \int x \cdot \nabla V(x)|\phi(x)|^{p+1}\,dx,
$$

$$
L(\phi) := \frac{1}{2} \int |\phi(x)|^2\,dx + \frac{1}{2} \int \left( |\nabla \phi(x)|^2 - \frac{2}{p+1} V(x)|\phi(x)|^{p+1} \right)\,dx.
$$

For a local solution $u$ of Eq. (1.1) with the initial data $u_0$, we put

$$
S(u, t) := \frac{1}{2} \int |\nabla u(t,x)|^2\,dx,
$$

$$
I(u, t) := \frac{1}{2} \int \left( |u(t,x)|^2 - \frac{2}{p+1} V(x)|u(t,x)|^{p+1} \right)\,dx,
$$

$$
L(u, t) := \frac{1}{2} Q(t) + E(t) = \frac{1}{2} \int |u(t,x)|^2\,dx + \frac{1}{2} \int \left( |\nabla u(t,x)|^2 - \frac{2}{p+1} V(x)|u(t,x)|^{p+1} \right)\,dx.
$$

For simplicity, we let $L(u_0) = L(u, 0)$, $S(u_0) = S(u, 0)$, $I(u_0) = I(u, 0)$ and $P(u_0) = P(u, 0)$. And $u(t, \cdot) \in \Gamma_1$ means that $u(t, \cdot) \in H^1_1(\mathbb{R}^2)$, $I(u, t) > 0$ and $L(u, t) < s$. It is similar that $u(t, \cdot) \in \Gamma_2$ and $u(t, \cdot) \in \Gamma_3$.

Then we state a result for the above three sets.

Theorem 3.1. Suppose that $u_0 \in \Gamma_j \ (j = 1, 2, 3)$, then the corresponding local solution $u$ of Eq. (1.1) satisfies $u(t, \cdot) \in \Gamma_j$ for every $t \in [0, T_{\max})$. That is to say, the sets, $\Gamma_1$, $\Gamma_2$ and $\Gamma_3$, are invariant under the flow generated by Schrödinger equation (1.1).

Proof. At first, we consider the set $\Gamma_1$. By (1.4) and (1.6), we see

$$
L(u_0) = L(u, t).
$$

Since $L(u_0) < s$, then

$$
L(u, t) < s
$$

for every $t \in [0, T)$.

Now we just need to prove $I(u, t) > 0$. It is proved by a contradiction argument. Assume that it does not hold that $I(u, t) > 0$ over the whole interval $[0, T)$. Since $I$ is continues in $t$, there exists one time $t_1 \in (0, T)$ such that

$$
I(u, t_1) = 0,
$$

which implies that $u(t_1, \cdot) \in M$, then by Theorem 2.1 we have

$$
S(u, t_1) := \frac{1}{2} \int |\nabla u(t_1, x)|^2\,dx \geq s.
$$
On the other hand, from (3.4) and (3.5) we obtain

$$S(u,t_1) < s.$$  

(3.7)

Thus between (3.6) and (3.7) there exists a contradiction, which implies that \( I(u,t) > 0 \) for every \( t \in [0,T) \).

Hereunto we have proved that \( \Gamma_1 \) is invariant under the flow generated by Schrödinger equation (1.1).

For the set \( \Gamma_2 \), according to the above proof we see that it is valid that \( L(u,t) < s \) and \( I(u,t) < 0 \) for every \( t \in [0,T) \). Thus, in order to prove \( u(t, \cdot) \in \Gamma_2 \) it is sufficient to prove \( P(u,t) < 0 \) for every \( t \in [0,T) \).

We here use a contradiction argument. Suppose that it is not true that \( P(u(t_0)) < 0 \), by the continuity of \( P(u,t) \) there exists a minimal time \( t_1 \in (0,T) \) such that \( P(u(t_1)) = 0 \) and \( I(u(t_1)) < 0 \).

