Skeletal Rigidity of Simplicial Complexes, II†

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This is the second part of a two-part paper, the first part of which appeared in an earlier issue of this journal. The notation and terminology follow those of the earlier part.

The paper concerns a generalization of infinitesimal rigidity from a graph (or one-dimensional simplicial complex) embedded in d-space to a higher-dimensional simplicial complex, again embedded in d-space. This part begins with a section on coning, an important construction which preserves rigidity and stress. Then we investigate the connections with the g-theorem, which characterizes the possible f-vectors of simplicial polytopes. This connection, and the possibility of a combinatorial proof of the g-theorem which it provides, was the original motivation behind the entire paper. Then we give two additional versions of r-rigidity and r-stress, which are equivalent to the three versions already given in part I. We conclude with a discussion of avenues for further work.

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1. CONE THEOREM

Let Δ be a simplicial complex, and let Δ′ = Δ * a be the cone, where a ∉ Δ(0). Suppose that we realize Δ′(r) in (d + 1)-space and centrally project Δ′ from the vertex q to a d-dimensional subspace H. This projection gives a realization of Δ(r−1) in d-space. We want to establish a correspondence between the r-rigidity of Δ and Δ′ in these realizations. For any vertex x (≠a) of Δ′, let Π(x) denote its (homogeneous) co-ordinates in H. For any extensor γ”, γH, Π(xy · · · z) denotes the extensor Π(xy · · · z) in H. We weight Π(x) so that x = Π(x) + αx a for some scalar αx. This means aΠ(ρ) = αx a for all ρ ∈ Δ(r−1).

First we find a injection of the r-motions of Δ into the r-motions of Δ′. We shall work with R,(Δ), since there are fewer trivial motions. The image of this injection will be the following space.

Let Fixa(Δ′) ⊆ Motiona(Δ′) be the subspace consisting of motions which are zero on πa, for all π ∈ Δ(r−3). Then, for all M ∈ Fixa(Δ′) and all σ ∈ Δ(r−2), M ○ Row(σ) = M(σ)a = 0. Since, in addition, M(σ)x = 0 for all x ∈ σ,

\[ M(σ) = \begin{cases} S_σ a & \text{if } σ ∈ Δ(r−2) \\ 0 & \text{otherwise}, \end{cases} \]

where Sσ is a step d + 1 − r tensor in H. (Since every step d + 1 − r tensor S can be written as S1 + S2a, S1 in H and S2a = S1 a, we may assume that Sσ is in H.)

THEOREM 1.1. (i) Motionr(Δ) = Fixa(Δ′).
(ii) Non Trivr(Δ) = Non Trivr(Δ').

PROOF. Let f: Motionr(Δ) → Fixa(Δ′) be the linear function defined by f(M) = M' where, for all σ ∈ Δ(r−2),

\[ M'(σ) = \begin{cases} M(σ)a & \text{if } σ ∈ Δ, \\ 0 & \text{otherwise}. \end{cases} \]

† Dedicated to the memory of Paul Filliman.

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We first show that \( f(M) = M' \in \text{Motion}^\tau_\Delta(\Delta') \). Since \( M(\sigma) = S_\sigma \Pi(\tilde{\sigma}) \) for some step \( d - r \) tensor \( S_\sigma \) in \( \mathbf{H} \), \( M'(\sigma) = S_\sigma \Pi(\tilde{\sigma}) \tilde{d} = S_\sigma \tilde{d} \tilde{a} \). Thus equation (6.1) of [17] is satisfied.

For all \( \rho \in (\Delta')^{(r-1)} \), we have

\[
M' \vee \text{Row}^\tau_\Delta(\rho) = \sum_{x: o \rho = \rho} M'(\sigma) \tilde{x} = \sum_{x: o \rho = \rho} S_\sigma \tilde{d} \tilde{x}.
\]

If \( a \parallel \rho \), then \( \tilde{\sigma} \tilde{x} = 0 \). If \( a \perp \rho \),

\[
\sum_{x: o \rho = \rho} M(\sigma) \tilde{d} \tilde{x} = \sum_{x: o \rho = \rho} M(\sigma) \tilde{\sigma} \tilde{a} \tilde{x} = 0.
\]

Thus in all cases, \( M' \vee \text{Row}^\tau_\Delta(\rho) = 0 \). Hence (6.2) of [17] is also satisfied. Clearly, \( M'(\pi a) = 0 \), so \( M' \in \text{Fix}_\rho(\Delta') \).

Next we prove that \( f \) is onto. For all \( M' \in \text{Fix}_\rho(\Delta') \), we have \( M'(\sigma) = S_\sigma \tilde{d} \tilde{a} \) if \( \sigma \in (\Delta')^{(r-2)} \) and \( 0 \) otherwise. Define \( M(\sigma) = S_\sigma \Pi(\tilde{\sigma}) \). We want to show that \( M \in \text{Motion}^\tau_\Delta(\Delta) \). For all \( \rho \in \Delta^{(r-1)} \),

\[
M' \vee \text{Row}^\tau_\Delta(\rho) = \sum_{x: o \rho = \rho} M'(\sigma) \tilde{x} = \sum_{x: o \rho = \rho} S_\sigma \tilde{d} \tilde{x} = \sum_{x: o \rho = \rho} S_\sigma \Pi(\tilde{\sigma}) \tilde{a} \tilde{x} = 0.
\]

Since, for tensors \( P \) in \( \mathbf{H} \), \( P \vee \tilde{d} = 0 \) iff \( P = 0 \), we have

\[
M \vee \text{Row}^\tau_\Delta(\rho) = \sum_{x: o \rho = \rho} M(\sigma) \Pi(\tilde{x}) = \sum_{x: o \rho = \rho} S_\sigma \Pi(\tilde{\sigma}) \tilde{a} \tilde{x} = 0,
\]

and \( M \in \text{Motion}^\tau_\Delta(\Delta) \). Thus \( f \) is one-to-one and the proof is complete for the first part.

For the second part, we first note that, for all \( \pi \in \Delta^{(r-3)} \), \( f(T^\tau_{\pi, \mu a}(\Delta)) = T^\tau_{\pi, \mu a}(\Delta') \). We call this image \( f(\text{Triv}^\tau_\Delta(\Delta)) = \text{Triv}(\text{Fix}_\mu(\Delta')) \). It is a simple exercise to see that these are all the trivial motions in \( \text{Fix}_\rho(\Delta') \).

Finally, we show that any \( M \in \text{Motion}^\tau_\Delta(\Delta') \) can be written as a linear combination of members in \( \text{Fix}_\rho(\Delta') \) and \( \text{Triv}^\tau_\Delta(\Delta') \).

For all \( \pi \in \Delta^{(r-3)} \), \( M(\pi a) = S_\pi \tilde{d} \tilde{a} \) for some step \( d + 2 - r \) tensor \( S_\pi \). As above, we can assume that \( S_\pi \) is in \( \mathbf{H} \). Using the trivial motions \( T^\tau_{\pi, S_\pi} \) for these extensors \( S_\pi \), we define \( N = M \vee \sum_{\pi \in \Delta^{(r-3)}} T^\tau_{\pi, S_\pi} \). For all \( \mu \in \Delta^{(r-3)} \),

\[
N(\mu a) = M(\mu a) - \left( \sum_{\pi \in \Delta^{(r-3)}} T^\tau_{\pi, S_\pi}(\mu a) = S_\mu \tilde{d} \tilde{a} = S_\mu \tilde{d} \tilde{a} = 0.
\]

This means that \( N \in \text{Fix}_\rho(\Delta') \).

We conclude that

\[
\text{Non Triv}^\tau_\Delta(\Delta') = \text{Fix}_\rho(\Delta')/\text{Triv}(\text{Fix}_\rho(\Delta')) = \text{Non Triv}^\tau_\Delta(\Delta').
\]

**Corollary 1.2.** \( \Delta' = \Delta \ast a \) is \( r \)-rigid in \( d + 1 \)-space if \( \Delta \), as a projection of \( \Delta' \) from the vertex \( a \), is \( r \)-rigid in \( d \)-space.

While the theorem shows the equivalence of static \( r \)-rigidity for \( \Delta' \) and its projection \( \Delta \), it does not demonstrate an isomorphism of the \( r \)-stresses.

**Theorem 1.3.** \( \text{Stress}^r_\Delta(\Delta) = \text{Stress}^r_\Delta(\Delta') \).

**Proof.** Let \( \lambda \) be an \( r \)-stress in \( \text{Stress}^r_\Delta(\Delta') \). Then, for all \( \sigma \in \Delta^{(r-2)} \),

\[
0 = \sum_{x: o \sigma x \in \Delta'} \lambda_{o \sigma x} \tilde{d} \tilde{x} = \sum_{x: o \sigma x \in \Delta} \lambda_{o \sigma x} \tilde{d} \tilde{x} + \lambda_{o \sigma} \tilde{d} \tilde{a}.
\]
Thus $\sum_{x:ox \in \Delta} \lambda_{ox} \Pi(\tilde{\sigma} \tilde{x}) = 0$, and $\lambda$ restricted to $\Delta$ is in $\text{Stress}_r^p(\Delta)$.

