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KLEENE CHAIN COMPLETENESS AND FIXEDPOINT PROPERTIES

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Introduction

Fixedpoint theorems play a fundamental role in denotational semantics of programming languages. In 1955, Tarski [6] and Davis [1] showed that a lattice L is complete if and only if every monotonic function $f: L \to L$ has a fixedpoint. Since then, other fixedpoint properties are considered, by imposing either the function to be ω-continuous or the fixedpoint to be the least fixedpoint. The key question is whether such fixedpoint properties can as well be characterized by some completeness properties of the given partially ordered set. In this direction, Markowsky [2] showed that a partially ordered set is chain-complete if and only if every monotonic function $f: D \to D$ has a least fixedpoint. Suppose we replace monotonic functions by ω-continuous functions which play a prominent role in Scott's theory of computation [4, 5]. A slight modification of Tarski's proof shows that if a partially ordered set D with a least element $\bot$ is ω-chain complete, then every ω-continuous function $f: D \to D$ has a least fixedpoint given by $\bigsqcup_{n\in\omega} f^n(\bot)$. The latter result is the well-known Tarski–Kleene–Knaster theorem. In 1978, Plotkin asked the validity of the converse of Tarski–Kleene–Knaster theorem because an affirmative answer would give us a characterization of ω-chain complete partially ordered sets in terms of the least fixedpoint property for ω-continuous functions.

Throughout the paper, D stands for a partially ordered set with a least element $\bot$. Let us consider Plotkin's puzzle in the following version: Given a D, if every ω-continuous function $f: D \to D$ has a least fixedpoint given by $\bigsqcup_{n\in\omega} f^n(\bot)$, is D ω-chain complete? In this paper, we answer the problem negatively (Mashburn obtained the same result independently in [3]). Despite this negative answer, we show a rather astonishing result, namely: If D is either countable or countably algebraic, then the converse of Tarski–Kleene–Knaster theorem is true. This positive answer is rather pleasing because most of the D's used in denotational semantics are either countable or countably algebraic.

During our studies, we introduce the notion of a Kleene chain as follows: an ω-chain $\{a_n\}_{n\in\omega}$ is Kleene if there exists some ω-continuous function $f: D \to D$ such that $a_n = f^n(\bot)$ for every n. D is Kleene-chain complete if every Kleene chain has a least upper bound. Using this notion of Kleene chains, we can rephrase the converse of Tarski–Kleene–Knaster theorem as follows: Is Kleene-chain completeness identical to ω-chain completeness? The following seems to be the key question: When is a strict ω-chain Kleene? Clearly, if every strict ω-chain is Kleene, then the two notions of chain completeness must coincide. To this end, we show

(a) if D is ω-chain complete, then every strict ω-chain is Kleene;

(b) if D is either countable or countably algebraic, then every strict ω-chain is Kleene; and

(c) non-Kleene ω-chains exist.

It follows from (b) that if D is either countable or countably algebraic, then the converse of Tarski–Kleene–Knaster theorem is valid.

We also study the following question: Suppose every ω-continuous function $f: D \to D$ has a least fixedpoint, does it follow that the least fixedpoint is given by $\bigsqcup_{n\in\omega} f^n(\bot)$? An affirmative answer would clearly give us a syntactic characterization of the least fixedpoint. However, we show that even if D is countable or countably algebraic, the answer may be negative.

1. Notations

An *ω-chain* $\{a_n\}_{n\in\omega}$ in D is any ascending chain; $\{a_n\}_{n\in\omega}$ is *strict* if it is strictly increasing and $a_0 = \bot$. D is *ω-chain complete* if every ω-chain has a least upper bound (l.u.b.). E is an *ω-chain semi-complete* subset of D if the l.u.b. (in D) of any ω-chain in E, whenever it exists, lies in E.

$f: D \to D$ is *ω-continuous* if $\bigsqcup_{n\in\omega} f(a_n) = f(\bigsqcup_{n\in\omega} a_n)$ for every ω-chain $\{a_n\}_{n\in\omega}$ whose l.u.b. exists. A stronger notion of continuity is *directed-continuity* which is $\bigsqcup f(H) = f(\bigsqcup H)$ for every directed set H whose l.u.b. exists.

x in D is an *algebraic element* if whenever $x \sqsubseteq \bigsqcup H$ for some directed set H, we must have $x \sqsubseteq h$ for some h in H. D is *algebraic* if every x in D is the l.u.b. of some directed set of algebraic elements. D is *countably algebraic* if D is algebraic and there are countably many algebraic elements. An ω-chain $\{a_n\}_{n\in\omega}$ is algebraic if all the a_n's are algebraic elements in D.

2. Kleene chains

An ω-chain $\{a_n\}_{n\in\omega}$ is *Kleene* if for some ω-continuous function $f: D \to D$, $a_n = f^n(\bot)$ for every n. In the study of chain completeness, we are only interested in infinite chains. Every infinite Kleene chain, according to our definition, must be strict. We can easily see that a strict ω-chain $\{a_n\}_{n\in\omega}$ is Kleene if and only if for some ω-continuous function $f: D \to D$, $f(a_n) = a_{n+1}$. In this section, we study the following problem: when is a strict ω-chain Kleene? First, we give some positive answers.

