The Maximum Principle and the Existence of Principal Eigenvalues for Some Linear Weighted Boundary Value Problems

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In this work we deal with the problem of the existence and uniqueness of principal eigenvalues for some linear weighted boundary value problems associated to a general second order uniformly elliptic operator. For a large class of sign definite weights, we characterize whether the boundary value problem admits a principal eigenvalue or not.

1. Introduction

In this work we study the eigenvalue problem

\[ L(x, D) \varphi = \lambda m(x) \varphi, \quad \text{in } \Omega, \quad \varphi|_{\partial \Omega} = 0, \]  

(1.1)

where \( \Omega \subset \mathbb{R}^N, N \geq 1 \), is a bounded domain with boundary \( \partial \Omega \) of class \( C^{2,\alpha} \), for some \( \alpha \in (0,1) \), \( L(x, D) \) is a strongly uniformly elliptic differential operator in \( \Omega \) of the form

\[ L(x, D) = - \sum_{i,j=1}^N \alpha_{ij}(x) D_i D_j + \sum_{i=1}^N \alpha_i(x) D_i + \alpha_0(x), \]  

(1.2)

with coefficients \( \alpha_{ij}, \alpha_i, \alpha_0 \in C^{\alpha}(\Omega) \), \( i, j \in \{1, \ldots, N\} \), and \( m(x) \in C(\overline{\Omega}) \) is a sign definite weight function. By strongly uniformly elliptic we mean that there exists a positive constant \( \lambda > 0 \) such that

\[ \sum_{i,j=1}^N \alpha_{ij}(x) \xi_i \xi_j \geq \lambda |\xi|^2 \]  

(1.3)

for all \( x \in \Omega \) and \( \xi = (\xi_1, \ldots, \xi_N) \in \mathbb{R}^N \). Without loss of generality we can assume \( \alpha_{ij} = \alpha_{ji} \) for all \( i, j \).
We are interested in the existence of principal eigenvalues of (1.1). By a principal eigenvalue we mean a value of \( \lambda \in \mathbb{R} \) for which (1.1) admits a positive solution \( \varphi \). Let \( \sigma_0^0(L) \) denote the principal eigenvalue of \( L \) in \( \Omega \) subject to homogeneous Dirichlet boundary conditions. In case \( \sigma_0^0(L) > 0 \) the existence of a principal eigenvalue of (1.1) was first shown by Manes and Micheletti [21] when \( L \) is selfadjoint. Then, Hess and Kato [15] extended the theorem of Manes and Micheletti to cover the case when \( L \) is not necessarily selfadjoint. Independently, Brown and Lin [6] obtained the theorem when \( L = -\Delta \). Basically, the following is known: If \( m \) does not change sign, then (1.1) admits one principal eigenvalue. If \( m \) changes sign, then (1.1) admits two principal eigenvalues; one negative and the other positive. The proofs of Brown and Lin and Manes and Micheletti are based on the variational characterization of the principal eigenvalue; the proof of Hess and Kato uses Krein-Rutman’s theorem. In [19] we found some sufficient conditions for the existence of a principal eigenvalue without assuming \( \sigma_0^0(L) > 0 \). This paper is a natural continuation of [19] partially motivated by a recent work by Berestycki, Nirenberg and Varadhan [5], where it was shown the existence of a principal eigenvalue for a general elliptic operator \( L \) in a general domain \( \Omega \). When the coefficients of \( L \) and \( \Omega \) are smooth we can estimate how small has to be the Lebesgue measure of the domain \( |\Omega| \) so that \( \sigma_0^0(L) > 0 \). Such estimate is a natural extension of Faber–Krahn inequality, [11, 17], which we use to get some simple explicit conditions in terms of the coefficients of \( L \) and the weight function \( m \) so that (1.1) admits a principal eigenvalue.

We now describe some of the results of this paper. It is very easy to see that if \( m \) is positive everywhere and bounded away from zero then (1.1) possesses a unique principal eigenvalue. By changing the signs of \( \lambda \) and \( m \) it is clear that the same happens if \( m \) is everywhere negative and bounded away from zero. However, things are far from simple when \( m \) vanishes somewhere, even when \( m \) has definite sign. Suppose \( m(x) \geq 0 \) for all \( x \in \Omega \), \( m \neq 0 \). Then, we obtain two types of results accordingly to the size of the region where \( m \) vanishes, say

\[ E := \{ x \in \Omega : m(x) = 0 \}. \]

If \( |E| = 0 \), then it follows from Theorem 6.4 that

\[ \lim_{\lambda \downarrow -\infty} \sigma_0^0[L - \lambda m] = \infty \]

and this implies that (1.1) admits a unique principal eigenvalue, because the mapping \( \lambda \rightarrow \sigma_0^0[L - \lambda m] \) is analytic, decreasing and

\[ \lim_{\lambda \uparrow \infty} \sigma_0^0[L - \lambda m] = -\infty. \]
On the contrary, if $\mathcal{C}$ has non-empty interior then the situation may change drastically as the following example illustrating the results of Section 6 shows. Assume $m$ is positive in some subdomain $\mathcal{O}_p$ of $\mathcal{O}$ such that $\mathcal{O}_p \subset \mathcal{O}$ with boundary sufficiently regular, and $\mathcal{C} = \mathcal{O} \setminus \mathcal{O}_p$. Suppose in addition that $\mathcal{C}$ is connected. This assumption excludes the one-dimensional problem, but all the results can be adapted to include also that case. Then,

$$\lim_{\lambda \to -\infty} \sigma_\lambda^n [L - \lambda m] = \sigma^n[L].$$

In particular, problem (1.1) has a principal eigenvalue if and only if

$$\sigma^n[L] > 0.$$  

Therefore, for sign definite weight functions the existence of a principal eigenvalue depends basically on the size and the shape of the region where the weight vanishes. If $\sigma^n[L] < 0$ and $\mathcal{C}$ is sufficiently close to $\mathcal{O}$ so that $\sigma^n[L] < 0$ then (1.1) does not admit a principal eigenvalue. On the contrary, if $|\mathcal{C}|$ is sufficiently small then $\sigma^n[L] > 0$ and hence (1.1) admits a unique principal eigenvalue, even if $\sigma^n[L] < 0$. We point out that these results do not depend on how large is $m$. In Section 6 we obtain general versions of these results and apply them to get some sufficient conditions for the existence of principal eigenvalues when $m$ is sign indefinite. It is very important to know how small has to be $|\mathcal{C}|$ so that (1.1) admits a principal eigenvalue, i.e., so that $\sigma^n[L] > 0$. This is why we are interested in finding out lower estimates of principal eigenvalues in terms of the Lebesgue measure of the support domain. These estimates are given in Section 5, where we obtain a generalization of Faber-Krahn inequality.

To prove the results of Section 6 we use the continuous dependence of the principal eigenvalue with respect to the domain and the characterization of the maximum principle in terms of the existence of a strict positive supersolution. Courant and Hilbert [7] observed that the continuous dependence with respect to the domain may fail when dealing with Neumann boundary conditions but it is true for selfadjoint operators under homogeneous Dirichlet boundary conditions. Additional information can be found in the papers of Arrieta, Hale and Han [2] and Babuska and Vyborny [3]. The continuous dependence of the principal eigenvalue with respect to the domain for the Dirichlet problem when dealing with a non-selfadjoint perturbation of the Laplacian was shown by Dancer in Theorem 1 of [10]. Nevertheless, we did not find a proof of the continuous dependence of the principal eigenpair for general operators $L$ (not necessarily selfadjoint), even when the coefficients and the domain are smooth. So, we include a proof of this result, which seems to be new, in Section 4. In Section 2 we give a short self-contained proof of the existence and uniqueness of $\sigma^n[L]$. 

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as well as the characterization of the strong maximum principle in terms of the positivity of the principal eigenvalue and in terms of the existence of a positive strict supersolution. Although we could adapt the general results of Berestycki, Nirenberg and Varadhan [5] to our situation here to get the characterization of the maximum principle in terms of the principle eigenvalue, our self-contained proof of this theorem provides us with a further result. Namely, the characterization of the maximum principle by means of the existence of a positive strict supersolution (which may vanish at the boundary). This result is a substantial improvement of the classical result by Protter and Weinberger [23] which says that if there is a positive supersolution that is positive on the boundary, then the strong maximum principle holds. In Section 3 we use these characterizations to give some short self-contained proofs of the main properties of the principal eigenvalue. Since some of these properties are not very well known, for instance the variational characterization (3.1) was not given in the book of Hess [14] and the proof of the concavity of the principal eigenvalue given by Beresticky, Niremberg and Varadhan in [5] is uncompleted, we think that it may be of interest for the reader to get them collected in a self-contained section. The continuous dependence of the principal eigenvalue with respect to the domain is not immediate in the general framework of [5]. So, it is not clear how to get the results of Section 6 for non-smooth coefficients and domains.

2. The Maximum Principle and the Principal Eigenvalue

The following generalized version of the maximum principle holds (cf. Theorem 2 of Walter [24]).

**Theorem 2.1.** Suppose \( h \in C(\bar{\Omega}) \cap C^2(\Omega) \) satisfies

\[
L(x, D) h \geq 0 \quad \text{and} \quad h(x) > 0 \quad \text{for all} \quad x \in \Omega.
\]

Then, for any \( u \in C^1(\bar{\Omega}) \cap C^2(\Omega) \) such that

\[
L(x, D) u \geq 0 \quad \text{in} \quad \Omega, \quad u \geq 0 \quad \text{on} \quad \partial \Omega,
\]

some of the following options occurs: Either \( u = \beta h \) in \( \Omega \) for some \( \beta < 0 \), or \( u \equiv 0 \) in \( \Omega \), or \( u(x) > 0 \) for all \( x \in \Omega \). If the last option occurs then \( (\partial u / \partial n)(x_0) < 0 \) for all \( x_0 \in \partial \Omega \) such that \( u(x_0) = 0 \), where \( n \) is the outward unit normal to \( \Omega \) at \( x_0 \).

This version drops the assumption \( h > 0 \) on \( \partial \Omega \) of Protter and Weinberger [23]. In the sequel given \( f, g \in C(\Omega) \) we shall write \( f > g \) if \( f(x) \geq g(x) \) for all \( x \in \Omega \) and there exists \( x_0 \in \Omega \) such that \( f(x_0) > g(x_0) \).
Definition 2.2. A function \( h \in C(\Omega) \cap C^2(\Omega) \) is said to be a strict positive supersolution of \( L(x, D) \) if \( h(x) > 0 \) for all \( x \in \Omega \) and either \( L(x, D)h > 0 \) in \( \Omega \), or \( L(x, D)h = 0 \) in \( \Omega \) and \( h > 0 \) on \( \partial \Omega \).