Next we show that every $r$-stress $\lambda$ of $\Delta$ can be extended (uniquely) to an $r$-stress of $\Delta'$. For any $\sigma \in \Delta^{(r-2)}$, we assume that $\sum_{x:ox \in \Delta} \lambda_{ox} \Pi(\tilde{\sigma} \tilde{x}) = 0$. But $\tilde{\sigma} \tilde{x} = \Pi(\tilde{\sigma} \tilde{x}) + S_\sigma \tilde{a}$ for some step $r-1$ tensor $S_\sigma$ in $\mathcal{H}$. So

$$\sum_{x:ox \in \Delta} \lambda_{ox} \tilde{\sigma} \tilde{x} = \sum_{x:ox \in \Delta} \lambda_{ox} (\Pi(\tilde{\sigma} \tilde{x}) + S_\sigma \tilde{a}) \Rightarrow \tilde{\sigma} \sum_{x:ox \in \Delta} \lambda_{ox} \tilde{x} = \sum_{x:ox \in \Delta} \lambda_{ox} S_\sigma \tilde{a}.$$

Take the join with $\tilde{a}$, to obtain $\tilde{\sigma} \tilde{a} \sum_{x:ox \in \Delta} \lambda_{ox} \tilde{x} = 0$. Since $\tilde{\sigma} \tilde{a} \neq 0$, we have

$$\tilde{\sigma} \sum_{x:ox \in \Delta} \lambda_{ox} \tilde{x} = \gamma_\sigma \tilde{a},$$

for some constant $\gamma_\sigma$. Now define $\lambda_{ox} = -\gamma_\sigma$. This extends $\lambda$ to $\Delta'$ with the property that, for all $\sigma \in \Delta^{(r-2)}$,

$$\sum_{x:ox \in \Delta'} \lambda_{ox} \tilde{\sigma} \tilde{x} = 0.$$

To show that this extension gives an $r$-stress of $\Delta'$, we need to show that, for every $\pi \sigma \in \Delta'^{(r-2)}$, we have

$$\sum_{x:ox \in \Delta'} \lambda_{max} \tilde{\pi} \tilde{x} = 0.$$

Since $\pi \sigma \in \Delta$, we have, from the previous paragraph,

$$\lambda_{max} \tilde{\pi} \tilde{x} \tilde{a} + \sum_{y:xy \in \Delta} \lambda_{max} \tilde{\pi} \tilde{x} \tilde{y} = 0.$$

Summing over all $x$, we have

$$\sum_{x:ox \in \Delta'} \lambda_{max} \tilde{\pi} \tilde{x} \tilde{a} + \sum_{x:ox \in \Delta'} \sum_{y:xy \in \Delta} \lambda_{max} \tilde{\pi} \tilde{x} \tilde{y} = 0.$$

Since every term in the double summation occurs twice with opposite signs, we have the desired result. 

For completeness, we state, without proof, the connection between the $r$-loads.

**Proposition 1.4.** Let $\text{L}(\Delta')$ be the subset of $\text{Load}_r^p(\Delta')$ where, for $L \in \text{L}(\Delta')$,

$$L(\sigma) = \begin{cases} \alpha_\sigma \tilde{\sigma} \tilde{a} & \text{if } \sigma \in \Delta^{(r-2)} \\ 0 & \text{otherwise} \end{cases}.$$

Then $\text{L}(\Delta')$ is the subspace generated by the rows $\{\sigma \alpha: \sigma \in \Delta^{(r-2)}\}$ of $\text{R}_r^p(\Delta')$.

$$\text{Load}_r^p(\Delta')/\text{L}(\Delta') = \text{Load}_r^p(\Delta).$$

As a corollary to the proofs of 1.1 and 1.3, we have the following theorem about general projections. A **general projection of $\Delta$ for $r$-rigidity** is a projection $\Pi'$ of the points of $\Delta$ into a hyperplane $H$ such that, for each $\sigma \in \Delta^{(r-1)}$, $\{\Pi x: x \in \sigma\}$ is projectively independent.

**Corollary 1.5.** If $\Delta$ is realized in $d + 1$-space, and $\Pi \Delta$ is a general projection into $d$-space, then there is an injection from $\text{NonTriv}_r(\Pi \Delta)$ into $\text{NonTriv}_r(\Delta)$ and an injection from $\text{Stress}_r(\Delta)$ into $\text{Stress}_r(\Pi \Delta)$. In particular:

(i) $\Delta$ is $r$-rigid implies that $\Pi \Delta$ is $r$-rigid;

(ii) $\Pi \Delta$ has only the trivial $r$-stress implies that $\Delta$ has only the trivial $r$-stress.
PROOF. Simply take a cone of $\Delta$ from the point of projection, creating $\Delta'$. (If this a point at infinity, a projective transformation will change this to a finite point if desired.) The proof of Theorem 1.1 shows that $\text{NonTriv}^{r}(\text{II} \Delta) = \text{Fix}_{a}(\Delta')/\text{Triv}(\text{Fix}_{a}(\Delta'))$. Because motions in $\text{Fix}_{a}(\Delta')$ are 0 on $(r-2)$-faces outside $\Delta$, the proof gives an injection from $\text{Fix}_{a}(\Delta')$ into $\text{Motion}^{r}_{a}(\Delta)$, which takes $\text{Fix}_{a}(\Delta')/\text{Triv}(\text{Fix}_{a}(\Delta'))$ into $\text{NonTriv}_{a}(\Delta)$.

If we simply omit the scalars for faces containing $a$, the proof of Theorem 1.3 clearly shows the required injection from $\text{Stress}^{r}_{a}(\Delta)$ into $\text{Stress}^{r}_{a}(\text{II} \Delta)$.

With these coning theorems, we can now confirm that $K_{r}$ is $r$-rigid for all meaningful $r$ and $d$. A general position realization in $d$-space has no $m + 1$ vertices in the projective span of $m$ of the vertices, $m \leq d$.

**Theorem 1.6.** (i) If $\Delta = K_{n}$ is in general position in $d$-space, $d \geq r - 1$, then $\Delta$ is $r$-rigid.

(ii) If $\Delta = K_{n}$ is in general position in $d$-space, $d + 1 \geq n$, then $\Delta$ has only the trivial $r$-stress.

**Proof.** (a) Assume that $d + 1 \geq n \geq r$. We can project down $n - r$ times from a sequence of vertices to obtain $K_{r}$ in dimension $(d - n + r) \geq r - 1$. Since this projection is also in general position, by Proposition 3.8 of [17] it is $r$-rigid. Since the matrix $R_{r}^{n}$ has at most one row (which will be non-zero) there are only the trivial $r$-stresses. When we now cone back up $n - r$ times, the original $K_{r}$ is also $r$-rigid, by the coning theorem.

(b) Assume that $n \leq r + 1$. By Remark 3.9 of [17], this is $r$-rigid, and has only the trivial $r$-stress.

(c) Assume that $n > d \geq r - 1$. Assume that $d - r + 1 = i \geq 0$. We can project down $i$ times to obtain $K_{n-i}$ in $(r - 1)$-space, with $n - i \geq r$. This projection is in general position in $(r - 1)$-space. This is $r$-rigid by Corollary 4.2 of [17]. When we cone back up $i$ times, the original $K_{r}$ is also $r$-rigid.

**Remark 1.7.** Since a general position $K_{r}$ is $r$-rigid in $d$-space, $d \geq r - 1$, we know that the trivial $r$-motions on a simplicial complex $\Delta$, in general position, are precisely the restrictions of the $r$-motions of $K_{r}$ on the same vertices. With some additional attention to details, we could show that:

(iii) $K_{r}$ is $r$-rigid in $d$-space, $n > d$, if the vertices of $K_{r}$ projectively span the space;

(iv) $K_{r}$ is $r$-rigid in $d$-space, $n \leq d$, if the vertices of $K_{r}$ are projectively independent.

Thus the trivial $r$-motions on any simplicial complex are the restriction of the $r$-motions of a complete $K_{r}$ on a possibly larger set of vertices which projectively span the space, without concern for 'general position'.

2. INTERPRETING THE $g$-THEOREM

In this section, we explain the connection between $r$-rigidity and the $g$-theorem of polyhedral combinatorics. It turns out that this is also connected with trying to count the dimension of the space of trivial motions. The results and conjectures of this section will be stated in terms of the truncated face-ring ridigity matrix, but dim $\text{Stress}$ and dim $\text{NonTriv}$, are independent of which version of the matrix we use (see Theorem 4.7). However, for dim $\text{Triv}$, we must use either the truncated face-ring matrix or the minimal matrix (see Section 3).

**Remark 2.1.** An heuristic count for the dimension of $\text{Triv}^{r}_{a}$ can be carried out as
follows. In $d$-space, a set $P$ of $d + 1$ independent points is a basis for the space of $1$-extensors. The $j$-subsets of $P$ give $\binom{d + 1}{j}$ independent $j$-extensors which generate the space of tensors of step $j$. If $Q$ is a $q$-element subset of $P$, then the space of $j$-tensors, modulo the space of $j$-tensors generated by $j$-extensors containing at least one element of $Q$ as a factor, has dimension $\binom{d + 1}{j - q}$ since it is determined by the $j$-subsets of $P - Q$.

For all $\pi \in \Delta(r-3)$, if $S$ is a $(d - r + 1)$ tensor such that $S \vee \pi = 0$, $T_{\pi,S}^T = 0$. Thus for $T_{\pi,S}^T$ to be non-zero, $S \vee \pi \neq 0$. So there are $\binom{d + 3 - r}{2}$ independent choices for $S$. This gives a count of $\binom{d + 3 - r}{2} f_{r-3}$ trivial motions.