Here we let \( u^{(1)}(t,x) = u(t, \gamma x) \). From the fact that \( I(u^{(1)}, t_1) < 0 \) at \( \gamma = 1 \) and \( I(u^{(1)}, t_1) > 0 \) when \( \gamma \to 0 \) and \( \gamma \neq 0 \), it follows that there exists \( \gamma_1 \in (0,1) \) such that

$$I(u^{(1)}, t_1) = 0.$$  

(3.8)

And for every \( u \in M \), we have

$$P(u^{(1)}, t) = \gamma^2 \left| \nabla u(t, x) \right|^2 dx - \frac{\gamma^{p-1}}{p+1} \int \left( (p-1)V\left( \frac{x}{\gamma} \right) - \frac{x}{\gamma} \nabla V\left( \frac{x}{\gamma} \right) \right) \left| u(t, x) \right|^{p+1} dx,$$

$$L(u^{(1)}, t) = \frac{1}{2} \gamma^2 \left| \nabla u(t, x) \right|^2 dx - \frac{\gamma^{p-1}}{p+1} \int V\left( \frac{x}{\gamma} \right) \left| u(t, x) \right|^{p+1} dx + \frac{1}{2} \int \left| u(t, x) \right|^2 dx.$$

Then by a direct calculation,

$$\gamma \frac{d}{d\gamma} L(u^{(1)}, t) = P(u^{(1)}, t).$$  

(3.9)

Now we consider the function \( L(u^{(1)}, t_1) \) when \( \gamma \in [\gamma_1, 1] \). By \( P(u^{(1)}, t_1) = 0 \) as \( \gamma = 1 \), we obtain

$$\gamma \frac{d}{d\gamma} L(u^{(1)}, t_1) = \gamma^2 \left( (p-1)V(x) - x \cdot \nabla V(x) \right) \left| u(t_1, x) \right|^{p+1} dx$$

$$- \frac{\gamma^{p-1}}{p+1} \int \left( (p-1)V\left( \frac{x}{\gamma} \right) - \frac{x}{\gamma} \cdot \nabla V\left( \frac{x}{\gamma} \right) \right) \left| u(t_1, x) \right|^{p+1} dx$$

$$> 0$$  

(3.10)

for every \( \gamma \in (\gamma_1, 1) \), where the last inequality uses Assumption 1.

Moreover, when \( \gamma = 1 \) we have

$$\gamma \frac{d}{d\gamma} L(u^{(1)}, t_1) = 0.$$  

(3.11)

Thus we know that

$$L(u^{(1)}, t_1) > L(u^{(1)}_1, t_1)$$  

(3.12)

for \( \gamma \in (\gamma_1, 1) \).

And from (3.8) it follows that \( u^{(1)}(t_1, \cdot) \in M \), then by Theorem 2.1 we have

$$L(u^{(1)}, t_1) = S(u^{(1)}, t_1) \geq s.$$  

(3.13)

By (3.12) and (3.13) we then have

$$L(u, t_1) > L(u^{(1)}, t_1) \geq s,$$  

(3.14)

which is absurd with \( L(u, t) < s \) for every \( t \in [0,T) \). Then we have obtained \( P(u, t) < 0 \) for every \( t \in [0,T) \).

Therefore, it has been proved that \( \Gamma_2 \) is invariant under the flow generated by Schrödinger equations (1.1).

By a similar argument as \( \Gamma_2 \), we can also show that \( \Gamma_3 \) is invariant under the flow generated by Eq. (1.1).
Remark 3.1. We now claim that
\[ \{ u \in H^1_r(\mathbb{R}^2) \mid L(u) < s, \ I(u) < 0, \ P(u) = 0 \} = \emptyset. \] (3.15)

Suppose the above statement is not true. Then there exists \( u_1 \in H^1_r(\mathbb{R}^2) \) satisfying that
\[ L(u_1) < s, \] (3.16)
\[ I(u_1) < 0, \] (3.17)
\[ P(u_1) = 0. \] (3.18)

Obviously, \( u_1 \neq 0 \). Thus as the argument in the proof of Theorem 3.1, by (3.17) and (3.18) we see that \( L(u_1) \geq s \), which induce a contradiction with (3.16). Then the statement (3.15) is valid.

