Coherent Algebras

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ABSTRACT

Coherent algebras are defined to be the subalgebras of the matrix algebras $M_n(\mathbb{C})$ closed under Hadamard (= coefficientwise) multiplication and containing the all 1 matrix, and are shown to be precisely the adjacency algebras of coherent configurations. Each such algebra has a type, which is a symmetric matrix with positive integer entries. The theory is illustrated by applications to quasisymmetric designs, which are essentially equivalent to coherent algebras of type $\begin{bmatrix} 2 & 2 \\ 2 & 3 \end{bmatrix}$.

INTRODUCTION

In Section 1 we define a *coherent algebra* to be a self-adjoint algebra of matrices over $\mathbb{C}$ which is closed under coefficientwise (= Hadamard) multiplication and contains the all 1 matrix. In Section 3 we observe that these are precisely the adjacency algebras of coherent configurations (c.c.'s), which means that we can identify coherent algebras with their underlying c.c.'s. The *type* of a coherent algebra or c.c., defined in Section 3, is a symmetric $t \times t$ matrix of positive integers, where t is the number of fibers of the underlying c.c. In particular, a 1×1 type (r) means homogeneous of rank r, which includes the association schemes. The possible 2×2 and 3×3 types are surveyed in Section 8. Some types of coherent algebras with $t = 2$ correspond to combinatorial objects of sufficient interest to warrent independent consideration. Type $\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$ corresponds to symmetric designs, where we have nothing new to say. Type $\begin{bmatrix} 2 & 2 \\ 2 & 3 \end{bmatrix}$ corresponds to quasisymmetric designs, introduced by Goethals and Seidel in [6]. To illustrate the theory we

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consider this case in Section 9, determining the intersection algebra and the irreducible representations explicitly in terms of the parameters of the corresponding quasisymmetric designs. We use the resulting parameter conditions to obtain a characterization of tight 4-designs closely related to work of P. J. Cameron [4]. Type $\begin{bmatrix} 3 & 2 \\ & 3 \end{bmatrix}$ and the corresponding designs are the subject of [9]; S. Hobart has investigated the designs corresponding to type $\begin{bmatrix} 2 & 2 \\ 2 & 4 \end{bmatrix}$ in [12].

Some convenient generalities about configurations are given in Section 2. Sections 4, 5, and 6 contain a discussion of characters of finite dimensional algebras over $\mathbf{C}$ designed for applications to coherent algebras and configurations and their weighted and generic versions. It is our intention to present applications to weighted algebras in another place, continuing the discussion begun in [7, 11]. The material on *feasible traces* will provide the background needed for filling in the details of the results described in [8] on *generic systems*.

There is a very extensive literature on association schemes and their applications, going back to the original papers of R. C. Bose and his associates (see [1], [4], and the bibliography of [13].) For references on c.c.'s see [7] and the references there.

One of the purposes of this paper is to provide a convenient reference for material used in [9], [10], [11], and [12], and related work in progress; for this reason, and because of some basic changes in notation, some overlap with [6] (in Sections 3 and 7) and some expository material on algebras have been included making our account essentially self-contained.

1. COHERENT ALGEBRAS

Let X be a finite set and K a field, and let $L(X) = L(X, K)$ be the algebras of K-valued functions on X under the pointwise operations. Because X is finite, we can identify $L(X)$ as a vector space with the vector space KX having basis X by identifying the standard basis $\{\chi_x \mid x \in X\}$, $\chi_x(y) = \delta_{xy}$, of $L(X)$ with X.

LEMMA 1.1. *The lattice of subalgebras of $L(X)$ is isomorphic with the lattice of partitions of X. ($\mathscr{P}_1 \geq \mathscr{P}_2$ for partitions means that $\mathscr{P}_1$ is a refinement of $\mathscr{P}_2$.)*

Proof. A subalgebra $\mathscr{A}$ of $L(X)$ has no nilpotent elements and so is semisimple. Let $E_1, \dots, E_r$ be the primitive idempotents of $\mathscr{A}$, and $X_i = \operatorname{supp}(E_i)$, the support of the function E_i. Then $(X_i)_{i=1,\dots,r}$ is a partition of

X, and $\mathscr{A}$ consists of the $F \in L(X)$ which are constant on X_i, $1 \leqslant i \leqslant r$. Conversely, given a partition of X, the K-valued functions on X which are constant on the parts form a subalgebra of $L(X)$. ■

We identify the algebra $M_X(K)$ of all matrices with coefficients in K having rows and columns indexed by X with $\mathrm{End}_K(L(X))$ by identifying each linear transformation with its matrix with respect to the standard basis, and we refer to $L(X)$ as the *standard module* for $M_X(K)$ and its subalgebras. On the other hand, we treat the algebras $L(X^2)$ and $M_X^\circ(K)$ as being identical, where $M_X^\circ(K)$ denotes $M_X(K)$ viewed as an algebra with the usual addition and multiplication by scalars but with matrix multiplication replaced by Hadamard (= coefficientwise) multiplication. By (1.1) we have

LEMMA 1.2. *The lattice of subalgebras of $M_X^\circ(K)$ is isomorphic with the lattice of partitions of X^2.*

From now on we take $K = \mathbb{C}$, the field of complex numbers. Then $L(X)$ has the inner product

$$(F,G) = \sum_{x \in X} F(x)\bar{G}(x) \qquad [F,G \in L(X)],$$

for which the standard basis is an orthonormal basis. Viewed as a matrix, the adjoint A^* of $A \in M_X(\mathbb{C})$, defined by $(AF,G) = (F, A^*G)$ for all $F,G \in L(X)$, is the conjugate transpose of A. A subalgebra $\mathscr{A}$ of $M_X(\mathbb{C})$ is *self-adjoint* if $A \in \mathscr{A}$ implies $A^* \in \mathscr{A}$. We define a *coherent algebra on* X to be a self-adjoint subalgebra of $M_X(\mathbb{C})$ which is also a subalgebra of $M_X^\circ(\mathbb{C})$. Thus a subalgebra of $M_X(\mathbb{C})$ is coherent if and only if it is closed under the adjoint map and Hadamard multiplication and contains the all 1 matrix J.

Before describing in Section 3 the partitions of X^2 which correspond to the coherent algebras on X, we consider in Section 2 some generalities about binary relations and configurations.

2. BINARY RELATIONS AND CONFIGURATIONS

If f is a binary relation on X, i.e., $f \subseteq X^2$, then we put

$$f(x) = \{y \in X | (x,y) \in f\} \qquad \text{for} \quad x \in X,$$

$$f^t = \{(y,x) | (x,y) \in f\},$$

$$\mathrm{pr}_1(f) = \{x \in X | (x,y) \in f \text{ for some } y \in X\},$$

$$\mathrm{pr}_2(f) = \{y \in X | (x,y) \in f \text{ for some } x \in X\}.$$

The *matrix* of f is the element A_f of $M_X(K)$ such that $A_f(x,y)=1$ or 0 according as (x,y) is in f or not; $f \mapsto A_f$ is a bijection of the set of binary relations on X onto the set of $(0,1)$ matrices in $M_X(K)$. The diagonal Δ of X^2 corresponds to the identity matrix, and X^2 corresponds to the all 1 matrix J. We can view A_f as the *adjacency matrix* of the graph (X,f) with X as its vertex set and f as its set of (directed) edges.

A *configuration* $\mathscr{C}=(X,(f_i)_{i\in\mathbf{I}})$ *on* X *over* a set $\mathbf{I}$ consists of a nonempty set X together with a family $(f_i)_{i\in\mathbf{I}}$ of nonempty binary relations on X. We assume throughout that X and $\mathbf{I}$ are finite sets, and call the cardinality n of X the *order* and the cardinality r of $\mathbf{I}$ the *rank* of $\mathscr{C}$. A configuration in this sense can be identified with its family $(\Gamma_i)_{i\in\mathbf{I}}$ of graphs $\Gamma_i=(X,f_i)$, or with the family $(A_i)_{i\in\mathbf{I}}$ of matrices of the f_i, which we call the *adjacency matrices* of the configuration. Our motivation for attaching a name to this very general concept comes from certain special examples including the following:

(1) We say that a group G acting on a set X *affords* the configuration $\mathscr{C}(G,X)=(X,X^2/G)$, where X^2/G denotes the set of *orbitals* for G, i.e., the set of orbits for G acting componentwise on X^2. We refer to this as the *group case*, and as the *transitive group case* if the action of G on X is transitive.

(2) $\mathscr{C}(G,X)$ is an example of a *coherent configuration*, and is an *association scheme* in the transitive case if the orbitals are self-paired (the definitions of coherent configuration and association scheme are repeated in Section 3 below).

(3) An important class of configurations which will not concern us here are the *chamber systems*, which can be defined as the configurations $(C,(e_i)_{i\in\mathbf{I}})$ consisting of a set C of *chambers* and a family $(e_i)_{i\in\mathbf{I}}$ of equivalence relations on C. (This definition of chamber system is clearly equivalent to the original one by Tits [14] in terms of a family of partitions rather than equivalence relations. In [10] we consider construction of chamber systems from coherent configurations and conversely.)

At this point we mention some convenient generalities about configurations $\mathscr{C}=(X,(f_i)_{i\in I})$ over $\mathbf{I}$.

(1) *Fusion and refinement.* Let $(\mathbf{I}_\alpha)_{\alpha\in\Omega}$ be a partial partition of $\mathbf{I}$ (i.e., a family of nonempty, pairwise disjoint subsets of $\mathbf{I}$), and put $g_\alpha=\bigcup_{i\in\mathbf{I}_\alpha} f_i$. Then we say that the configuration $\mathscr{D}=(X,(g_\alpha)_{\alpha\in\Omega})$ is obtained from $\mathscr{C}$ by *fusion*, or is a *fusion* of $\mathscr{C}$, and that $\mathscr{C}$ is a *refinement* of $\mathscr{D}$. In this case we write $\mathscr{C}\geqslant\mathscr{D}$, which defines a partial order $\geqslant$ on the set of configurations on X. This induces a partial order on the set of those configurations for which $(f_i)_{i\in\mathbf{I}}$ is a partition of X^2, which is consistent with the partial order of

partitions by refinement. There is a maximum configuration X for which the relations are just the singletons in X^2, and a minimum configuration having just the single relation X^2.