Theorem 1. *Given a strict ω-chain $\{a_n\}_{n\in\omega}$ in D, if the* l.u.b. *of $\{a_n\}_{n\in\omega}$ exists, then $\{a_n\}_{n\in\omega}$ is Kleene.*

Proof. Let a be the l.u.b. of $\{a_n\}_{n\in\omega}$. Define $f: D \to D$ as follows:

$$f(x) = \begin{cases} a & \text{if } x \not\sqsubseteq a_n \text{ for every } n, \\ a_{n+1} & \text{if otherwise and } n \text{ is the smallest integer} \\ & \text{satisfying } x \sqsubseteq a_n. \end{cases}$$

The ω-continuity of f is obvious; in fact, f is directed-continuous. Clearly $f(a_n) = a_{n+1}$ for every n, hence $\{a_n\}_{n\in\omega}$ is Kleene. □

Corollary 1. *If D is ω-chain complete, then every ω-chain is Kleene.*

Theorem 2. *If D is countable, then every strict ω-chain is Kleene.*

Proof. Let $\{a_n\}_{n\in\omega}$ be any strict ω-chain in D. Without loss of generality, assume that $\{a_n\}_{n\in\omega}$ does not have any l.u.b. (otherwise, $\{a_n\}_{n\in\omega}$ is Kleene by Theorem 1). Define the set C as follows:

$$C = \{x \in D \mid x \sqsubseteq \text{every upper bound of } \{a_n\}_{n\in\omega}\}.$$

Note that all the a_n's are in C, C is an ω-chain semi-complete set, and furthermore, C does not contain any upper bound of the chain $\{a_n\}_{n\in\omega}$. If $\{a_n\}_{n\in\omega}$ does not have any upper bound at all, then C is equal to the entire set D. Since D is countable, we can find some countable enumeration of C, let it be $\{c_n\}_{n\in\omega}$. For every pair (i, j) in $\omega \times \omega$, we define the set $B_{i,j}$ as follows:

$$B_{i,j} = \begin{cases} \emptyset & \text{if } a_i \sqsubseteq c_j, \\ \downarrow c_j & \text{otherwise} \end{cases}$$

where $\downarrow c_j$ is the principal ideal $\{x \in D \mid x \sqsubseteq c_j\}$ determined by c_j. Now we define a partition $\{C_i\}_{i\in\omega}$ of the set C.

$$C_0 = \{\bot\}, \qquad C_{i+1} = \Big[\downarrow a_{i+1} \cup \bigcup_{j \le i+1} B_{i+1,j}\Big] \Big\backslash \bigcup_{l \le i} C_l.$$

Claim 1. $\{C_n\}_{n\in\omega}$ defines a partition of the set C.

Proof. Given an arbitrary c_j in the set C, there exists some a_{i+1} with $j \leq i+1$ such that $a_{i+1} \not\sqsubseteq c_j$ (because the set C does not contain any upper bound of the chain $\{a_n\}_{n\in\omega}$). Hence $c_j \in B_{i+1,j}$. So if c_j is not in $\bigcup_{t\leq i} C_t$, it must be in C_{i+1}

Claim 2. If $x \sqsubseteq y$ and $y \in C_{i+1}$, then $x \in C_t$ for some $t \leq i+1$.

Proof. Since $y \in C_{i+1}$, y belongs to the set $\downarrow a_{i+1} \cup \bigcup_{j\leq i+1} B_{i+1,j}$ which is closed downwards. Hence x must also belong to the set $\downarrow a_{i+1} \cup \bigcup_{j\leq i+1} B_{i+1,j}$. So if x is not in $\bigcup_{t\leq i} C_t$, it must be in C_{i+1}.

Claim 3. $a_i \in C_i$ for every i.

Proof. We prove by induction. Clearly $a_0(=\bot) \in C_0$. Assume $a_i \in C_i$. If $a_{i+1} \notin C_{i+1}$, then a_{i+1} must belong to C_i by Claim 2, hence $a_{i+1} \in B_{i,j}$ for some $j \leq i$. The non-emptiness of $B_{i,j}$ implies $a_i \not\sqsubseteq c_j$. However, $a_{i+1} \in B_{i,j}(=\downarrow c_j)$ implies $a_{i+1} \sqsubseteq c_j$, contradicting $a_i \not\sqsubseteq c_j$.

Claim 4. Each C_i is an ω-chain semi-complete set.

Proof. Note that ω-chain semi-complete sets are closed under Boolean operations. Since each C_i is some Boolean combination of ω-chain semi-complete sets, the claim immediately follows.

We are now ready to demonstrate that $\{a_n\}_{n\in\omega}$ is Kleene. If $\{a_n\}_{n\in\omega}$ does have an upper bound, let a be any one of them. Now define $f: D \to D$ as follows:

$$f(x) = \begin{cases} a & \text{if } x \notin C, \\ a_{i+1} & \text{if } x \in C_i. \end{cases}$$

That f is well-defined follows from Claim 1. Claim 2 says that f is monotonic. The ω-continuity of f follows from Claim 4. Finally $f(a_i) = a_{i+1}$ follows from Claim 3, hence $\{a_n\}_{n\in\omega}$ is Kleene. □

Now we turn to countably algebraic domains. First, we show the following.

Theorem 3. *Every strict algebraic ω-chain is Kleene.*

Proof. Let $\{a_n\}_{n\in\omega}$ be any strict algebraic ω-chain. Without loss of generality, assume that $\{a_n\}_{n\in\omega}$ does not have any l.u.b. If $\{a_n\}_{n\in\omega}$ does have an upper bound, let a be any one of them. As in Theorem 2, we define C to be the set $\{x \in D \mid x \sqsubseteq \text{every upper bound of } \{a_n\}_{n\in\omega}\}$. We remind the reader that C is ω-chain semi-complete and does not contain any upper bound of the chain $\{a_n\}_{n\in\omega}$. Now define the function $f: D \to D$ as follows:

$$f(x) = \begin{cases} a & \text{if } x \notin C, \\ a_{i+1} & \text{if otherwise and } i \text{ is the largest integer satisfying } a_i \sqsubseteq x. \end{cases}$$

f is well-defined because when $x \in C$, there is always a largest integer i satisfying $a_i \sqsubseteq x$, otherwise such an x would be an upper bound of the chain $\{a_n\}_{n\in\omega}$. To check the ω-continuity of f, suppose $f(\bigsqcup_{n\in\omega} d_n) = a_{i+1}$ for some ω-chain $\{d_n\}_{n\in\omega}$. Since $a_i \sqsubseteq \bigsqcup_{n\in\omega} d_n$ and a_i is compact, we must have $a_i \sqsubseteq d_k$ for some k, hence $f(d_k) = a_{i+1}$. Thus, the ω-continuity of f follows from the ω-chain semi-completeness of C. □

Theorem 4. *If D is countably algebraic, then every strict ω-chain is Kleene.*

Proof. Let $\{b_i\}_{i\in\omega}$ be any strict ω-chain. Without loss of generality, assume that $\{b_i\}_{i\in\omega}$ does not have any l.u.b. Since D is countably algebraic, we can find an algebraic ω-chain $\{b_{i,n}\}_{n\in\omega}$ for each i satisfying the following properties:

(i) each b_i is the l.u.b. of the chain $\{b_{i,n}\}_{n\in\omega}$;

(ii) $b_{i,n} \sqsubseteq b_{i+1,n}$ and

(iii) $b_{i+1,n} \not\sqsubseteq b_i$ for every n.