We next state the main result for global existence and blow up of solutions of inhomogeneous nonlinear Schrödinger equation (1.1) for \( p \geq 3 \) on \( \mathbb{R}^2 \).

Theorem 3.2. For \( p \geq 3 \), let \( u(t, x) \) be a local solution of Eq. (1.1) on \( [0, T) \) with \( u_0 \in H^1(\mathbb{R}^2) \), as in Proposition 1.1.

(i) If \( u_0 \in \Gamma_1 \), then the solution \( u \) of the inhomogeneous nonlinear Schrödinger equation (1.1) globally exists in time on \( [0, \infty) \).

Furthermore, the solution \( u \) satisfies
\[ \| \nabla u(t, \cdot) \|_2^2 < 2s \] (3.19)
for every \( t \in [0, \infty) \), where \( s \) is defined as (2.4).

(ii) If \( u_0 \in \Gamma_2 \) and \( |x|u_0 \in L^2(\mathbb{R}^2) \), then the local solution \( u \) of the inhomogeneous nonlinear Schrödinger equation (1.1) blows up in finite time \( T \), that is,
\[ \lim_{t \to T^-} \| \nabla u(t, \cdot) \|_2^2 = \infty. \] (3.20)

(iii) If \( u_0 \in \Gamma_3 \) and \( |x|u_0 \in L^2(\mathbb{R}^2) \), then the local solution \( u \) of the inhomogeneous nonlinear Schrödinger equation (1.1) blows up in a maximal time \( T \), that is to say,
\[ \lim_{t \to T^-} \| \nabla u(t, \cdot) \|_2^2 = \infty, \] (3.21)
where the maximal time \( T \) can be either finite or infinite.

Proof. (i) Since \( u_0 \in \Gamma_1 \), from Theorem 3.1 it follows that \( u(t, \cdot) \in \Gamma_1 \), that is to say,
\[ L(u, t) < s, \] (3.22)
and
\[ I(u, t) > 0. \] (3.23)

Then by (3.22) and (3.23) we have
\[ \frac{1}{2} \int |\nabla u(t, x)|^2 \, dx < s \] (3.24)
for every \( t \in [0, T) \), which implies that the solution \( u \) of the inhomogeneous nonlinear Schrödinger equation (1.1) globally exists in time on \( [0, \infty) \).

(ii) We now put
\[ \Gamma^\varepsilon_2 := \{ \phi \in H^1_r(\mathbb{R}^2) : L(\phi) < s, \ I(\phi) < 0, \ P(\phi) < -\varepsilon < 0 \}, \] (3.25)
where \( \varepsilon \) is sufficiently small. We here need to claim that if \( u_0 \in \Gamma_2 \) then there exists a small number \( \varepsilon > 0 \) such that \( \Gamma^\varepsilon_2 \) is invariant under the flow generated by the inhomogeneous nonlinear Schrödinger equation (1.1).
Noting the fact that \( u(t, x) \) is continuous in \( t \) on \( [0, T) \), we see that the functions \( P(u, t), I(u, t) \) and \( L(u, t) \) are also continuous in \( t \) on \( [0, T) \), which implies that there exist the limits of \( P(u, t), I(u, t) \) and \( L(u, t) \) as \( t \to T^- \).

By Theorem 3.1 and \( u_0 \in \Gamma_2 \), we see that the local solution \( u \) satisfies \( P(u, t) < 0 \) for \( t \in [0, T) \). Thus,

\[
\lim_{t \to T^-} P(u, t) \leq 0. \tag{3.26}
\]

Similarly,

\[
\lim_{t \to T^-} I(u, t) \leq 0. \tag{3.27}
\]

In addition, by \( u_0 \in \Gamma_2, (1.4) \) and \( (1.6) \) we see that \( L(u, t) = L(u_0) \) for \( t \in [0, T) \). Then we have that

\[
\lim_{t \to T^-} L(u, t) < s. \tag{3.28}
\]

