Explosion Time of Second-Order Ito Processes

K. Narita

Department of Mathematics, Faculty of Technology, Kanagawa University, 3-27 Rokkakubashi, Kanagawa-ku, Yokohama 221, Japan

Submitted by Harold Kushner

In this paper a sufficient condition is given for the existence of the global solution as is a sufficient condition for the non-existence of the global solution of the second-order stochastic differential equation with the random disturbance of the so-called white noise. © 1984 Academic Press, Inc.

1. INTRODUCTION

Let us consider the second-order stochastic differential equation which is written formally as an ordinary differential equation

$$\ddot{y} + g(t, y, \dot{y}) \dot{y} + a(t)f(y) = h(t, y, \dot{y}) \dot{w}$$

(1)

with the white noise $\dot{w}$, where by $\dot{}$ we mean the symbolic derivative $d/dt$. We are interested in whether Eq. (1) has a global solution or not. For this purpose we treat the two-dimensional non-linear Ito equation and investigate the explosion criteria for the solution of the stochastic differential equation. In the following we introduce the precise formulation of the problem.

Let $(\Omega, F, P)$ be a probability space with an increasing family $\{F_t; t \geq 0\}$ of sub-$\sigma$-algebras of $F$ and let $w(t)$ be a one-dimensional Brownian motion process adapted to $F_t$. Then we consider the system of the stochastic differential equations:

$$dX_1(t) = X_2(t) \, dt,$$

$$dX_2(t) = \{-g(t, X_1(t), X_2(t)) X_2(t) - a(t)f(X_1(t))\} \, dt$$

$$+ h(t, X_1(t), X_2(t)) \, dw(t).$$

(2)

Throughout this paper we assume the following conditions;

$$a: [0, \infty) \rightarrow (-\infty, \infty)$$

is continuously differentiable,

$$f: (-\infty, \infty) \rightarrow (-\infty, \infty)$$

is continuously differentiable,

$g$ and $h$: $[0, \infty) \times (-\infty, \infty) \times (-\infty, \infty) \rightarrow (-\infty, \infty)$ have continuous first partials.
The system (2) is one of the formulations such that $X_1(t)$ may correspond to the response of the oscillator (1) with the restoring force $f$ and the damping $g$ to the formal white noise $\dot{w}$. In general, the solution $X(t) = (X_1(t), X_2(t))$ of (2) with the initial condition $X(t_0) = x_0 \in \mathbb{R}^2$ ($t_0 \geq 0$) is defined up to the random time $e(t_0, x_0)$, where $e(t_0, x_0) = \lim_{t \to \infty} e_n(t_0, x_0)$ and $e_n(t_0, x_0) = n \wedge \inf\{t; |X(t)| \geq n\}$ (here and hereafter $\mathbb{R}^* = (-\infty, \infty) \times (-\infty, \infty)$ and $a \wedge b$ stands for the smaller of $a$ and $b$). This random time $e(t_0, x_0)$ is called the explosion time of the solution $X(t)$ of (2) with the initial condition $X(t_0) = x_0$. The following remark enables us to understand the meaning of the explosion time (see [4, Sect. 2, Chap. IV; 5, Sect. 4.3; and 6, Sect. 3]).

**Remark.** $\lim_{t \to e(t_0, x_0)} |X(t)| = \infty$ for $e(t_0, x_0) < \infty$, almost surely. Hence, if $e(t_0, x_0) < \infty$, then the explosion occurs.

In particular, the process $X_1(t)$ of (2) is called the second-order Ito process. When $h(t, x_1, x_2) \neq 0$ for each $t$, $x_1$, and $x_2$, we know by Goldstein [3, Corollary 4.4] that $X_2(t)$ is of unbounded variation in every finite interval with probability one, up to the explosion time.

Therefore the existence problem of the global solution of (1) can be restated in terms of the explosion problem of the solution of (2). Namely, in Section 2 we show a sufficient condition for the non-occurrence of the explosion (Theorem 1), and in Section 3 we show a sufficient condition for the occurrence of the explosion (Theorem 2).