Remark 2.3. If \( K > 0 \) is large enough so that \( z_0 + K > 0 \) in \( \Omega \), then any positive constant is a strict positive supersolution of \( L + K \). Due to Theorem 2.1 if \( h \) is a strict positive supersolution of \( L(x, D) \) then \( (\partial h/\partial n)(x_0) < 0 \) for all \( x_0 \in \partial \Omega \) such that \( h(x_0) = 0 \).

Corollary 2.4. Suppose \( L(x, D) \) admits a strict positive supersolution \( h \). Then, the following assertions are true:

(i) If \( u \in C^1(\Omega) \cap C^2(\Omega) \) satisfies

\[
L(x, D)u \geq 0 \quad \text{in} \, \Omega, \quad u \geq 0 \quad \text{on} \, \partial \Omega,
\]

with some of these inequalities strict, then \( u(x) > 0 \) for all \( x \in \Omega \) and \( (\partial u/\partial n)(x_0) < 0 \) for all \( x_0 \in \partial \Omega \) such that \( u(x_0) = 0 \).

(ii) \( u \equiv 0 \) is the unique classical solution to

\[
L(x, D)u = 0 \quad \text{in} \, \Omega, \quad u = 0 \quad \text{on} \, \partial \Omega.
\]

Proof. (i) Suppose \( u \) satisfies (2.1) with some of the inequalities strict. Then, \( u \not\equiv 0 \). Moreover, if \( u = \beta h \) with \( \beta < 0 \) then

\[
L(x, D)u = \beta L(x, D)h \leq 0
\]

and hence

\[
L(x, D)u = L(x, D)h = 0.
\]

Thus, since \( h \) is a strict positive supersolution, \( h > 0 \) on \( \partial \Omega \) and so \( u = \beta h < 0 \) on \( \partial \Omega \), which is impossible because (2.1) says that \( u \geq 0 \) on \( \partial \Omega \).

Therefore, it follows from Theorem 2.1 that \( u(x) > 0 \) for all \( x \in \Omega \) and that \( (\partial u/\partial n)(x_0) < 0 \) for all \( x_0 \in \partial \Omega \) such that \( u(x_0) = 0 \). The proof of (i) is completed.

(ii) Let \( u \in C^1(\Omega) \cap C^2(\Omega) \) be a solution of (2.2). Since \( h \) is a strict positive supersolution, \( u \not\equiv \beta h \) for any \( \beta < 0 \). Indeed, if \( u = \beta h \) for some \( \beta < 0 \) then \( 0 = Lu = \beta Lh \) and so \( Lh = 0 \). Hence, \( h > 0 \) on \( \partial \Omega \) and \( u < 0 \) on \( \partial \Omega \), which contradicts the fact that \( u \) vanishes at the boundary. Therefore, it follows from Theorem 2.1 that either \( u \equiv 0 \) or \( u(x) > 0 \) for all \( x \in \Omega \). Moreover, if the second option occurs then \( (\partial u/\partial n)(x) < 0 \) for all \( x \in \partial \Omega \).

To show that \( u \equiv 0 \) we argue by contradiction. Suppose \( u(x) > 0 \) for all \( x \in \Omega \). Due to Remark 2.3 \( h - \delta u \geq 0 \) in \( \Omega \) for \( \delta > 0 \) sufficiently small. Let \( \delta_0 > 0 \) be the largest real number such that \( h - \delta_0 u \geq 0 \) in \( \Omega \). We have

\[
L(h - \delta_0 u) = Lh \geq 0 \quad \text{in} \, \Omega, \quad h - \delta_0 u = h \geq 0 \quad \text{on} \, \partial \Omega.
\]
Moreover, since $h$ is a strict positive supersolution some of these inequalities is strict. Thus, it follows from part (i) that $h(x) - \delta_x u(x) > 0$ for all $x \in \Omega$ and that $(\partial h/\partial n)(x_0) < \delta_x (\partial u/\partial n)(x_0)$ for all $x_0 \in \partial \Omega$ such that $h(x_0) = \delta_x u(x_0)$. This contradicts the definition of $\delta_x$. The proof is completed.

Due to this corollary if $L(x, D)$ admits a strict positive supersolution then for any $f \in C(\overline{\Omega})$ and $g \in C^1(\overline{\Omega})$ the linear boundary value problem

$$L(x, D) u = f \quad \text{in } \Omega, \quad u = g \quad \text{on } \partial \Omega,$$

has a unique classical solution $u \in C^1(\overline{\Omega}) \cap C^2(\Omega)$. Moreover, if $f \in C^1(\overline{\Omega})$ and $g \in C^{2+ \gamma}(\overline{\Omega})$ then $u \in C^{2+ \gamma}(\overline{\Omega})$ (cf. [12]). Furthermore, if $f \geq 0$ in $\Omega$ and $g \geq 0$ in $\partial \Omega$ with some of these inequalities strict then $u(x) > 0$ for all $x \in \Omega$ and $(\partial u/\partial n)(x) < 0$ for all $x \in \partial \Omega$ such that $g(x) = 0$, i.e., $L(x, D)$ satisfies the strong maximum principle.

Suppose $L$ admits a strict positive supersolution and let $L^{-1}$ denote the solution operator of

$$L(x, D) u = f \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial \Omega. \quad \text{(2.3)}$$

The operator $L^{-1}: C(\overline{\Omega}) \to C^1_0(\overline{\Omega})$ is bounded. Moreover, the inclusion mapping $J: C^1_0(\overline{\Omega}) \to C^1(\overline{\Omega})$ is compact and hence $L^{-1} J: C^1_0(\overline{\Omega}) \to C^1_0(\overline{\Omega})$ is a compact endomorphism of $X := C^1_0(\overline{\Omega})$. If we order $X$ by the cone of non-negative functions in $\Omega$, $P_0 := \{ u \in X : u \geq 0 \}$, then the interior of $P_X$, denoted by int $P_X$, is the set of functions $u \in X$ such that $u(x) > 0$ for all $x \in \Omega$ and $(\partial u/\partial n)(x) < 0$ for all $x \in \partial \Omega$. We have just seen that $L^{-1} J P_0 \setminus \{ 0 \} \subset \text{int } P_X$. In other words, $L^{-1} J$ is strongly positive. Therefore, it follows from the sharp version of Krein-Rutman’s theorem given by Amann in Theorem 3.2 of [1] that $\text{spr}(L^{-1} J)$ is the unique eigenvalue of $L^{-1} J$ to a positive eigenfunction $\varphi \in \text{int } P_X$. Moreover, it is algebraically simple and by elliptic regularity $\varphi \in C^{2+ \gamma}(\overline{\Omega})$. In the sequel we shall consider the spaces $U := C_0^{2+ \gamma}(\overline{\Omega})$ and $V := C^1(\overline{\Omega})$ ordered by their cones of non-negative functions $P_U$ and $P_V$, respectively, and given an ordered Banach space $(E, P)$ and $f, g \in E$, we write $f \geq g$ if $f - g \in P$, $f > g$ if $f - g \in P \setminus \{ 0 \}$, and $f \geq g$ if $f - g \in \text{int } P$.

Let $\mathscr{L}: U \to V$ denote the operator induced by $L(x, D)$ subject to homogeneous Dirichlet boundary conditions. We have just seen that if $L(x, D)$ admits a strict positive supersolution then

$$\sigma_H[\mathscr{L}] := \frac{1}{\text{spr}(L^{-1} J)} > 0$$
is the unique eigenvalue of $\mathcal{L}$ to a positive eigenfunction. Moreover, since $\text{spr}(L^{-1}J)$ is an algebraically simple eigenvalue of $L^{-1}J$ it follows easily that $\sigma_1^{\mathcal{L}}[\mathcal{L}]$ is a simple eigenvalue of $\mathcal{L}$.

For arbitrary $L(x, D)$, the positive constants are strict positive supersolutions of $L(x, D) + K$ for large $K > 0$ and hence

$$
\sigma_1^{\mathcal{L}}[\mathcal{L}] := \frac{1}{\text{spr}(L + K)^{-1}J} - K
$$

is the unique eigenvalue of $\mathcal{L}$ to a positive eigenfunction. Moreover, $\sigma_1^{\mathcal{L}}[\mathcal{L}]$ is a simple eigenvalue. Note that $\sigma_1^{\mathcal{L}}[\mathcal{L}] < 0$ is not excluded.

The eigenvalue $\sigma_1^{\mathcal{L}}[\mathcal{L}]$ is called the principal eigenvalue of $\mathcal{L}$. Its associated eigenfunction $\varphi \geq 0$ is known as the principal eigenfunction, unique up to multiplicative constants. The following result characterizes the maximum principle in terms of the sign of the principal eigenvalue.

**Theorem 2.5.** The following assertions are equivalent:

(i) $\sigma_1^{\mathcal{L}}[\mathcal{L}] > 0$;

(ii) $L$ admits a strict positive supersolution;

(iii) For any $u \in C^1(\bar{\Omega}) \cap C^2(\Omega)$ satisfying

$$
L(x, D) u \geq 0 \quad \text{in } \Omega, \quad u \geq 0 \quad \text{on } \partial \Omega,
$$

with some of these inequalities strict, it follows that $u(x) > 0$ for all $x \in \Omega$ and $(\partial u/\partial n)(x_0) < 0$ for all $x_0 \in \partial \Omega$ such that $u(x_0) = 0$. In other words, $\mathcal{L}$ satisfies the strong maximum principle.

**Proof.** If (i) is satisfied then the principal eigenfunction itself provides us with a strict positive supersolution. Thus, (i) implies (ii). Moreover, Corollary 2.4 shows that (ii) implies (iii). The fact that (iii) implies (i) follows from Krein-Rutman theorem. The proof is completed. 

Very recently, Berestycki, Nirenberg and Varadhan obtained a general version of the equivalence between (i) and (iii) of Theorem 2.5 for general operators on general bounded domains [5]. Our characterization of the maximum principle by means of (ii) is essential for the rest of this work.

For the sequel, if there is not ambiguity we shall write $L$, instead of $\mathcal{L}$.