Next we claim that it is impossible that

\[
\lim_{t \to T^-} P(u, t) = 0. \tag{3.29}
\]

Suppose that (3.29) is true, then by (3.27) and (3.28) we need to consider the following cases:

**Case I:** \( \lim_{t \to T^-} I(u, t) < 0 \). Here we can suppose that \( \lim_{t \to T^-} \|\nabla u(t, \cdot)\|^2 < \infty \). (Otherwise, we have obtained the blow-up result for Eq. (1.1).) Using a similar argument as in the proof for the case \( \Gamma_2 \) of Theorem 3.1, by (3.29) we obtain that

\[
\lim_{t \to T^-} L(u, t) \geq s,
\]

which obviously contradicts (3.28). Thus we see that it is impossible that \( \lim_{t \to T^-} P(u, t) = 0 \) in this case.

**Case II:** \( \lim_{t \to T^-} I(u, t) = 0 \). By (3.28) we have

\[
\lim_{t \to T^-} \frac{1}{2} \int |\nabla u(t, x)|^2 \, dx < s.
\]

On the other hand, from the above inequality and \( \lim_{t \to T^-} I(u, t) = 0 \) it follows that

\[
\lim_{t \to T^-} \|\nabla u(t, \cdot)\|^2_2 < \infty.
\]

Thus obviously \( \lim_{t \to T^-} u(t, \cdot) \in M \). Whence by Theorem 2.1 we see that

\[
\lim_{t \to T^-} \frac{1}{2} \int |\nabla u(t, x)|^2 \, dx \geq s.
\]

Then it is obvious that Case II is also impossible.

Thus, by (3.26) we see that

\[
\lim_{t \to T^-} P(u, t) < 0. \tag{3.30}
\]

By the continuity of \( P(u, t) \) in \( t \), obviously there exists a small number \( \epsilon > 0 \) such that \( P(u, t) < -\epsilon \). Then as the proof of Theorem 3.1 we can similarly claim that \( \Gamma_2^\epsilon \) is also invariant under the flow generated by inhomogeneous nonlinear Schrödinger equation (1.1). Therefore, using Taylor expansion of \( \int |x|^2 |u(t, x)|^2 \, dx \) in \( t \), by (1.8) we see

\[
\int |x|^2 |u(t, x)|^2 \, dx \leq \int |x|^2 |u_0(x)|^2 \, dx + 4t \Im \int \bar{u}_0 x \cdot \nabla u_0 \, dx - 8\epsilon t^2,
\]

where \( \Im u \) means the imaginary part of a complex-valued function \( u(t, x) \), which implies that \( \int |x|^2 |u(t, x)|^2 \, dx < 0 \) when \( t \to \infty \). Thus there exists a contradiction. This concludes that the solution \( u \) of Schrödinger equation blows up in finite time \( T \), that is,

\[
\lim_{t \to T^-} \|\nabla u(t, \cdot)\|^2_2 = \infty.
\]
(iii) To proceed the statement (iii) of Theorem 3.2, for \( u_0 \in \Gamma_3 \) we will first prove
\[
\int \left| \nabla u(t,x) \right|^2 dx \leq c \frac{2}{p+1} \int V(x) \left| u(t,x) \right|^{p+1} dx
\]  
(3.32)
where \( c \) is some constant.

By Theorem 3.1, we see that \( u(t,\cdot) \in \Gamma_3 \), here \( u \) is the corresponding solution of Eq. (1.1) with the initial data \( u_0 \in \Gamma_3 \).