There are some obvious linear relations on these trivial motions. For each $\mu \in \Delta(r-4)$, we have, if $\sigma = \mu ab$,

$$\sum_{x : x \in A^{(r-3)}} \text{Sign}[\mu, x] T_{\mu x, S}(\sigma) = \text{Sign}[\mu, a] T_{\pi a, S}(\mu ab) + \text{Sign}[\mu, b] T_{\mu b, S}(\mu ab)$$

$$= \text{Sign}[\mu, a] S \mu \bar{ab} + \text{Sign}[\mu, b] S \mu \bar{ba}$$

$$= S \mu \bar{a} \bar{b} + S \mu \bar{b} \bar{a} = 0$$

and $\sum_{x : x \in A^{(r-3)}} T_{\mu x, S}(\sigma) = 0$ if $\mu \not\equiv \sigma$. Thus $\sum_{x : x \in A^{(r-3)}} T_{\mu x, S} = 0$. We need $S \vee \mu \bar{x} \neq 0$ for at least one $x$, so that this relation involves at least one $T_{\pi,S}^T$ counted above. For this, it is sufficient that $S \vee \mu \neq 0$; hence there are $\binom{d + 4 - r}{d - r + 1} = \binom{d + 4 - r}{d - r + 1}$ independent choices for $S$, and the total count is now $\binom{d + 3 - r}{2} f_{r-3} - \binom{d + 4 - r}{2} f_{r-4}$. Now we add back in relations among these relations, one for each face in $\Delta^{(r-5)}$, etc. Thus we have the following heuristic count for the dimension of the space of trivial motions $\text{Triv}^r(\Delta)$:

$$\dim(\text{Triv}^r(\Delta)) = \sum_{i=3}^{r+1} (-1)^{i-1} \binom{d + 1 - i}{d - 2} f_{r-i}$$

$$= \sum_{j=-1}^{r-3} (-1)^{r+j+1} \binom{d - j}{d - r + 1} f_j.$$
Since
\[ f_{r-1} - (d + 2 - r) f_{r-2} = \#\text{rows}(R^r_\Delta) - \#\text{columns}(R^r_\Delta) \]
\[ = \text{rank}(R^r_\Delta) + \dim \text{Stress}_r(\Delta) - \text{rank}(R^r_\Delta) - \dim \text{Motion}^r_\Delta(\Delta), \]
the proposition follows. \[\square\]

**Corollary 2.4.** Let \( \Delta \) be realized in dimension \( d \). Then
\[ g_r(\Delta, d) = \dim \text{Stress}_r(\Delta) - \dim \text{Non Triv}^*_r(\Delta) \]
if
\[ \dim \text{Triv}^*_r(\Delta) = \sum_{j=-1}^{r-3} (-1)^{r+j+1} \binom{d - j}{d - r + 1} f_j. \]

**Corollary 2.5.** Suppose that \( K_n \) is realized in general position in \( d \)-space, for \( n \geq r \)
and \( d = n - 1, \cdots, n + r - 2 \). Then we have
\[ \dim(\text{Triv}^*_r(K_n)) = \sum_{j=-1}^{r-3} (-1)^{r+j+1} \binom{d - j}{d - r + 1} \binom{n}{j+1}. \]

**Proof.** By Theorem 1.6, \( K_n \), in general position, is \( r \)-rigid and \( r \)-stress-free in \( d \)-space for all \( d \geq n - 1, n \geq r \geq 1 \). Hence, by Corollary 2.4, it suffices to prove that \( g_r(K_n, d) = 0 \) for \( n \geq r \) and \( n - 1 \leq d \leq n + r - 2 \). But \( g_r(K_n, n - 1) = h_r(K_n, n) = (1, 0, 0, \cdots, 0) \). Now, inducting on \( d \), using \( g_r(\Delta, i + 1) = g_r(\Delta, i) - g_{r-1}(\Delta, i) \) for all \( \Delta \), we obtain the desired result. \( \square \)

**Corollary 2.6.** If \( \Delta \) is realized in dimension \( d \) so that it has only the trivial \( r \)-stress and only trivial \( r \)-motions, then
\[ g_r(\Delta, d) = \sum_{j=-1}^{r-3} (-1)^{r+j+1} \binom{d - j}{d - r + 1} f_j - \dim \text{Triv}^*_r(\Delta). \]

**Proposition 2.7.** For any Cohen–Macaulay \( d \)-complex \( \Delta \) realized in sufficiently general position in \( d \)-space, and \( r \leq d + 1 \),
\[ \dim(\text{Motion}^r_\Delta(\Delta)) = \sum_{i=3}^{r+1} (-1)^{i-1} \binom{d + i - r}{i - 1} f_{r-i}. \]
Thus this sum is an upper bound for \( \dim \text{Triv}^*_r(\Delta) \), with equality if such a realization is \( r \)-rigid.

**Proof.** Lee [12] proves that \( \Delta \) is Cohen–Macaulay iff the dimension of the space of what he calls linear \( r \)-stresses is \( h_r(\Delta, d + 1) \), for \( r = 1, 2, \cdots, d \) for a generic realization of \( \Delta \) in dimension \( d + 1 \). Lee's linear \( r \)-stress for \( \Delta \) realized in dimension \( d + 1 \) is the same as Lee's affine \( r \)-stress for \( \Delta \) realized in dimension \( d \), which is also the same as our \( r \)-stress for \( \Delta \) realized in dimension \( d \). But \( h_r(\Delta, d + 1) = g_r(\Delta, d) \) for any simplicial complex; thus for a generic (or sufficiently general) realization
of $\Delta$ in dimension $d$, $\dim(\text{Stress}_r(\Delta)) = g_r(\Delta, d)$. The proposition now follows from Proposition 2.3.

As we have just seen, $K_n$ is an example where the trivial $r$-motions attain the upper bound. In our sequel [16], we will show that this bound is also met for any shellable, $d$-complex in $d$ and $(d + 1)$-space, for all $r \leq d$. Tay [15] also proved that $d$-dimensional PL-spheres are $r$-rigid in $d$-space for all $r \leq d + 1$, thus providing another example. Finally, in [16] we show that all homology $d$-spheres realized in general position in $d$-space are $r$-rigid for all $r$, extending the result of Tay.

**Conjecture 2.8.** Let $\Delta$ be a $(d' - 1)$-dimensional Cohen–Macaulay complex realized in general position in $d$-space, with $d \leq d' + r - 1$, $r \leq d' + 1$. Then

$$g_r(\Delta, d) = \dim \text{Stress}_r(\Delta) - \dim \text{Non Triv}_r(\Delta).$$

Equivalently,

$$\dim \text{Triv}^T_r(\Delta) = \sum_{i=3}^{r+1} (-1)^{i-1} \binom{d + i - r}{i - 1} f_i.$$

**Theorem 2.9 ([16]),** Conjecture 2.8 is true for $\Delta$ shellable.

**Conjecture 2.10.** Let $\Delta$ be a $(d - 1)$-sphere in $d$-space realized in generic position, and $r \leq (d + 1)/2$. Then:

1. $\Delta$ is $r$-rigid; and
2. $g_r(\Delta, d) = \dim \text{Stress}_r(\Delta)$.

The second conclusion of this conjecture, if proved for some class of spheres, would, using Corollary 12 of Lee [12], give a combinatorial proof of the full $g$-theorem for that class of spheres. The first conclusion of the conjecture, if proved for some class of shellable spheres, such as simplicial polytopes, would immediately imply the second conclusion, using Theorem 2.9, and hence the full $g$-theorem for that class. In fact, it suffices to prove the first conclusion only in the case $r = (d + 1)/2$, $d$ odd.

**Conjecture 2.11.** Let $\Delta$ be a $(d - 1)$-sphere in $d$-space realized in generic position, and $r \geq (d + 1)/2$. Then $\Delta$ has only trivial $r$-stresses, and $-g_r(\Delta, d) = \dim \text{Motion}^T_r(\Delta)$.

We have an even stronger version of Conjecture 2.10, which is, like the previous ones, true for $r = 2$ and is due in this case to Fogelsanger [8].

**Conjecture 2.12.** If $\Delta$ is a triangulated $(d - 1)$-pseudo-manifold realized in generic position in $d$-space, and $r \leq (d + 1)/2$, then $\Delta$ is $r$-rigid and $g_r(\Delta, d) = \dim \text{Stress}_r(\Delta)$.

### 3. The Minimal Rigidity Matrix

We already have three matrices for $r$-rigidity. Why do we need another one? If we examine the available matrices, it appears that $R^T_r(\Delta)$ is very good for $r$-stresses and $r$-loads. It was the matrix that we used for the statics of $K_n$, and the statics of coning. However, it has a large set of trivial $r$-motions, so we did not use it for kinematic arguments. On the other hand, $R^T_r(\Delta)$, while awkward for statics because of its extra loads, is good for kinematics, because we eliminated the trivial motions of the first kind. Also, the remaining trivial motions have a very simple form. Accordingly, this
was the matrix we used for the kinematics of coning. The matrix that we introduce here seeks the best of both worlds—the simple statics of $R^T(\Delta)$ and the reduced trivial motions of $R_{r}(\Delta)$. Moreover, for 2-rigidity it will match the standard euclidean matrix, but for larger $r$ will have fewer columns (and no trivial motions of the first kind).

With $R^T(\Delta)$, we eliminated the trivial motions of the first kind by adding new independent rows, $Row^T(\sigma a)$, for each $a \in \Delta^{(r-2)}$. We can reduce the number of rows and columns by row reducing the entire matrix, block by block on these added rows $Row^T(\sigma a)$ and then remove both these rows and the columns of their leading entries. This will leave the dimensions of the cokernel (the space of $r$-stresses) and the kernel (the space of $r$-motions) unchanged. This creates the ‘minimal’ matrix $R^M(\Delta)$, with $f_{r-1}$ rows and $(d + 2 - r)f_{r-2}$ columns. While the pattern of non-zero blocks remains unchanged, the particular form of the new non-zero entries is less clear.