Throughout this paper, we shall use the differential generator

$$L = \frac{\partial}{\partial t} + x_2 \frac{\partial}{\partial x_1} - \{g(t, x_1, x_2) x_2 + a(t)f(x_1)\} \frac{\partial}{\partial x_2}$$

$$+ \frac{1}{2} h^2(t, x_1, x_2) \frac{\partial^2}{\partial x_2^2}$$

(3)

associated with the system (2). Also, we shall use the function

$$V(t, x) = a(t) F(x_1) + x_2^2/2$$

(4)

where $t \geq 0$, $x = (x_1, x_2) \in \mathbb{R}^2$ and $F(x_1) = \int_0^{x_1} f(s) \, ds$. Then it is easy to see that

$$LV(t, x) = a'(t) F(x_1) - g(t, x_1, x_2) x_2 + h^2(t, x_1, x_2)/2$$

(5)

for $t \geq 0$ and $x = (x_1, x_2) \in \mathbb{R}^2$. Moreover, let $U(t, x)$ be a scalar function which is twice continuously differentiable with respect to $x \in \mathbb{R}^2$ and once
with respect to \( t \geq 0 \), and let \( K(u) \) be a twice continuously differentiable function on \( u \in (-\infty, \infty) \). Then we notice that

\[
L K(U(t, x)) = (L U(t, x)) K'(U(t, x)) \\
+ \frac{1}{2} h^2(t, x_1, x_2) \left( \frac{\partial U(t, x)}{\partial x_2} \right)^2 K''(U(t, x))
\]

(6)

for \( t \geq 0 \) and \( x = (x_1, x_2) \in \mathbb{R}^2 \). We shall use (5) and (6) in the proof of our theorems.

2. NON-EXPLOSION CRITERION

In this section we give a sufficient condition for the infinite explosion time with probability one. At the beginning of the discussion we consider the deterministic case. For many special equations such as the Liénard equation

\[
\dot{y} + g(y, \dot{y}) \dot{y} + f(y) = p(t)
\]

it is known by Bushaw [1, p. 36] that all solutions are continuabel to \( t = \infty \) whenever \( g, f \) and \( p \) are continuous, \( g \geq 0 \) and \( yf(y) > 0 \) for \( y \neq 0 \).

For the stochastic case of (1), consider the system (2) and let \( e(t, x_0) \) be the explosion time of the solution \( X(t) \) of (2) with the initial condition \( X(t_0) = x_0 \in \mathbb{R}^2 \). Then we obtain the next theorem.

**Theorem 1.** Suppose that the following conditions hold:

(i) \( a(t) > 0 \) for all \( t \geq 0 \),

(ii) \( x_1 f(x_1) > 0 \) for all \( x_1 \neq 0 \),

(iii) \( g(t, x_1, x_2) \geq 0 \) for all \( t \geq 0 \) and \( (x_1^2 + x_2^2)^{1/2} \geq r \) with a constant \( r > 0 \),

(iv) \( c \leq h^2(t, x_1, x_2) \leq k(t)(1 + F(x_1) + x_1^2) \) for all \( t \geq 0 \) and \( (x_1, x_2) \in \mathbb{R}^2 \) with a constant \( c > 0 \) and a continuous function \( k(t) \), where

\[
F(x_1) = \int_0^{x_1} f(s) \, ds.
\]

Then, \( P(e(t_0, x_0) = \infty) = 0 \) for all \( t_0 \geq 0 \) and \( x_0 \in \mathbb{R}^2 \).