For a fixed time \( t \in [0,T) \), we let \( u^\lambda(t,x) = \lambda u(t,x) \). And we know the fact \( I(u^\lambda,t) < 0 \) with \( \lambda = 1 \) and \( \lambda^{-2} I(u^\lambda,t) > 0 \) with \( \lambda \to 0 \) and \( \lambda > 0 \), then by the continuity of \( I(u^\lambda,t) \) as a function of \( \lambda \) we can find \( \lambda^* \in (0,1) \) such that
\[
2I(u^{\lambda^*},t) = \lambda^*_2 \int \left| \nabla u(t,x) \right|^2 dx - \frac{2\lambda^*_2}{p+1} \int V(x) \left| u(t,x) \right|^{p+1} dx = 0.
\]  
(3.33)

And by Theorem 2.1 we have
\[
S(u^{\lambda^*},t) \geq s.
\]  
(3.34)

Noting \( u(t,\cdot) \in \Gamma_3 \), we then have \( \mathcal{L}(u,t) < s \leq S(u^{\lambda^*},t) \), i.e.,
\[
(\lambda^*_2 - 1) \int \left| \nabla u(t,x) \right|^2 dx \geq \left( \lambda^*_2 - 1 \right) \frac{2}{p+1} \int V(x) \left| u(t,x) \right|^{p+1} dx.
\]  
(3.35)

Using (3.33), we see
\[
(\lambda^*_2 - 1) \int \left| \nabla u \right|^2 dx \geq \left( \lambda^*_2 - 1 \right) \frac{2}{p+1} \int V(x) \left| u \right|^{p+1} dx.
\]  
(3.36)

Since \( 0 < \lambda^* < 1 \), it is valid that \( \lambda^*_2 - 1 < 0 \). We then obtain that there exists a constant \( c > 0 \) such that (3.32) is valid.

Now let
\[
\Gamma^*_3 := \{ \phi \in H^1(\mathbb{R}^2) \mid L(\phi) < s, \ I(\phi) < 0, \ P(\phi) > \epsilon > 0 \}.
\]  
(3.37)

here \( \epsilon \) is a small positive number. As the proof of \( \Gamma^*_2 \) in (ii), similarly we can obtain that if \( u_0 \in \Gamma_3 \) then
\[
\lim_{t \to T^{-}} P(u,t) > 0.
\]

By the continuity of \( P(u,t) \) in \( t \) we easily see that there exists a small \( \epsilon > 0 \) such that the set \( \Gamma^*_3 \) is invariant under the flow generated by inhomogeneous nonlinear Schrödinger equation (1.1). Thus we have
\[
\frac{d^2}{dt^2} \int \left| x \right|^2 \left| u(t,x) \right|^2 dx = 8P(u,t) > 8\epsilon > 0.
\]  
(3.38)

From Assumption 1 it follows that
\[
P(u,t) \leq \int \left| \nabla u(t,x) \right|^2 dx.
\]  
(3.39)

Then by (3.32) we have that
\[
P(u,t) \leq c \frac{2}{p+1} \int V(x) \left| u(t,x) \right|^{p+1} dx.
\]  
(3.40)

By Assumption 1, Hölder’s inequality, Young’s inequality and (1.4) we obtain
\[
\int V(x) \left| u(t,x) \right|^{p+1} dx \leq k \int \left| u(t,x) \right|^{p+1} dx
\]
\[
\leq \frac{k}{2} \int \left( \left| u(t,x) \right|^2 + \left| u(t,x) \right|^{2p} \right) dx
\]
\[
= \frac{k}{2} \left( \| u_0 \|^2 + \int \left| u(t,x) \right|^{2p} dx \right).
\]  
(3.41)
Combining (3.38), (3.40) and (3.41), we have

\[ 8\chi + c \int |u(t,x)|^{2p} \, dx > \frac{d^2}{dt^2} \int |x|^2 |u(t,x)|^2 \, dx > 8\epsilon, \tag{3.42} \]

where \( \chi \) is some constant depending on \( \|u_0\|_2^2 \) and \( V(x) \).

In addition, (3.38) implies that \( \frac{d}{dt} \| \nabla u(t, \cdot) \|_2^2 \) is strictly increasing in \( t \) over \([0, T)\). Thus by Taylor expansion of \( \int |x|^2 |u(t,x)|^2 \, dx \) in \( t \) we then obtain

\[
\int |x|^2 |u(t,x)|^2 \, dx \geq \| \nabla u_0 \|_2^2 + 4t \Im \int \bar{u}_0(x)x \cdot \nabla u_0(x) \, dx + 8\epsilon t^2, \tag{3.43}
\]

\[
\int |x|^2 |u(t,x)|^2 \, dx \leq \| \nabla u_0 \|_2^2 + 4t \Im \int \bar{u}(t,x)x \cdot \nabla u(t,x) \, dx + \left( 8\chi + c \int |u(t,x)|^{2p} \, dx \right) t^2, \tag{3.44}
\]

where \( \Im u \) means the imaginary part of a complex-valued function \( u(t,x) \).