We could present $R^M(\Delta)$ in terms of the above modifications of $R^T(\Delta)$, much as we presented $R^T(\Delta)$ in terms of $R^F(\Delta)$. However, we prefer to a more geometric presentation of $R^M(\Delta)$, as follows. Take a subspace $U$ in general position of dimension $d + 1 - r$ in projective space. Then, for each $\rho \in \Delta^{(r-1)}$, $(\rho)_{\rho} \cap U$ is a single point $u_{\rho}$, and for each $\sigma \in \Delta^{(r-2)}$, $(\sigma)_{\rho} \cap U = \emptyset$. For $\rho = \sigma x$, we replace (by row reduction) the entry $R^M(\rho, \sigma) = \bar{x}$ by the entry $Sign[\sigma, x]u_{\rho}$. We now drop the additional rows in $R^T$, and reduce the columns by a switch to homogeneous co-ordinates in the space $U$. Clearly, this leaves the dimension of the space of $r$-stresses and the dimension of the kernel (the space of motions) unchanged. This coincides with the previous analysis, for example, if we transform $\Delta$ so that the first $r - 1$ columns of the non-zero block for $a \in \sigma$ are independent, for all $\sigma \in \Delta^{(r-2)}$, and choose $U$ to be the space spanned by a basis for the last $d + 2 - r$ columns of a block.

We combine these approaches in an explicit way. Take a $(d + 2 - r)$-extensor $\tilde{U}$ for points spanning the space $U$, such that $[\tilde{\sigma} \tilde{U}] = \tilde{\sigma} \vee \tilde{U} \neq 0$ for all $\sigma \in \Delta^{(r-2)}$. (By a projective transformation, the reader can assume that $U = \langle e_{1}, \ldots, e_{d+1} \rangle_{\rho}$ and $\tilde{U} = \bar{e}_{r} \vee \cdots \vee e_{d+1}$, for a standard basis $\{e_{1}, \ldots, e_{d+1}\}$ of the vector space.) Now replace each entry in the projective rigidity matrix $R^F(\rho, \sigma) = Sign[\sigma, x]u_{\rho}$, by

$$R^M(\rho, \sigma) = Sign[\sigma, \rho](\tilde{\rho} \wedge \tilde{U}) = Sign[\sigma, x]\tilde{u}_{\rho},$$

where $\wedge$ is the Grassmann–Cayley operation on extensors spanning the space (see [6]), which produces a 1-extensor representing the unique point of intersection. (Equivalently, $\tilde{u}_{\rho}$ is defined below, in equation (3.1); see comments following equation (3.3).) When we switch to homogeneous co-ordinates for $U$, this gives us a version of the minimal $r$-rigidity matrix $R^M(\Delta)$.

The above remarks on the row reductions show that the space of (minimal) $r$-stresses $Stress^T(\Delta)$ is indeed isomorphic to the space

$$Stress^T(\Delta) = Stress^F(\Delta) = Stress^T(\Delta)$$

and there are no new $r$-stresses. However, we offer a constructive proof which clarifies the exact relationships among the $r$-stress spaces. We will then define minimal $r$-motions, and minimal $r$-loads and show that all the usual correspondences hold.

First we establish some notation. For any point $\bar{x} \in U$ (the first $r - 1$ co-ordinates are 0), we let $\bar{\bar{x}}$ be the homogeneous co-ordinates of $\bar{x}$ in $U$ (the last $d + 2 - r$ co-ordinates in the specific model). Conversely, for any point $\bar{x} \in U$, let $\bar{x}$ be its co-ordinates back in the full space (add $r - 1$ zeros at the beginning, in the specific model). Finally, for any $\{a_{1}, \ldots, a_{r-1}\} = \sigma \in \Delta^{(r-2)}$, let $\sigma_{i} = \{a_{1}, \ldots, a_{i-1}, a_{i+1} \cdots a_{r-1}\}$. 

T.-S. Tay et al.
PROPOSITION 3.1. For all $\sigma \in \Delta^{(r-2)}$ and any $x \not\parallel \sigma$, define

$$\bar{x}_\sigma = [\bar{\sigma} \bar{U}] \bar{x} + \sum_{a_i \in \sigma} (-1)^{r-1} [\bar{\sigma}_i \bar{\sigma} \bar{U}] \bar{a}_i = (\bar{\sigma} \bar{x}) \wedge \bar{U}. \quad (3.1)$$

Then we have $\bar{x}_\sigma \in U$ and

$$\bar{\sigma} \bar{x} = \bar{\sigma} \bar{x}_\sigma/[\bar{\sigma} \bar{U}]. \quad (3.2)$$

PROOF. Equation (3.1) gives the explicit formula for calculation of the Grassman $\wedge$ (or interior product); see Doubilet et al. [6]. This gives projective co-ordinates for the point of intersection of the spaces $\langle \sigma x \rangle_p$ and $\langle U \rangle_p$. Direct verification shows that $\bar{\sigma} \bar{x}_\sigma = [\bar{\sigma} \bar{U}] \bar{x}$, since $\bar{\sigma} \bar{a}_i = 0$ for $a_i \in \sigma$. Since $\bar{x}_\sigma = (\bar{\sigma} \bar{x}) \wedge \bar{U}$, $\bar{x}_\sigma \in U$. □

The minimal $r$-rigidity matrix $R^M_r$ has its rows indexed by $\Delta^{(r-1)}$ and its columns indexed by $\Delta^{(r-2)}$, with

$$R^M_r(\rho, \sigma) = \begin{cases} \bar{x}_\sigma & \text{if } \rho = \sigma x, \\ 0 & \text{if } \sigma \not\parallel \rho, \end{cases} \quad (3.3)$$

where $\bar{x}_\sigma$ is defined in (3.1), and $\bar{x}_\sigma$ gives the co-ordinates in $U$. Note that all entries in the row for $\rho$ are the same point, up to sign, since they are $\text{Sign}[\sigma, \rho](\bar{\sigma} \bar{U})$, expressed in co-ordinates for $U$. We also write this entry as $\text{Sign}[\sigma, \rho] \bar{u}_\rho$, where $\bar{u}_\rho = \bar{\sigma} \bar{U}$. (For comparison with the euclidean matrix, we note that $\bar{w}_x, \bar{u}$ projects to $x, u$ from $\langle \sigma \rangle_E$, if we have chosen $U$ at infinity.)

EXAMPLE 3.2. Consider the simplicial complex consisting of the four triangles of a triangular pyramid in 3-space (see Figure 1). Letting $r = 3$ and since $d = 3$, $U$ is a 1-dimensional space. Then the minimal rigidity matrix is as follows:

<table>
<thead>
<tr>
<th>$R^M_3$</th>
<th>$ab$</th>
<th>$ac$</th>
<th>$ae$</th>
<th>$af$</th>
<th>$bc$</th>
<th>$ce$</th>
<th>$ef$</th>
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</tr>
</thead>
<tbody>
<tr>
<td>$abc$</td>
<td>$\bar{u}_{abc}$</td>
<td>$\bar{u}_{abc}$</td>
<td>$\bar{u}_{abc}$</td>
<td>$\bar{u}_{abc}$</td>
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</table>

For this matrix, we can see that there are no non-trivial stresses. (For example, $\lambda_{abc} \bar{u}_{abc} + \lambda_{abf} \bar{u}_{abf} = 0$ requires that $\lambda_{abc} = \lambda_{abf} = 0$, since the points are distinct.) The rows $M_1$, $M_2$ and $M_3$ represent 'minimal motions' (with entries $\bar{x} \in U$ and $\bar{u}_{ab} \in U$ for

![Figure 1](image-url)
an arbitrary $y$, they are orthogonal to the rows of the rigidity matrix). We will see below that $M_1$ and $M_2$ are trivial, but that $M_3$ is not:

<table>
<thead>
<tr>
<th></th>
<th>$ab$</th>
<th>$ac$</th>
<th>$ae$</th>
<th>$af$</th>
<th>$bc$</th>
<th>$ce$</th>
<th>$ef$</th>
<th>$bf$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M_1$</td>
<td>$\bar{u}$</td>
<td>$\bar{u}$</td>
<td>$\bar{u}$</td>
<td>$\bar{u}$</td>
<td>$\bar{u}$</td>
<td>$\bar{u}$</td>
<td>$\bar{u}$</td>
<td>$\bar{u}$</td>
</tr>
<tr>
<td>$M_2$</td>
<td>$-\bar{u}_{ab}$</td>
<td>$\bar{u}_{bc}$</td>
<td>$\bar{u}_{ef}$</td>
<td>$\bar{u}_{bf}$</td>
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<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$M_3$</td>
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</tbody>
</table>

We can now provide explicit isomorphisms among the $\text{Stress}_r(\Delta)$.

**Theorem 3.3.**

$\text{Stress}_r^p(\Delta) = \text{Stress}_r^E(\Delta) = \text{Stress}_r^M(\Delta) = \text{Stress}_r^T(\Delta)$.

**Proof.**

(i) $\text{Stress}_r^p(\Delta)$ to $\text{Stress}_r^M(\Delta)$. For each $\lambda \in \text{Stress}_r^p(\Delta)$, and $\sigma \in \Delta^{(r-2)}$,

$$\sum_{x, \sigma \in \Delta^{(r-2)}} \lambda_{\sigma x} \bar{x} = \sum_{x, \sigma \in \Delta^{(r-2)}} \lambda_{\sigma x} (\bar{\sigma} \bar{U}) \land \bar{U} = 0 \land \bar{U} = 0.$$

This gives the identity as a map from $\text{Stress}_r^p(\Delta)$ to $\text{Stress}_r^M(\Delta)$.