(2) *Partitions.* By a *partition* of $\mathscr{C}$ we mean a partition $\mathscr{P} = (X_\alpha)_{\alpha \in \Omega}$ of X such that for all $i \in \mathbf{I}$, $f_i \subseteq X_\alpha \times X_\beta$ for some $\alpha, \beta \in \Omega$. Given such a partition of $\mathscr{C}$, we put

$$\mathbf{I}^{\alpha\beta} = \left\{ i \in I \middle| f_i \subseteq X_\alpha \times X_\beta \right\},$$

so that $(\mathbf{I}^{\alpha\beta})_{\alpha,\beta \in \Omega}$ is a partition of I. The configurations $\mathscr{C}^\alpha = (X_\alpha, (f_i)_{i \in \mathbf{I}^{\alpha\alpha}})$ are the *fibers* of the partition. We put $r_{\alpha\beta} = |\mathbf{I}^{\alpha\beta}|$, so that $r = \sum_{\alpha,\beta \in \Omega} r_{\alpha\beta}$ and $\mathscr{C}^\alpha$ has rank $r_\alpha = r_{\alpha\alpha}$. We call the matrix $(r_{\alpha\beta})$ the *type* of the partition $\mathscr{P}$. In the group case, the orbits of G on X form a partition of $\mathscr{C}(G, X)$, and the fibers are afforded by the action of G on the orbits. We refer to the rank and type of this partition as the *rank* and *type* respectively of the action of G on X.

(3) *Induced configurations.* Given $Z \subseteq X$, $Z \neq \varnothing$, put

$$\mathbf{I}_Z = \left\{ i \in \mathbf{I} \mid f_i \cap Z^2 \neq \varnothing \right\}.$$

Then the configuration $(Z, (f_i \cap Z^2)_{i \in \mathbf{I}_Z})$ is the *induced* configuration on Z, denoted by $\langle Z \rangle$, or $\langle Z \rangle_{\mathscr{C}}$. More generally, if $\mathscr{Q} = (Z_\alpha)_{\alpha \in \Omega}$ is a partial partition of X, put $Z = \bigcup_{\alpha \in \Omega} Z_\alpha$ and $f_i^{\alpha\beta} = f_i \cap Z_\alpha \times Z_\beta$. The configuration $(Z, (f_i^{\alpha\beta}))$ over all triples (i, α, β) such that $f_i^{\alpha\beta} \neq \varnothing$ is the *induced* configuration on $\mathscr{Q}$, denoted by $\langle \mathscr{Q} \rangle$ or $\langle \mathscr{Q} \rangle_{\mathscr{C}}$. If $\mathscr{Q} = (Z_1, \ldots, Z_t)$, then we write $\langle Z_1, \ldots, Z_t \rangle$ for $\langle \mathscr{Q} \rangle$. Clearly $\mathscr{Q}$ is a partition of $\langle \mathscr{Q} \rangle$ with fibers $\langle Z_i \rangle$, and a partition $\mathscr{P}$ of X is a partition of $\mathscr{C}$ if and only if $\langle \mathscr{P} \rangle = \mathscr{C}$.

As a natural extension of the matrix method for graphs we consider the *adjacency algebra* of $\mathscr{C}$, which is the subalgebra $\mathscr{A} = \mathscr{A}(\mathscr{C})$ of $M_X(\mathbf{C})$ generated by the adjacency matrices A_i. In the group case, $\mathscr{A}$ is the centralizer algebra of the permutation representation. At the root of the consideration of adjacency algebras are the following familiar interpretations of matrix and Hadamard multiplication: The (x, y) entry of the product $A_i A_j$ is

$$P_{ij}(x, y) = \text{the number of } (f_i, f_j) \text{ paths from } x \text{ to } y$$

$$= \left| f_i(x) \cap f_j^t(y) \right|,$$

while the (x, y) entry of the Hadamard product $A_i \circ A_j$ is 1 or 0 according as (x, y) is in $f_i \cap f_j$ or not. In particular, the A_i are idempotents with respect to Hadamard multiplication, and A_i and A_j are orthogonal if and only if $f_i \cap f_j = \varnothing$.

3. COHERENT ALGEBRAS AND COHERENT CONFIGURATIONS

A coherent algebra $\mathscr{A}$ on X as defined at the end of Section 1 has a *standard basis* $(A_i)_{i\in\mathbf{I}}$ consisting of the primitive idempotents of $\mathscr{A}$ viewed as a subalgebra of $M_X^\circ(\mathbf{C})$. Clearly A_i is the matrix of f_i, where $(f_i)_{i\in\mathbf{I}}$ is the partition of X^2 corresponding to $\mathscr{A}$ under the correspondence of (1.2). That is, the A_i are the adjacency matrices of the configuration $\mathscr{C}=(X,(f_i)_{i\in\mathbf{I}})$, which we refer to as the configuration *underlying* $\mathscr{A}$, and $\mathscr{A}$ is the adjacency algebra of $\mathscr{C}$. The assumption that $\mathscr{A}$ is a coherent algebra translates into the following.

3.1. *The family $(A_i)_{i\in\mathbf{I}}$ of nonzero $n\times n$ $(0,1)$ matrices is the standard basis of a coherent algebra if and only if the following four conditions hold*:

(A) $\Sigma_{i=0}^{r}A_i=J$ $(r=|\mathbf{I}|)$,
(B) $\Sigma_{\alpha\in\Omega}A_\alpha=I$, $\Omega\subseteq\mathbf{I}$,
(C) $A_i^*=A_{i^*}$, $i^*\in\mathbf{I}$ $(i\in\mathbf{I})$,
(D) $A_iA_j=\Sigma_{k\in\mathbf{I}}p_{ij}^kA_k$ $(i,j\in\mathbf{I})$,

where (D) is interpreted as meaning that $(A_i)_{i\in\mathbf{I}}$ span a subalgebra of $M_X(\mathbf{C})$.

In turn these conditions translate into

3.2. *The configuration $\mathscr{C}=(X,(f_i)_{i\in\mathbf{I}})$ is the underlying configuration of a coherent algebra if and only if*

(I) $(f_i)_{i\in\mathbf{I}}$ *is a partition of* X^2,
(II) $f_i^t=f_{i^*}$, $i^*\in\mathbf{I}$ $(i\in\mathbf{I})$,
(III) $f_i\cap\Delta\neq\varnothing$ *implies* $f_i\subseteq\Delta$ $(i\in\mathbf{I})$, *and*
(IV) $p_{ij}(x,y)$, $(x,y)\in f_k$, *is independent of the choice of* $(x,y)\in f_k$ $(i,j,k\in\mathbf{I})$.

A configuration which satisfies conditions (I) through (IV) is called *coherent*. Thus the coherent configurations on X are precisely the configurations which underly coherent algebras on X, with $p_{ij}^k=p_{ij}(x,y)$, $(x,y)\in f_k$. From now on we refer to coherent configurations as c.c.'s. If $f\colon X^2\to\mathbf{I}$ is a surjective map, then $(X,f,\mathbf{I})$ will denote the configuration over $\mathbf{I}$ such that $f_i=f^{-1}(i)$, $i\in\mathbf{I}$. These are precisely the configurations such that $(f_i)_{i\in\mathbf{I}}$ is a partition of X^2 and include in particular the c.c.'s. We have

PROPOSITION 3.3. *The c.c's on X form a lattice isomorphic with the lattice of coherent algebras on X.*

The maximum configuration is coherent, and its adjacency algebra is $M_X(\mathbf{C})$. The minimum c.c. $(X,(\Delta, X^2-\Delta))$ has adjacency algebra $\mathbf{C}I \oplus \mathbf{C}(J-I)$. We can define the *coherent closure* of a subalgebra of $M_X(\mathbf{C})$ to be the intersection of all coherent algebras on X which contain it. The *coherent closure* of a configuration on X can then be defined to be the c.c. underlying the coherent closure of its adjacency algebra. There is an algorithm [15] for constructing the coherent closure of a configuration as follows. First reduce in the obvious way to the case in which (I), (II), and (III) hold. Then let $B_1, \ldots, B_s$ be the adjacency matrices, and put $B = t_1 B_1 + \cdots + t_s B_s$, where $t_1, \ldots, t_s$ are distinct noncommuting indeterminates. The entries of B^2 are sums of products $t_i t_j$. If the distinct entries in $B_i \circ B^2$ are $p_1, p_2, \ldots,$ then $B_i = p_1 B_{i1} + p_2 B_{i2} + \cdots$, where the B_{ij} are $(0,1)$ matrices. For each i, replace B_i by the $B_{i1}, B_{i2}, \ldots$. Repeat this process until no further refinement is possible.

Consider a coherent algebra $\mathscr{A}$ and its underlying c.c. $\mathscr{C}$ (we frequently identify $\mathscr{A}$ with $\mathscr{C}$). The subset Ω of $\mathbf{I}$ appearing in (B) of 3.1, i.e., such that $\Delta = \bigcup_{\alpha \in \Omega} f_\alpha$, is uniquely determined, and $\mathscr{A}$ and $\mathscr{C}$ are called *homogeneous* if $|\Omega| = 1$, i.e., if A_i is the identity matrix for some $i \in \mathbf{I}$. We obtain the *standard partition* $(X_\alpha)_{\alpha \in \Omega}$ of $\mathscr{C}$ by putting $X_\alpha = \mathrm{pr}_1 f_\alpha = \mathrm{pr}_2 f_\alpha$, $\alpha \in \Omega$. That this is a partition of $\mathscr{C}$ in the sense of Section 2 follows from

3.4 [7]. *For $i \in \mathbf{I}$, $\mathrm{pr}_1 f_i = X_\alpha$ and $\mathrm{pr}_2 f_i = X_\beta$ for some $\alpha, \beta \in \Omega$.*

The *fibers* and *type* of $\mathscr{C}$ are defined to be the fibers and type as defined in Section 3 of the standard partition. Using the notation of Section 3, the fiber $\mathscr{C}^\alpha$ is a homogeneous c.c. of order $n_\alpha = |X_\alpha|$ and rank r_α. Put $\mathscr{A}^{\alpha\beta} = \langle A_i \mid i \in \mathbf{I}^{\alpha\beta} \rangle$, and call the $\mathscr{A}^\alpha = \mathrm{A}^{\alpha\alpha}$ the *fibers* of $\mathscr{A}$. Then:

3.5.