### 3. Some Properties of the Principal Eigenvalue

As an immediate consequence of Theorem 2.5 the following min–max characterization of the principal eigenvalue holds.
Theorem 3.1. Let \( \mathcal{P} \) denote the set of functions \( \phi \in C^2(\Omega) \) such that \( \phi(x) > 0 \) for all \( x \in \Omega \). Then,

\[
\sigma_1^\Omega[L] = \sup_{\phi \in \mathcal{P}} \inf_{x \in \Omega} \frac{L\phi(x)}{\phi(x)}.
\]

Proof. Let \( \lambda < \sigma_1^\Omega[L] \) be. Then \( \sigma_1^\Omega[L - \lambda] > 0 \) and the unique solution of

\[(L - \lambda)\psi = 1 \text{ in } \Omega, \quad \psi|_{\partial \Omega} = 1,
\]
satisfies \( \psi(x) > 0 \) for all \( x \in \Omega \). In particular,

\[
\lambda < \inf_{x \in \Omega} \frac{L\psi(x)}{\psi(x)} \leq \sup_{\phi \in \mathcal{P}} \inf_{x \in \Omega} \frac{L\phi(x)}{\phi(x)}.
\]

Since this inequality holds for any \( \lambda < \sigma_1^\Omega[L] \) we find that

\[
\sigma_1^\Omega[L] \leq \sup_{\phi \in \mathcal{P}} \inf_{x \in \Omega} \frac{L\phi(x)}{\phi(x)}.
\]

To complete the proof we argue by contradiction. Suppose

\[
\sigma_1^\Omega[L] < \sup_{\phi \in \mathcal{P}} \inf_{x \in \Omega} \frac{L\phi(x)}{\phi(x)}.
\]

Then, there exist \( \varepsilon > 0 \) and \( \phi \in \mathcal{P} \) such that

\[
\sigma_1^\Omega[L] + \varepsilon < \inf_{x \in \Omega} \frac{L\phi(x)}{\phi(x)}
\]

and hence

\[(L - \sigma_1^\Omega[L] - \varepsilon)\phi > 0.
\]

Thus, \( \phi \) is a strict positive supersolution of \( L - \sigma_1^\Omega[L] - \varepsilon \) and due to Theorem 2.5

\[
\sigma_1^\Omega[L - \sigma_1^\Omega[L] - \varepsilon] > 0.
\]

This is impossible. The proof is completed. \( \blacksquare \)
**Proposition 3.2.** (i) Let \( p_1, p_2 \in C(\Omega) \) be such that \( p_1 < p_2 \). Then \( \sigma_1^0[L + p_1] < \sigma_1^0[L + p_2] \). (ii) The mapping \( p \rightarrow \sigma_1^0[L + p] \), from \( C(\Omega) \) into \( \mathbb{R} \), is continuous. (iii) If \( \Omega_0 \) is a proper subdomain of \( \Omega \) with \( \partial \Omega \) of class \( C^{2+\nu} \) then \( \sigma_1^0[\mathcal{L}] > \sigma_1^0[\mathcal{L}'] \).

**Proof.** (i) Let \( \varphi \geq 0 \) the principal eigenfunction associated to \( \sigma_1^0[L + p_1] \). Then

\[
L \varphi_1 + p_2 \varphi_1 > L \varphi_1 + p_1 \varphi_1 = \sigma_1^0[L + p_1] \varphi_1
\]

and hence \( \varphi_1 \) is a strict positive supersolution of \( L + p_2 - \sigma_1^0[L + p_1] \). Thus, due to Theorem 2.5 we find that \( \sigma_1^0[L + p_2 - \sigma_1^0[L + p_1]] > 0 \). This completes the proof of part (i).

(ii) Let \( p \in C(\Omega) \) be and consider a sequence \( p_n \in C(\Omega), n \geq 1 \), such that

\[
\lim_{n \to \infty} \| p_n - p \| = 0.
\]

Then, given \( \varepsilon > 0 \) there exists a natural number \( n_0 \geq 1 \) such that \( p - \varepsilon < p_n < p + \varepsilon \), for \( n \geq n_0 \), and hence it follows from part (i) that

\[
\sigma_1^0[L + p] - \varepsilon < \sigma_1^0[L + p_n] < \sigma_1^0[L + p] + \varepsilon
\]

for all \( n \geq n_0 \). This completes the proof of part (ii).

(iii) Let \( \Omega_0 \) be a proper subdomain of \( \Omega \). Let \( \varphi \) denote the principal eigenfunction corresponding to \( \sigma_1^0[L] \). Then \( L \varphi - \sigma_1^0[L] \varphi = 0 \) in \( \Omega_0 \) and \( \varphi > 0 \) on \( \partial \Omega_0 \). Thus, \( \varphi \) is a strict positive supersolution of \( L - \sigma_1^0[L] \) in \( \Omega_0 \) and it follows from Theorem 2.5 that \( \sigma_1^0[L - \sigma_1^0[L]] = \sigma_1^0[L] - \sigma_1^0[L] > 0 \). This completes the proof of part (iii).

We now show the concavity of \( \sigma_1^0[L + p] \) with respect to \( p \). This result was obtained by Kato [16]. Hess [14] gave a proof of the concavity of the principal eigenvalue for Neumann and Robbin boundary conditions; in 14 was not proven the concavity for the case of Dirichlet boundary conditions. The proof we include here is based upon the proof given by Berestycki, Nirenberg and Varadhan [5]. We point out that the proof of Proposition 2.1 in page 68 of [5] is incomplete; although, it may be completed with some of the calculations in pages 70, 71.

**Theorem 3.3.** The mapping \( p \rightarrow \sigma_1^0[L + p] \), from \( C(\Omega) \) into \( \mathbb{R} \), is concave; that is,

\[
\sigma_1^0[L +.tp_1 + (1-t)p_2] \geq t\sigma_1^0[L + p_1] + (1-t)\sigma_1^0[L + p_2]
\]

for all \( p_1, p_2 \in C(\Omega) \) and \( t \in (0, 1) \).
Proof. Since $L$ is strongly uniformly elliptic in $\Omega$, for any $x \in \overline{\Omega}$

$$\langle a, b \rangle := \sum_{i, j=1}^{N} a_j(x) a_i b_j$$

defines an scalar product in $\mathbb{R}^N$ and Hölder’s inequality shows that

$$2 \sum_{i, j=1}^{N} a_j(x) a_i b_j \leq \sum_{i, j=1}^{N} a_j(x) a_i a_j + \sum_{i, j=1}^{N} a_j(x) b_i b_j,$$

for all $a = (a_1, \ldots, a_N) \in \mathbb{R}^N$, $b = (b_1, \ldots, b_N) \in \mathbb{R}^N$ and $x \in \overline{\Omega}$. From this inequality it follows easily that the mapping $G: C^2(\Omega) \to C(\overline{\Omega})$ defined by

$$G(u) = (L - \alpha_0) u + \sum_{i, j=1}^{N} \alpha_{ij} D_i u D_j u$$

is concave; that is, $G(tu_1 + (1-t)u_2) \geq t G(u_1) + (1-t) G(u_2)$ for all $u_1, u_2 \in C^2(\Omega)$ and $t \in (0, 1)$. Note that given $\phi \in \mathcal{P}$, the following relation holds

$$\frac{L \phi}{\phi} = (L - \alpha_0) \psi + \alpha_0 - \sum_{i, j=1}^{N} \alpha_{ij} D_i \psi D_j \psi = G(\psi), \quad \psi = \log \phi.$$ 

Consider $p_1, p_2 \in C(\overline{\Omega})$, $t \in (0, 1)$, and $\phi_1, \phi_2 \in \mathcal{P}$ arbitrary. Set $\psi_i = \log \phi_i$, $i = 1, 2$. Then, $\phi_i^t \phi_2^{1-t} \in \mathcal{P}$ and it follows that

$$\frac{[L + t p_1 + (1-t) p_2] (\phi_1^t \phi_2^{1-t})}{\phi_1^t \phi_2^{1-t}}$$

$$= t p_1 + (1-t) p_2 + \frac{L (\phi_1^t \phi_2^{1-t})}{\phi_1^t \phi_2^{1-t}}$$

$$= t p_1 + (1-t) p_2 + G(\phi_1^t \phi_2^{1-t})$$

$$= t p_1 + (1-t) p_2 + G(t \psi_1 + (1-t) \psi_2)$$

$$\geq t p_1 + (1-t) p_2 + t G(\psi_1) + (1-t) G(\psi_2)$$

$$= t \frac{(L + p_1) \phi_1}{\phi_1} + (1-t) \frac{(L + p_2) \phi_2}{\phi_2}$$

$$\geq t \inf_{\Omega} \frac{(L + p_1) \phi_1}{\phi_1} + (1-t) \inf_{\Omega} \frac{(L + p_2) \phi_2}{\phi_2}.$$
Therefore, due to Theorem 3.1, we obtain
\[ \sigma_0^\alpha[L + tp_1 + (1-t) p_2] \geq t \inf_{\Omega} \frac{(L + p_1) \phi_1}{\phi_1} + (1-t) \inf_{\Omega} \frac{(L + p_2) \phi_2}{\phi_2}. \]

This inequality is satisfied for all \( \phi_1, \phi_2 \in \mathcal{P} \). So, we can take supremums with respect to \( \phi_1 \) and \( \phi_2 \) separately to get
\[ \sigma_0^\alpha[L + tp_1 + (1-t) p_2] \geq t \sigma_0^\alpha[L + p_1] + (1-t) \sigma_0^\alpha[L + p_2]. \]

The proof is completed.

4. Continuous Dependence of \( \sigma_0^\alpha[L] \) with Respect \( \Omega \)

In this section we show that \( \sigma_0^\alpha[L] \) depends continuously on \( \Omega \) if \( \sigma_y \in C^2(\bar{\Omega}) \) and \( \sigma \in C^1(\bar{\Omega}) \).

**Definition 4.1.** Let \( \Omega_0 \) be a subdomain of \( \Omega \) with boundary \( \partial \Omega_0 \) of class \( C^{2+} \). Let \( \Omega_k, k \geq 1, \) be a sequence of subdomains of \( \Omega \) with boundaries \( \partial \Omega_k, k \geq 1, \) of class \( C^{2+} \). It is said that
\[ \lim_{k \to \infty} \Omega_k = \Omega_0 \]
if the following two conditions are satisfied:

(i) There exists a sequence \( \Omega_k^I, k \geq 1, \) of subdomains of \( \Omega \) with boundaries of class \( C^{2+} \) such that
\[ \Omega_k^I \subset \Omega_{k+1}^I, \quad \Omega_k^I \subset \Omega_0 \cap \Omega_k, \quad k \geq 1, \]
and
\[ \bigcup_{k=1}^\infty \Omega_k^I = \Omega_0. \]

(ii) There exists a sequence \( \Omega_k^E, k \geq 1, \) of subdomains of \( \Omega \) with boundaries of class \( C^{2+} \) such that
\[ \Omega_{k+1}^E \subset \Omega_k^E, \quad \Omega_0 \cup \Omega_k \subset \Omega_k^E, \quad k \geq 1, \]
and
\[ \bigcap_{k=1}^\infty \Omega_k^E = \bar{\Omega}_0. \]
Theorem 4.2. Suppose \( \pi_i \in C^1(\Omega) \) and \( \pi_j \in C^1(\overline{\Omega}) \) for all \( i,j \). Let \( \Omega_n \) be a subdomain of \( \Omega \) with boundary \( \partial \Omega_n \) of class \( C^{2+r} \). Let \( \Omega_k, k \geq 1 \), be a sequence of subdomains of \( \Omega \) with boundaries \( \partial \Omega_k \) of class \( C^{2+r} \) such that

\[
\lim_{k \to \infty} \Omega_k = \Omega_n.
\]