(ii) $\text{Stress}_r^M(\Delta)$ to $\text{Stress}_r^T(\Delta)$. Expanding by (3.1), we have, for each $\lambda \in \text{Stress}_r^M$ and $\sigma \in \Delta^{(r-2)}$,

$$0 = \sum_{x, \sigma \in \Delta^{(r-2)}} \lambda_{\sigma x} \bar{x} = \sum_{x, \sigma \in \Delta^{(r-2)}} \lambda_{\sigma x} (\bar{\sigma} \bar{U}) + \sum_{a \in \sigma} (-1)^{r-i} (\bar{\sigma} \bar{U}) \lambda_{\sigma x}.$$

If we define

$$\lambda_{\sigma a} = \frac{1}{\bar{\sigma} \bar{U}} (-1)^{r-i} \sum_{x, \sigma \in \Delta^{(r-2)}} (\bar{\sigma} \bar{U}) \lambda_{\sigma x},$$

we have an $r$-stress for $R_r^T(\Delta)$. This gives the unique extension of $\lambda \in \text{Stress}_r^M$ to $\lambda \in \text{Stress}_r^T$.

(iii) $\text{Stress}_r^T(\Delta)$ to $\text{Stress}_r^E(\Delta)$. We already know these are isomorphic, but we wish to be explicit. For each $\lambda \in \text{Stress}_r^T$, and for each $\sigma \in \Delta^{(r-2)}$,

$$\sum_{x, \sigma \in \Delta^{(r-2)}} \lambda_{\sigma x} \bar{x} = \bar{\sigma} \lor \left( \sum_{x, \sigma \in \Delta^{(r-2)}} \lambda_{\sigma x} \bar{x} + \sum_{a \in \sigma} \lambda_{\sigma a} \bar{a} \right) = 0.$$

Thus all of these injections are bijections. We already know that $\text{Stress}_r^E = \text{Stress}_r^p$, by Proposition 5.2 of [17].


Let $A_{r,d}^M$ (resp. $B_{r,d}^M$) be the space all functions which assign to every member of $\Delta^{(r-2)}$ a 1-extensor of length $d + 2 - r$ (resp. a step $(d + 1 - r)$ extensor) in $\Delta U$. As usual, we have an isomorphism $\ast$ between $A_{r,d}^M$ and $B_{r,d}^M$, extending the isomorphism for $\Delta U$, and we write $N \lor Q = \sum_{\sigma} N(\sigma) \lor Q(\sigma)$.

An $r$-motion with respect to $R_r^M(\Delta)$ is a function $M \in B_{r,d}^M$ satisfying, for all $\rho \in \Delta^{(r-1)}$,

$$M \lor \text{Row}_{r,d}^M(\rho) = \sum_{x: \sigma \in \rho} M(\sigma) \bar{x} = 0.$$

The space of (minimal) $r$-motions is written $\text{Motion}_{r,d}^M(\Delta)$. Clearly, $(\text{Motion}_{r,d}^M(\Delta))^\perp = (\text{Row}_{r,d}^M(\rho))^\ast$ in $B_{r,d}^M$.

The space of trivial $r$-motions $\text{Triv}_{r,d}^M(\Delta)$ is generated by the following. For every $\pi \in \Delta^{(r-3)}$, every tensor $\bar{S}$ of step $d - r$ in $U$, and a basis $\{ \bar{y} \}$ for the overall $d$-dimensional projective space, define the motion $T_{\bar{x}, \pi, S, \bar{y}}^M(\sigma)$ by

$$T_{\bar{x}, \pi, S, \bar{y}}^M(\sigma) = \begin{cases} \text{Sign}(\pi, \bar{y}) \bar{S} \bar{y}_\sigma & \text{if } \sigma = \pi x, \\ 0 & \text{if } \pi \notin \sigma. \end{cases}$$
It is a simple exercise to check that these trivial motions are \( r \)-motions. If \( y \in U \), then all \( \bar{y}_\sigma = \bar{y} \) and the formula simplifies to:

\[
T_{\pi, S, \bar{y}}(\sigma) = \begin{cases} 
\text{Sign} [\pi, x] (\bar{S} \bar{y}) & \text{if } \sigma = \pi x, \\
0 & \text{if } \pi \not\| \sigma.
\end{cases}
\]

so we could choose an arbitrary \( (\bar{S} \bar{y}) \) of step \((d - r + 1)\). These are the analogues of translations. However, for \( y \not\in U \) (say, for \( y \) in a basis of the complement \( U^* \) of \( U \)), the \( y_\sigma \)'s are distinct, and we have the analogue of a rotation (i.e. a cross-product) which appeared for the euclidean matrix with \( r = 2 \).

We define an \( r \)-load \( \text{with respect to } \mathbf{R}^M_{\pi} \) as a function \( L^M \in A^e_r \) (i.e. \( L^M(\sigma) \) is a 1-extensor) satisfying, for all \( \pi \in \Delta^{(r-3)} \) and for a spanning set \( \{ \cdots, y, \cdots \} \) in the original space,

\[
\sum_{x:xx = \sigma \in \Delta^{(r-3)}} \text{Sign} [\pi, x] L^M(\pi x) \bar{y}_\sigma = 0. 
\]  \hspace{1cm} (3.4)

**Proposition 3.4.** \( (\text{Load}^M_{\pi}(\Delta))^* = \text{Triv}^M_{\pi}(\Delta)^\perp. \)

**Proof.** \( L^M \in \text{Load}^M_{\pi}(\Delta) \) if, for all tensors \( S \) of step \( d - r \),

\[
\left( \sum_{x:xx = \sigma \in \Delta^{(r-2)}} \text{Sign} [\pi, x] L^M(\pi x) \bar{y}_\sigma \right) \vee \bar{S} = 0.
\]

This is equivalent to

\[
(-1)^{d-r} \sum_{x:xx = \sigma \in \Delta^{(r-2)}} \text{Sign} [\pi, x] L^M(\pi x) \vee \bar{S} = 0,
\]

which means that \( L^M* \in (T^M_{\pi}(\Delta))^\perp. \)

**Corollary 3.5.** (i) \( B^M_{\pi, d} = (\text{Row}^M_{\pi})^* \oplus \text{Triv}^M_{\pi} \oplus \text{NonTriv}^M_{\pi} \), with these three spaces mutually orthogonal.

(ii) \( \text{Load}^M_{\pi} = \text{Row}^M_{\pi} \oplus (\text{NonTriv}^M_{\pi})^* \).

(iii) \( \Delta \) is kinematically \( r \)-rigid with respect to \( \mathbf{R}^M_{\pi}(\Delta) \) iff it is statically \( r \)-rigid with respect to \( \mathbf{R}^M_{\pi}(\Delta) \).

**Example 3.6.** Consider the complex of Example 3.5 in [17] and the motions, transferred to the minimal form, with its equilibria \( E^M \), unresolved load \( L^M \), trivial motions \( T^M \) and non-trivial motion \( M^M \). Since \( r = 3 \) and \( d = 3 \), \( U \) is a 1-dimensional space. Moreover, since we assumed that all triangles were coplanar in this example, the points \( \bar{u}_p \) are all weighted versions of the same point \( \bar{q} \) on the line \( U \). For trivial motions \( S \) is a step \( 3 - 3 = 0 \) extensor (a scalar). Let \( \bar{y} \) be any point off this line, and \( \bar{z} \neq \bar{q} \) be another point on the line:

<table>
<thead>
<tr>
<th>( \mathbf{R}^M_3 )</th>
<th>( \lambda )</th>
<th>( ab )</th>
<th>( ac )</th>
<th>( ae )</th>
<th>( bc )</th>
<th>( be )</th>
<th>( ce )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( abc )</td>
<td>( \alpha \beta \gamma )</td>
<td>( \bar{u}_{abc} )</td>
<td>( -\bar{u}_{abc} )</td>
<td>( \bar{u}_{abc} )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( abe )</td>
<td>( \alpha \beta \delta )</td>
<td>( \bar{u}_{abe} )</td>
<td>( -\bar{u}_{abe} )</td>
<td>( \bar{u}_{abe} )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( ace )</td>
<td>( \alpha \gamma \delta )</td>
<td>( \bar{u}_{ace} )</td>
<td>( -\bar{u}_{ace} )</td>
<td>( \bar{u}_{ace} )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( bce )</td>
<td>( \beta \gamma \delta )</td>
<td>( \bar{u}_{bce} )</td>
<td>( -\bar{u}_{bce} )</td>
<td>( \bar{u}_{bce} )</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
\begin{align*}
\begin{array}{|c|c|c|c|}
\hline
E^M_{b,\bar{y}} & \bar{y}_{ab} & \bar{y}_{bc} & \bar{y}_{be} \\
E^M_{b,\bar{z}} & \bar{z} & \bar{z} & \bar{z} \\
\hline
L^M & \alpha \beta \bar{y}_{ab} & \alpha \gamma \bar{y}_{ac} & \alpha \delta \bar{y}_{ae} & \beta \gamma \bar{y}_{bc} & \beta \delta \bar{y}_{be} & \gamma \bar{y}_{ce} \\
\hline
T^M_{a,1,\bar{y}} & \bar{y}_{ab} & \bar{y}_{bc} & \bar{y}_{ad} & \bar{y}_{be} \\
T^M_{a,1,\bar{z}} & \bar{z} & \bar{z} & \bar{z} & \bar{z} \\
\hline
M^M & \bar{q} \\
\hline
\end{array}
\end{align*}

\textbf{Theorem 3.7.} (i) $Motion^r_T(\Delta) = Motion^M_T(\Delta)$.
(ii) $\text{Triv}^r_T(\Delta) = \text{Triv}^M_T(\Delta)$.