**Proof.** Let \( X(t) = (X_1(t), X_2(t)) \) be the solution of (2) with the initial condition \( X(t_0) = x_0 \in \mathbb{R}^2 \) and assume that \( P(e(t_0, x_0) < \infty) > 0 \) for some \( (t_0, x_0) \), to the contrary. For notational simplicity we put \( e = e(t_0, x_0) \). In the following we take a sample such that \( e < \infty \). Then we notice by the Remark in Section 1 that \( |X(e-)| = \infty \) for such a sample and hence put \( \rho = \sup\{t; |X(t)| = r\} \).
Let $L$ and $V(t, x)$ be the differential generator and the function defined by (3) and (4), respectively. Then it follows from the conditions that

$$LV(t, x) \leq a'(t) F(x_1) + \frac{1}{2} k(t)(1 + F(x_1) + x_2^2)$$

$$\leq \frac{|a'(t)|}{a(t)} V(t, x) + \frac{1}{2} k(t) \left( 1 + \frac{1}{a(t)} V(t, x) + 2V(t, x) \right)$$

$$= A(t) V(t, x) + k(t)/2$$

for all $t \geq 0$ and $|x| = (x_1^2 + x_2^2)^{1/2} \geq r$, where

$$A(t) = \frac{|a'(t)|}{a(t)} + \frac{1}{2} \frac{k(t)}{a(t)} + k(t).$$

Now, set

$$U(t, x) = \exp \left\{ - \int_0^t A(s) ds \right\} V(t, x).$$

Then the above inequality implies that

$$LU(t, x) \leq \frac{1}{2} k(t) \exp \left\{ - \int_0^t A(s) ds \right\}$$

for all $t \geq 0$ and $|x| = (x_1^2 + x_2^2)^{1/2} \geq r$. So, we obtain by Ito's formula concerning stochastic differentials that

$$U(t, X(t)) \leq U(\rho, X(\rho)) + \frac{1}{2} \int_0^t k(s) \exp \left\{ - \int_0^s A(u) du \right\} ds$$

$$+ M(t) - M(\rho)$$

for all $\rho \leq t < e$, where

$$M(t) = \int_{t_0}^t \exp \left\{ - \int_0^s A(u) du \right\} X_2(s) h(s, X_1(s), X_2(s)) dw(s).$$

By the time substitution rule (see McKean [5, Sect. 2.5]), we know that $M(t) = z(\phi(t))$, where $z$ is a new Brownian motion process run with the clock

$$\phi(t) = \int_{t_0}^t \exp \left\{ - 2 \int_0^s A(u) du \right\} X_2^2(s) h^2(s, X_1(s), X_2(s)) ds.$$
and hence

\[ U(t, X(t)) \leq U(\rho, X(\rho)) + \frac{1}{2} \int_{\rho}^{t} k(s) \exp \left\{ -\int_{0}^{s} A(u) \, du \right\} \, ds \]

\[ + z(\phi(t)) - z(\phi(\rho)) \]

for all \( \rho \leq t < e \). Namely, we get

\[ V(t, x) \leq \exp \left\{ \int_{0}^{t} A(s) \, ds \right\} \left[ U(\rho, X(\rho)) \right. \]

\[ + \frac{1}{2} \int_{\rho}^{t} k(s) \exp \left\{ -\int_{0}^{s} A(u) \, du \right\} \, ds \]

\[ \left. + z(\phi(t)) - z(\phi(\rho)) \right] \quad (7) \]

for all \( \rho \leq t < e \). Since \( dX_1(t) = X_2(t) \, dt \) and since \( x_2^2 \leq c^{-1}x_2^2 h^2(t, x_1, x_2) \) by (iv), it follows from Schwartz's inequality that

\[ (X_1(t) - X_1(\rho))^2 \]

\[ \leq (t - \rho) \int_{\rho}^{t} X_2^2(s) \, ds \]

\[ \leq c^{-1}(t - \rho) \int_{\rho}^{t} X_2^2(s) \, h^2(s, X_1(s), X_2(s)) \, ds \]

\[ \leq c^{-1}(t - \rho) \exp \left\{ 2 \int_{0}^{t} A(s) \, ds \right\} \]