Consider the strict algebraic chain $\{a_i(=b_{i,i})\}_{i\in\omega}$ – that $\{a_i\}_{i\in\omega}$ forms a chain follows from property (ii) above. Note that the chains $\{a_i\}_{i\in\omega}$ and $\{b_i\}_{i\in\omega}$ both have the same set of upper bounds; consequently, $\{a_i\}_{i\in\omega}$ does not have any l.u.b. in D. By Theorem 3, we can find some ω-continuous function $f: D \to D$ such that

(1) $f(a_i) = a_{i+1}$ for every i;

(2) the range of f is given by the set $\{a_i \mid i \in \omega\} \cup \{a\}$ where a is some upper bound of the chain $\{a_i\}_{i\in\omega}$ whenever it exists; and

(3) $f(b_i) = a_{i+1}$ – this follows from our definition of f in Theorem 3 and property (iii) above.

Now define the function $g: \{a_i \mid i \geq 1\} \cup \{a\} \to \{b_i \mid i \geq 1\} \cup \{a\}$ as follows:

$$g(a) = a, \qquad g(a_i) = b_i.$$

Clearly the composition function $g \circ f: D \to D$ is ω-continuous. Also $g \circ f(b_i) = b_{i+1}$ for every i, hence $\{b_i\}_{i\in\omega}$ is Kleene. □

Corollary 2. *If D is either countable or countably algebraic, then ω-chain completeness coincides with Kleene chain completeness, hence the validity of the converse of Tarski's fixedpoint theorem.*

Next we show that non-Kleene strict ω-chains exist. We define the following partially ordered set D_1. The underlying set of D_1 consists of $\{a_i \mid i \in \omega\} \cup \{(i+1, j) \mid i, j \in \omega\} \cup [\omega \to N]$ where N is the 'flat' partially ordered set of natural numbers with the least element $\bot_N$ and $[\omega \to N]$ stands for the set of all functions from ω to N. The partial order $\sqsubseteq$ on D_1 is defined as follows:

(1) a_0 is the bottom element $\bot$ of D_1;

(2) $a_i \sqsubseteq a_j$ if and only if $i \leq j$;

(3) $(i, j) \sqsubseteq a_k$ if and only if $i \leq k$;

(4) $(i, s) \sqsubseteq (i, t)$ if and only if $s \leq t$;

(5) $(i, j) \sqsubseteq f$ for $f \in [\omega \to N]$ if and only if $f(i) \neq \bot_N$ and $j \leq f(i)$; and

(6) for $f, g \in [\omega \to N]$, $f \sqsubseteq g$ if and only if f is 'less defined than' g, i.e., for every $i \in \omega$, $f(i) \sqsubseteq_N g(i)$ where $\sqsubseteq_N$ is the partial ordering on N.

It should be clear that this relation $\sqsubseteq$ is indeed a partial order. The strict ω-chain $\{a_i\}_{i\in\omega}$ has no upper bound in D_1, but for each $i>0$, the chain $(i, 0)$, $(i, 1), \ldots,$ $(i, j), \ldots$ has a least upper bound a_i. Figure 1 illustrates this domain D_1.

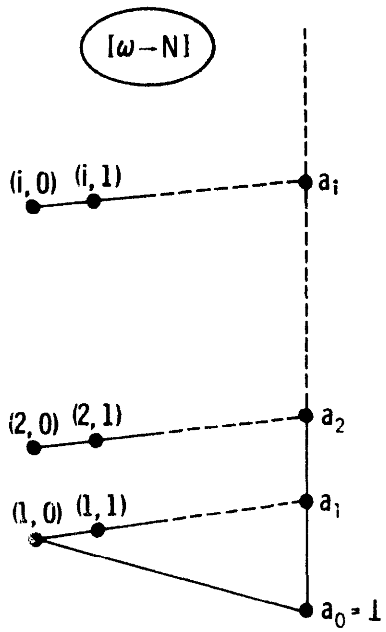

Fig. 1.

D_1 is not countably algebraic because elements like (i, j) which have to belong to any basis of D_1 are not algebraic – the ascending chain $\{(i+1, k) \mid k \in \omega\}$ with the l.u.b. a_{i+1} is $\sqsupseteq(i, j)$ but none of $(i+1, k)$ is $\sqsupseteq(i, j)$.

Theorem 5. *There exists a strict ω-chain which is not Kleene.*

Proof. Consider D_1 in Fig. 1. We claim that any strict ω-chain in D_1 which has some a_k $(k>0)$ in it cannot be Kleene. We prove by contradiction. Suppose $\{f^n(\perp) \mid n \in \omega\} \cap \{a_i \mid i \in \omega\} \neq \emptyset$ for some ω-continuous function $f: D_1 \to D_1$ and $\{f^n(\perp)\}_{n\in\omega}$ is strictly increasing. There are two cases to consider.

Case 1: $f(\perp) \in \{a_i \mid i>0\}$. In this case, the chain $\{f^n(\perp)\}_{n\in\omega}$ must be of the form $\perp, a_{p(0)}, a_{p(1)}, \ldots, a_{p(i)}, \ldots$ for some strictly increasing function $p: \omega \to \omega$. For each i, since $f(a_{p(i)}) = a_{p(i+1)}$ and $a_{p(i)}$ is the l.u.b. of the increasing chain $\{(p(i), s)\}_{s\in\omega}$, there exists some $q(i) \in \omega$ such that $f[(p(i), q(i))] = a_{p(i+1)}$. Consider the following function $h \in [\omega \to N]$:

$$h(x) = \begin{cases} q(i) & \text{if } x = p(i) \text{ for some } i, \\ \perp_N & \text{otherwise.} \end{cases}$$

From our definition of the partial ordering on D_1, we can see that h is an upper bound of the set $\{(p(i), q(i)) \mid i \in \omega\}$. Since f is monotonic and f maps each $(p(i), q(i))$

to $a_{p(i+1)}$, f must map h to some upper bound of the chain $\{a_i\}_{i\in\omega}$ which, however, does not exist in D_1. Contradiction.