(i) *$\mathscr{A}^\alpha$ can be identified with the adjacency algebra of $\mathscr{C}^\alpha$,*
(ii) *$\mathscr{A} = \bigoplus_{\alpha, \beta \in \Omega} \mathscr{A}^{\alpha\beta}$ (vector space direct sum), and*
(iii) *$\mathscr{A}^{\alpha\beta} \mathscr{A}^{\gamma\varepsilon} = \delta_{\beta\gamma} \mathscr{A}^{\alpha\varepsilon}$.*

For $i \in \mathbf{I}^{\alpha\beta}$, put $m_i = |f_i|$ and $v_i = p^\alpha_{ii*} = |f_i(x)|$, $x \in X_\alpha$. Then:

3.6 [6]. *For $i, j, k \in \mathbf{I}$ and $\alpha, \beta, \gamma \in \Omega$,*

(i) $m_i = n_\alpha v_i = n_\beta v_{i*} = m_{i*}$,
(ii) $n_\alpha = \sum_{i \in \mathbf{I}^{\alpha\beta}} v_{i*}$,

(iii) $p_{\alpha j}^{k} = \begin{cases} \delta_{jk} & \text{if } j \in \bigcup_{\beta \in \Omega} \mathbf{I}^{\alpha\beta}, \\ 0 & \text{otherwise}, \end{cases}$

(iv) $p_{ij}^{\alpha} = \begin{cases} \delta_{ij^*} v_i & \text{if } i \in \bigcup_{\beta \in \Omega} \mathbf{I}^{\alpha\beta}, \\ 0 & \text{otherwise}, \end{cases}$

(v) $p_{ij}^{k} = p_{j^*i^*}^{k^*}$,

(vi) $p_{ij}^{k} v_k = p_{kj^*}^{i} v_i$,

(vii) $p_{ij}^{k} \neq 0$ *implies that* $i \in \mathbf{I}^{\alpha\beta}$, $j \in \mathbf{I}^{\beta\gamma}$, *and* $k \in \mathbf{I}^{\alpha\gamma}$ *for some* $\alpha, \beta, \gamma \in \Omega$,

(viii) *if* $i \in \mathbf{I}^{\alpha\beta}$ *and* $k \in \mathbf{I}^{\alpha\gamma}$, *then* $\sum_{i \in \mathbf{I}} p_{ij}^{k} = v_i$,

(ix) *if* $j \in \mathbf{I}^{\beta\gamma}$ *and* $k \in \mathbf{I}^{\alpha\gamma}$, *then* $\sum_{i \in \mathbf{I}} p_{ij}^{k} = v_{j^*}$, *and*

(x) $\sum_{t \in \mathbf{I}} p_{si}^{t} p_{tj}^{u} = \sum_{k \in \mathbf{I}} p_{ij}^{k} p_{sk}^{u}$ $(s, u, i, j \in \mathbf{I})$.

The regular representation $A_i \mapsto M_i = (p_{si}^{t})_{s,t \in I}$ maps $\mathscr{A}$ isomorphically onto a subalgebra $\mathscr{M}$ of $M_{\mathbf{I}}(\mathbf{C})$ called the *intersection algebra*. The *intersection matrices* M_i are blocked according to the partition $(\mathbf{I}^{\alpha\beta})_{\alpha,\beta \in \Omega}$ of $\mathbf{I}$. For $i \in \mathbf{I}^{\alpha\beta}$ and $a \in \Omega$, let M_i^a denote the $(\mathbf{I}^{a\alpha}, \mathbf{I}^{a\beta})$ block of M_i; $M_i^a = (p_{si}^{t})_{s \in \mathbf{I}^{a\alpha},\, t \in \mathbf{I}^{a\beta}}$ is an $r_{a\alpha} \times r_{a\beta}$ matrix, and the other blocks of M_i are all zero. From the above we have

3.7. *For* a, α, β, γ, $\sigma \in \Omega$ *and* $i \in \mathbf{I}$,

(a) $M_{\alpha}^{a} = \mathbf{I}$,

(b) M_i^a *has column sum* v_{i^*},

(c) $\sum_{i \in I^{\alpha\beta}} M_i^a$ *has sth row* $(v_s, \ldots, v_s)$, $s \in \mathbf{I}^{a\alpha}$,

(d) $[M_i^a D^{a\beta}]^t = M_{i^*}^a D^{a\alpha}$, *where* $D^{a\sigma}$ *is the diagonal matrix whose sth diagonal entry is* v_s, $s \in \mathbf{I}^{a\sigma}$, *and*

(e) $M_i^a M_j^a = \sum_{k \in \mathbf{I}^{\alpha\gamma}} p_{ij}^{k} M_k^a$ $(i \in \mathbf{I}^{\alpha\beta},\ j \in \mathbf{I}^{\beta\gamma})$.

It is natural to refer to a coherent algebra $\mathscr{A}$ and its underlying c.c. $\mathscr{C}$ as *symmetric* if A_i is symmetric for all i, i.e., f_i is symmetric for all i. There is not much danger of confusion with the usual concept of symmetric algebra, since coherent algebras are always symmetric in that sense, being semisimple (see Section 7). We call $\mathscr{C}$ *commutative* if $\mathscr{A}$ is, and observe that

3.8. *For coherent algebras and configurations,*

symmetric $\Rightarrow$ *commutative* $\Rightarrow$ *homogeneous*.

In particular, *association schemes* as originally defined by Bose and Shimamoto are precisely the symmetric c.c.'s, and the adjacency algebra of such an association scheme is its *Bose-Mesner algebra* [1]. Nowadays the term *association scheme* frequently means commutative c.c. in our terms.

3.9 [7]. *Homogeneous c.c.'s of rank* $\leqslant 5$ *are commutative.*

In the group case, $\mathscr{C}(G, X)$ is always coherent, i.e., the centralizer algebra of a group action is a coherent algebra, and the standard partition is the partition X/G of X into orbits under G. We define the *type* of the action of G on X to be the type of $\mathscr{C}(G, X)$. Transitivity of G on X is equivalent to homogeneity of $\mathscr{C}(X, G)$.

4. FEASIBLE TRACES

In this and the next two sections we give a discussion of algebras designed for applications to coherent algebras and their weighted and generic versions [7, 8]. In applications of coherent algebras we typically reach the point of having on hand a candidate for an intersection algebra $\mathscr{M}$ and a candidate ζ for the character afforded by the standard module. The problem at that stage is to decide whether ζ is a character of $\mathscr{M}$ or not, that is, since a feasible $\mathscr{M}$ is semisimple, whether ζ is an integral linear combination of irreducible characters. There are analogues of this for weighted and generic algebras. We abstract this situation in the following notion of *feasible trace*.

We begin with an r-dimensional algebra $\mathscr{A}$ over a field F and define a *feasible trace* to be a linear function $\zeta \in \operatorname{Hom}_F(\mathscr{A}, F)$ such that $\zeta(xy) = \zeta(yx)$ for all $x, y \in \mathscr{A}$. We say that ζ is *nondegenerate* if $\operatorname{rad}\zeta = \varnothing$, where $\operatorname{rad}\zeta = (x \in \mathscr{A} \mid \zeta(x\mathscr{A}) = 0)$. A linear functional $\zeta : \mathscr{A} \mapsto F$ determines an associative bilinear form on $\mathscr{A}$ according to $(x, y) = \zeta(xy)$, and conversely [associative means $(xy, z)] = (x, yz)]$. The linear functional ζ is a feasible trace if and only if the corresponding bilinear form is symmetric, and in that case $\operatorname{rad}\zeta$ coincides with the radical of the bilinear form. (Thus an algebra equipped with a nondegenerate feasible trace is a *symmetric algebra* in the classical sense, not to be confused with the completely different notion of symmetric for coherent algebras and configurations as defined in Section 3.)

4.1. *Let ζ be a nondegenerate feasible trace on $\mathscr{A}$, $w_1, \ldots, w_r$ a basis of $\mathscr{A}$, and $\hat{w}_1, \ldots, \hat{w}_r$ the dual basis defined by $\zeta(w_i \hat{w}_j) = \delta_{ij}$. Then*

(1) $w_i = \sum_{j=1}^{r} \zeta(w_i w_j) \hat{w}_j$,
(2) $\hat{\hat{w}}_i = w_i$, *and*
(3) $w_i x = \sum_{j=1}^{r} a_{ij} w_j$, $x \in \mathscr{A}$, *implies* $x \hat{w}_i = \sum_{j=1}^{r} a_{ji} \hat{w}_j$.

Proof. (1) and (2) are clear. If $x\hat{w}_i = \Sigma b_{ij}\hat{w}_j$, then $a_{ij} = \zeta(w_i x\hat{w}_j) = b_{ji}$. ∎

A nondegenerate feasible trace ζ induces a linear isomorphism $T: \mathscr{A} \to \mathrm{Hom}_F(\mathscr{A}, F)$ according to $T(x)(y) = \zeta(xy)$, $x, y \in \mathscr{A}$, and so a nondegenerate bilinear form is defined on $\mathrm{Hom}_F(\mathscr{A}, F)$ by $(T(x), T(y)) = \zeta(xy)$, $x, y \in \mathscr{A}$. For this *dual form* we have

$$(\phi, \psi) = \sum_{i=1}^{r} \phi(w_i)\psi(\hat{w}_i) \qquad [\phi, \psi \in \mathrm{Hom}_F(\mathscr{A}, F)].$$

For any extension field K of F, ζ extends uniquely to a nondegenerate feasible trace on $\mathscr{A}_K = \mathscr{A} \otimes_F K$, and the dual form induced by this extension is the unique extension to $\mathrm{Hom}_K(\mathscr{A}_K, K)$ of the dual form induced by ζ.