\[ \times \int_{\rho}^{t} \exp \left\{ -2 \int_{0}^{s} A(u) \, du \right\} X_2^2(s) \, h^2(s, X_1(s), X_2(s)) \, ds \]

and hence

\[ (X_1(t) - X_1(\rho))^2 \leq c^{-1}(t - \rho) \exp \left\{ 2 \int_{0}^{t} A(u) \, du \right\} (\phi(t) - \phi(\rho)) \quad (8) \]

for all \( \rho \leq t < e \). On the other hand, we have by the definition of \( V(t, x) \) that

\[ X_2^2(t)/2 \leq V(t, X(t)) \quad (9) \]

for all \( \rho \leq t < e \). We discuss this according to each of two cases:

\[ \phi(e) < \infty, \quad \phi(e) = \infty. \]
First, let $\phi(e) < \infty$. Then it follows from (7), (8) and (9) that both $X_1^2(t)$ and $X_2^2(t)$ stay bounded as $t$ tends to $e$, which is a contradiction since $|X(e^-)| = \infty$.

Next, let $\phi(e) = \infty$. Then we obtain by (7) that

$$0 \leq \liminf_{t \to e} V(t, X(t)) \leq \exp \left\{ \int_0^e A(s) \, ds \right\} \left[ U(\rho, X(\rho)) \right.$$

$$\left. + \frac{1}{2} \int_0^e k(s) \, ds \exp \left\{ - \int_0^s A(u) \, du \right\} ds \right.$$

$$\left. + \liminf_{t \to e} z(\phi(t)) - z(\phi(\rho)) \right]$$

$$= -\infty,$$

which is also absurd. Thus we get that $P(e(t_0, x_0) = \infty) = 1$ for all $t_0 \geq 0$ and $x_0 \in \mathbb{R}^2$, and hence the proof is completed.

Theorem 1 is a generalization of McKean's result [5, Problem 5, Sect. 4.51, where the system of the stochastic differential equations

$$dX_1(t) = X_2(t) \, dt,$$

$$dX_2(t) = -f(X_1(t)) \, dt + dw(t)$$

is treated under the assumption that $x_i f(x_i) > 0$ for $x_i \neq 0$.

3. EXPLOSION CRITERION

In this section we give a sufficient condition for the finite explosion time by using the restriction on the growth of the coefficient $f(y)$ of (2). At the beginning of the discussion we consider the deterministic non-linear second-order differential equation

$$\ddot{y} + a(t)f(y) = 0 \tag{10}$$

where $a$ is continuous, $f$ is continuous and $y f(y) > 0$ for $y \neq 0$, and suppose that $a(t) < 0$ on an interval $t_0 \leq t < t_1$ with $a(t_1) \leq 0$. Then it follows from the theorem of Burton and Grimmer [2] that (10) has a solution $y(t)$ defined for $t = t_0$ satisfying $|\lim_{t \to T} |y(t)| = \infty$ for some $T \in (t_0, t_1]$ if and only if either

$$\int_0^\infty (1 + F(u))^{-1/2} \, du < \infty \tag{11}$$
or
\[
\int_{0}^{-\infty} (1 + F(u))^{-1/2} \, du > -\infty
\]
(12)
holds, where \( F(u) = \int_{0}^{u} f(s) \, ds \).

For the stochastic case of (1), consider the system (2), and let \( e(t_0, x_0) \) be the explosion time of the solution \( X(t) \) of (2) with the initial condition \( X(t_0) = x_0 \in \mathbb{R}^2 \). Then we show that the convergence of the integrals of (11) and (12) plays a role on the occurrence of the explosion.