Case 2: $f(\bot)\notin\{a_i \mid i>0\}$. In this case, $f(\bot)$ must be equal to some pair (i, j_0) and the chain $\{f^n(\bot)\}_{n\in\omega}$ must be of the form $\bot$, (i, j_0), $(i, j_1), \ldots, (i, j_t)$, $a_{p(0)}$, $a_{p(1)}$, $\ldots, a_{p(t)}, \ldots$ where we have $j_0<j_1<\cdots<j_t$ and p is some strictly increasing function from ω to ω with $i\leq p(0)$. Also note that the range of f must be a subset of all the upper bounds of the pair (i, j_0). As in Case 1, we can find some function $q:\omega\to\omega$ such that f maps $(p(i), q(i))$ to $a_{p(i+1)}$ for each $i\in\omega$. The rest of the proof is similar to the proof in Case 1. □

Theorem 6. *ω-chain completeness is not identical to Kleene chain completeness, hence the converse of Tarski's fixedpoint theorem is not valid.*

Proof. Consider D_1 in Fig. 1. Clearly D_1 is not ω-chain complete. We claim that D_1 is Kleene-chain complete. Take any strictly increasing Kleene chain $\{f^n(\bot)\}_{n\in\omega}$ in D_1. Our proof in Theorem 5 says that $\{f^n(\bot)\}_{n\in\omega}$ cannot contain any $a_k (k>0)$. There are two possible cases:

Case 1: Some $f^n(\bot)$ belongs to the function space $[\omega\to N]$; in this case, $\{f^n(\bot)\}_{n\in\omega}$ must have a l.u.b. in D_1 because $[\omega\to N]$ is ω-chain complete.

Case 2: Otherwise, the l.u.b. of $\{f^n(\bot)\}_{n\in\omega}$ must be a_i, assuming $f(\bot)=(i, k)$ for some $k\in\omega$.

In either case, the l.u.b. of $\{f^n(\bot)\}_{n\in\omega}$ exists. □

Remark. Say that x in D is *weakly algebraic* if whenever $x=\bigsqcup H$ for some directed set H, then $x=h$ for some h in H. Obviously, if x is algebraic, then x is weakly algebraic. Now consider our D_1 in Fig. 1. One can see that all the (i, j)'s are weakly algebraic although they are not algebraic, as we pointed out earlier. Hence D_1 has a countable basis of weakly algebraic elements. Therefore, the converse of Tarski's fixedpoint theorem may be invalid in a D which has a countable basis of weakly algebraic elements (see Corollary 2).

In our formulation of a Kleene chain, we imposed the function f to be ω-continuous. It is known that the directed-continuity of a function coincides with the ω-continuity for countably algebraic partially ordered sets. However, these two notions differ in general. In fact, we can show the following.

Theorem 7. *There exists a Kleene chain $\{a_n\}_{n\in\omega}$ in some D such that for no directed-continuous function $f:D\to D$, we have $a_n=f^n(\bot)$ for every $i\in\omega$.*

Proof. First, let us define our partially ordered set D_2. The underlying set of D_2 consists of $\{a_i \mid i\in\omega\}\cup\{(i+1,\beta)\mid\beta<\alpha, i\in\omega\}\cup\{c_\beta\mid\beta<\alpha\}$ where α is some uncountable regular ordinal. The partial order on D_2 is defined as follows: (see Fig. 2)

(1) a_0 is the bottom element $\bot$ of D_2;

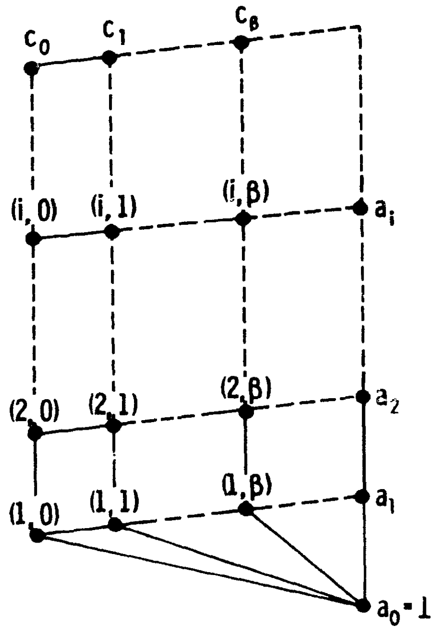

Fig. 2.

(2) $(i, \lambda) \sqsubseteq (j, \eta)$ if and only if $\lambda \leq \eta$ and $i \leq j$;
(3) $a_i \sqsubseteq a_j$ if and only if $i \leq j$; and
(4) $(i, \beta) \sqsubseteq a_k$ if and only if $i \leq k$ for all $\beta < \alpha$.

Note that c_β is the l.u.b. of the ω-chain $\{(i, \beta\}_{i>0}$ for each $\beta < \alpha$, a_i is the l.u.b. of the α-chain $\{(i, \beta)\}_{\beta<\alpha}$, and the α-chain $\{c_\beta\}_{\beta<\alpha}$ does not have any upper bound in D_2. We claim that for no directed-continuous function $f: D_2 \to D_2$, we have $a_n = f^n(\bot)$ for every $n \in \omega$. Assume not. Then because f is directed-continuous, we can find $p(i)$ for each $i > 0$ such that $f[(i, p(i))] = a_{i+1}$; we may also assume that p, as a function from ω to α, is strictly increasing. Now the ω-chain $\{(i, p(i))\}_{i>0}$ has a l.u.b. in D_2, namely, c_λ for some $\lambda < \alpha$, because α is an uncountable regular ordinal, hence the cofinality of α cannot be ω. Since f is monotonic, f must map c to some upper bound of the chain $\{a_i\}_{i\in\omega}$, which, however, does not exist in D_2. Contradiction. It remains to show that $\{a_i\}_{i\in\omega}$ is Kleene. Define the following function $f: D_2 \to D_2$:

$$f(x) = \begin{cases} a_{i+1} & \text{if } x = a_i, \\ a_1 & \text{if otherwise.} \end{cases}$$

Obviously, f is ω-continuous and $f^n(\bot) = a_n$ for every $n \in \omega$. □

3. Notions weaker than Kleene-chain completeness

In this section, we study two properties of D which are weaker than Kleene-chain completeness. They are:

Property 1. Every ω-continuous function $f:D \to D$ has a least fixedpoint.