\textbf{Proof.} (i) For any $M \in Motion^r_T(\Delta)$, $M(\sigma)\vec{a} = 0$ for all $a \in \sigma$. Since $\sigma \cup U$ spans the space, we can rewrite $M(\sigma) = S_\sigma \vec{a}$ for some

$$S_\sigma = \frac{1}{[\sigma U]} (M(\sigma) \wedge \vec{U}),$$

a $d + 1 - r$ extensor (which may be decomposed as an exterior product of points of $U$). Define $f$: $Motion^r_T(\Delta) \to Motion^M_T(\Delta)$ by $f(M) = W$, where for all $\sigma \in \Delta^{(r-2)}$,

$$W(\sigma) = \bar{S}_\sigma = \frac{1}{[\sigma U]} (M(\sigma) \wedge \vec{U}).$$

Then, for each $\rho \in \Delta^{(r-1)}$, working directly in the larger space,

$$\sum_{\sigma | \rho} \text{Sign}[\sigma, \rho/\sigma] W(\sigma) \cdot \bar{a}_\rho = \sum_{\sigma | \rho} \text{Sign}[\sigma, \rho/\sigma] S_\sigma \vee (\vec{\rho} \wedge \vec{U}) = \sum_{\sigma | \rho} \text{Sign}[\sigma, \rho/\sigma] [S_\sigma \vee \vec{\rho}] \vec{U} = (\sum_{\sigma | \rho} \text{Sign}[\sigma, \rho/\sigma] [S_\sigma \vee \vec{\rho}]) \vec{U} = \left(\sum_{\sigma | \rho} M(\sigma) \vec{x}\right) \vec{U} = 0.$$

Thus $f$ is well defined.

Conversely, for all $W \in Motion^M_T(\Delta)$ and for all $\sigma \in \Delta^{(r-2)}$, define $M(\sigma) = \bar{W}(\sigma) \vee \vec{\sigma}$. Then, for each $\rho \in \Delta^{(r-1)}$, working directly in the larger space, the previous argument shows that if $W \in Motion^r_T$, then

$$\left(\sum_{\sigma | \rho} \text{Sign}[\sigma, \rho/\sigma] [S_\sigma \vee \vec{\rho}]\right) \vec{U} = 0.$$

Since $\vec{U}$ is a non-zero extensor, we conclude that the scalar

$$\sum_{\sigma | \rho} \text{Sign}[\sigma, \rho/\sigma] [S_\sigma \vee \vec{\rho}] = 0,$$

and $M(\sigma) \in Motion^r_T$. Thus $f$ is an isomorphism.

(ii) Consider a $T^r_{x,S} \in \text{Triv}^r_T$, with $\pi \in \Delta^{(r-3)}$, and $S$ a $(d + 1 - r)$-tensor, which, for convenience, we may assume is an extensor. Let $S\vec{\pi} = S'\vec{\pi}\vec{\tau}$ for some $(d - r)$-extensor $S'$ spanned by points in $U$. If $S$ is contained in $U$, we have many choices of $S = S'\vec{y}$ and $\vec{y}$ is in $U$. Otherwise,

$$S' = \frac{1}{[\vec{\pi} \vec{\tau} \vec{U}]} S\vec{\pi} \wedge \vec{U}$$
is a \((d - r)\)-extensor of points in \(U\) and in the space of \(S\), so \(S\bar{x} = S'\bar{y}\bar{x}\) for some 1-extensor \(\bar{y}\) not in \(U\). We conclude that \(T_{\bar{x},S}'(\bar{x}) = S'\bar{y}(\bar{x}\bar{x})\) for all appropriate \(x\).

Therefore \(f(T_{\bar{x},S}'(\bar{x}) = \bar{W}(\bar{x})\), where

\[
\bar{W}(\bar{x}) = \frac{1}{[\bar{x}\bar{x}\bar{x}]} (S'\bar{y}(\bar{x}\bar{x})) \bar{x} = (-1)^{r-2}S'\bar{y}\bar{x}.
\]

We conclude that \(f(T_{\bar{x},S}' = \pm T_{\bar{x},S}'\bar{y}\bar{x}\) as required. Since this is true for all generators, \(f\) is a map from \(\text{Triu}_{r}^{T}\) to \(\text{Triu}_{r}^{M}\). (Note that, for all the allowed choices of \(y\), each \(\bar{y}\bar{x}\) will be the same.)

Conversely, for any \(T_{\bar{x},S}'\bar{y} \in \text{Triu}_{r}^{M}, f^{-1}(T_{\bar{x},S}'\bar{y}) = T_{\bar{x},S}'\bar{y}\) is the appropriate inverse, so the map is an isomorphism.

\textbf{Corollary 3.8.} \(\Delta\) is kinematically \(r\)-rigid with respect to \(R_{r}^{M}(\Delta)\) if it is \(r\)-rigid with respect to \(R_{r}^{f}(\Delta)\).

The following is implicit from the row reductions, and the previous results, so we do not offer a direct proof.

\textbf{Proposition 3.9.} (i) \(\text{Row}_{r}^{M}(\Delta) \subseteq \text{Load}_{r}^{M}(\Delta)\).

(ii) \(\text{Load}_{r}^{r}(\Delta) = \text{Load}_{r}^{M}(\Delta)\).

(iii) \(\text{Row}_{r}^{r}(\Delta) = \text{Row}_{r}^{M}(\Delta)\).

\textbf{Remark 3.10.} Another algebraic vision of the move from the truncated matrix to the minimal matrix runs as follows. Because of the extra rows, all calculations for a column of \(R_{r}^{T}\), including the motions there, are carried out 'modulo \(\bar{x}\)'. Two \(i\)-tensors \(P\) and \(Q\) are equivalent modulo \(\sigma\), written \(P \equiv \bar{a} Q\), if \(P_{a} = Q_{a}\) for all \(a \in \sigma\).

This is the presentation adopted for the sequel [16]. Rather than calculate directly with this equivalence relation, the minimal matrix selects uniform representatives for these equivalence classes.

\section{4. The Face-Ring Rigidity Matrix}

We now consider the full face-ring rigidity matrix. Let \(R_{r}^{f}(\Delta)\) be a matrix with 1-extensor entries the rows of which are indexed by arbitrary monomials of \(M_{r}\), and the columns of which are indexed by monomials in \(M_{r-1}\). The entry in row \(\rho\) and column \(\sigma\) is

\[
R_{r}^{f}(\rho, \sigma) = \begin{cases} \bar{x} & \text{if } \rho = \sigma x, \\ 0 & \text{if } \sigma \nmid \rho. \end{cases}
\]

We call \(R_{r}^{f}(\Delta)\) the face-ring rigidity matrix, because its rows and columns (for various \(r\)) are indexed by all the non-zero monomials in the Stanley–Reisner ring of \(\Delta\).

\textbf{Example 4.1.} We show here the matrix \(R_{r}^{f}(\Delta)\) for \(\langle a, b, c, e \rangle\), realized in the plane with its four vertices satisfying the projective equation \(a\bar{a} + b\bar{b} + c\bar{c} + e\bar{e} = 0\). We also
display the 3-stress present in this realization:

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Suppose that $\mathbf{A}_{r,d}^F$ (resp. $\mathbf{B}_{r,d}^F$) is the space of all step-1-extensor-valued (resp. step-d-extensor-valued) functions on $\mathcal{M}_{r-1}$. As usual, we have a duality map $\ast$ between $\mathbf{A}_{r,d}^F$ and $\mathbf{B}_{r,d}^F$, extending the duality map for $\Lambda U$, and we write $N \vee Q = N^\ast \cdot Q = \sum_{\sigma} N(\sigma) \vee Q(\sigma)$.

An $r$-motion of $\Delta$ with respect to $\mathbf{R}_r^F$ is a function $M^F \in \mathbf{B}_{r,d}^F$ satisfying, for all $\rho \in \mathcal{M}_r$,

$$\sum_{x: x \mid \rho} M^F(\rho/x) \bar{x} = M^F \vee Row_r^F(\rho) = 0. \quad (4.1)$$

We denote the space of all motions by $\text{Motion}_r^F(\Delta)$. Thus $(\text{Motion}_r^F)^\ast$ is the kernel of $\mathbf{R}_r^F$.

For every monomial $\pi \in \mathcal{M}_{r-2}$, and every tensor $S$ of co-step 2, let $T_{\pi,S}^F$ be the trivial $r$-motion

$$T_{\pi,S}^F(\sigma) = \begin{cases} S \bar{x} & \text{if } \sigma = \pi x, \\ 0 & \text{if } \pi \nmid \sigma. \end{cases} \quad (4.2)$$

Then, for all $\rho \in \mathcal{M}_r$,

$$T_{\pi,S}^F \vee Row_r^F(\rho) = \begin{cases} T_{\pi,S}^F(\pi x) \bar{y} + T_{\pi,S}^F(\pi y) \bar{x} = 0 & \text{if } \rho = \pi xy, \\ 0 & \text{if } \pi \nmid \rho. \end{cases}$$

So $T_{\pi,S}^F$ is an $r$-motion. Let $\text{Triv}_r^F(\Delta)$ denote the subspace of $\text{Motion}_r^F(\Delta)$ generated by the trivial motions.

An $r$-load of $\Delta$ with respect to $\mathbf{R}_r^F$ is a function $L^F \in \mathbf{A}_{r,d}^F$ satisfying, for all $m \in \mathcal{M}_{r-2}$,

$$\sum_{x: x \mid \rho} L^F(mx) \vee \bar{x} = 0. \quad (4.3)$$
Proposition 4.2. (i) $\mathbf{B}_{r,d}^{F} = (\text{Row}_{r}^{F})^{*} \oplus \text{Triv}_{r}^{F} \oplus \text{NonTriv}_{r}^{F}$, with these three subspaces mutually orthogonal.