**Theorem 2.** Suppose that the following conditions hold:

(i) \( a(t) \leq -m \) and \( a'(t) \geq 0 \) for all \( t \geq 0 \) with a constant \( m > 0 \),

(ii) \( x_1 f(x_1) > 0 \) for all \( x_1 \neq 0 \),

(iii) \( g(t, x_1, x_2) \leq -\varepsilon_0 h^2(t, x_1, x_2) \) for all \( t \geq 0 \) and \( (x_1, x_2) \in \mathbb{R}^2 \) with a constant \( \varepsilon_0 > 0 \),

(iv) \( h^2(t, x_1, x_2) \geq k(t) \) for all \( t \geq 0 \) and \( (x_1, x_2) \in \mathbb{R}^2 \) with a non-negative and continuous function \( k(t) \) satisfying
\[
\int_{0}^{\infty} k(t) \, dt = \infty.
\]

Further, suppose that both (11) and (12) hold.

Then, \( P(e(t_0, x_0) < \infty) > 0 \) for all \( t_0 \geq 0 \) and \( x_0 \in \mathbb{R}^2 \).

**Proof:** Under the assumption on the coefficients, suppose that both (11) and (12) hold and let \( e(t_0, x_0) \) be the explosion time of the solution \( X(t) = (X_1(t), X_2(t)) \) of (2) with the initial condition \( X(t_0) = x_0 \in \mathbb{R}^2 \). To the contrary, in the following we assume that \( P(e(t_0, x_0) = \infty) = 1 \) for some \((t_0, x_0)\). Now choose positive numbers \( p \) and \( q \) so that
\[
0 < p \leq 2\varepsilon_0 \quad \text{and} \quad 0 < q \leq \frac{1}{2}
\]
and then put \( K(u) = \exp(\mu u) \) for \( u \in (-\infty, \infty) \) and
\[
U(t, x) = -V(t, x) + q \int_{t_0}^{t} k(s) \, ds
\]
for \( t \geq t_0 \) and \( x \in \mathbb{R}^2 \) with the function \( V(t, x) \) defined by (4). Then it follows from (5) and (6) that
\[
LK(U(t, x)) = [-LV(t, x) + qk(t) + ph^2(t, x_1, x_2)x_2^2/2] \, pK(U(t, x))
\]
\[
= [-a'(t) F(x_1) + \{g(t, x_1, x_2) + ph^2(t, x_1, x_2)/2\} x_2^2
\]
\[
- \{h^2(t, x_1, x_2)/2 - qk(t)\}] \, pK(U(t, x))
\]
for \( t \geq t_0 \) and \( x = (x_1, x_2) \in \mathbb{R}^2 \). Notice the conditions of the coefficients and the choice of the numbers \( p \) and \( q \). Then we see that

\[
-a'(t) F(x_1) + \{g(t, x_1, x_2) + ph^2(t, x_1, x_2)\}/2 \leq 0
\]

and hence

\[
LK(U(t, x)) \leq 0
\] (13)

for all \( t \geq t_0 \) and \( x = (x_1, x_2) \in \mathbb{R}^2 \). Since \( X(t) \) is defined for all \( t \geq t_0 \) almost surely by the assumption, we notice that \( \lim_{n \to \infty} e_n = \infty \) almost surely, where \( e_n = n \wedge \inf\{t; |X(t)| \geq n\} \). Then, Ito's formula concerning stochastic differentials and (13) imply that for arbitrarily fixed \( n \)

\[
\{K(U(t \wedge e_n, X(t \wedge e_n))); t \geq t_0\}
\]

is a positive super-martingale. Therefore, the super-martingale inequality yields that for any number \( N > 0 \)

\[
P(\sup_{t_0 < t \leq e_n} K(U(t, X(t))) > N) \leq N^{-1} K(U(t_0, x_0)).
\]

Let \( n \) tend to infinity in the above equation and then notice that \( e_n \uparrow \infty \) almost surely as \( n \uparrow \infty \). Then we obtain that

\[
P(\sup_{t_0 < t} K(U(t, X(t))) > N) \leq N^{-1} K(U(t_0, x_0)),
\]

from which follows

\[
P(\sup_{t_0 < t} K(U(t, X(t))) < \infty) = 1.
\]