Property 2. Every ω-continuous function $f:D \to D$ has a fixedpoint.

Obviously, the following implications hold:

$$\omega\text{-chain completeness} \Rightarrow \text{Kleene-chain completeness} \Rightarrow \text{Property 1} \Rightarrow \text{Property 2}.$$

Theorem 6 shows that the reverse of the first implication is invalid. The following example shows that Property 2 does not imply Property 1.

Description of D_3

We define the partially ordered set D_3 (see Fig. 3). Essentially, D_3 has a top element T, a strict ω-chain $\{a_n\}_{n\in\omega}$ and two incomparable upper bounds b, c of $\{a_n\}_{n\in\omega}$.

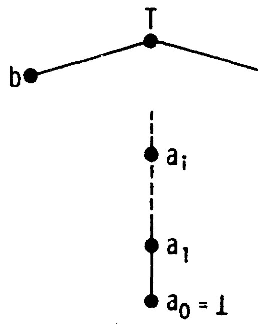

Fig. 3.

First we argue that every ω-continuous function $f: D_3 \to D_3$ has a fixedpoint. Consider the chain $\{f^n(\bot)\}_{n\in\omega}$. If this chain has a l.u.b., then this l.u.b. must be the least fixedpoint of f. Now assume that this chain does not have a l.u.b. Then it must be a strict ω-chain, hence some subchain of $\{a_n\}_{n\in\omega}$. Since b, c are both upper bounds of $\{a_n\}_{n\in\omega}$, f must map them to one of b, c or T. Hence at least one of them must be a fixedpoint of f because $f(b) \sqsubseteq f(\text{T})$ and $f(c) \sqsubseteq f(\text{T})$. This proves Property 2. Property 1 does not hold because the following function f does not have a least fixedpoint:

$$f(x) = \begin{cases} a_{i+1} & \text{if } x = a_i, \\ x & \text{otherwise.} \end{cases}$$

Next we show that Property 1 does not imply Kleene-chain completeness. It would be nice were it the case for the reason that Property 1 gives us no syntactical characterization of the least fixedpoint whereas Kleene-chain completeness says that the least fixedpoint is given by $\bigsqcup_{n\in\omega} f^n(\bot)$. It is not difficult to find a counter-example for this.

Theorem 8. *Property* 1 *does not imply Kleene-chain completeness.*

Proof. First we define D_4. D_4 has

(1) a strict ω-chain $\{a_n\}_{n\in\omega}$ with two incomparable upper bounds b and c;

(2) two ascending chains $\{b_n\}_{n\in\omega}$ and $\{c_n\}_{n\in\omega}$ whose l.u.b.'s are b and c respectively;

(3) a l.u.b. of b_i and c_i for every i where d_j and d_k are incomparable for $j \neq k$.

The diagram of D_4 is given in Fig. 4.

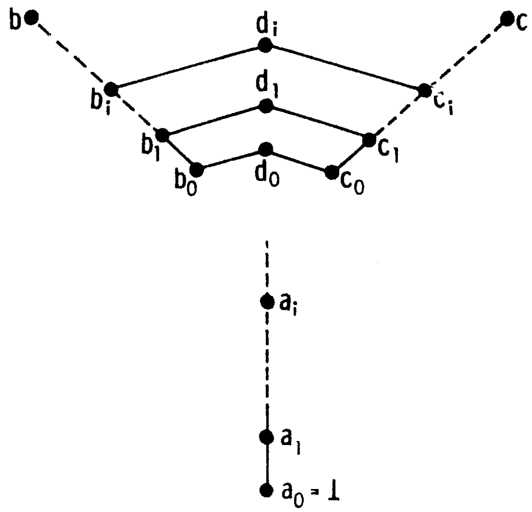

Fig. 4.

Next we show that every ω-continuous function f has a least fixedpoint. Let us consider the chain $\{f^n(\bot)\}_{n\in\omega}$. As argued in the previous theorem, if this chain has a l.u.b., then $\bigsqcup_{n\in\omega} f^n(\bot)$ is the least fixedpoint of f. Thus let us assume that this chain does not have a l.u.b. Then it must be some strict subchain of $\{a_n\}_{n\in\omega}$, since strictly increasing subchains of $\{a_n\}_{n\in\omega}$ are the only chains in D_4 with no l.u.b.'s. As b and c are the only upper bounds of $\{a_n\}_{n\in\omega}$ in D_4, we must have $f(b) = b$ or c and $f(c) = b$ or c. Assume $f(b) = b$. Since ω-continuity of f implies $b = \bigsqcup_{n\in\omega} f(b_n)$ and $f(b_n) \sqsupseteq f(\bot) = a_l$ for some $l > 0$ for all n, there must exist some $k \in \omega$ such that $f(b_n) = b$ for all $n \geq k$. Hence $f(d_n) = b$ for all $n \geq k$ and therefore $f(c_n) \sqsubseteq b$ for all $n \geq k$, concluding that $f(c) = b$ must hold. Similarly, from $f(b) = c$, we can show $f(c) = c$. In either case, f has only one fixedpoint which must be the least fixedpoint. Therefore D_4 satisfies Property 1. Note that D_4 is countable, hence it is not Kleene-chain complete by Theorem 2. □

In summary, we have shown:

Property 2 $\not\Rightarrow$ Property 1 $\not\Rightarrow$ Kleene-chain completeness $\not\Rightarrow$ ω-chain completeness.