Then,

\[
\lim_{k \to \infty} \sigma_1^{\partial \Omega_k}[L] = \sigma_1^{\partial \Omega}[L].
\]

Proof. Let \( \Omega_k^E, \Omega_k^I, k \geq 1 \), be two sequences of subdomains of \( \Omega \) satisfying the requirements of Definition 4.1. Then,

\[
\Omega_k^I \subset \Omega_k \subset \Omega_k^E \quad \text{and} \quad \Omega_k^I \subset \Omega_0 \subset \Omega_k^E \quad \text{for all} \quad k \geq 1.
\]

Hence, due to Proposition 3.2(iii),

\[
\sigma_1^{\partial \Omega_k^I}[L] \geq \sigma_1^{\partial \Omega_k}[L] \geq \sigma_1^{\partial \Omega_k^E}[L] \quad \text{and} \quad \sigma_1^{\partial \Omega_k}[L] \geq \sigma_1^{\partial \Omega_0}[L] \geq \sigma_1^{\partial \Omega}[L]
\]

for all \( k \geq 1 \), and we are done if we show that

\[
\lim_{k \to \infty} \sigma_1^{\partial \Omega_k^I}[L] = \lim_{k \to \infty} \sigma_1^{\partial \Omega_k^E}[L] = \sigma_1^{\partial \Omega}[L].
\]

By construction \( \{ \sigma_1^{\partial \Omega_k^I}[L] \} \) is decreasing and bounded below by \( \sigma_1^{\partial \Omega_0}[L] \), and \( \{ \sigma_1^{\partial \Omega_k^E}[L] \} \) is increasing and bounded above by \( \sigma_1^{\partial \Omega}[L] \). Hence, both sequences are convergent and

\[
\lim_{k \to \infty} \sigma_1^{\partial \Omega_k^I}[L] \leq \sigma_1^{\partial \Omega_0}[L] \leq \lim_{k \to \infty} \sigma_1^{\partial \Omega_k^E}[L].
\]

We now prove that

\[
\lim_{k \to \infty} \sigma_1^{\partial \Omega_k^I}[L] = \sigma_1^{\partial \Omega_0}[L]. \quad (4.1)
\]

Let \( \varphi_k^I \in C^{2+r}_0(\overline{\Omega_k^I}), k \geq 1 \), denote the principal eigenfunctions associated with \( \sigma_1^{\partial \Omega_k^I}[L] \), unique up to multiplicative constants. Normalize them so that

\[
\| \varphi_k^I \|_{H^1_0(\Omega_k^I)} = 1
\]

for all \( k \geq 1 \). Let \( \varphi_k \), \( k \geq 1 \), denote the extension of \( \varphi_k^I \) by zero to \( \Omega_0 \). Then

\[
\varphi_k \in H^1_0(\Omega_0) \quad \text{and} \quad \| \varphi_k \|_{H^1_0(\Omega_0)} = 1 \quad \text{for all} \quad k \geq 1.
\]
Since $H^1_0(\Omega_0)$ is compactly imbedded in $L^2(\Omega_0)$ we can extract a subsequence of $\{\varphi_k\}$, again labelled by $k$, such that

$$\lim_{k \to \infty} \varphi_k = \varphi_0 \quad \text{in} \quad L^2(\Omega_0),$$

for some $\varphi_0 \in L^2(\Omega_0)$. We now show that $\{\varphi_k\}$ is also a Cauchy sequence in $H^1_0(\Omega_0)$, and therefore it converges in $H^1_0(\Omega_0)$ to $\varphi_0$.

Suppose $l \leq k$. Then $\Omega_l^I \subset \Omega_k^I$ and due to the fact that $L$ is strongly uniformly elliptic in $\Omega$, we find that

$$\alpha \int_{\Omega_0} \left| \nabla (\varphi_k - \varphi_l) \right|^2 \leq \sum_{i,j=1}^N \int_{\Omega_0} \alpha_{ij} D_i (\varphi_k - \varphi_l) D_j (\varphi_k - \varphi_l)$$

$$= \sum_{i,j=1}^N \int_{\Omega_k^I} \alpha_{ij} D_i \varphi_k D_j \varphi_k + \sum_{i,j=1}^N \int_{\Omega_l^I} \alpha_{ij} D_i \varphi_l D_j \varphi_l$$

$$- 2 \sum_{i,j=1}^N \int_{\Omega_k^I} \alpha_{ij} D_i \varphi_k D_j \varphi_l$$

$$= - \sum_{i,j=1}^N \int_{\Omega_k^I} D_i (\alpha_{ij} D_i \varphi_k) \varphi_k - \sum_{i,j=1}^N \int_{\Omega_l^I} D_i (\alpha_{ij} D_i \varphi_l) \varphi_l$$

$$+ 2 \sum_{i,j=1}^N \int_{\Omega_k^I} D_i (\alpha_{ij} D_i \varphi_k) \varphi_l.$$
where
\[ a_j := \sigma_j + \sum_{i=1}^{N} D_{ij}(\sigma_i), \quad 1 \leq j \leq N. \]

Rearranging terms gives
\[
\alpha \int_{\Omega_0} |\nabla(\varphi_k - \varphi_j)|^2 \leq \sigma_{11}^{(1)} \int_{\Omega_0} \varphi_k(\varphi_k - \varphi_j) + \sigma_{11}^{(2)} \int_{\Omega_0} \varphi_j(\varphi_j - \varphi_k) \\
+ (\sigma_{11}^{(1)} - \sigma_{11}^{(2)}) \int_{\Omega_0} \varphi_j \varphi_k - \int_{\Omega_0} \alpha_0(\varphi_k - \varphi_j)^2 \\
+ \sum_{j=1}^{N} \int_{\Omega_0} a_j(\varphi_j - \varphi_k) D_j \varphi_k + \sum_{j=1}^{N} \int_{\Omega_0} a_j \varphi_j D_j (\varphi_k - \varphi_j). \quad (4.2)
\]

Now, taking into account that
\[ \|\varphi_n\|_{H^1(\Omega_0)} = 1 \]
for all \( n \geq 1 \), applying Hölder’s inequality and integrating by parts, we obtain the following
\[
\sigma_{11}^{(1)} \int_{\Omega_0} \varphi_k(\varphi_k - \varphi_j) \leq C_1 \|\varphi_k - \varphi_j\|_{L^2(\Omega_0)},
\]
\[
\sigma_{11}^{(2)} \int_{\Omega_0} \varphi_j(\varphi_j - \varphi_k) \leq C_1 \|\varphi_j - \varphi_k\|_{L^2(\Omega_0)},
\]
\[
(\sigma_{11}^{(1)} - \sigma_{11}^{(2)}) \int_{\Omega_0} \varphi_j \varphi_k \leq |\sigma_{11}^{(2)}| \int_{\Omega_0} \varphi_j \varphi_k - |\sigma_{11}^{(1)}| \int_{\Omega_0} \varphi_j \varphi_k,
\]
\[
- \int_{\Omega_0} \alpha_0(\varphi_k - \varphi_j)^2 \leq -\inf_{\Omega_0} \alpha_0 \|\varphi_k - \varphi_j\|_{L^2(\Omega_0)},
\]
\[
\sum_{j=1}^{N} \int_{\Omega_0} a_j(\varphi_j - \varphi_k) D_j \varphi_k \leq C \|\varphi_j - \varphi_k\|_{L^2(\Omega_0)}.
\]
\[
\sum_{j=1}^{N} \int_{\Omega_0} a_j \varphi_j D_j (\varphi_k - \varphi_j) = -\sum_{j=1}^{N} \int_{\Omega_0} (\varphi_k - \varphi_j) D_j (a_j \varphi_j) \leq C \|\varphi_j - \varphi_k\|_{L^2(\Omega_0)}.
\]
where $C_1 > 0$ is an upper bound of $|\sigma_1^{\text{up}}(L)|$ and $C > 0$ only depends on the coefficients of $L$. Thus, since $\{\varphi_k\}$ is a Cauchy sequence in $L^2(\Omega_0)$, $\{\sigma_1^{\text{up}}(L)\}$ is a Cauchy sequence in $\mathbb{R}$, and $x > 0$, it follows from (4.2) that $\{\varphi_k\}$ is also a Cauchy sequence in $H^1_0(\Omega_0)$. Therefore, $\varphi_0 \in H^1_0(\Omega_0)$ and

$$\lim_{k \to \infty} \varphi_k = \varphi_0 \quad \text{in } H^1_0(\Omega_0).$$

In particular, it follows that

$$\|\varphi_0\|_{H^1_0(\Omega_0)} = 1$$

and so $\varphi_0 \neq 0$. We now show that $\varphi_0$ is a weak solution of

$$L\varphi_0 = \sigma_1^{\text{up}}\varphi_0 \quad \text{in } \Omega_0,$$  

(4.3)

where

$$\sigma_1^{\text{up}} := \lim_{k \to \infty} \sigma_1^{\text{up}}(L).$$

This completes the proof of (4.1). Indeed, by elliptic regularity $\varphi_0$ is a classical solution of (4.3) and it follows from the uniqueness of the principal eigenvalue that $\sigma_1^{\text{up}} = \sigma_1^{\text{up}}(L)$. To prove that $\varphi_0$ is a weak solution of (4.3) consider a test function $\psi \in C^\infty_0(\Omega_0)$. Then, there exists an integer number $k_0 = k(\psi) \geq 1$ such that $\text{supp } \psi \subset \Omega_0$ for all $k \geq k_0$. Hence, multiplying $L\varphi_k = \sigma_1^{\text{up}}(L)\varphi_k$ by $\psi$, integrating in $\Omega_k$ and applying the formula of integration by parts gives

$$\sum_{i,j=1}^{N} \int_{\Omega_k} a_{ij} D_i \psi D_j \varphi_k + \sum_{j=1}^{N} \int_{\Omega_k} a_j \psi D_j \varphi_k + \int_{\Omega_k} \sigma_0 \psi \varphi_k = \sigma_1^{\text{up}}(L) \int_{\Omega_k} \psi \varphi_k.$$  

(4.4)

As $\varphi_k \equiv 0$ in $\Omega_0 \setminus \Omega_k$ and $\lim_{k \to \infty} \|\varphi_k - \varphi_0\|_{H^1_0(\Omega_0)} = 0$, letting $k \to \infty$ in (4.4) yields

$$\sum_{i,j=1}^{N} \int_{\Omega_0} a_{ij} D_i \psi D_j \varphi_0 + \sum_{j=1}^{N} \int_{\Omega_0} a_j \psi D_j \varphi_0 + \int_{\Omega_0} \sigma_0 \psi \varphi_0 = \sigma_1^{\text{up}}(L) \int_{\Omega_0} \psi \varphi_0.$$  

Therefore, $\varphi_0$ is a weak solution of (4.3) and the proof of (4.1) is concluded.