(ii) $(\text{Load}_{r}^{F}(\Delta))^{*} = \text{Triv}_{r}^{F}(\Delta)^{\perp}$.

(iii) $\text{Load}_{r}^{F} = \text{Row}_{r}^{F} \oplus (\text{NonTriv}_{r}^{F})^{*}$.

(iv) Every row of $\mathbf{R}_{r}^{F}$ is an $r$-load.

Proof. (i) Suppose that $L^{F} \in \mathbf{A}_{r,d}^{F}$. Then, for any generator $T_{r,s}^{F}$ of $\text{Triv}_{r}^{F}(\Delta)$, we have

$$L^{F} \vee T_{r,s}^{F} = \sum_{x: r, x \in \mathcal{M}} L^{F}(\pi x)M_{r,s}(\pi x) = -S \vee \left( \sum_{x: r, x \in \mathcal{M}} L^{F}(\pi x)x \right).$$

Thus if $(L^{F})^{*} \in \text{Triv}_{r}^{F}(\Delta)^{\perp}$, then $S \vee (\sum_{x: r, x \in \mathcal{M}} L^{F}(\pi x)x) = 0$ for all $S$. Therefore $\sum_{x: r, x \in \mathcal{M}} L^{F}(\pi x)x = 0$. This means that $L^{F} \in \text{Load}_{r}^{F}(\Delta)$.

Conversely, suppose that $L^{F} \in \text{Load}_{r}^{F}(\Delta)$. Then $L^{F} \vee T_{r,s}^{F} = 0$ for all $x$ and all $S$. Thus $(L^{F})^{*} \in \text{Triv}_{r}^{F}(\Delta)^{\perp}$.

(ii) and (iii) follow immediately.

Theorem 4.3. $\Delta$ is statically $r$-rigid in $d$-space with respect to $\mathbf{R}_{r}^{F}(\Delta)$ if it is infinitesimally $r$-rigid in $d$-space with respect to $\mathbf{R}_{r}^{F}(\Delta)$.

Theorem 4.4. $\text{Stress}_{r}^{F}(\Delta) = \text{Stress}_{r}^{T}(\Delta)$.

Proof. Let $\lambda \in \text{Stress}_{r}^{F}(\Delta)$. Then its restriction to $\mathcal{M}_{r}^{(r)} \cup \mathcal{M}_{r}^{(r-1)}$ is in $\text{Stress}_{r}^{T}(\Delta)$, since columns in $\mathbf{R}_{r}^{F}$ indexed by elements of $\mathcal{M}_{r}^{(r-1)}$ have non-zero entries only in rows indexed by elements of $\mathcal{M}_{r}^{(r)} \cup \mathcal{M}_{r}^{(r-1)}$.

Conversely, suppose that $\lambda$ is an $r$-stress of $\mathbf{R}_{r}^{T}$. We shall show, by induction, that it extends uniquely to an $r$-stress of $\mathbf{R}_{r}^{F}$. Suppose that $\lambda$ is defined on $\mathcal{M}_{r}^{(j)}$, for all $s$ such that $r \geq s > j$, where $j < r - 2$. Suppose, moreover, for all $\sigma \in \mathcal{M}_{r}^{(j)}$, $\lambda_{\sigma X}(x) = 0$. Clearly, the above is true for $j = r - 2$. We shall show that $\lambda$ extends uniquely to $\mathcal{M}_{r}^{(j+1)}$, and that, for all $\sigma \in \mathcal{M}_{r}^{(j+1)}$, $\sum_{x: r, x \in \mathcal{M}} \lambda_{\sigma x}x = 0$. We need a lemma, adapted from Filliman [7].

Lemma 4.5. Under the induction hypothesis stated above, for any monomial $\sigma = x_{1}^{k_{1}}x_{2}^{k_{2}} \cdots x_{j}^{k_{j}} \in \mathcal{M}_{r}^{(j+1)}$ supported on $\mu = x_{1} \cdots x_{j}$, we have

$$\sum_{x: r, x \in \mathcal{M}} \lambda_{\sigma x}x = 0$$

and

$$\sum_{x: r, x \in \mathcal{M}} \lambda_{\sigma x}x = 0,$$

where $m$ and $n$ are any two distinct indices in $\{1, 2, \ldots, j\}$, with $k_{m} > 1$ and $\nu = \mu / x_{m}x_{n}$.

Proof of Lemma. We assume that $x_{1} < x_{2} < \cdots < x_{j}$, and for (4.4) we also assume, without loss of generality, that $k_{1} > 1$. For each $x \uparrow \mu$, $\sigma x / x_{1} \in \mathcal{M}_{r-1}^{(j+1)}$. Thus, by the induction hypothesis,

$$\sum_{y: \sigma x y / x_{1} \in \mathcal{M}} \lambda_{\sigma x y / x_{1}}y = 0.$$
Summing over all $x$ we have
\[ \sum_{x: \alpha \in \mathcal{M}, \mu} \lambda_{\alpha x} \tilde{x} = - \sum_{(x, y): \alpha \in \mathcal{M}, \mu, \nu} \lambda_{\alpha xy} \tilde{y} \tilde{x}_2 \cdots \tilde{x}_2 \tilde{x} \cdot \tilde{x} \cdot \tilde{x} \cdot \tilde{x} \cdot \tilde{x}. \]

The summation on the right-hand side is over all ordered pairs $(x, y)$ such that $\mu xy \in \mathcal{M}_{j+2}$. So every term occurs twice with opposite signs, whence the right-hand side is 0. This completes the proof of (4.4).

With $x_m$ replacing $x_1$, we obtain, from (4.6),
\[ \lambda_{\alpha x} \tilde{x}_m + \lambda_{\alpha xx_m/x_m} \tilde{x}_n = - \sum_{y: \alpha y \in \mathcal{M}, y \neq x_m} \lambda_{\alpha xy} \tilde{y}. \]

Thus
\[ \lambda_{\alpha x} \tilde{x}_m \tilde{x} + \lambda_{\alpha xx_m/x_m} \tilde{x}_n \tilde{x} = - \sum_{y: \alpha y \in \mathcal{M}, y \neq x_m} \lambda_{\alpha xy} \tilde{y} \tilde{x}. \]

As before, summing over all $x$ yields the result.

We now complete the proof of the theorem. We obtain, from (4.4), for any monomial $\sigma = x_1^k \cdots x_j^k \in \mathcal{M}_{j-1}$ supported on $\mu = x_1 \cdots x_j$,
\[ \sum_{x: \alpha \in \mathcal{M}, \mu} \lambda_{\alpha x} \tilde{x} = - \sum_{z \mid \mu} \alpha_{z \mu} \tilde{z}. \quad (4.7) \]

For all $z \mid \mu$ define $\lambda_{\alpha z} = \alpha_{z \mu}$. Since the $\alpha$'s are unique, this extends $\lambda$ uniquely to the factored monomials $\alpha z$. But the monomial $\sigma z$ may be factored in more than one way, so we must check that $\lambda$ is well-defined on each monomial. If $k_m > 1$ and $\sigma x_m = \sigma' x_m$, then $\text{Supp}(\sigma') = \mu$. If we perform the preceding calculation with $\sigma'$ instead of $\sigma$, then we obtain, as in (7), $\lambda_{\sigma' x_m} = \alpha_{\sigma' z_m}$. We need to show that $\alpha_{\sigma x_m} = \alpha_{\sigma' x_m}$. Let $\nu = \mu / x_m x_n$. From (4.7) we have
\[ \sum_{x: \sigma x \in \mathcal{M}, \mu} \lambda_{\sigma x} \tilde{x}_m \tilde{x}_n = - \alpha_{\sigma x_m} \tilde{\alpha}_{x_n} \tilde{x}_m \tilde{x}_n \]
and
\[ \sum_{x: \sigma' x \in \mathcal{M}, \mu} \lambda_{\sigma' x} \tilde{x}_m \tilde{x}_n = - \alpha_{\sigma' x_m} \tilde{\alpha}_{x_m} \tilde{x}_m \tilde{x}_n \]

It follows from (4.5) that $\alpha_{\sigma x_m} = \alpha_{\sigma' x_m}$.

**Proposition 4.6.** $\text{NonTriv}_F^T(\Delta) = \text{NonTriv}_F^T(\Delta)$.

We omit the proof of this proposition, as it is longer than the corresponding result for the other matrices.

The following theorem summarizes the main results that we have concerning the rigidity matrices.

**Theorem 4.7.** Let $\Delta$ be simplicial complex realized in $d$-space. Then:
1. $\text{Stress}_F^T = \text{Stress}_T^T = \text{Stress}_T^T = \text{Stress}_M^T$.
2. $\text{NonTriv}_F^T(\Delta) = \text{NonTriv}_T^T(\Delta) = \text{NonTriv}_T^T(\Delta) = \text{NonTriv}_M^T(\Delta)$.
(3) All the concepts of r-rigidity are equivalent.

(4) \( Load^R_r(\Delta) = Load^E_r(\Delta) = Load^P_r(\Delta) = \frac{Load^T_r(\Delta)}{L(\Delta)} \), where \( L(\Delta) \) is defined in (6.4) of [17].

(5) \( Motion^T_r(\Delta) = Motion^M_r(\Delta) = \frac{Motion^E_r(\Delta)}{Triv^E_r(\Delta)} = \frac{Motion^P_r(\Delta)}{Triv^P_r(\Delta)} \).

(6) \( Triv^T_r(\Delta) = Triv^M_r(\Delta) = \frac{Triv^E_r(\Delta)}{Triv^E_r(\Delta)} = \frac{Triv^P_r(\Delta)}{Triv^P_r(\Delta)} \).

(7) The dimensions of all of these spaces are projectively invariant.