Because of the results we obtained on countable or countably algebraic D's in Corollary 2, we ask the validity of the first two of these implications restricted to such D's. However, we note that D_3 is countable and countably algebraic, hence the invalidity of the first implication is immediate. The question on the validity of the second implication is more difficult to settle. The following theorem gives us a negative answer.

Theorem 9. *There exists a countable and algebraic D such that every ω-continuous function $f: D \to D$ has a least fixedpoint but the least fixedpoint may not be given by* $\bigsqcup_{n\in\omega} f^n(\bot)$.

Let us describe D_5 as in Fig. 5.

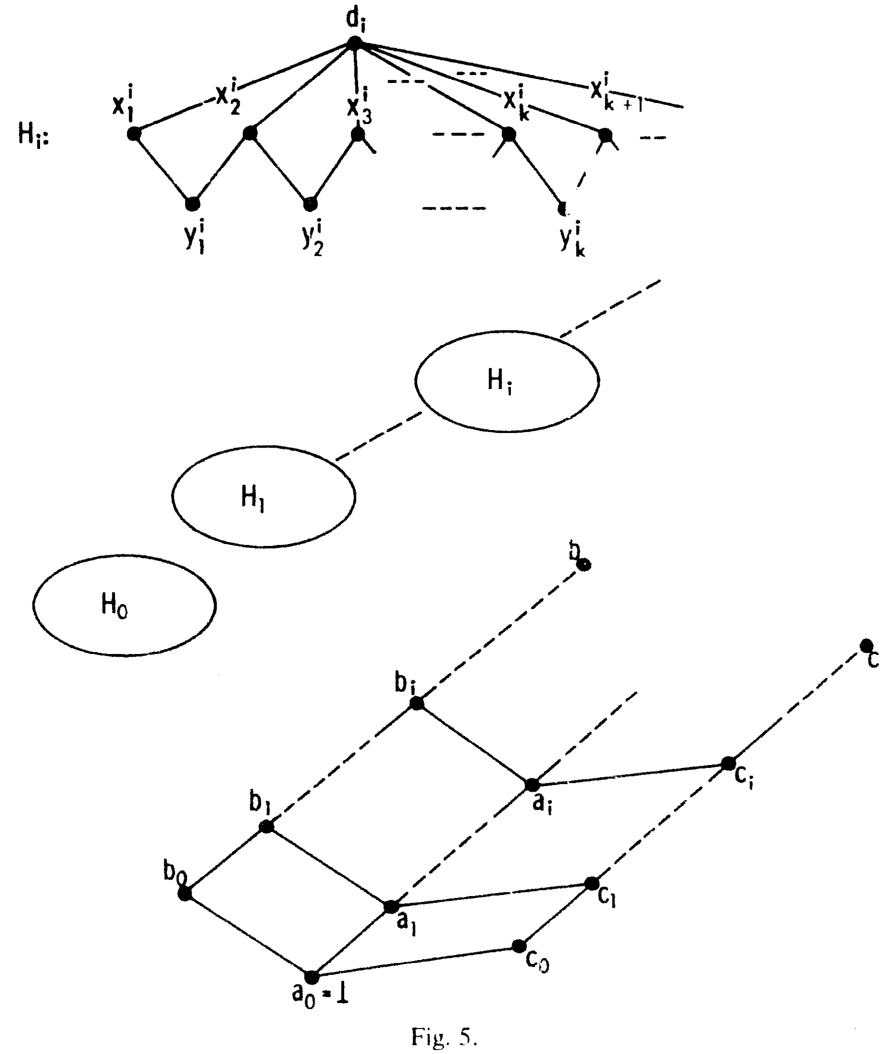

Fig. 5.

Description of D_5

D_5 has three ascending chains $\{a_n\}_{n\in\omega}$ with $a_0 = \bot$, $\{b_n\}_{n\in\omega}$ and $\{c_n\}_{n\in\omega}$; $b = \bigsqcup_{n\in\omega} b_n$ and $c = \bigsqcup_{n\in\omega} c_n$ while $\{a_n\}_{n\in\omega}$ has no l.u.b.; $a_i \sqsubseteq b_i$ and $a_i \sqsubseteq c_i$ for every $i \in \omega$ while $a_i \not\sqsubseteq b_j$ and $a_i \not\sqsubseteq c_j$ if $j < i$; the b_i's and the c_j's are incomparable. For each i, there is an infinite set of elements

$$H_i = \{d_i, x_1^i, x_2^i, \ldots, y_1^i, y_2^i, \ldots\}$$

such that $b_i \sqsubseteq z$ and $c_i \sqsubseteq z$ for every $z \in H_i$. Elements of H_i are ordered in the following way:

(i) $c_{i+k} \sqsubseteq x^i_k$ for every $k \in \omega$;
(ii) d_i is the largest element in H_i;
(iii) $y^i_k \sqsubseteq d_{i+1}$, $y^i_k \sqsubseteq x^i_k$ and $y^i_k \sqsubseteq x^i_{k+1}$ for $i \in \omega$ and $k > 0$; and
(iv) $x^i_k \sqsubseteq d_i + u_i(k)$ where $u_i(k) = i * \lfloor k/i \rfloor + i - \mathrm{Rem}(k, i)$;

$\mathrm{Rem}(k, i)$ is the remainder of integer division of k by i. Note that u_i is a bijection from ω to ω for each i. The ordering on D_5 is obtained by taking the reflexive-transitive closure of the relations defined above.

The following properties of D_5 are rather easy to verify:

Properties of D_5

(i) D_5 is countable and algebraic; the only non-algebraic elements are b and c.
(ii) The upper bounds of the chain $\{a_n\}_{n\in\omega}$ consist of b, c and d_i for every $i \in \omega$.
(iii) For each $z \in H_i$, there exists a largest j such that H_j contains some upper bound of z in it.
(iv) If x^i_k and $x^{i'}_{k'}$ have a common lower bound in some H_j ($j > 0$), then $i = i'$ and $k = k'$ or $k' + 1$ or $k' - 1$.