To show that

$$\lim_{k \to \infty} \sigma_1^{\text{up}}(L) = \sigma_1^{\text{up}}(L)$$
we can argue as before but this time we have to work with the extensions of the principal eigenfunctions $\phi_k^E$ associated with $\sigma_1^{O_e}[L]$ by zero to $\Omega_1$. Now, we have
\[
\lim_{k \to \infty} \|\phi_k - \phi_0\|_{H(\Omega_1)} = 0.
\]

To complete the proof it suffices to show that $\phi_0 \in H_0^1(\Omega_0)$. Indeed, if this occurs then $\phi_0$ is a principal eigenfunction associated to $\lim_{k \to \infty} \sigma_1^{O_e}[L]$ and since $\sigma_1^{O_e}[L]$ is unique necessarily
\[
\sigma_1^{O_e}[L] = \lim_{k \to \infty} \sigma_1^{O_e}[L].
\]

To show $\phi_0 \in H_0^1(\Omega_0)$ we argue as follows. Given $l \geq 1$ for any $k \geq l$ we have $\Omega_k^e \subset \Omega_l^e$ for all $k \geq l$ and so
\[
\phi_k \in H_0^1(\Omega_l^e), \quad k \geq l \geq 1.
\]
Hence,
\[
\lim_{k \to \infty} \|\phi_k - \phi_0\|_{H(\Omega_l)} = 0,
\]
for any $l \geq 1$ and thus
\[
\phi_0 \in \bigcap_{l=1}^{\infty} H_0^1(\Omega_l^e).
\]

Therefore, the proof is completed if we show that
\[
H_0^1(\Omega_0) = \bigcap_{l=1}^{\infty} H_0^1(\Omega_l^e). \tag{4.5}
\]
It is clear that $H_0^1(\Omega_0) \subset \bigcap_{l=1}^{\infty} H_0^1(\Omega_l^e)$. So, it suffices to show that $\bigcap_{l=1}^{\infty} H_0^1(\Omega_l^e) \subset H_0^1(\Omega_0)$. On the other hand, for any domain $\Omega$ with smooth boundary it follows from Theorem 3.7 of Wloka [25] that
\[
H_0^1(\Omega) = \{ u \in H^1(\mathbb{R}^N) : \text{supp} \, u \subset \bar{\Omega} \}. \tag{4.6}
\]
Pick up $u \in \bigcap_{l=1}^{\infty} H_0^1(\Omega_l^e)$. Then, $u \in H_0^1(\Omega_l^e)$ for all $l \geq 1$ and so
\[
\text{supp} \, u \subset \bar{\Omega}_l^e, \quad \forall l \geq 1.
\]
Thus,
\[
\text{supp} \, u \subset \bigcap_{l=1}^{\infty} \Omega_l^e = \Omega_0
\]
and therefore $u \in H_0^1(\Omega_0)$. The proof is completed. \[\square\]
Suppose $\Omega_0$ to be a general domain, without any regularity requirement on $\partial \Omega_0$, which can be approximated in the sense of Definition 4.1 by a sequence of subdomains with smooth boundary $\Omega_k' \subset \Omega_0$ such that $\Omega_k' \subset \Omega_l'$ for $k \leq l$. Then, it follows from the proof of Theorem 4.2 that

$$\sigma_1^{\Omega_0}(L) := \lim_{k \to \infty} \sigma_1^{\Omega_k}(L)$$

(4.7)

is well defined and that there exists a principal eigenfunction associated with it. Hence, the existence of a principal eigenvalue for $L$ on a general domain $\Omega_0$ follows. This provides us with a direct proof of some results of Berestycki, Nirenberg and Varadhan [5]. In fact approaching $\Omega_0$ by regular domains from its interior was the basic technical tool in the proofs of the main theorems of [5]. Nevertheless, if no regularity assumption is assumed on $\Omega_0$ we can not be sure about the uniqueness of the principal eigenvalue nor about the continuous dependence of the principal eigenvalue with respect to the support domain. It seems this fact was not observed in [5].

Condition (4.6) is the key so that (4.5) and hence $\varphi_0 \in H^1(\Omega_0)$ hold. Theorem 3.7 of Wloka [25] shows that if $\Omega_0$ satisfies the segment property then (4.6) and so (4.5) are satisfied. It is said that $\Omega_0$ satisfies the segment property if for each $x \in \partial \Omega_0$ there exists a neighbourhood $U_x$ of $x$ and a vector $y_x \neq 0$ such that for each $z \in \Omega_0 \cap U_x$ the point $z + ty_x$ belongs to $\Omega_0$ for all $0 < t < 1$. When condition (4.5) occurs for any sequence of domains $\Omega_k'$ containing $\Omega_0$ and converging to $\Omega_0$ in the sense of Definition 4.1 it is said that $\Omega_0$ is stable. This concept goes back to Babuska and Vyborny [3]. In the last part of the proof of Theorem 4.2 we have seen that if $\Omega_0$ is stable and it has a unique principal eigenvalue, then $\sigma_1^{\Omega_k}(L) \to \sigma_1^{\Omega_0}(L)$, as $k \to \infty$, for any sequence of domains converging to $\Omega_0$ from outside.

As Courant and Hilbert [7] and Babuska and Vyborny [3] treated exclusively the case of selfadjoint operators, and Dancer [10] dealt with a perturbation of $-\Delta$, it seems that Theorem 4.2 is new in its full generality. Moreover, our proof differs substantially from those of [3], [7] and [10], providing us in addition with the continuous dependence of the principal eigenfunction with respect to the domain.

### 5. A SUFFICIENT CONDITION FOR THE MAXIMUM PRINCIPLE

It is well known that if the Lebesgue measure of $\Omega$, $|\Omega|$, is small enough then the operator $L$ is coercive in the sense that it satisfies the strong maximum principle, i.e., $\sigma_1^{\Omega_0}(L) > 0$. This fact was observed by Gilbarg and Trudinger after the proof of Lemma 8.4 in [12]. This result is also true for
general operators and general domains and has been used extensively by Berestycki and Nirenberg in [4]; in fact, it was the motivation of [5], as claimed by the authors in the bottom of page 52, before the statement of Proposition 1.1. In this section we shall obtain an explicit lower estimate of $\sigma_1^0(L)$ in terms of the Lebesgue measure of $\Omega$ and the several coefficient functions of $L$. Our estimate measures explicitly how small has to be $|\Omega|$ so that $L$ satisfy the strong maximum principle.

Note that if $x_0 \in C^{1+\gamma}(\bar{\Omega})$, then the operator $L$ can be written in the form

$$L = -\sum_{i,j=1}^{N} D_i(x_0, D_j) + \sum_{j=1}^{N} a_j D_j + x_0,$$

where

$$a_j := x_j + \sum_{i=1}^{N} D_i(x_0), \quad 1 \leq j \leq N.$$

Set

$$a := (a_1, ..., a_N), \quad |a| := \sup_{x \in \Omega} \left( \sum_{j=1}^{N} a_j(x)^2 \right)^{1/2}.$$

A well known result is the inequality of Faber [11] and Krahn [17] which states that when $L = -A$ among all domains $\Omega$ with a fixed Lebesgue measure, $|\Omega|$, the ball has the smallest principal eigenvalue. Therefore,

$$\sigma_1^0[-A] \geq \frac{\sigma_1^{B_1}[-A] \cdot |B_1|^{2/N}}{|\Omega|^{2/N}},$$

where $B_1 = \{ x \in \mathbb{R}^N : |x| \leq 1 \}$. The following result is a generalization of this inequality, for general operators with smooth coefficients.

**Theorem 5.1.** Suppose $x_0 \in C^{1+\gamma}(\bar{\Omega})$. Then the following estimates hold:

(i) If $L$ is selfadjoint then

$$\sigma_1^0[L] \geq \frac{\alpha \cdot \sigma_1^{B_1}[-A] \cdot |B_1|^{2/N}}{|\Omega|^{2/N}} + \inf_{\Omega} x_0.$$

(ii) If $L$ is not selfadjoint; that is, if $|a|_{\infty} > 0$, then

$$\sigma_1^0[L] \geq \frac{\alpha \cdot \sigma_1^{B_1}[-A] \cdot |B_1|^{2/N}}{|\Omega|^{2/N}} + \frac{|a|_{\infty} \cdot (\sigma_1^{B_1}[-A])^{1/2} \cdot |B_1|^{1/N}}{|\Omega|^{1/N}} + \inf_{\Omega} x_0,$$
provided
\[ |\Omega| \leq \frac{\chi^N(\sigma_0^R[-A])^{N/2}}{|a|_\infty^N} |B_1|. \]

**Proof.** Let \( \varphi \geq 0 \) denote the principal eigenfunction corresponding to \( \sigma_0^R[L] \). Multiplying the following relation by \( \varphi \)
\[ L\varphi = \sigma_0^R[L] \varphi, \]
integrating in \( \Omega \) and applying the formula of integration by parts, we find that
\[
\sigma_0^R[L] \int_\Omega \varphi^2 = \sum_{i,j=1}^N \int_\Omega \varphi \partial_i D_j \varphi D_j \varphi + \sum_{j=1}^N \int_\Omega a_j \varphi \partial_j \varphi + \int_\Omega \varphi \partial_\partial \varphi. \tag{5.1}
\]
Since \( L \) is strongly uniformly elliptic,
\[
\sum_{i,j=1}^N \int_\Omega \varphi \partial_i D_j \varphi D_j \varphi \geq a \int_\Omega |\nabla \varphi|^2. \tag{5.2}
\]
Moreover, it follows from Hölder’s inequality that
\[
\left| \sum_{j=1}^N \int_\Omega a_j \varphi \partial_j \varphi \right| \leq \int_\Omega |\varphi| |\nabla \varphi| \leq |a|_\infty \int_\Omega |\varphi| \|\nabla \varphi\|_2.
\]
Hence,
\[
\sum_{j=1}^N \int_\Omega a_j \varphi \partial_j \varphi \geq -|a|_\infty \|\varphi\|_2 \|\nabla \varphi\|_2. \tag{5.3}
\]
Substituting (5.2) and (5.3) into (5.1) yields
\[
\sigma_0^R[L] \geq \left( \frac{\int_\Omega |\nabla \varphi|^2}{\int_\Omega \varphi^2} \right)^{1/2} \left( \frac{\int_\Omega |\nabla \varphi|^2}{\int_\Omega \varphi^2} \right)^{1/2} - |a|_\infty \inf_\Omega \varphi. \tag{5.4}
\]
On the other hand, using the variational characterization of \( \sigma_0^R[-A] \) it follows from Faber–Krahn’s inequality that
\[
\frac{\int_\Omega |\nabla \varphi|^2}{\int_\Omega \varphi^2} \geq \frac{\sigma_0^R[-A] - |B_1|^{2N}}{|\Omega|^{2N}}. \tag{5.5}
\]
If $L$ is selfadjoint then $a \equiv 0$ and it follows from (5.4) and (5.5) that
\[\sigma_1^0[L] \geq \frac{\pi \cdot \sigma_1^0[-A] \cdot |B_1|^{2/N}}{|\Omega|^{2/N}} + \inf_{\Omega} a_0.\]
This completes the proof of (i). Suppose $L$ is not selfadjoint. Then $|a|_{\infty} > 0$ and due to (5.5) the following condition
\[|\Omega| \leq \frac{\pi^N (\sigma_1^0[-A])^{N/2} |B_1|}{|a|^{N}_{\infty}}\]
implies
\[\pi \left( \int_{\Omega} |\nabla \varphi|^2 \right)^{1/2} - |a|_{\infty} \geq 0.\]
Therefore, it follows from (5.4) that
\[\sigma_1^0[L] \geq \frac{\pi \cdot \sigma_1^0[-A] \cdot |B_1|^{2/N}}{|\Omega|^{2/N}} - \frac{|a|_{\infty} \cdot (\sigma_1^0[-A])^{1/2} \cdot |B_1|^{1/N}}{|\Omega|^{1/N}} + \inf_{\Omega} a_0.\]
The proof is completed.