5. FEASIBLE TRACES ON SEMISIMPLE ALGEBRAS

Throughout this section $\mathscr{A}$ will be a semisimple algebra of finite dimension r over a field F of characteristic 0, and K will be a splitting field of $\mathscr{A}$. Then $\mathscr{A}_K = \mathscr{A} \otimes_F K$ decomposes into a direct sum of simple (two-sided) ideals

$$\mathscr{A}_K = \bigoplus_{s=1}^{m} B_s.$$

Each B_s is isomorphic with a full matrix algebra over K, of degree e_s, say, so

$$\sum_{s=1}^{m} e_s^2 = r.$$

Let $1 = \Sigma_{s=1}^{m} 1_s$, $1_s \in B_s$, be the identity element of $\mathscr{A}_K$; then 1_s is the identity element of B_s and $\{1_s \mid s = 1, \dots, m)$ is the set of central primitive idempotents of $\mathscr{A}_K$. Let $\Delta_1, \dots, \Delta_m$ be the inequivalent absolutely irreducible representations of $\mathscr{A}$, written in K, and let $\zeta_1, \dots, \zeta_m$ be the corresponding characters. The same notation will be used for the extensions of Δ_s and ζ_s to $\mathscr{A}_K$, and the notation will be chosen so that Δ_s corresponds to 1_s in the sense that

$$\Delta_s(1_t) = \delta_{st}\Delta_s(1) \qquad (1 \leqslant s, t \leqslant m).$$

This implies that Δ_s has degree $\zeta_s(1) = \zeta_s(1_s) = e_s$.

The feasible traces on $\mathscr{A}$ are easily described, namely

PROPOSITION 5.1. *A linear functional $\zeta: \mathscr{A} \to F$ is a feasible trace on $\mathscr{A}$ if and only if it has the form*

$$(*) \qquad \zeta = \sum_{s=1}^{m} z_s \zeta_s$$

with $z_s \in K$, and in that case the extension of ζ to $\mathscr{A}_K$ has radical $\oplus_{s \in \Phi} B_s$, where $\Phi = \{ s \mid z_s = 0\}$; in particular ζ is nondegenerate if and only if $z_s \neq 0$ for all $s = 1, 2, \ldots, m$.

Proof. Assume that ζ is a feasible trace on $\mathscr{A}$, and consider first the case in which $\mathscr{A}$ is a full matrix algebra over F. Then for all i, j, k,

$$\zeta(E_{ij}) = \zeta(E_{ik}E_{kj}) = \zeta(E_{kj}E_{ik}) = \delta_{ij}\zeta(E_{kk}),$$

where the E_{ij} are the usual matrix units. Therefore, ζ is a multiple of the trace function on $\mathscr{A}$ and is nondegenerate if and only if it is nonzero. In the general case, ζ extends to a feasible trace on $\mathscr{A}_K$, which we still call ζ, and we define $\eta_s(x) = \zeta(x_s)$ for $x = x_1 + \cdots + x_m \in \mathscr{A}$, $x_s \in B_s$. Then η_s is a feasible trace on $\mathscr{A}_K$, $\sum_{t \neq s} B_s \subseteq \operatorname{rad} \eta_s$, and $\eta_s | B_s$ is a feasible trace on B_s. Hence $\eta_s = z_s \zeta_s$, $z_s \in K$, by the first paragraph of this section, and ζ has the form $(*)$.

Conversely, a linear functional $\zeta: \mathscr{A} \to F$ which can be written in the form $(*)$ with $z_s \in K$ is certainly a feasible trace. For the extension to $\mathscr{A}_K$ we have $\operatorname{rad} \zeta = \oplus_{s \in \Phi} B_s$ for some Φ, and $\operatorname{rad} \zeta$ is an ideal. Moreover $B_s \subseteq \operatorname{rad} \zeta$ if and only if $\zeta(x) = 0$ for all $x \in B_s$ if and only if $z_s = 0$. ■

If ζ is a feasible trace given by $(*)$, then we refer to z_s as the *feasible multiplicity* of ζ_s in ζ. Thus ζ will be a virtual character (ordinary character) if and only if its feasible multiplicities are all rational integers (nonnegative rational integers).

For the rest of this section we assume that ζ is a nondegenerate feasible trace on $\mathscr{A}$, $\zeta=\sum_{s=1}^{m} z_s\zeta_s$, $z_s\in K^*$. Write

$$\Delta_s(x)=\left(a_{ij}^s(x)\right) \qquad (x\in\mathscr{A},\quad 1\leqslant s\leqslant m).$$

There is a basis $(\varepsilon_{ij}^s)_{1\leqslant i,j\leqslant e_s;\, 1\leqslant s\leqslant m}$ of $\mathscr{A}$ defined by

$$\Delta_t\left(\varepsilon_{ij}^s\right)=\delta_{st}E_{ij}^s,$$

where E_{ij}^s is the $e_s\times e_s$ matrix unit. Then $\zeta_t(\varepsilon_{ij}^s)=\delta_{st}\delta_{ij}$, so $\zeta(\varepsilon_{ij}^s)=\delta_{ij}z_s$ and $\varepsilon_{ij}^s\varepsilon_{kl}^t=\delta_{st}\delta_{jk}\varepsilon_{il}^s$. Hence

$$\text{(5.2)} \qquad T\left(\varepsilon_{ij}^s\right)\left(\varepsilon_{kl}^t\right)=\zeta\left(\varepsilon_{ij}^s\varepsilon_{kl}^t\right)=\delta_{st}\delta_{jk}\delta_{il}z_s.$$

To simplify the notation, list the r linear functions a_{ij}^s in some order $a_1,\dots,a_r$, and if $a_\lambda=a_{ij}^s$, write

$$a_{\bar\lambda}=a_{ji}^s,$$

$$h_\lambda=z_s,$$

$$\varepsilon_\lambda=\varepsilon_{ij}^s.$$

Then (5.2) becomes

$$T(\varepsilon_\lambda)(\varepsilon_\mu)=\zeta(\varepsilon_\lambda\varepsilon_\mu)=\delta_{\lambda\bar\mu}h_\lambda \qquad (1\leqslant\lambda,\mu\leqslant r)$$

and therefore, using the dual form, we have

$$\text{(5.3)} \qquad (a_\lambda,a_\mu)=\delta_{\lambda\bar\mu}\frac{1}{h_\lambda} \qquad (1\leqslant\lambda,\mu\leqslant r).$$

These equations are referred to as the *Schur relations*. Since $\zeta_s=\sum_{i=1}^{e_s}a_{ii}^s$, the Schur relations imply the *orthogonality relations*

$$\text{(5.4)} \qquad (\zeta_s,\zeta_t)=\delta_{st}\frac{e_s}{z_s} \qquad (1\leqslant s,t\leqslant m)$$

for the characters.

The meaning of the Schur relations is that the linear functionals $a_1, \dots, a_r$ form a basis of $\mathrm{Hom}_F(\mathscr{A}, K)$ with dual basis $a_1^*, \dots, a_r^*$ with respect to the dual form given by $a_\lambda^* = (1/h_\lambda)a_{\bar\lambda}$. Another way of saying the same thing is that the matrix $A = (a_\lambda(w_i))$ is nonsingular with inverse $A^{-1} = [h_\lambda a_{\bar\lambda}(\hat{w}_i)]^t$. Then $A^{-1}A = I$ gives

$$\sum_{\lambda=1}^{r} h_\lambda a_{\bar\lambda}(\hat{w}_i) a_\lambda(w_j) = \delta_{ij}. \tag{5.5}$$

We have

$$w_i = \sum_{\lambda=1}^{r} a_\lambda(w_i)\varepsilon_\lambda \tag{5.6}$$

so

$$\varepsilon_\lambda = h_\lambda \sum_{i=1}^{r} a_{\bar\lambda}(\hat{w}_i) w_i. \tag{5.7}$$

Since $1_s = \sum_{i=1}^{e_s} \varepsilon_{ii}^s$,

$$1_s = z_s \sum_{i=1}^{r} \zeta(\hat{w}_i) w_i. \tag{5.8}$$

The algebra $\mathscr{A}$ is commutative if and only if $m = r$ if and only if $e_1 = e_2 = \cdots = e_m = 1$. In this case (5.5) and (5.6) become respectively

$$\sum_{s=1}^{r} z_s \zeta_s(\hat{w}_i)\zeta_s(w_j) = \delta_{ij} \tag{5.5$'$}$$

and

$$w_i = \sum_{s=1}^{r} \zeta_s(w_i) 1_s. \tag{5.6$'$}$$

6. SELF-ADJOINT ALGEBRAS

Now we return to the notation of Sections 1 through 3 and consider a self-adjoint subalgebra $\mathscr{A}$ of $M_X(\mathbf{C})$. The standard A-module $L(X)$ is faithful, and if W is an $\mathscr{A}$-submodule, so is its orthogonal complement $W^{\perp}$. Hence $L(X)$ is an orthogonal direct sum of A-submodules. In particular, therefore, $\mathscr{A}$ is semisimple, and we may apply the discussion and notation of Section 5, taking $F = K = \mathbf{C}$ and ζ to be the character afforded by $L(X)$. In this case z_s is the actual multiplicity of Δ_s in the representation Δ afforded by $L(X)$, i.e., of ζ_s in ζ. Because $L(X)$ is an orthogonal direct sum of irreducible submodules, we may assume that

$$\Delta_i(A^*) = \Delta_i(A)^* \qquad (A \in \mathscr{A}, \quad 1 \leqslant i \leqslant m),$$

i.e., that

$$a_\lambda(A^*) = \overline{a_{\bar{\lambda}}(A)} \qquad (A \in \mathscr{A}, \quad 1 \leqslant \lambda \leqslant r),$$

and hence $\zeta_i(A^*) = \overline{\zeta_i(A)}$, $A \in \mathscr{A}$.