Obviously the following ω-continuous function f has a least fixedpoint (namely, b) different from $\bigsqcup_{n\in\omega} f^n(\bot)$: $f(x) = a_{i+1}$ if $x = a_i$; otherwise $f(x) = b$. It remains to prove that every ω-continuous function from D_5 to D_5 has a least fixedpoint.

Let f be any ω-continuous function from D_5 to D_5. Let us consider the chain $\{f^n(\bot)\}_{n\in\omega}$. If this chain has a l.u.b., then this l.u.b. must be the least fixedpoint of f. Thus let us assume that this chain does not have a l.u.b., hence it must be some subchain of $\{a_n\}_{n\in\omega}$. For simplicity, let us assume that this chain is $\{a_n\}_{n\in\omega}$; hence $f(a_n) = a_{n+1}$ for every $n \in \omega$. The case in which this chain is a proper subchain of $\{a_n\}_{n\in\omega}$ is proved essentially in the same way with a little more complex terminology. Under this assumption, the following lemma is easy to establish and proofs are omitted.

Lemma 1. (a) *Any fixedpoint of f must be an upper bound of the chain* $\{a_n\}_{n\in\omega}$.
(b) *If z is an upper bound of the chain* $\{a_n\}_{n\in\omega}$, *so must be* $f(z)$.

A corollary of Lemma 1 is that any fixedpoint of f, if it exists, must be either b, or c, or some d_i ($i \in \omega$). We now distinguish three cases.

Case 1: $f(c) \neq c$. In this case, $f(c)$ must be either b or some d_i. If $f(c) = b$, then $f(d_i) = b$ for every i because $f(c) = b \sqsubseteq f(d_i)$. Hence $f(b_i) \sqsubseteq f(d_i) = b$ for every i; therefore $f(b) = f(\bigsqcup_{i\in\omega} b_i) = \bigsqcup_{i\in\omega} f(b_i) = b$ by Lemma 1(b). By Lemma 1(a), b must be the only fixedpoint of f, hence the least fixedpoint. Now if $f(c) = d_i$ for some i, then $f(d_j) = d_i$ for every j since $d_i = f(c) \sqsubseteq f(d_j)$. Hence $f(b_j) \sqsubseteq f(d_j) = d_i$ and then

$f(b) = f(\bigsqcup_{j \in \omega} b_j) = \bigsqcup_{j \in \omega} f(b_j) \sqsubseteq d_i$, concluding that $f(b) \neq b$. Thus d_i is the only fixedpoint of f, hence the least fixedpoint.

Case 2: $f(c) = c$ but $f(b) \neq b$. If z is any fixedpoint of f other than c, then z must be equal to some d_i (by Lemma 1) which is greater than c in the partial ordering. Hence c is the least fixedpoint of f.

Case 3: $f(b) = b$ and $f(c) = c$. We want to show that Case 3 is impossible. This will, of course, conclude our claim that every ω-continuous function $f: D_5 \to D_5$ has a least fixedpoint. To this end, we need to establish a number of lemmas. First we note that there exists some integer $n \in \omega$ such that $b_i \sqsubseteq f(b_i) \sqsubseteq b$ and $c_i \sqsubseteq f(c_i) \sqsubseteq c$ for every $i \geqslant n$. (It is possible that $f(b_i)$ or $f(c_i) = a_j$ for some j if $i < n$.) Throughout the rest of the proof, we fix this integer n.

Lemma 2. (a) $\forall i \geqslant n \forall z \in H_i \exists j > i [f(z) \in H_j]$.

(b) $f(\{x^i_k \mid k > 0\})$ *is infinite for every* $i \geqslant n$.

(c) $\forall i \geqslant n \forall k > 0 \forall j, j' > i [f(x^i_k) = x^j_{k'}$ *and* $f(x^i_{k+1}) = x^{j'}_{k''} \Rightarrow j = j'$ *and* $k'' = k'$ *or* $k' + 1$ *or* $k' - 1]$.

Proof. (a) Since $a_{i+1} = f(a_i) \sqsubseteq f(b_i) \sqsubseteq b$ for $i \geqslant n$, we must have $b_{i+1} \sqsubseteq f(b_i)$. Similarly, $\sqsubseteq f(c_i)$. Take any z in H_i for $i \geqslant n$. Then $b_i \sqsubseteq z$ and $c_i \sqsubseteq z$. Hence $b_{i+1} \sqsubseteq f(z)$ and $c_{i+1} \sqsubseteq f(z)$. Hence $f(z)$ belongs to H_j for some $j > i$.

(b) Consider any h distinct elements $f(x^i_{k_1}), \ldots, f(x^i_{k_h})$ for $h > 0$. By the definition of D_5, there exists a largest $j > i$ and some z in H_j such that z is an upper bound of some of $f(x^i_{k_s})$'s where $1 \leqslant s \leqslant h$. Let $k = u_i^{-1}(j - i)$. Since $x^i_k \sqsubseteq d_j$ by our definition of the partial ordering, we have $f(x^i_k) \sqsubseteq f(d_j)$. From (a) above, $f(d_j)$ belongs to $H_{j'}$ for some $j' > j$. From the way we pick the integer j, we conclude that $f(x^i_k)$ must be different from all the $f(x^i_{k_s})$'s, $1 \leqslant s \leqslant h$. Hence $f(\{x^i_k \mid k > 0\})$ is infinite.

(c) Let $f(x^i_k) = x^j_{k'}$ and $f(x^i_{k+1}) = x^{j'}_{k''}$. Since $y^i_k \sqsubseteq x^i_k$ and $y^i_k \sqsubseteq x^i_{k+1}$, we have $f(y^i_k) \sqsubseteq f(x^i_k) = x^j_{k'}$ and $f(y^i_k) \sqsubseteq f(x^i_{k+1}) = x^{j'}_{k''}$. Since $f(y^i_k)$ belongs to D_s for some $s > i$, we conclude $j = j'$ and $k'' = k'$, or $k' + 1$, or $k' - 1$ by Property (iv) of D_5. □

The following is the key lemma of the whole proof.

Lemma 3. *For every* $i \geqslant n$, *there exist some* $j > i$ *and* $m, m' \in \omega$ *such that* $f(d_i) = d_j$ *and* $f(\{x^i_k \mid k \geqslant m\}) = \{x^j_{k'} \mid k' \geqslant m'\}$.