5. OTHER AREAS OF WORK

Remark 5.1. The r-rigidity matrix, in any form, depends on both the abstract simplicial complex and the geometric realization. We have already seen that a general realization may be r-rigid, while a realization in special position may not be r-rigid. If the abstract complex is realized with algebraic indeterminants for the co-ordinates, the rigidity matrix, say \( R^M_r \), will have its maximum rank. This will record the generic behaviour of the complex.

If the generic realization is r-rigid, then almost all realizations (an open dense subset of \( \mathbb{R}^{d/2} \)) are r-rigid. The special positions which may not be rigid can only occur when some minor loses rank, i.e. when some polynomial in the co-ordinates is 0. This is an extension of results for 2-rigidity [18]. If this generic realization has a non-trivial motion, then all realizations will have non-trivial motions. Thus an abstract complex is either almost always r-rigid, or never r-rigid.

Remark 5.2. Using an induction based on bistellar operations, PL d-spheres in d-space have been shown to be generically r-rigid for all r [11, 15]. The induction also shows that the count of trivial motions matches the heuristic value, for generic realizations. Recent work of Lee [12], Tay [15] and of Filliman [7] also studies bistellar operations applied to r-rigidity of generic realizations of PL \((d - 1)\)-spheres in d-space for \( r \leq (d + 1)/2 \). However, there is an essential gap left in this effort, for the critical case of \( r = (d + 1)/2 \), when \( d \) is odd.

The classical theorem of Dehn proved the 2-rigidity of all convex realizations of 3-polytopes in 3-space. There is an active search for similar geometric and combinatorial arguments for r-rigidity of d-polytopes in d-space. McMullen [13] offers a proof of the g-theorem for polytopes using convexity and bistellar operations. It is likely that versions of these arguments will demonstrate that all convex realizations of a simplicial d polytope in \((d + 1)\)-space are r-rigid for all \( r \leq (d + 1)/2 \).

In another inductive approach, Fogelsanger [8] combines the combinatorial technique of ‘vertex splitting’ (Whiteley [24]) with an inductive topological decomposition to prove the generic 2-rigidity of simplicial d-pseudo-manifolds in d-space (and a broader class of minimal homology cycles).

Remark 5.3. The matrices \( R^P_r, R^E_r, R^T_r \) and \( R^M_r \) generalize to non-simplicial polyhedral complexes. The idea is simple, and we outline it for \( R^P_r \). The rows are indexed by \( \Delta^{r-1} \) and the columns are indexed by \( \Delta^{r-2} \). We choose, arbitrarily, an orientation for each face, and the usual boundary operator then defines a sign...
We choose a non-zero $i$-extensor $\pi$ of points in the space $\langle \sigma \rangle_t$ for each face $\sigma \in \Delta^{(i-1)}$, and we define the $r$-rigidity matrix

$$R^M_r(\rho, \sigma) = \begin{cases} \text{Sign}[\sigma, \rho] \bar{\rho} & \text{if } \rho = \sigma x, \\ 0 & \text{if } \sigma \nmid \rho. \end{cases}$$

We can define $r$-stresses, $r$-motions, trivial $r$-motions and $r$-loads, using the $(r-3)$-faces. Most of the results given here, such as the equivalence of static and kinematic $r$-rigidity, the equivalence of $r$-rigidity with respect to the various matrices, the connections with reduced homology and coning, generalize to this context. In particular, the volume interpretation of $r$-motions generalizes, and $r$-motions are, in some sense, the changes which preserve the $(r-1)$-volume of $(r-1)$-faces. We have no 'natural' extension to $R^F_r$ for non-simplicial complexes, because an appropriate analogue of the 'face-ring' is lacking for non-simplicial complexes.

**Remark 5.4.** We can dualize the entire presentation given in this paper. If we take a simple pure $d$-complex in $d$-space, and use hyperplane co-ordinates for the facets, we can use this dual representation to construct appropriate matrices and prove their equivalence. This gives a dual $(r^*)$-rigidity of the simple cell complex. In general, this dual structure is not a cell complex. However, if the original cell complex has the topology of a manifold, we can identify a dual cell structure, in the sense of this manifold.

Whiteley [22, 23] studied what we now call 2*-rigidity in 3-space, and in the plane. This dual $r^*$-rigidity is relevant for people who prefer to work with simple polytopes to prove results such as the lower bound theorem and the $g$-theorem (McMullen [13] and Oda [14]), where analogues of Lee's $r$-stresses have appeared for simple convex polytopes.

**Remark 5.5.** There is a classical extension of 2-rigidity to non-simplicial polytopes. Triangulate each 2-face of the polytope with additional bars, and study the 2-rigidity of the resulting skeleton. Then there are the following theorems.

**Theorem (Alexandrov [1]).** Any strictly convex 2-polytope in 3-space, with all 2-faces triangulated, is 2-rigid, and has only the trivial 2-stress.

**Theorem (Whiteley [21]).** Any strictly convex $(d-1)$-polytope in $d$-space, with all 2-faces triangulated, is 2-rigid.

If we count the added bars for the triangulations this proves that

$$g^*_2(\Delta) = f_1 + \sum_{F_i \in \Delta^{(2)}} (|F_i| - 3) - d_{f_0} + \left(\frac{d + 1}{2}\right) \geq 0$$

or

$$f_1 + \sum_{F_i \in \Delta^{(2)}} (|F_i| - 3) \geq d_{f_0} - \left(\frac{d + 1}{2}\right).$$

This gives the generalized lower bound theorem (Kalai [10]), related to the generalized $h$-vector of the non-simplicial cell complex. The issue of rigidity after re-triangulations of appropriate faces of non-simplicial polytopes is another area for future research. For example, the 'shallow retriangulations' of Bayer [2] are promising, when appropriate triangulations exist.
REMARK 5.6. Any (possibly non-simplicial) cell \((d - 1)\)-manifold \(\Delta\) in \(d\)-space has a dual cell complex \(\Delta^*\) in the sense of the manifold. We can look for connections between \(r\)-rigidity and \(r^*\)-rigidity of the same cell complex. For \(2\)-rigidity in \(3\)-space, there is an explicit connection.

**Theorem (Whiteley [21]).** For any cell complex \(\Delta\) realized as a 2-manifold in 3-space, and its dual cell complex \(\Delta^*\), within the manifold, realized in a projectively polar realization, the following are equivalent:

(i) \(\Delta\), with all its 2-faces triangulated, is 2-rigid;
(ii) \(\Delta\) is 2*-rigid;
(iii) \(\Delta^*\), with all its 2-faces triangulated, is 2-rigid;
(iv) \(\Delta^*\) is 2*-rigid.

We anticipate that there will be more connections to be explored.

REMARK 5.7. The concept of 'bar-and-body frameworks' for 2-rigidity [19] can be generalized to 'body-and-face' structures for \(r\)-rigidity. These structures reveal the basic matroid pattern of all the \(r\)-rigidity matrices as truncations of a matroid union of copies of the homology matroid.

REMARK 5.8. The \(g\)-theorem says that, for \(\Delta\) the boundary of a simplicial convex \(d\)-polytope in \(d\)-space, \(g_r(\Delta, d) = -g_{d-1-r}(\Delta, d)\). We have conjectured, for generic realizations of this polytope, \(r \leq (d + 1)/2\), \(g_r = \dim(\text{Stress}_r)\) and \(g_{d+1-r} = \dim(\text{NonTriv}_{d+1-r})\). This suggests that there is an explicit isomorphism between \(\text{Stress}_r(\Delta)\) and \(\text{NonTriv}_{d+1-r}(\Delta)\).

For example, for a triangulated 2-sphere in 3-space, there is an explicit geometric construction giving a correspondence between 2-motions and 2-stresses, which holds for all realizations, including those which are not 2-rigid. (Gluck [9], and Crapo and Whiteley [3]). This correspondence generalizes to 2-motions and \((d - 1)\)-stresses on a \(d\)-polytope in \(d\)-space.

We conjecture that such a correspondence exists, for all \(d\), and for all realizations, not just the 'generic' (or convex) realizations in \(d\)-space. In McMullen's work [13] there are such correspondences implicitly for convex realizations.

**Conjecture 5.9.** For \(\Delta\) the boundary of a \(d\)-polytope realized in \(d\)-space, there is an isomorphism between \(\text{NonTriv}_r(\Delta)\) and \(\text{Stress}_{d-r+1}(\Delta)\).

We have also verified this conjecture for the elementary case of 1-motions and \(d\)-stresses on any oriented \((d - 1)\)-manifold in \(d\)-space (including degenerate realizations).

REMARK 5.10. The classical theorem of Maxwell, and its converses, give a correspondence between 2-stresses in the plane and projections of spherical polyhedra (simplicial or non-simplicial) in 3-space [4,20]. This result can be generalized to a correspondence between \(r\)-stresses in \(r\)-space and projections of spherical \(r\)-polytopes from \((r + 1)\)-space. The recent paper of Crapo and Whiteley [5] used this correspondence for \(r = 3\) to explore projections of 4-polytopes and related surfaces.

REMARK 5.11. Maxwell's theorem, modified to discs in place of spheres, gives a basic correspondence between 2-stresses and piecewise linear, globally continuous
functions over a polygonal decomposition of the plane. This is the simplest case of a series of correspondences and analogies between $r$-stresses and $C^m_r$-splines (piecewise polynomial, of degree at most $k$, globally $C^m$ functions) over cell decompositions of $n$-space. The strong analogy between 2-stresses in 3-space, and $C^2$-splines over plane decompositions has been very fruitful for both rigidity and spline theory [25]. We anticipate that analogies between splines and $r$-rigidity will arise which will contribute to both fields of study.

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REFERENCES


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