Now assume that $(X_\alpha)_{\alpha \in \Omega}$ is a partition of X, and let

$$\mathscr{A}^{\alpha\beta} = \left\{ A \in \mathscr{A} \mid AL(X_\gamma) \subseteq \delta_{\alpha\gamma} L(X_\beta) \right\},$$

where $L(X_\alpha)$ has been identified in the natural way with a subspace of $L(X)$ so that $L(X) = \oplus_{\alpha \in \Omega} L(X_\alpha)$. Assume that $\mathscr{A} = \oplus_{\alpha, \beta \in \Omega} \mathscr{A}^{\alpha\beta}$ (vector space direct sum). Then

$$\mathscr{A}^{\alpha\beta} \mathscr{A}^{\gamma\delta} \subseteq \delta_{\beta\gamma} \mathscr{A}^{\alpha\delta},$$

and $\mathscr{A}^{\alpha\beta}$ can be identified with a subspace of $\mathrm{Hom}_{\mathbf{C}}(L(X_\beta), L(X_\alpha))$. In particular, $\mathscr{A}^\alpha = \mathscr{A}^{\alpha\alpha}$ is then identified with a self-adjoint subalgebra of $M_{X_\alpha}(\mathbf{C})$. (This is the situation associated with the standard partition in Section 3.)

Let σ_α be the identity element of $\mathscr{A}^\alpha$, and put $\Lambda(\alpha) = \{ s \mid \sigma_\alpha 1_s \neq 0 \}$. By a straightforward application of centralizer ring theory we see that

$$\{ \sigma_\alpha 1_s \mid s \in \Lambda(\alpha) \}$$

is the set of central primitive idempotents of $\mathscr{A}^\alpha$. We number the irreducible representation, character, degree, and multiplicity of $\mathscr{A}^\alpha$ corresponding to $\sigma_\alpha 1_s$, $s \in \Lambda(\alpha)$, accordingly: $\Delta_{\alpha s}, \zeta_{\alpha s}, e_{\alpha s}, z_{\alpha s}$. If $A = \sum_{\alpha, \beta \in \Omega} A_{\alpha\beta}$, $A_{\alpha\beta} \in \mathscr{A}^{\alpha\beta}$, then

$$\zeta_s(A) = \sum_{\alpha \in M(s)} \zeta_{\alpha s}(A_{\alpha\alpha}), \tag{6.1}$$

where $M(s) = \{\alpha \in \Omega \mid s \in \Lambda(\alpha)\}$. In particular

$$e_s = \sum_{\alpha \in M(s)} e_{\alpha s}. \tag{6.2}$$

Moreover,

$$z_s = z_{\alpha s} \quad \text{for} \quad s \in \Lambda(\alpha). \tag{6.3}$$

Let $r_{\alpha\beta} = \dim \mathscr{A}^{\alpha\beta}$, so that $r = \sum_{\alpha, \beta \in \Omega} r_{\alpha\beta}$, $r_{\alpha\beta} = r_{\beta\alpha}$. Since

$$\mathscr{A}^{\alpha\beta} \approx \bigoplus_{s \in \Lambda(\alpha) \cap \Lambda(\beta)} M_{e_{\alpha s} \times e_{\beta s}}(\mathbf{C}), \tag{6.4}$$

where $M_{a \times b}(\mathbf{C})$ is the space of $a \times b$ matrices over $\mathbf{C}$, we have

$$r_{\alpha\beta} = \sum_{s \in \Lambda(\alpha) \cap \Lambda(\beta)} e_{\alpha s} e_{\beta s}. \tag{6.5}$$

7. APPLICATION TO COHERENT ALGEBRAS

The notation and discussion of Section 5 are immediately applicable to a coherent algebra $\mathscr{A}$, and in particular $\mathscr{A}$ is semisimple. We take for (w_i) the standard basis $(A_i)_{i \in \mathbf{I}}$; then by (3.6)

$$(A_i, A_j^*) = \delta_{ij} m_i, \qquad (i, j \in \mathbf{I}), \tag{7.1}$$

so the dual basis is given by

$$\hat{A}_i = \frac{1}{m_i} A_i^*. \qquad (i \in \mathbf{I}). \tag{7.2}$$

We assume, as we may, that the irreducible representations Δ_s are written so that $\Delta_s(A^*) = \Delta_s(A)^*$, $A \in \mathscr{A}$. Then $a_{\bar{\lambda}}(\hat{A}_i) = (1/m_i)\overline{a_\lambda(A_i)}$ and

$\zeta_s(\hat{A}_i) = (1/m_s)\overline{\zeta_s(A_i)}$, so (5.3) through (5.6′) become

$$(7.3) \qquad \sum_{i \in \mathbf{I}} \frac{1}{m_i} a_\lambda(A_i)\, \overline{a_\mu(A_i)} = \delta_{\lambda\mu} \frac{1}{h_\lambda},$$

$$(7.4) \qquad \sum_{i \in \mathbf{I}} \frac{1}{m_i} \zeta_s(A_i)\, \overline{\zeta_t(A_i)} = \delta_{st} \frac{e_s}{z_s},$$

$$(7.5) \qquad \sum_{\lambda=1}^{r} h_\lambda \overline{a_\lambda(A_i)}\, a_\lambda(A_j) = \delta_{ij} m_j,$$

$$(7.6) \qquad A_i = \sum_{\lambda=1}^{r} a_\lambda(A_i) \varepsilon_\lambda,$$

$$(7.7) \qquad \varepsilon_\lambda = h_\lambda \sum_{i \in \mathbf{I}} \frac{1}{m_i} \overline{a_\lambda(A_i)}\, A_i,$$

$$(7.8) \qquad 1_s = z_s \sum_{i \in \mathbf{I}} \frac{1}{m_i} \overline{\zeta_s(A_i)}\, A_i.$$

Moreover, $\mathscr{A}$ is commutative if and only if $m = r$ if and only if $e_1 = e_2 = \cdots = e_m = 1$, and in that case

$$(7.9) \qquad \sum_{s=1}^{r} z_s \overline{\zeta_s(A_i)}\, \zeta_s(A_j) = \delta_{ij} m_i$$

and

$$(7.10) \qquad \varepsilon_\lambda = z_s \sum_{i \in \mathbf{I}} \frac{1}{m_i} \overline{\zeta_s(A_i)}\, A_i.$$

Similarly, the notation and discussion of Section 7 can be applied at once, taking $(X_\alpha)_{\alpha \in \Omega}$ to be the standard partition. We then have

$$(7.11) \qquad \zeta_s(A_i) = \begin{cases} \delta_{\alpha\beta} \zeta_{\alpha s}(A_i) & \text{if} \quad \alpha \in M(s), \\ 0 & \text{otherwise} \end{cases} \qquad (i \in \mathbf{I}^{\alpha\beta}).$$

Here (6.2) through (6.5) apply without change, and we can add the following. The elements $\sum_{x \in X_\alpha} x$ span an irreducible submodule of $L(X)$. Let us call this the *principal submodule* and refer to the central primitive idempotent, irreducible representation, character, degree, and multiplicity associated with it as *principal*, labeling them with a subscript 1: for example, 1_1 and ζ_1 are the principal central primitive idempotent and character of $\mathscr{A}$, $e_1 = \zeta_1(1) = |\Omega|$ is the number of fibers of $\mathscr{C}$, and $1_{\alpha 1}$ and $\zeta_{\alpha 1}$ are the principal idempotent and character of $\mathscr{A}^\alpha$, $e_{\alpha 1} = \zeta_{\alpha 1}(1_{\alpha 1}) = 1$. Since $1 \in \Lambda(\alpha)$ for all $\alpha \in \Omega$, we have $1_{\alpha 1} = \sigma_\alpha 1_1$ and $e_{\alpha 1} = z_{\alpha 1} = 1$.

We have the canonical decomposition

$$L = \mathop{\perp}_{s=1}^{m} L_s$$

of $L = L(X)$ into an orthogonal direct sum of components $L_s = 1_s L$, and the decomposition

$$L_s = \mathop{\perp}_{t=1}^{m_s} L_{s,t}$$

of L_s into an orthogonal direct sum of isomorphic simple components $L_{s,t} = \varepsilon_{tt}^s L = \varepsilon_{tt}^s L_s$. With the notational convention above, the principal submodule is $L_1 = L_{1,1}$. If $(x, y) \in f_k$, then

$$(\varepsilon_{tt}^s x, \varepsilon_{tt}^s y) = \varepsilon_{tt}^s(x, y) = \frac{z_s}{m_k} \overline{a_{tt}^s(A_k)}\,.$$

Hence if for some given s and t these r numbers are distinct, then $\mathscr{C}$ can be recovered from the angles between the projections of the standard basis X of L onto the irreducible submodule $L_{s,t}$.

The one additional general fact that we have about coherent algebras is that they satisfy the *Krein conditions* discovered by L. Scott in the context of multiplicity free permutation representations. By (7.7) and (7.6) we have

$$\text{(7.12)} \qquad \varepsilon_\lambda \circ \varepsilon_\mu = h_\lambda h_\mu \sum_{i \in \mathbf{I}} \frac{1}{m_i^2} \overline{a_\lambda(A_i) a_\mu(A_i)}\, A_i$$

$$= h_\lambda h_\mu \sum_{\nu=1}^{r} \sum_{i \in \mathbf{I}} \frac{1}{m_i^2} \overline{a_\lambda(A_i) a_\mu(A_i)}\, a_\nu(A_i) \varepsilon_\nu,$$

so $\varepsilon_\lambda \circ \varepsilon_\mu = \sum_{\nu=1}^{r} q_{\lambda\mu}^\nu \varepsilon_\nu$ with

$$\text{(7.13)} \qquad q_{\lambda\mu}^\nu = h_\lambda h_\nu \sum_{i \in \mathbf{I}} \frac{1}{m_i^2} \overline{a_\lambda(A_i) a_\mu(A_i)}\, a_\nu(A_i).$$

In case $\mathscr{A}$ is commutative and $|X| = n$, (7.13) becomes

$$(7.13') \quad q_{uv}^{s} = \frac{z_u z_v}{n^2} \sum_{i \in \mathbf{I}} \frac{1}{v_i^2} \overline{\zeta_u(A_i)\zeta_v(A_i)}\, \zeta_s(A_i) \qquad (1 \leqslant u, v, s \leqslant m).$$

Fix $\lambda = \bar{\lambda}$, $\mu = \bar{\mu}$. Then ε_λ and ε_μ are positive semidefinite Hermitian matrices, and therefore so is $\varepsilon_\lambda \circ \varepsilon_\mu$. But this means that:

7.14. *For $1 \leqslant s \leqslant m$ and $\lambda = \bar{\lambda}$ and $\mu = \bar{\mu}$, the matrices*

$$\Delta_s(\varepsilon_\lambda \circ \varepsilon_\mu) = \left(q_{\lambda\mu}^{\nu}\right)_{1 \leqslant i, j \leqslant e_s}, \qquad a_\nu = a_{ij}^{s},$$

are positive semidefinite.