Proof. By Lemma 2(a) and (b), there is $j > i$ for every $i \geqslant n$ such that $f(d_i) = d_j$. Since d_i is the largest element in H_i, for every z in H_i, $f(z)$ belongs to $H_{j'}$ for some $i < j' \leqslant j$. Let m be the smallest integer satisfying $m > j - i$ and $u_i(k) \geqslant j - i$ for every $k \geqslant m$ (existence of m follows from the definition of $u_i(k)$). Hence $x^i_k \sqsupseteq c_j$ and $d_h \sqsupseteq x^i_k$ for some $h \geqslant j$ for every $k \geqslant m$. Since $f(c_j) \sqsupseteq c_{j+1}$, $f(x^i_k) \sqsupseteq c_{j+1}$ and $d_{h'} \sqsupseteq f(x^i_k)$ for some $h' > j$ for every $k \geqslant m$. Thus for every $k \geqslant m$, $f(x^i_k) = x^s_t$ for some $t > 0$ and $i < s \leqslant j$. However by Lemma 2(c), this s does not depend on k. On the other hand, $f(d_i) = d_j$ is an upper bound of the set $f(\{x^i_k \mid k \geqslant m\})$ which is an infinite set by Lemma 2(b). Thus $s=j$. Let $m'=\min\{t|x^j_t=f(x^i_k)$ for some $k\geq m\}$. Then the lemma follows from Lemma 2(b) and (c). □

Using this lemma, let us define a partial function $F_{ij}:\omega\to\omega$ for $i\geq n$ and corresponding j as follows: if $f(x^i_k)=x^j_{k'}$ for some k and k', then $F_{ij}(u_i(k))=u_j(k')$, otherwise $F_{ij}(u_i(k))$ is undefined. We show the following properties of F_{ij}.

Lemma 4. (a) *If $F_{ij}(t)$ is defined, then $F_{ij}(t)>t+i-j$.*

(b) *There exist $p,q\in\omega$ such that $F_{ij}(t)$ is defined and monotonic for every $t\geq p$ and $\{t|t\geq q\}\subseteq F_{ij}(\{t|t\geq p\})$.*

(c) $\exists r,s\in\omega\forall t\geq r[F_{ij}(t)=t+s]$.

Proof. (a) If $F_{ij}(u_i(k))=u_j(k')$, then $f(d_{i+u_i(k)})=d_h$ with $h\leq j+u_j(k')$ by Lemma 3. Hence $i+u_i(k)<h\leq j+u_j(k')$ by Lemma 2(a); thus $u_j(k')>u_i(k)+i-j$.

(b) Define $p=\max_{1\leq k\leq m}u_i(k)+1$ and $q'=\max_{1\leq k\leq m'}u_j(k')+1$ where m and m' are the integers given in Lemma 3. Then by Lemma 3, $\{t|t\geq q'\}$ is a subset of $F_{ij}(\omega)$. Since $u_i(k)\geq p$ implies $k>m$, $F_{ij}(t)$ is defined for every $t\geq p$. Now:

$$F_{ij}(\{t|t\geq p\})\supseteq F_{ij}(\omega)\backslash F_{ij}(\{t|0\leq t<p\})\supseteq\{t|t\geq q'\}\backslash F_{ij}(\{t|0\leq t<p\}).$$

Since $F_{ij}(\{t|0\leq t<p\})$ is finite and u_j is a bijection, there exists some q to give the required result. Finally we show that F_{ij} is monotonic for $t\geq p$. By Lemma 3, if $u_i(k)\geq p$, $k>m$ and therefore $F_{ij}(u_i(k))=u_j(k')$ if and only if $f(d_{i+u_i(k)})=d_{j+u_j(k')}$ for some k'. To demonstrate monotonicity of F_{ij}, it suffices to show that if $f(d_s)=d_h$ and $f(d_{s+1})=d_{h'}$, then $h\leq h'$ for every $s>i$. Now if $f(d_s)=d_h$, there exists some r such that $f(y^s_r)$ belongs to H_h. Since $y^s_r\sqsubseteq d_{s+1}$, we have $f(y^s_r)\sqsubseteq f(d_{s+1})=d_{h'}$, concluding $h\leq h'$.

(c) By (b) of this lemma, it can be shown that $F_{ij}(t+p-q+\sum^t_{v=q}(|M_v|-1))=t$ $(t\geq q)$ where $M_v=\{t'\geq p|F_{ij}(t')=v\}$ and $|M_v|$ is the cardinality of M_v. By (a) of this lemma, a constant K exists such that for every t, $\sum^t_{v=q}(|M_v|-1)\leq K$, proving the required result. □

Now we are ready to complete the proof. Let r' be the smallest integer such that $r'\geq m$ and $u_i(k)\geq r$ for every $k\geq r'$ where m and r are integers given in Lemmas 3 and 4 respectively. By Lemma 3(c), F_{ij} is injective for all $t\geq r$; thus if $f(x^i_k)=x^j_{k'}$ for $k\geq r'$, then $f(x^i_{k+1})=x^j_{k'+1}$ or $x^j_{k'-1}$ by Lemma 2(c). However, if $f(x^i_{k+1})=x^j_{k'-1}$, then $f(x^i_{k''})=x^j_{k'}$ for some $k''>k$ by Lemmas 2(c) and 3, which is impossible since F_{ij} is injective for $t\geq r$. Hence $f(x^i_{k+1})=x^j_{k'+1}$. This inductively shows $f(x^i_{k+v})=x^j_{k'+v}$ for all $v\in\omega$. Thus $F_{ij}(u_i(k+v))=u_j(k'+v)$ for all $v\in\omega$ with fixed k (s.t. $u_i(k)\geq r$) and k'. However, $F_{ij}(u_i(k+v))=u_i(k+v)+s$ by Lemma 4(c). Thus $u_j(k'+v)=u_i(k+v)+s$ for every $v\in\omega$ with constants $k\geq r'$, k' and s. This obviously contradicts the definitions of u_i and u_j. □

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