From this result it is straightforward to calculate the constant $\eta$ of the statement of Theorem 2.6 of Berestycki, Nirenberg and Varadhan [5]. In this reference no explicit estimate for $\eta$ was given, even in the case when the coefficients of $L$ and $\partial \Omega$ are smooth. Instead of that, a general version of the main theorem of Lieb [18] was found to show that for general operators and domains the maximum principle holds provided $|\Omega|$ is sufficiently small.

6. Characterizing the Existence of Principal Eigenvalues for Some Linear Weighted Boundary Value Problems

In this section we discuss the existence of principal eigenvalues for some linear weighted boundary value problems of the form
\[L(x, D) \varphi = \lambda m(x) \varphi, \quad \varphi \in U,\]
where $m \in C'(\Omega)$ is arbitrary. We may take $m \in C(\overline{\Omega})$, but then the solutions of (6.1) will be in $C^1(\Omega) \cap C^2(\Omega)$, instead of $U$. By a principal eigenvalue we mean a value of $\lambda \in \mathbb{R}$ such that (6.1) admits a positive eigenfunction. Note that a $\lambda \in \mathbb{R}$ is a principal eigenvalue of (6.1) if and only if
\[\sigma_1^0[L - \lambda m] = 0,\]
where $m: U \to V$ stands for the multiplication operator induced by $m$. Thus, the problem of analyzing the existence of principal eigenvalues of (6.1) is equivalent to the problem of the search for zeros of the function $\mu(\lambda)$ defined by

$$
\mu(\lambda) := \sigma^m_0[L - \lambda m], \quad \lambda \in \mathbb{R}.
$$

(6.2)

The following result collects some well known properties of $\mu(\lambda)$ which are the key to analyze the existence of principal eigenvalues for (6.1). The proof can be found in the book of Hess [14]. For the sake of completeness we include a very short proof of it.

**Lemma 6.1.** For any $m \in C^{\infty}(\bar{\Omega})$, the following assertions are true:

(i) The function $\mu(\lambda)$ defined by (6.2) is analytic and concave. In particular, $\mu'(\lambda) \leq 0$ for all $\lambda \in \mathbb{R}$ and either $\mu'(\lambda) \equiv 0$ or $\mu'(\lambda) < 0$ except at most for a discrete set of values of $\lambda$.

(ii) If $m(x_0) > 0$ for some $x_0 \in \Omega$ then $\lim_{\lambda \to -\infty} \mu(\lambda) = -\infty$.

(iii) If $m(x_1) < 0$ for some $x_1 \in \Omega$ then $\lim_{\lambda \to -\infty} \mu(\lambda) = -\infty$.

**Proof.** (i) The family of operators $L(\lambda)$ defined by $L(\lambda) := L - \lambda m: U \to V$, is analytic in $\lambda$. Moreover, $\mu(\lambda)$ is a simple eigenvalue of $L(\lambda)$ and so due to the perturbation result of Crandall and Rabinowitz [8] the mapping $\lambda \to \mu(\lambda)$ is analytic in $\lambda$. The concavity of $\mu(\lambda)$ follows as a consequence of Theorem 3.3. It is straightforward to see that

$$
\mu'(\lambda) \leq 0 \quad \text{for all} \quad \lambda \in \mathbb{R}.
$$

The alternative follows from the fact that $\lambda \to \mu'(\lambda)$ is analytic. Note that if $m$ is a constant then $\mu(\lambda) = \sigma^m_0[L] - \lambda m$ for all $\lambda \in \mathbb{R}$ and so $\mu'(\lambda) \equiv 0$. The proof of part (i) is completed.

(ii) Let $B \subset \Omega$ a ball centered at $x_0$ such that $m \geq m_B$ in $B$, with $m_B > 0$. It follows from Proposition 3.2(iii) that $\sigma^m_0[L - \lambda m] \leq \sigma^m_0[L - \lambda m_B]$. Moreover, for any $\lambda > 0$ we have $-\lambda m \leq -\lambda m_B$ in $B$ and hence, due to Proposition 3.2(i), we have

$$
\sigma^m_0[L - \lambda m] \leq \sigma^m_0[L - \lambda m_B] = \sigma^m_0[L] - \lambda m_B.
$$

Thus, $\sigma^m_0[L - \lambda m] < \sigma^m_0[L] - \lambda m_B$ for all $\lambda > 0$ and letting $\lambda \to \infty$, the proof of part (ii) is completed.

(iii) Let $B \subset \Omega$ be a ball centered at $x_1$ such that $m \leq m_B$ in $B$, with $m_B < 0$. It follows from Proposition 3.2(iii) that $\sigma^m_0[L - \lambda m] < \sigma^m_0[L - \lambda m_B]$. 


Moreover, for any \( \lambda < 0 \) we have \(-\lambda m \leq -\lambda m_S \) in \( B \) and hence it follows from Proposition 3.2(i) that
\[
\sigma^0_\lambda[L - \lambda m] \leq \sigma^0_\lambda[L - \lambda m_S] = \sigma^0_\lambda[L] - \lambda m_S.
\]
Thus, \( \sigma^0_\lambda[L - \lambda m] < \sigma^0_\lambda[L] - \lambda m_S \) and since \( m_S < 0 \), letting \( \lambda \to -\infty \) in this relation, the proof of part (iii) is completed.

For the sequel we shall restrict ourselves to the case \( N \geq 2 \), because the statements of the results should be slightly modified to be adapted to the one-dimensional case. As we shall use Theorem 4.2 as a basic tool, we assume throughout this section that \( \phi_{ij} \in \mathcal{C}^2(\overline{\Omega}) \) and \( \phi_{ij} \in \mathcal{C}^1(\overline{\Omega}) \) for all \( i, j \).

We first consider the case of sign defined weight functions \( m(x) \).

**Theorem 6.2.** Let \( m \in \mathcal{C}^0(\overline{\Omega}) \) be such that \( m > 0 \) in \( \Omega \). Suppose there are \( h \) subdomains of \( \Omega \), say \( \Omega_j, 1 \leq j \leq h \), with \( \partial \Omega_j \), of class \( \mathcal{C}^{2+\varepsilon} \) for \( 1 \leq j \leq h \), such that \( \Omega_j \subset \Omega, 1 \leq j \leq h, \Omega_i \cap \Omega_j \) is empty for \( i \neq j \),

\[
m(x) > 0 \quad \text{if and only if} \quad x \in \bigcup_{j=1}^h \Omega_j
\]

and

\[
\Omega_0 := \Omega - \bigcup_{j=1}^h \Omega_j
\]

is a proper subdomain of \( \Omega \). Then,

\[
\lim_{\lambda \to -\infty} \sigma^0_\lambda[L - \lambda m] = \sigma^0[L]. \quad (6.3)
\]

In particular, \( (6.1) \) admits a principal eigenvalue if, and only if,

\[
\sigma^0_1[L] > 0. \quad (6.4)
\]

Moreover, if \( (6.4) \) is satisfied then \( (6.1) \) has a unique principal eigenvalue, denoted by \( \sigma^0_1[L; m] \), which is a simple eigenvalue of the pair \((L - \sigma^0_1[L; m], m, m)\).

**Proof.** Let \( \varphi \) denote the principal eigenfunction associated to \( \sigma^0_1[L - \lambda m] \). Since \( m \equiv 0 \) in \( \Omega_0 \),

\[
L \varphi = (L - \lambda m) \varphi = \sigma^0_1[L - \lambda m] \varphi
\]

in \( \Omega_0 \). Moreover, \( \varphi > 0 \) on \( \partial \Omega_0 \) and hence \( \varphi \) is a strict positive supersolution of

\[
L - \sigma^0_1[L - \lambda m]
\]
in $\Omega_0$. Thus, it follows from Theorem 2.5 that
\[ \sigma_1^{D_0}[L - \sigma_1^{D_0}[L - \lambda m]] > 0. \]
Hence,
\[ \sigma_1^{D_0}[L - \lambda m] < \sigma_1^{D_0}[L] \]
for all $\lambda \in \mathbb{R}$ and so
\[ \lim_{\lambda \to -\infty} \sigma_1^{D_0}[L - \lambda m] \leq \sigma_1^{D_0}[L]. \]

To complete the proof of (6.3) we have to show that for any $\epsilon > 0$ there exists $\lambda(\epsilon) \in \mathbb{R}$ such that
\[ \sigma_1^{D_0}[L - \lambda m] \geq \sigma_1^{D_0}[L] - \epsilon \]
for all $\lambda \leq \lambda(\epsilon)$. Let $\epsilon > 0$ be. Consider the family of subdomains of $\Omega$ defined by
\[ \Omega_\delta := \Omega_0 \cup \{ x \in \Omega : d(x, \partial \Omega_\delta) < \delta \} \]
for $\delta > 0$. It is clear that for all $\delta > 0$
\[ \Omega_0 \subset \Omega_\delta \]
and that the family $\Omega_\delta$ converges to $\Omega_0$ in the sense of Definition 4.1 as $\delta \to 0$. Therefore, it follows from Theorem 4.2 that
\[ \lim_{\delta \to 0} \sigma_1^{D_0}[L] = \sigma_1^{D_0}[L] \]
and hence
\[ \sigma_1^{D_0}[L] < \sigma_1^{D_0}[L] < \sigma_1^{D_0}[L] + \epsilon \]
for $\delta > 0$ small enough. Since $m(x) > 0$ for all $x \in \bigcup_{j=1}^J \Omega_j$, there exists a constant $m_L > 0$ such that
\[ m(x) \geq m_L \]
for all $x \in \Omega - \Omega_0$. Let $\psi$ denote the principal eigenfunction associated to $\sigma_1^{D_0}[L]$. Consider the function $\Psi$ defined in $\Omega$ by
\[ \Psi(x) := \begin{cases} \psi(x) & \text{if } x \in \Omega_{0,2}, \\ \tilde{\psi}(x) & \text{if } x \in \Omega - \Omega_{0,2}. \end{cases} \]
where \( \hat{\psi} \) is any smooth extension of \( \psi \) outside \( \Omega_{\delta/2} \) with the property
\[
\hat{\psi}(x) \geq c > 0 \quad \text{for all} \quad x \in \Omega - \Omega_{\delta/2},
\]
for some constant \( c > 0 \).