We refer to this and to the following two consequences as the *Krein conditions*:

$$(7.15) \qquad q_{\lambda\mu}^{\nu} \geqslant 0 \qquad (\lambda = \bar{\lambda}, \quad \mu = \bar{\mu}, \quad \nu = \bar{\nu}),$$

from which follows

$$(7.16) \qquad \sum_{i \in \mathbf{I}} \frac{1}{m_i^2} \overline{\zeta_u(A_i)\zeta_v(A_i)}\, \zeta_s(A_i) \geqslant 0 \qquad (1 \leqslant u, v, s \leqslant m).$$

We can summarize the necessary conditions that we know in general for intersection matrices of coherent algebras in a definition as follows. Suppose given a finite set $\mathbf{I}$, a subset Ω of $\mathbf{I}$, and a partition $(\mathbf{I}^{\alpha\beta})_{\alpha,\beta \in \Omega}$ of $\mathbf{I}$ with $\alpha \in \mathbf{I}^{\alpha\alpha}$. Assume given for each $i \in \mathbf{I}$ a matrix $M_i = (p_{si}^{t})_{s,t \in \mathbf{I}}$ of nonnegative integers satisfying the conditions of 3.6 or equivalently 3.7. Then $(M_i)_{i \in \mathbf{I}}$ is a basis of a semisimple subalgebra $\mathscr{M}$ of $M_I(\mathbf{C})$, and

$$\zeta(M_i) = \begin{cases} n_i & \text{if } \ i \in \Omega, \\ 0 & \text{otherwise} \end{cases} \qquad (i \in \mathbf{I})$$

defines a feasible trace on $\mathscr{M}$. We say that $(M_i)_{i \in \mathbf{I}}$ is a *feasible set of intersection matrices* or that $\mathscr{M}$ is a *feasible intersection algebra* if in addition (1) the multiplicities for ζ are positive integers, i.e., ζ is the character of a faithful representation of $\mathscr{M}$, and (2) the Krein conditions hold for $\mathscr{M}$ and for $\mathscr{M}^{\alpha} = \langle (M_i)_{i \in \mathbf{I}^{\alpha\alpha}} \rangle$, $\alpha \in \Omega$ (where the standard basis is replaced by the appropriate intersection matrices).

8. CONFIGURATIONS WITH FIBERS OF SMALL RANK

Because the type $(r_{\alpha\beta})$ of a coherent algebra is a symmetric matrix, we omit entries below the main diagonal. The conditions of Section 7 imply that there are restrictions on the matrices $(r_{\alpha\beta})$ which can occur as types of coherent algebras. For instance

$$\begin{bmatrix} 2 & 2 & 1 \\ & 2 & 2 \\ & & 2 \end{bmatrix}$$

cannot be a type. Notice that if $\mathscr{A}^{\alpha}$ is commutative, e.g., if $r_{\alpha} \leqslant 5$, then $e_{\alpha s} = 1$ for all $s \in \Lambda(\alpha)$ and $r_{\alpha} = |\Lambda(\alpha)|$. Hence if $\mathscr{A}^{\beta}$ is commutative too, then $r_{\alpha\beta} = |\Lambda(\alpha) \cap \Lambda(\beta)| \leqslant \min(r_{\alpha}, r_{\beta})$.

Here is a list of the possible 2×2 and 3×3 types with all entries $\leqslant 3$. For obvious reasons those with a diagonal entry 1 and those 3×3 cases with more than two nondiagonal entries 1 are omitted:

$$(1)\ \begin{bmatrix} 2 & 2 \\ & 2 \end{bmatrix}, \quad (2)\ \begin{bmatrix} 2 & 2 \\ & 3 \end{bmatrix}, \quad (3)\ \begin{bmatrix} 3 & 2 \\ & 3 \end{bmatrix}, \quad (4)\ \begin{bmatrix} 3 & 3 \\ & 3 \end{bmatrix},$$

$$(5)\ \begin{bmatrix} 2 & 2 & 2 \\ & 2 & 2 \\ & & 2 \end{bmatrix}, \quad (6)\ \begin{bmatrix} 2 & 2 & 1 \\ & 3 & 2 \\ & & 2 \end{bmatrix}, \quad (7)\ \begin{bmatrix} 2 & 2 & 2 \\ & 3 & 2 \\ & & 2 \end{bmatrix},$$

$$(8)\ \begin{bmatrix} 2 & 2 & 1 \\ & 3 & 2 \\ & & 3 \end{bmatrix}, \quad (9)\ \begin{bmatrix} 2 & 2 & 2 \\ & 3 & 2 \\ & & 3 \end{bmatrix}, \quad (10)\ \begin{bmatrix} 2 & 2 & 2 \\ & 3 & 3 \\ & & 3 \end{bmatrix},$$

$$(11)\ \begin{bmatrix} 3 & 2 & 1 \\ & 3 & 2 \\ & & 3 \end{bmatrix}, \quad (12)\ \begin{bmatrix} 3 & 2 & 2 \\ & 3 & 2 \\ & & 3 \end{bmatrix},$$

$$(13)\ \begin{bmatrix} 3 & 2 & 3 \\ & 3 & 2 \\ & & 3 \end{bmatrix}, \quad (14)\ \begin{bmatrix} 3 & 3 & 3 \\ & 3 & 3 \\ & & 3 \end{bmatrix},$$

In case (1) we have $n_1 - 1 = z_{12} = z_{22} = n_2 - 1$, so $n_1 = n_2$, and the c.c.'s of this type correspond to complementary pairs of symmetric designs. The correspondence between c.c.'s of the type of case (3) and quasisymmetric designs is exploited in Section 9 below, and that between coherent algebras of the type of case (3) and what we call *strongly regular designs* is the starting point of [8].

The action of $\mathrm{PGL}_d(q)$ on the subspaces of $\mathrm{PG}_{d-1}(q)$, $d \geqslant 4$, has type

$$\begin{bmatrix} 2 & 2 & 2 & \cdots & & 2 & 2 \\ & 3 & 3 & \cdots & & 3 & 2 \\ & & 4 & \cdots & & 4 & 2 \\ & & & \ddots & & \vdots & \vdots \\ & & & & 4 & 4 & 2 \\ & & & & & 3 & 2 \\ & & & & & & 2 \end{bmatrix},$$

which provides examples for cases (2), (4), (7), and (10).

The action of $\mathrm{PSp}_{2d}(2)$, $d \geqslant 2$, on the two types of quadrics and the points of symplectic $\mathrm{PG}_{2d-1}(2)$ has type

$$\begin{bmatrix} 2 & 1 & 2 \\ & 2 & 2 \\ & & 3 \end{bmatrix},$$

giving examples of cases (2) and (6).

The action of $O^{\varepsilon}_{2d}(2)$, $d \geqslant 2$, on the singular and nonsingular points of orthogonal $\mathrm{PG}_{2d-1}(2)$ has type

$$\begin{bmatrix} 3 & 2 \\ & 3 \end{bmatrix},$$

which is case (3).

The c.c.'s with at least three fibers and $r_{\alpha\beta} = 2$ for all α, β correspond to the linked symmetric designs studied by P. J. Cameron [3].

Corresponding to the isomorphisms $U_4(2) \approx O_6^-(2) \approx \mathrm{Sp}_4(3)$ we have an example of case (11) with $n_1 = 27$, $n_2 = 36$, $n_3 = 40$, and multiplicities

$$\begin{matrix} 1 & 20 & 6 & & \\ 1 & 20 & & 15 & \\ 1 & & & 15 & 24 \end{matrix}$$

The degrees are therefore $e_1 = 3$, $e_2 = e_4 = 2$, and $e_3 = e_5 = 1$.

If $(r_{\alpha\beta})$ is the type of a coherent algebra, then "repeating the first fiber" produces a coherent algebra of type

$$\left[\begin{array}{c|c} & r_{11} \\ r_{\alpha\beta} & \vdots \\ & r_{1t} \\ \hline & r_{11} \end{array}\right].$$

Cases (4), (7), (10), (13), and (14) can be realized in this way. This gives the only examples we know of cases (13) and (14).

The referee has pointed out that examples of cases (8) and (12) can be obtained as follows, showing that all cases are realized. Let K be a maximal k-arc in a projective plane Π (i.e., K is a set of points such that $|l \cap K| = 0$ or k for every line l of Π). Then the points of K, the lines intersecting K, and the points of Π not on K give an example of (8). The objects corresponding to the outer nodes of a D_4 diagram provide an example of (12).

9. QUASI-SYMMETRIC DESIGNS AND C.C.'S OF TYPE (2,2;3)

A basic reference for quasisymmetric designs is Cameron and van Lint [4]. We repeat the definition here primarily to introduce our notation, which differs from that of [4]. Our goals are to establish the equivalence between c.c.'s of type (2,2;3)—here we write

$$(2,2;3) \quad \text{in place of} \quad \begin{bmatrix} 2 & 2 \\ & 3 \end{bmatrix}$$

—and complementary pairs of quasisymmetric designs, to record and analyze the parameter conditions which result from consideration of the corresponding coherent algebra, and to prove the results 9.7 and 9.11 for quasisymmetric designs. Starting from the parameter conditions in 9.4, the discussion of quasisymmetric designs is self-contained. A reader wishing to avoid c.c.'s can supply the needed formulas from 9.4 by counting and matrix theoretic methods.

9A. *Quasisymmetric Designs*

We consider incidence structures (X_1, X_2, F) consisting of disjoint sets X_1 and X_2, whose elements are called *points* and *lines* respectively, and a subset F of the Cartesian product $X_1 \times X_2$, whose elements are called *flags*.

A point x and a block y are *incident* if (x, y) is a flag. The incidence matrix C of the structure will have rows indexed by the points and columns by the clocks. The *dual* structure (X_1, X_2, F^t) has incidence matrix C^t.