Thus we see that, as \( t \to \infty \), the right-hand side of (3.43) must tend to \(+\infty\).

Furthermore, obviously it depends on the right-hand side of (3.44) whether the maximal existence time \( T \) can be finite or infinite.

Hereunto we have gotten the blow-up result when \( u_0 \in \Gamma_3 \).

We next state a sufficient condition for the global existence of solutions of the inhomogeneous nonlinear Schrödinger equation (1.1).

**Theorem 3.3.** Suppose that \( u_0 \in H^1_1(\mathbb{R}^2) \) satisfies

\[ \int \left( |\nabla u_0(x)|^2 + |u_0(x)|^2 \right) \, dx < 2s, \tag{3.45} \]

then the corresponding solution \( u \) of inhomogeneous nonlinear Schrödinger equation (1.1) globally exists in time on \([0, \infty)\), and satisfies that

\[ \int |\nabla u(t,x)|^2 \, dx < 2s. \tag{3.46} \]

for every \( t \in [0, \infty) \).

**Proof.** If \( u_0 = 0 \), then the result is obvious. Next we suppose that \( u_0(x) \neq 0 \) on \( \mathbb{R}^2 \).

Obviously by (3.45) we see that

\[ L(u_0) = \frac{1}{2} \int \left| u_0(x) \right|^2 \, dx + \frac{1}{2} \int \left( \left| \nabla u_0(x) \right|^2 - \frac{2}{p+1} V(x) \left| u_0(x) \right|^{p+1} \right) \, dx < s, \tag{3.47} \]

and

\[ \frac{1}{2} \int |\nabla u_0(x)|^2 \, dx < s. \tag{3.48} \]

Since here we will use the statement (i) of Theorem 3.2 to prove this theorem, it is sufficient to prove \( I(u_0) > 0 \).

This will be proved by contradiction argument.

Firstly we suppose that

\[ I(u_0) = \frac{1}{2} \int \left( |u_0(x)|^2 - \frac{2}{p+1} V(x) |u_0(x)|^{p+1} \right) \, dx \leq 0. \tag{3.49} \]

Case I: \( I(u_0) = 0 \). Noting \( u_0 \neq 0 \), by Theorem 2.1 we see that \( S(u_0) \geq s \). But from (3.48) it follows that \( S(u_0) < s \).

Then obviously there exists a contradiction.

Case II: \( I(u_0) < 0 \). With \( u_0^\lambda = \lambda u_0(x) \) we see that \( \lambda^{-2} I(u_0^\lambda) > 0 \) as \( \lambda \to 0 \) and \( \lambda > 0 \), then by the continuity of \( I(u_0^\lambda) \) as a function of \( \lambda \) we can find \( \lambda_1 \in (0, 1) \) such that
\[
\frac{1}{2} \int \left( \lambda_1^2 |u_0(x)|^2 - \frac{2}{p+1} \lambda_1^{p+1} V(x) |u_0(x)|^{p+1} \right) dx = 0. \tag{3.50}
\]

Thus by Theorem 2.1 we see that
\[
S(u_0^{\lambda_1}) = s. \tag{3.51}
\]

On the other hand, by (3.48) we see
\[
\frac{1}{2} \int \lambda_1^2 |\nabla u_0(x)|^2 < \frac{1}{2} \int |\nabla u_0(x)|^2 dx < s, \tag{3.52}
\]
which implies that there exists a contradiction between (3.51) and (3.52).