Since \( K(u) = \exp(pu) \) with \( p > 0 \), it must be that

\[
P(\sup_{t_0 < t} U(t, X(t)) < \infty) = 1.
\]

Accordingly, we get that

\[
U(t, X(t)) = -a(t) F(X_1(t)) - X_2^2(t)/2 + q \int_{t_0}^{t} k(s) \, ds
\]

\[
\leq \alpha
\]
for all \( t \geq t_0 \), where \( a = \sup_{t_0 \leq t} U(t, X(t)) \). Since \( mF(x_1) \leq -a(t) F(x_1) \) by (i) and (ii), the above inequality implies that

\[
X^2(t)/2 - mF(X_1(t)) \geq q \int_{t_0}^{t} k(s) \, ds - a
\]

for all \( t \geq t_0 \). Moreover, since \( \int_0^{\infty} k(t) \, dt = \infty \) by the condition (iv), for any number \( \beta > 0 \) we can find a time \( t' > t_0 \) such that \( q \int_{t_0}^{t} k(s) \, ds - a > \beta \) for all \( t \geq t' \). In the following we fix such \( \beta \) and \( t' \) and consider the time interval \( t' \leq t < \infty \). Then it is easy to see that

\[
X^2(t) \geq 2\beta + 2mF(X_1(t))
\]

for all \( t \geq t' \), from which follows

\[
(\delta + F(X_1(t)))^{-1/2} X_1(t) \geq (2m)^{1/2}
\]
or

\[
(\delta + F(X_1(t)))^{-1/2} X_1(t) \leq - (2m)^{1/2}
\]

for all \( t \geq t' \), where \( \delta = \beta/m \) (>0). Integrate both sides of the above inequalities from \( t' \) to \( t \) (>\( t' \)), and then consider that \( dX_1(t) = X_1(t) \, dt \) and that \( dY(X_1(t)) = Y'(X_1(t)) \, dX_1(t) \) for \( Y(x_1) = \int_{0}^{x_1} (\delta + F(u))^{-1/2} \, du \). Then we obtain that

\[
\int_{X_1(t')}^{X_1(t)} (\delta + F(u))^{-1/2} \, du \geq (2m)^{1/2} (t - t')
\]
or

\[
\int_{X_1(t')}^{X_1(t)} (\delta + F(u))^{-1/2} \, du \leq - (2m)^{1/2} (t - t')
\]

for all \( t \geq t' \), and hence

\[
\int_{X_1(t')}^{\infty} (\delta + F(u))^{-1/2} \, du \geq (2m)^{1/2} (t - t') \quad (14)
\]
or

\[
\int_{X_1(t')}^{-\infty} (\delta + F(u))^{-1/2} \, du \leq - (2m)^{1/2} (t - t') \quad (15)
\]

for all \( t \geq t' \). Notice that

\[
(\delta + F(u))^{-1/2} \leq \delta^{-1/2}(1 + F(u))^{-1/2} \quad \text{for } 0 < \delta \leq 1
\]
and that

\[(\delta + F(u))^{-1/2} \leq (1 + F(u))^{-1/2} \quad \text{for} \quad \delta > 1.\]

Then, combining these facts with (14) and (15) and then letting \( t \) tend to infinity in (14) and (15), we obtain that

\[
\int_{\mathcal{X}_1(t')}^{\infty} (1 + F(u))^{-1/2} \, du = \infty
\]

or

\[
\int_{\mathcal{X}_1(t')}^{-\infty} (1 + F(u))^{-1/2} \, du = -\infty
\]

respectively, which is a contradiction. Hence the proof is completed.

ACKNOWLEDGMENT

The author would like to thank Professor M. Motoo for valuable suggestions on the explosion problem.

REFERENCES

1. D. W. Bushaw, "The Differential Equation \( x'' + g(x, x') + h(x) = e(t) \)," Terminal Report on Contract AF 29 (600)-1003, Holloman Air Force Base, New Mexico, 1958.