A *quasisymmetric design* (q.s.d.) is an incidence structure such that

(a) each block is incident with S points and each point is incident with T blocks,

(b) two points are incident with Λ blocks, and

(c) two blocks are incident with either a or b points, $a > b$, and both possibilities occur.

We denote by m the number of points and by n the number of blocks, so that $mT = nS$.

The *block graph* Γ of a q.s.d. is the graph with the blocks as vertices, to being adjacent if and only if they are incident with a common points.

Associated with a q.s.d. (X_1, X_2, F) is the configuration $\mathscr{C} = (X, (f_i)_{i \in \mathbf{I}})$ over $\mathbf{I} = \{1, 2, \ldots, 9\}$ defined by $X = X_1 \amalg X_2$ and

$$f_1 = \operatorname{diag} X_1^2, \qquad f_2 = \operatorname{diag} X_2^2, \qquad f_3 = X_1^2 - f_1,$$

$$f_4 = \left\{(x, y) \in X_2^2 \middle| x \text{ and } y \text{ are incident with } a \text{ common points}\right\},$$

$$f_5 = \left\{(x, y) \in X_2^2 \middle| x \text{ and } y \text{ are incident with } b \text{ common points}\right\},$$

$$f_6 = F, \qquad f_7 = X_1 \times X_2 - F, \qquad f_8 = f_6^t, \quad \text{and} \quad f_9 = f_7^t.$$

We claim that:

9.1. *$\mathscr{C}$ is coherent (and hence is the coherent closure of (X, F)).*

Proof. It is clear that $\mathscr{C}$ satisfies axioms 3.2(I) and (II) for coherence. Thus it suffices to show that the linear span of the matrices of the relations $(f_i)_{i \in \mathbf{I}}$ is a subalgebra of the algebra of $m + n$ by $m + n$ matrices.

Let C be the incidence matrix of (X_1, X_2, F). The conditions (a), (b), and (c) defining quasisymmetric designs are equivalent to

$$JC = SJ, \qquad CJ = TJ,$$

$$CC^t = (T - \Lambda)I + \Lambda J,$$

$$C^tC = (S - b)I + (a - b)A + bJ,$$

where A is the adjacency matrix of the block graph. Hence $CC^tC = (T - \Lambda)C + \lambda SJ = (S - b)C + (a - b)CA + bTJ$, and therefore $CA \in \langle C, J \rangle$. Moreover, $A^2 \in \langle (C^tC)^2, C^tC, I, J \rangle \subseteq \langle (C^tC)^2, A, I, J \rangle$ and $(C^tC)^2 = C^t(CC^t)C \in \langle C^tC, J \rangle$; hence $A^2 \in \langle A, I, J \rangle$. The result now follows easily. (The argument shows that the adjacency algebra of $\mathscr{C}$ is generated as a coherent algebra by the $m + n$ by $m + n$ matrix having C as its upper right hand block and all other entries 0.) ■

As corollaries we have the facts (cf. [4]) that:

9.2. *The block graph of a quasisymmetric design is strongly regular.*

9.3. *The number of blocks incident with a point x and adjacent to a block y depends only on whether or not x and y are incident.*

The strongly regular parameters of the block graph will be denoted by n, k, l, λ, μ, r, s, f, and g as usual. The respective numbers defined by (9.3) will be denoted by N and P, so that

$$CA = (N - P)C + PJ.$$

9B. Quasisymmetric Designs and Coherent Configurations

The following two results, 9.4 and 9.5, are the objectives of this subsection.

9.4. *The parameters of a quasisymmetric design satisfy the following conditions*:

(1) $m = f + 1$,
(2) $mT = nS$,
(3) $P(n - T) = (k - N)T$,
(4) $(S - 1)T = \Lambda(m - 1)$,
(5) $ak = NS$,
(6) $bl = (T - N - 1)S$,
(7) $S + aN + b(T - N - 1) = T + \Lambda(S - 1)$,
(8) $aP + b(T - P) = \Lambda S$,
(9) $N^2 + P(k - N) = k + \lambda N + \mu(T - N - 1)$,
(10) $NP + P(k - P) = \lambda P + \mu(T - P)$,
(11) $Na + P(S - a) = S + a\lambda + b(k - \lambda - 1)$,

(12) $Nb + P(S - b) = a\mu + b(k - \mu)$,
(13) $ar - b(r+1) = T - \Lambda - S$,
(14) $as - b(s+1) = -S$,
(15) $mP = S(k - r)$,
(16) $P = N - r$.

Now consider a c.c. $\mathscr{C} = (X,(f_i)_{i \in I})$ *of type* (2,2;3) *over* $\mathbf{I} = \{1,2,\ldots,9\}$, *with standard partition* $X = (X_\alpha)_{\alpha \in \Omega}$, $\Omega = \{1,2\}$, *such that* $\mathbf{I}^{11} = \{1,3\}$, $\mathbf{I}^{22} = \{2,4,5\}$, $\mathbf{I}^{12} = \{6,7\}$, *and*, $\mathbf{I}^{21} = \{8,9\}$, $6* = 8$, $7*4 = 9$. *Then*:

9.5. (X_1, X_2, f_6) *and* (X_1, X_2, f_7) *are complementary quasisymmetric designs.*

According to 9.1 and 9.5, a c.c. of type (2,2;3) is equivalent to a complementary pair of quasisymmetric designs.

We prove 9.4 and 9.5 together, starting with a c.c. $\mathscr{C}$ as in the paragraph above 9.5 and putting $n_1 = m$ and $n_2 = n$. For the multiplicities of the fibers we have $z_{11} = z_{21} = 1$, $z_{12} = z_{22} = m - 1$, and $z_{23} = n - m$. In particular, $n > m$.

It will be convenient to consider the set system $(X_1, \mathscr{B})$ with $\mathscr{B} = \{f_8(x) \mid x \in X_2\}$. Suppose that the fiber $\mathscr{C}^2$ is primitive, i.e., that the graphs (X_2, f_4) and (X_2, f_5) are connected. Then if $f_8(x) = f_8(y)$ for some x and y, we must have $f_8(x) = X_1$, a contradiction. Therefore, if $\mathscr{C}^2$ is primitive, then $|\mathscr{B}| = n$.

Now suppose that $f_4 = f_5^t$. Then $\mathscr{C}^2$ is primitive and $p_{86}^4 = p_{86}^5$. It follows that $(X_1, \mathscr{B})$ is a symmetric design and hence that $m = |\mathscr{B}| = n$, a contradiction. Thus $\mathscr{C}^2$ is symmetric and (X_2, f_4) is a strongly regular graph. We take the parameters of this graph to be n, k, l, λ, μ, r, s, f, and g as usual, so that the character multiplicity tables for $\mathscr{C}^1$ and $\mathscr{C}^2$ are

$$\begin{array}{cc|c} 1 & m-1 & 1 \\ 1 & -1 & m-1 \end{array} \quad \text{and} \quad \begin{array}{ccc|c} 1 & k & 1 & 1 \\ 1 & r & -(r+1) & f \\ 1 & s & -(s+1) & g \end{array}$$

The degrees for $\mathscr{C}$ are $e_1 = e_{11} + e_{21} = 2$, $e_2 = e_{12} + e_{22} = 2$, $e_3 = e_{23} = 1$, and the multiplicities are $z_1 = z_{11} = z_{21} = 1$, $z_2 = z_{12} = z_{22}$, $z_3 = z_{23}$. Now $z_{12} = m - 1$, and we number f_4 and f_5 so that $z_{22} = f$, $z_{23} = g$, and then $m = f + 1$.

We have $v_1 = v_2 = 1$, $v_3 = m - 1$, $v_4 = k$, and $v_5 = l$. We put $v_6 = T$ and $v_8 = S$; then $v_7 = n - T$, $v_9 = m - S$, and $mT = nS$. Further we put $p_{64}^6 = N$, $p_{64}^7 = P$, $p_{86}^4 = a$, $p_{86}^5 = b$, and $p_{68}^3 = \Lambda$. These choices are consistent with 9.1 given the above convention on the numbering of f_4 and f_5. The relations

$p_{ij}^k v_k = p_{kj*}^i v_i$ give (3), (4), (5), and (6) of (9.4) for the parameters of $\mathscr{C}$. In particular, $T > \Lambda$.

We now write out explicitly the matrices M_i^a,

$$M_1^1 = I, \qquad M_1^2 = I,$$

$$M_2^1 = I, \qquad M_2^2 = I,$$

$$M_3^1 = \begin{bmatrix} 0 & 1 \\ m-1 & m-2 \end{bmatrix}, \qquad M_3^2 = \begin{bmatrix} S-1 & S \\ m-S & m-S-1 \end{bmatrix},$$

$$M_4^1 = \begin{bmatrix} N & P \\ k-N & k-P \end{bmatrix}, \qquad M_4^2 = \begin{bmatrix} 0 & 1 & 0 \\ k & \lambda & \mu \\ 0 & k-\lambda-1 & k-\mu \end{bmatrix},$$

$$M_5^1 = \begin{bmatrix} T-N-1 & T-P \\ l-T+N+1 & l-T+P \end{bmatrix},$$

$$M_5^2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & k-\lambda-1 & k-\mu \\ l & \tilde{\mu} & \tilde{\lambda} \end{bmatrix} \qquad (\tilde{\lambda} = l-k+\mu-1, \quad \tilde{\mu} = l-k+\lambda+1),$$

$$M_6^1 = \begin{bmatrix} 1 & 0 \\ S-1 & S \end{bmatrix}, \qquad M_6^2 = \begin{bmatrix} S & a & b \\ 0 & S-a & S-b \end{bmatrix},$$

$$M_7^1 = \begin{bmatrix} 0 & 1 \\ m-S & m-S-1 \end{bmatrix}, \qquad M_7^2 = \begin{bmatrix} 0 & S-a & S-b \\ m-S & m+a-2S & m+b-2S \end{bmatrix},$$

$$M_8^1 = \begin{bmatrix} T & \Lambda \\ 0 & T-\Lambda \end{bmatrix}, \qquad M_8^2 = \begin{bmatrix} 1 & 0 \\ N & P \\ T-N-1 & T-P \end{bmatrix},$$

$$M_9^1 = \begin{bmatrix} 0 & T-\Lambda \\ n-T & n-2T+\Lambda \end{bmatrix}, \qquad M_9^2 = \begin{bmatrix} 0 & 1 \\ k-N & k-P \\ l-T+N+1 & l-T+P \end{bmatrix}.$$

Substituting into 3.7 (e), we obtain (7) through (12) of 9.4.