Thus we see that \( I(u_0) > 0 \), which implies that \( u_0 \in \Gamma_1 \). Then by Theorem 3.2 the solution \( u \) of the inhomogeneous nonlinear Schrödinger equation (1.1) with the initial data \( u_0 \in H^1_p(\mathbb{R}^n) \) satisfying (3.45) globally exists in \( t \) on \([0, \infty)\). □

4. Final remarks

This section begins with \( V(x) \) satisfying that \( 0 < V(x) \leq v \) (here \( v \) is some constant) and the following global condition on \((x - x_0) \cdot \nabla V(x)\). There exists \( x_0 \in \mathbb{R}^2 \) such that
\[
(x - x_0) \cdot \nabla V(x) \leq 0, \quad \forall x \in \mathbb{R}^2, \tag{4.1}
\]
where the point \( x_0 \) is called as a global maximum of \( V(x) \), which was considered by Merle [15]. Without loss of generality we assume that \( x_0 = 0 \). Then for the above \( V(x) \), we next claim that \( \Gamma_3 = \emptyset \) with \( p = 3 \) by a contradiction argument.

Assume that \( \Gamma_3 \neq \emptyset \), that is to say, there exists at least one \( u_0 \in \Gamma_3 \). And by Theorem 3.1 we see that the corresponding solution \( u \) of Eq. (1.1) is also in \( \Gamma_3 \). By (3.36) and \( p = 3 \) we then see that
\[
E(t) = E(0) < 0 \tag{4.2}
\]
for every \( t \in [0, T) \).

And by (1.9) and (4.1) we have
\[
P(u_0) = 2E(0) + \frac{1}{4} \int x \cdot \nabla V(x) |u_0(x)|^4 dx \leq 2E(t) < 0, \tag{4.3}
\]
which is absurd with \( u_0 \in \Gamma_3 \). Thus we see that \( \Gamma_3 = \emptyset \).

If \( |x| |u_0| \in L^2(\mathbb{R}^2) \) and \( u_0 \in \Gamma_2 \), then by (4.2) and (4.3) we easily see that
\[
\frac{d^2}{dt^2} \int |x|^2 |u(t, x)|^2 dx \leq 16E(0) < 0 \tag{4.4}
\]
for all \( t \in [0, T) \).

Thus we can prove a similar blow-up result as the statement (ii) of Theorem 3.2. In [15] Merle showed a blow-up result under the assumption that the initial energy is sufficiently negative. But here we obtained the blow-up result without sufficiently negative initial energy.

In this paper we consider only 2-dimension space. Because of the following technical reason we cannot extend our results to \( n \)-dimension space with \( n \geq 3 \). Indeed, considering the case \( \mathbb{R}^n \) with \( n \geq 2 \) in the proof of Theorem 2.1 with \( p = 1 + \frac{4}{n} \), by Gagliardo–Nirenberg inequality for \( \phi \in M \) we have
\[
\int |\phi(x)|^2 dx \leq \int |\phi(x)|^{p+1} dx \leq c \left( \int |\nabla \phi(x)|^2 dx \right)^{\frac{n(p-1)}{4}} \left( \int |\phi(x)|^2 dx \right)^{\frac{n+2-(n-2)p}{4}}. \tag{4.5}
\]
It is obvious that \( \frac{n+2-(n-2)p}{4} = 1 \) if and only if \( n = 2 \). Then by (4.5) we see
\[
\left( \int |\nabla \phi(x)|^2 dx \right)^{\frac{n(p-1)}{4}} \left( \int |\phi(x)|^2 dx \right)^{\frac{n+2-(n-2)p}{4}} \geq c > 0, \tag{4.6}
\]
which does not show that \( \int |\nabla \phi(x)|^2 dx > C > 0 \) when \( n > 2 \), that is, we cannot determine that \( \phi \neq 0 \).

In another words, for the case \( n \geq 3 \) we need to construct some other variational problem.
Acknowledgments

The author wishes to express his deep gratitude to Professor Hitoshi Kitada for his constant encouragement and kind guidance. Thanks also to the referees for their comments and careful reading the manuscript.

The study is supported by Japanese Government Scholarship.

References