Since \( \sigma_1^\Omega[L] < \sigma_1^\Omega[L] + \varepsilon \) and \( \Psi' > 0 \) in \( \Omega \), we find that
\[
(L - \lambda m + \varepsilon - \sigma_1^\Omega[L]) \Psi > (L - \lambda m - \sigma_1^\Omega[L]) \Psi \quad \text{in} \ \Omega.
\]
Moreover, since \( \Psi = \psi \) in \( \Omega_{\delta/2} \) and
\[
L \psi = \sigma_1^\Omega[L] \psi
\]
in \( \Omega_\delta \), we obtain that
\[
(L - \lambda m - \sigma_1^\Omega[L]) \Psi = -\lambda m \psi > 0 \quad \text{in} \ \Omega_{\delta/2},
\]
provided \( \lambda < 0 \). Furthermore, in \( \Omega - \Omega_{\delta/2} \) we have
\[
(L - \lambda m - \sigma_1^\Omega[L]) \Psi = (L - \sigma_1^\Omega[L]) \hat{\psi} - \lambda m \hat{\psi}
\]
and hence
\[
(L - \lambda m - \sigma_1^\Omega[L]) \Psi > (L - \sigma_1^\Omega[L]) \hat{\psi} - \lambda m c,
\]
provided \( \lambda < 0 \). Since \( (L - \sigma_1^\Omega[L]) \hat{\psi} \) does not depend on \( \hat{\lambda} \) there exists \( \hat{\lambda}(\varepsilon) < 0 \) such that
\[
(L - \sigma_1^\Omega[L]) \Psi - \lambda m c > 0
\]
in \( \Omega - \Omega_{\delta/2} \) for all \( \lambda \leq \hat{\lambda}(\varepsilon) \). Thus, for any \( \lambda \leq \hat{\lambda}(\varepsilon) \) the function \( \Psi \) is a strict positive supersolution of the operator
\[
L - \lambda m + \varepsilon - \sigma_1^\Omega[L]
\]
in \( \Omega \). Hence, due to Theorem 2.5 we find that
\[
\sigma_1^\Omega[L - \lambda m + \varepsilon - \sigma_1^\Omega[L]] > 0.
\]
Therefore,
\[
\sigma_1^\Omega[L] < \sigma_1^\Omega[L - \lambda m] + \varepsilon
\]
for all \( \lambda < \hat{\lambda}(\varepsilon) \). The proof of (6.3) is completed.

The mapping \( \lambda \to \sigma_1^\Omega[L - \lambda m] \) is strictly decreasing and we know that
\[
\lim_{\lambda \to -\infty} \sigma_1^\Omega[L - \lambda m] = \sigma_1^\Omega[L], \quad \lim_{\lambda \to \infty} \sigma_1^\Omega[L - \lambda m] = -\infty.
\]
Therefore, there exists $\lambda_1 \in \mathbb{R}$ such that $\sigma_1^{0\epsilon}[L - \lambda_1 m] = 0$ if and only if (6.4) holds. Moreover, it is unique if it exists. Suppose (6.4) and let $\sigma_1^{0\epsilon}[L; m]$ denote the unique principal eigenvalue of (6.1). The fact that $\sigma_1^{0\epsilon}[L; m]$ is a simple eigenvalue of $(L - \sigma_1^{0\epsilon}[L; m]) m, m)$ follows easily arguing as Hess [14]. The proof is completed.

Remark 6.3. If we assume $m < 0$, instead of $m > 0$, we obtain the analogue of Theorem 6.2. In this case, the following holds

$$\lim_{\lambda \to -\infty} \sigma_1^{0\epsilon}[L - \lambda m] = -\infty, \quad \lim_{\lambda \to -\infty} \sigma_1^{0\epsilon}[L - \lambda m] = \sigma_1^{0\epsilon}[L],$$

and hence (6.1) admits a principal eigenvalue if, and only if, $\sigma_1^{0\epsilon}[L] > 0$. In both cases, independently of the values of $m$ on its support, Theorem 5.1 provides us with some sufficient easily computable conditions in terms of the several coefficients of $L$ and the measure of $\Omega$ so that (6.1) admits a principal eigenvalue.

The next result provides us with a sufficient conditions so that

$$\lim_{\lambda \to -\infty} \sigma_1^{0\epsilon}[L - \lambda m] = \infty, \quad \text{(6.5)}$$

for $m > 0$.

Theorem 6.4. Suppose $m \in C'(\overline{\Omega})$ satisfies $m(x) > 0$ almost everywhere in $\Omega$. Then, condition (6.5) holds. In particular, (6.1) admits a unique principal eigenvalue $\sigma_1^{0\epsilon}[L; m]$ which is a simple eigenvalue of the pair $(L - \sigma_1^{0\epsilon}[L; m]) m, m)$.

Proof. We will use some ideas taken from [9] and [20]. Note that we are assuming that

$$|\{ x \in \Omega : m(x) = 0 \}| = 0.$$

It suffices to show that for any $C > 0$ there exists $\lambda(C)$ such that

$$C = \sigma_1^{0\epsilon}[L - \lambda(C) m]. \quad \text{(6.6)}$$

Let $K > 0$ be sufficiently large so that

$$C + K + \lambda m > 0, \quad \sigma_1^{0\epsilon}[L + K] > 0. \quad \text{(6.7)}$$

Then, the operator

$$\mathcal{T}_2 := (L + K)^{-1} [(C + K + \lambda m) \cdot \cdot \cdot]$$
is well defined and it is compact and strongly order preserving. A standard
calculation shows that

\[ C = \sigma_0^1 [L - \lambda m] \Leftrightarrow \text{spr } \mathcal{T}_i = 1. \]

We claim that when \( \text{spr } \mathcal{T}_i < 1 \) or \( \text{spr } \mathcal{T}_i > 1 \) it is independent for \( K \) satisfying (6.7). To show this claim let \( \varphi, \psi \) be the unique positive functions, up
to a constant, such that \( \varphi, \psi \in U \) and

\[
(L - \lambda m) \varphi = \sigma_0^1 [L - \lambda m] \varphi, \quad (6.8)
\]

\[
\mathcal{T}_i \psi = \text{spr } \mathcal{T}_i \psi. \quad (6.9)
\]

By the definition of \( \mathcal{T}_i \), (6.9) can be written as

\[
(L + K) \psi = \frac{1}{\text{spr } \mathcal{T}_i} (C + K + \lambda m) \psi. \quad (6.10)
\]

Let \( \varphi^*, \psi^* \) be such that

\[
(L^* - \lambda m) \varphi^* = \sigma_0^1 [L^* - \lambda m] \varphi^*, \quad (6.11)
\]

\[
(L^* + K) \psi^* = \frac{1}{\text{spr } \mathcal{T}_i} (C + K + \lambda m) \psi^*. \quad (6.12)
\]

Multiplying (6.8) by \( \psi^* \), integrating on \( \Omega \) and applying the formula of integration by parts we find that

\[
\int_{\Omega} \varphi (L^* - \lambda m) \psi^* = \sigma_0^1 [L - \lambda m] \int_{\Omega} \varphi \psi^*. \]

Hence,

\[
\int_{\Omega} \varphi (L^* + K) \psi^* = \int_{\Omega} (K + \lambda m) \varphi \psi^* + \sigma_0^1 [L - \lambda m] \int_{\Omega} \varphi \psi^*
\]

and substituting (6.12) into this relation it follows that

\[
\text{spr } \mathcal{T}_i = \frac{(K + C) \int_{\Omega} \varphi \psi^* + \lambda \int_{\Omega} m \varphi \psi^*}{(K + \sigma_0^1 [L - \lambda m]) \int_{\Omega} \varphi \psi^* + \lambda \int_{\Omega} m \varphi \psi^*}. \quad (6.13)
\]

Under condition (6.7) we have

\[
\text{spr } \mathcal{T}_i < 1 \iff C < \sigma_0^1 [L - \lambda m]
\]

and

\[
\text{spr } \mathcal{T}_i > 1 \iff C > \sigma_0^1 [L - \lambda m].
\]
This completes the proof of the claim above. Without lost of generality we can assume $C > \sigma_1^0[L]$. Then,

$$\text{spr} \, \mathcal{J}_0 = \frac{K + C}{K + \sigma_1^0[L]} > 1.$$  

Moreover, for any $K$ satisfying (6.7) the mapping $\lambda \rightarrow C + K + \lambda m$ is decreasing as $\lambda \downarrow -\infty$ and so thanks to Theorem 3.2 of Amann [1] the mapping $\lambda \rightarrow \text{spr} \, \mathcal{J}_0$ is decreasing as $\lambda \downarrow -\infty$ as long as (6.7) holds. It suffices to show that $\text{spr} \, \mathcal{J}_0 < 1$ for $\lambda < 0$ with $|\lambda|$ sufficiently large. Since

$$C + K + \lambda m = K + \sigma_1^0[L] - 1 + C - \sigma_1^0[L] + 1 + \lambda m$$
we have

$$C + K + \lambda m \leq K + \sigma_1^0[L] - 1 + [C - \sigma_1^0[L] + 1 + \lambda m]^+,$$

the positive part, and hence

$$\text{spr} \, \mathcal{J}_0 \leq \text{spr}(L + K)^{-1} [K + \sigma_1^0[L] - 1 + [C - \sigma_1^0[L] + 1 + \lambda m]^+].$$

It suffices to show that

$$\text{spr}(L + K)^{-1} [K + \sigma_1^0[L] - 1 + [C - \sigma_1^0[L] + 1 + \lambda m]^+] < 1$$

for $\lambda < 0$ with $|\lambda|$ sufficiently large. Since $m(x) > 0$ a.e. in $\Omega$

$$\lim_{\lambda \downarrow -\infty} [C - \sigma_1^0[L] + 1 + \lambda m]^+ = 0$$
in $L^p(\Omega)$ for all $p \in (1, \infty)$. Thus, by using the continuity of the spectral radius we find that

$$\lim_{\lambda \downarrow -\infty} \text{spr}(L + K)^{-1} [K + \sigma_1^0[L] - 1 + [C - \sigma_1^0[L] + 1 + \lambda m]^+] = \text{spr}(L + K)^{-1} [K + \sigma_1^0[L] - 1].$$

Finally, an easy calculation shows that

$$\text{spr}(L + K)^{-1} [K + \sigma_1^0[L] - 1] = \frac{K + \sigma_1^0[L] - 1}{K + \sigma_1^0[L]} < 1.$$  

The proof is completed.  