The irreducible representation can be written as shown in Table 1, where the E_{ij} are the 2×2 matrix units and $\alpha_1 = (ST)^{1/2}$, $\alpha_2 = P^{-1}(k-N)(ST)^{1/2}$, and $\beta_1 = -\beta_2 = (T-\Lambda)^{1/2}$. From this we obtain (13) through (16) of 9.4.

All the conditions of 9.4 are now established for the parameters of $\mathscr{C}$. It remains to prove that $a > b$. But $a = b$ implies $S = b$ by (14) of 9.4, and hence $T = \Lambda$, a contradiction. Hence $a > b$ by (14).

TABLE 1

	Δ_1	Δ_2	Δ_3
A_1	E_{11}	E_{11}	0
A_2	E_{22}	E_{22}	1
A_3	$(m-1)E_{11}$	$-E_{11}$	0
A_4	kE_{22}	rE_{22}	s
A_5	lE_{22}	$-(r+1)E_{22}$	$-(s+1)$
A_6	$\alpha_1 E_{12}$	$\beta_1 E_{12}$	0
A_7	$\alpha_2 E_{12}$	$\beta_2 E_{12}$	0
A_8	$\alpha_1 E_{21}$	$\beta_1 E_{21}$	0
A_9	$\alpha_2 E_{21}$	$\beta_2 E_{21}$	0

9C. *Analysis of Parameters*

Throughout this subsection we consider a q.s.d. with notation as in Section 9A and without repeated blocks, i.e., we assume $S > a$. From 9.4 it is easy to obtain expressions for the parameters in terms of the design parameters m, S, Λ, a, and b, namely

$$T = \frac{(m-1)\Lambda}{S-1},$$

$$n = \frac{m(m-1)\Lambda}{S(S-1)},$$

$$(a-b)k = \frac{(m-S)[S(S-1)-b(m-1)]\Lambda}{(S-a)(S-1)},$$

$$(a-b)^2\mu = \frac{(S^2-bm)[S(S-1)-b(m-1)]\Lambda}{S(S-1)},$$

$$(a-b)r = \frac{(m-S)\Lambda}{S-1} - (S-b),$$

$$(a-b)s = -(S-b),$$

$$f = m-1,$$

$$(a-b)P = \frac{[S(S-1)-b(m-1)]\Lambda}{S-1},$$

$$(a-b)N = \frac{a(m-S)[S(S-1)-b(m-1)]\Lambda}{S(S-a)(S-1)}.$$

In particular, the formulas for P and N are obtained as follows. We have $P(n-T)=PT(m-S)/S$, while $(k-N)T=(S-a)NT/a$, and therefore $N=a(m-S)P/S(S-a)$. But $(a-b)P=\Lambda S-bT=[S(S-1)-b(m-1)]\Lambda/(S-1)$, giving the formulas for P and N above.

On the other hand,

$$(a-b)N=(1-b)T+\Lambda(S-1)-(S-b)$$
$$=\frac{\left[(S-1)^2-(b-1)(m-1)\right]\Lambda}{S-1}-(S-b).$$

Equating the two expressions for $(a-b)N$ gives

$$\text{(9.6)}\qquad \left[S^4-2S^3-\left[(a+b-1)(m-1)-1\right]S^2+abm(m-1)\right]\Lambda$$
$$=S(S-1)(S-a)(S-b),$$

which determines Λ in terms of S, m, a, and b.

9D. *Tight 4-Designs*

We consider a quasisymmetric design D which satisfies the following condition $T(x)$ for some point x:

$T(x)$: Each triple x, u, v of distinct points is incident with the same number Λ_x of blocks.

An obvious count gives $(m-2)\Lambda_x=(S-2)\Lambda$.

It is easy to see that $T(x)$ holds if and only if D^x is symmetric or quasisymmetric if and only if D_x is symmetric or quasisymmetric. Here D^x is the *residual* design consisting of the points different from x and the blocks not through x, and D_x is the *derived* design consisting of the points different from x and the blocks through x. Since $T(x)$ implies $T(x)$ for the complement and since the quasisymmetric designs with $S \leqslant 3$ are known, we assume that $4 \leqslant S \leqslant m/2$.

The design D^x has parameters $m^x=m-1$, $n^x=n-T$, $S^x=S$, $T^x=T-\Lambda$, and

$$\Lambda^x=\Lambda-\Lambda_x=\frac{m-S}{m-2}\Lambda.$$

Moreover, two blocks of D^x have either a or b common points, with D^x symmetric if just one of these occurs and quasisymmetric if both occur.

9.7. *A quasisymmetric design with symmetric residual design is a Hadamard 3-design.*

Proof. Assume that D^x is symmetric; then $S = T - \Lambda$ and $(m - S)\Lambda = \delta(m - 2)$, where δ is one of a or b. By (4) of 9.4, $(S - 1)(S + \Lambda) = \Lambda(m - 1)$, and hence

$$(9.8) \qquad (m - S)\Lambda = \delta(m - 2) = S(S - 1).$$

Moreover, by (13) of 9.4 we have

$$(9.9) \qquad (a - b)r = b.$$

Substituting $\Lambda = S(S - 1)/(m - S)$ in (9.6) and simplifying, we obtain

$$S^4 - S^3 - (a + b)mS^2 + [(a + b)m + ab]S + abm(m - 2) = 0.$$

Using $S^2 = S + \delta(m - 2)$, we reduce this to $2\delta S = \delta m + ab$, which, in view of our assumption that $S \leqslant m/2$, implies that $b = 0$ and hence that $r = 0$ by (9.9). This means that the block graph Γ is the complement of a ladder graph. There are at most two vertices on each rung of the ladder, because for a given block y not through x, the blocks through x not meeting y form a coclique in Γ. Now the result of Goethals and Seidel [6; 4, (3.3)] implies that D is a Hadamard 3-design. ■

9.10. *If D^x is quasisymmetric, then*

$$S^4 - 4S^3 - [(a + b - 1)(m - 2) - 5]S^2$$
$$+ [(a + b - 1)(m - 2) - 2]S + ab(m - 1)(m - 2) = 0.$$

Proof. The equation (9.6) for D^x is

$$[S^4 - 2S^3 - [(a + b - 1)(m - 2) - 1]S^2 + ab(m - 1)(m - 2)]\frac{S - 2}{m - 2}\Lambda$$
$$= S(S - 1)(S - a)(S - b).$$

Equating the left hand side of this with that of (9.6) yields 9.10. ■

Now assume that for some point y different from x the following condition $T(x, y)$ holds in addition to $T(x)$:

$T(x, y)$: The number of blocks through each quadruple x, y, u, v of distinct points is a constant Λ_{xy}.

Clearly $(m-2)(m-3)\Lambda_{xy} = (S-2)(S-3)\Lambda$.

9.11. *If a quasisymmetric design with m points and S points per block such that $4 \leqslant S \leqslant m/2$ satisfies $T(x)$ and $T(x, y)$ for some pair x and y of distinct points, then it is isomorphic with the unique Steiner system $4-(23, 7, 1)$.*

Proof. Assume $T(x)$ and $T(x, y)$; then D^x satisfies $T(y)$. If D^x is symmetric, then so is D^{xy}, and then $S = T^x = T^{xy}$, that is, $T - \Lambda = T - 2\Lambda + \Lambda_x$. But this implies that $m = S$, which is impossible. Hence D^x is quasisymmetric.

If D^{xy} is symmetric, then D^x is a Hadamard 3-design by 9.7. But then $S = 2a$, $m - 1 = 4a$, and $(m-S)\Lambda/(m-2) = 2a - 1$, whence $a = 1$ and $S = 2$, contrary to assumption. It follows that D^{xy} must be quasisymmetric.

Finally, therefore, we must consider the case in which all three of the designs D, D^x, and D^{xy} are quasisymmetric. By [5] and [2] it will suffice to show that D is a tight 4-design, and by [4, (3.6)] it will suffice for this to show that $n = m(m-1)/2$, or equivalently, that $\Lambda = S(S-1)/2$, which we proceed to do. We have the equation in 9.10 for D^x, namely

$$S^4 - 4S^3 - [(a+b-1)(m-3) - 5]S^2 + [(a+b-1)(m-3) - 2]S + ab(m-2)(m-3) = 0.$$

Subtracting (9.10) from this gives

$$(a+b-1)S^2 - (a+b-1)S - 2ab(m-2) = 0.$$

If $a + b - 1 = 0$, then D is a nonsymmetric 2-$(m, S, 1)$-design, and the only ones of these which satisfy the hypotheses of 9.11 are the pair designs, which are excluded here. We have, therefore, that $a + b - 1 \neq 0$ and hence

$$\text{(9.12)} \qquad S^2 - S - \varkappa = 0, \qquad \varkappa = \frac{2ab(m-2)}{a+b-1}.$$

Substituting in (9.10) gives

$$S^2 - 3S - \nu = 0, \qquad \nu = \frac{(a+b-1)(m-3)-4}{2}$$

and hence $S = (\varkappa - \nu)/2$. Now using (9.6), 9.10, and (9.12), we verify that the following equalities are mutually equivalent:

$$\Lambda = \frac{S(S-1)}{2},$$

$$S^4 - 2S^3 - [(a+b-1)(m-1)+1]S^2 + 2(a+b)S + ab(m+1)(m-2) = 0,$$

$$2S^3 - (a+b+5)S^2 - [(a+b-1)(m-4)-4]S + 2ab(m-2) = 0,$$

$$4\varkappa = (2\varkappa - 2\nu - 4)S,$$

$$4(S-1) = 2\varkappa - 2\nu - 4,$$

and

$$S = \frac{\varkappa - \nu}{2}.$$

This completes the proof of 9.11. ■

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