**Remark 6.5.** If $m(x) < 0$ a.e. in $\Omega$ then

$$\lim_{\lambda \to -\infty} \sigma_1^0[L - \lambda m] = -\infty, \quad \lim_{\lambda \to -\infty} \sigma_1^0[L - \lambda m] = \infty.$$
The proof of Theorem 6.2 can be adapted to obtain the following “dual version” of Theorem 6.2.

**Theorem 6.6.** Let \( m \in C^1(\bar{\Omega}) \) be such that \( m > 0 \) in \( \Omega \). Suppose there exists a subdomain \( \Omega_o \) of \( \Omega \) such that \( \bar{\Omega}_o \subset \bar{\Omega} \), \( m \equiv 0 \) in \( \bar{\Omega}_o \) and \( m(x) > 0 \) for all \( x \in \bar{\Omega} - \bar{\Omega}_o \). Then,

\[
\lim_{\lambda \to -\infty} \sigma_{1}\left[L - \lambda m\right] = \sigma_{1}\left[L\right].
\]

From this result, the next theorem follows easily.

**Theorem 6.7.** Let \( m \in C^1(\bar{\Omega}) \) be such that \( m > 0 \) in \( \Omega \). Suppose there exist \( h \) subdomains of \( \Omega \), say \( \Omega_i \), \( 1 \leq j \leq h \), with \( \partial \Omega_i \) of class \( C^{2,\alpha} \) for \( 1 \leq j \leq h \), such that \( \bar{\Omega}_i \subset \bar{\Omega} \), \( 1 \leq j \leq h \), \( \bar{\Omega}_j \cap \bar{\Omega}_i \) is empty for \( i \neq j \), \( m \equiv 0 \) in \( \bigcup_{i=1}^{h} \Omega_i \setminus \bigcup_{i=1}^{h} \Omega_i \). Then,

\[
\lim_{\lambda \to -\infty} \sigma_{1}\left[L - \lambda m\right] = \min_{1 \leq j \leq h} \sigma_{1}\left[L\right]. \tag{6.14}
\]

**Proof.** Rearrange the \( \Omega_i \)'s, if necessary, so that

\[
\sigma_{1}\left[L\right] \leq \cdots \leq \sigma_{h}\left[L\right]. \tag{6.15}
\]

Let \( \phi \) denote the principal eigenfunction associated to \( \sigma_{1}\left[L - \lambda m\right] \). Since \( m \equiv 0 \) in \( \Omega_i \),

\[
L \phi = (L - \lambda m) \phi = \sigma_{1}\left[L - \lambda m\right] \phi
\]

in \( \Omega_i \). Moreover, \( \phi > 0 \) on \( \partial \Omega_i \) and hence \( \phi \) is a strict positive supersolution of

\[
L - \sigma_{1}\left[L - \lambda m\right]
\]

in \( \Omega_i \). Thus, it follows from Theorem 2.5 that

\[
\sigma_{1}\left[L - \sigma_{1}\left[L - \lambda m\right]\right] > 0.
\]

Hence,

\[
\sigma_{1}\left[L - \lambda m\right] < \sigma_{1}\left[L\right]
\]

for all \( \lambda \in \mathbb{R} \) and so

\[
\lim_{\lambda \to -\infty} \sigma_{1}\left[L - \lambda m\right] \leq \sigma_{1}\left[L\right]. \tag{6.16}
\]
To complete the proof of (6.14), consider a family $\Omega_{\varepsilon}, \varepsilon > 0$, of subdomains of $\Omega$ such that $\Omega_{\varepsilon} \subset \Omega$, $\bigcup_{j=1}^{\infty} \Omega_{j} \subset \Omega_{\varepsilon}$, for all $\varepsilon > 0$, $\Omega_{\varepsilon_{1}} \subset \Omega_{\varepsilon_{2}}$ if $\varepsilon_{1} < \varepsilon_{2}$, and

$$\bigcap_{\varepsilon > 0} \Omega_{\varepsilon} - \bigcup_{j=1}^{\infty} \Omega_{j} = 0.$$ 

The idea to construct the family $\Omega_{\varepsilon}$ is to connect the $\Omega_{j}$'s by means of narrow strips of width $\varepsilon$. Now, for any $\varepsilon > 0$ consider $m_{\varepsilon} \in C^{\infty}(\Omega)$, $m_{\varepsilon} > 0$, such that $m_{\varepsilon}(x) \equiv 0$ in $\Omega_{\varepsilon}$, $m_{\varepsilon}(x) > 0$ for all $x \in \Omega - \Omega_{\varepsilon}$, and $m > m_{\varepsilon}$ in $\Omega$. Then, for any $\lambda < 0$ and $\varepsilon > 0$ we have

$$\sigma_{0}[L - \lambda m] > \sigma_{0}[L - \lambda m_{\varepsilon}].$$

Moreover, due to Theorem 6.6,

$$\lim_{\lambda \to -\infty} \sigma_{0}^{\frac{1}{2}}[L - \lambda m_{\varepsilon}] = \sigma_{0}^{\frac{1}{2}}[L].$$

Hence, using (6.16), we obtain

$$\sigma_{0}^{\frac{1}{2}}[L] \geq \lim_{\lambda \to -\infty} \sigma_{0}^{\frac{1}{2}}[L - \lambda m] \geq \sigma_{0}^{\frac{1}{2}}[L]$$

(6.17)

for all $\varepsilon > 0$. Since $\sigma_{0}^{\frac{1}{2}}[L]$ decreases as $\varepsilon \to 0$, $\sigma_{0}^{\frac{1}{2}}[L]$ increases as $\varepsilon \to 0$. Moreover, (6.17) implies that $\sigma_{0}^{\frac{1}{2}}[L]$ is bounded above. Thus, the limit $\lim_{\lambda \to 0} \sigma_{0}^{\frac{1}{2}}[L] \in \mathbb{R}$ is well defined. Furthermore, the same argument as in the proof of Theorem 4.2 shows that

$$\lim_{\varepsilon \to 0} \sigma_{0}^{\frac{1}{2}}[L] \in \{ \sigma_{0}^{\frac{1}{2}}[L], \ldots, \sigma_{0}^{\frac{1}{2}}[L] \}.$$

Therefore, it follows from (6.15) and (6.17) that

$$\lim_{\varepsilon \to 0} \sigma_{0}^{\frac{1}{2}}[L] = \sigma_{0}^{\frac{1}{2}}[L]$$

and that

$$\lim_{\lambda \to -\infty} \sigma_{0}^{\frac{1}{2}}[L - \lambda m] = \sigma_{0}^{\frac{1}{2}}[L].$$

The proof is completed. ■

Remark 6.8. If $m < 0$, instead of $m > 0$, we obtain the analogue of Theorem 6.7. In this case,

$$\lim_{\lambda \to -\infty} \sigma_{0}^{\frac{1}{2}}[L - \lambda m] = -\infty, \quad \lim_{\lambda \to -\infty} \sigma_{0}^{\frac{1}{2}}[L - \lambda m] = \min_{1 \leq j \leq h} \sigma_{0}^{\frac{1}{2}}[L].$$
We now use the previous results to construct a class of sign indefinite weights $m(x)$ for which problem (6.1) admits two principal eigenvalues, being $\sigma_1^0(L) < 0$. So improving substantially the main theorem of Hess and Kato [15].

Suppose $\sigma_1^0(L) < 0$ and let $\Omega_p, \Omega_n$ be two subdomains of $\Omega$ such that $\bar{\Omega}_p \subset \Omega, \bar{\Omega}_n \subset \Omega, \bar{\Omega}_p \cap \bar{\Omega}_n$ is empty and

$$\sigma_1^{0^-}(L) > 0.$$ 

Let $m^+, m^- \in C'(\bar{\Omega})$ be such that $m^+(x) > 0$ for all $x \in \Omega_p$, $m^-(x) > 0$ for all $x \in \Omega_n$, $m^+ \equiv 0$ in $\Omega - \Omega_p$, and $m^- \equiv 0$ in $\Omega - \Omega_n$. Consider

$$m := m^+ - m^-.$$ 

Then, due to Theorem 6.2, we find that

$$\lim_{\lambda \to -\infty} \sigma_1^0(L - \lambda m^+ ) = \sigma_1^{0^-}(L) > 0, \quad \lim_{\lambda \to -\infty} \sigma_1^0(L - \lambda m^+ ) = -\infty.$$ 

In particular, there is a unique $\sigma_1^0[L; m^+] \in \mathbb{R}$ such that

$$\sigma_1^0[L - \sigma_1^0[L; m^+] m^+] = 0.$$ 

Moreover, since $\sigma_1^0(L) < 0$, we have

$$\sigma_1^0[L; m^+] < 0.$$ 

Note that

$$\sigma_1^0[L - \lambda m^+] > 0 \quad \text{for all} \quad \lambda < \sigma_1^0[L; m^+] .$$ 

Set $m^- = \sup_{\Omega} m^-$. Then, for any $\lambda < \sigma_1^0[L; m^+]$ we obtain

$$\sigma_1^0[L - \lambda m] = \sigma_1^0[L - \lambda m^+ + \lambda m^- ] > \sigma_1^0[L - \lambda m^+] + \lambda m^- ,$$ 

and hence, if we assume

$$m^- \leq \sup_{\lambda < \sigma_1^0[L; m^+]} \frac{\sigma_1^0[L - \lambda m^+]}{-\lambda} \tag{6.18}$$ 

then $\sigma_1^0[L - \lambda m] > 0$ for some $\lambda < \sigma_1^0[L; m^+]$. Therefore, due to the fact that

$$\lim_{\lambda \to -\infty} \sigma_1^0[L - \lambda m] = -\infty, \quad \lim_{\lambda \to -\infty} \sigma_1^0[L - \lambda m] = -\infty, $$
and since the mapping \( \lambda \mapsto \sigma_1^2[L - \lambda m] \) is concave, (6.1) admits two principal eigenvalues, which are negative and simple in the sense of the theorem of Hess and Kato [15]. Note that the function \( f(\lambda) \) defined by
\[
f(\lambda) := \frac{\sigma_1^2[L - \lambda m^+] - \lambda}{\sigma_1^2[L; m^+]},
\]
is positive and satisfies \( f(\sigma_1^2[L; m^+]) = 0, \lim_{\lambda \to -\infty} f(\lambda) = 0 \). So, the right hand side of (6.18) is a positive constant.

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