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Journal of Pure and Applied Algebra 186 (2004) 219–231

JOURNAL OF PURE AND APPLIED ALGEBRA

www.elsevier.com/locate/jpaa

Weak n-categories: comparing opetopic foundations

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Received 26 October 2002
Communicated by C.A. Weibel

Abstract

We construct for each tidy symmetric multicategory Q a cartesian monad $(\mathscr{E}_Q, T_Q)$, and extend this assignation to a functor. We exhibit a relationship between the slice construction on symmetric multicategories, and the 'free operad' monad construction on suitable monads. We use this to give an explicit description of the relationship between Baez–Dolan and Leinster opetopes.

MSC: 18D05; 18A99; 20G42; 03G30

0. Introduction

The present paper follows from [4] and we refer the reader to that paper for introductory and motivating remarks, and definitions concerning the Baez–Dolan theory.

In this paper we address the definition of weak n-category given by Tom Leinster in [10]. This approach is based on $(\mathscr{E}, T)$-multicategories; these structures were defined by Burroni [2] and have also been treated by Hermida [5].

The role that these multicategories play is not explicitly analogous to that of operads and multicategories in the opetopic and multitopic versions respectively, so the comparison is more subtle than in [4]. In fact, rather than comparing the role of symmetric multicategories with that of $(\mathscr{E}, T)$-multicategories, we compare it with the role of cartesian monads. This is the subject of Section 1.2.

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0022-4049/03/$ - see front matter
doi:10.1016/S0022-4049(03)00140-3

To study this relationship, we can restrict our attention to *tidy* symmetric multicategories. In Section 1.2 we construct a functor

$$\zeta : \mathbf{TidySymMulticat} \to \mathbf{CartMonad}$$

where **CartMonad** is the category of cartesian monads and cartesian monad opfunctors. Given a tidy symmetric multicategory Q we construct a cartesian monad $(\mathscr{E}_Q, T_Q)$ which acts on sets of 'labelled Q-objects' to give sets of 'source-labelled Q-arrows'.

The idea is that, for an $(\mathscr{E}, T)$-multicategory, much information about an arrow is given by its domain, that is, by the action of T; in the Baez–Dolan setting the domain of an arrow is just a list of objects, and the information is captured elsewhere. So, the functor part of T_Q is constructed from the collection of arrows itself, the unit from the identities, and multiplication from the reduction laws of Q.

In Section 2 we examine the construction of opetopes. First we examine the process of constructing k-cells from $(k-1)$-cells. In [1] Baez and Dolan define the 'slicing' process for this purpose. Leinster does define a slicing process on $(\mathscr{E}, T)$-multicategories, but since we are considering a comparison between symmetric multicategories and *monads*, we seek an analogous process defined on these monads, rather than on $(\mathscr{E}, T)$-multicategories. That is, given a *suitable* monad $(\mathscr{E}, T)$, the monad $(\mathscr{E}, T)' = (\mathscr{E}', T')$ is defined to be the 'free $(\mathscr{E}, T)$-operad monad' [10]. We show that

$$\zeta(Q^+) \cong \zeta(Q)'.$$

In this sense, the processes are analogous.

Finally, we apply these results to the construction of opetopes. Having established a relationship between the underlying theories, it is straightforward to compare these constructions.

Leinster constructs 'opetopes' which are not a priori the same, but have an analogous role, based on a series $(\mathbf{Set}/S_n, T_n)$ of cartesian monads. We show that, for each $n \geqslant 0$

$$\zeta(I^{n+}) \cong (\mathbf{Set}/S_n, T_n)$$

and deduce that

$$\mathrm{o}(I^{n+}) \simeq S_n$$

for each $n \geqslant 0$, where o denotes the object-category. Informally, we see that Baez–Dolan opetopes and Leinster opetopes are the same up to isomorphism.

Throughout this paper we repeatedly find that the details of proofs are fiddly but uninteresting. So we include some informal comments about how the constructions may be interpreted, as a gesture towards demonstrating that the notions are in fact naturally arising. The aim is to shed some light on the relationship between the various structures involved, a relationship which has previously remained unclear.

1. The theory of multicategories

In this section we examine the underlying theories involved, and we construct a way of relating these theories to one another; this relationship provides subsequent equivalences between the definitions. We adopt a concrete approach here; certain aspects of the definitions suggest a more abstract approach but this will require further work beyond the scope of this work.

1.1. $(\mathscr{E}, T)$-multicategories

In [10] opetopes are constructed using $(\mathscr{E}, T)$-multicategories. These are defined by Burroni in [2] as 'T-categories'.

Definition 1.1. Let T be a cartesian monad on a cartesian category $\mathscr{E}$. An $(\mathscr{E}, T)$-multicategory is given by an 'objects-object' C_0 and an 'arrows-object' C_1, with a diagram

$$TC_0 \xleftarrow{d} C_1 \xrightarrow{c} C_0$$

in $\mathscr{E}$ together with maps $C_0 \xrightarrow{\text{ids}} C_1$ and $C_1 \circ C_1 \xrightarrow{\text{comp}} C_1$ satisfying associative and identity laws. (See [10] for full details.)

We write **CartMonad** for the category of cartesian monads and cartesian monad opfunctors (see [11,10] for definitions).

1.2. Relationship between symmetric multicategories and cartesian monads

We begin the comparison by constructing a functor

$$\zeta : \mathbf{TidySymMulticat} \to \mathbf{CartMonad}.$$

This is enough since we have seen [4] that all the symmetric multicategories involved in the construction of opetopes are tidy.

Given any tidy symmetric multicategory Q, we construct a cartesian monad $\zeta(Q) = (\mathscr{E}_Q, T_Q)$, say.

Write $o(Q) = \mathbb{C}$. Q is tidy, so $\mathbb{C} \simeq S$, say, where S is a discrete category. For various of the constructions which follow, we assume that we have chosen a specific functor $S \xrightarrow{\sim} \mathbb{C}$. However, when isomorphism classes are taken subsequently, we observe that the construction in question does not depend on the choice of this functor.

Put $\mathscr{E}_Q = \mathbf{Set}/S$ and observe immediately that this is cartesian. (This is sufficient here, though of course **Set**/S has much more structure than this.)

Informally, an element $(X, f) = (X \xrightarrow{f} S)$ of $\mathbf{Set}/S$ may be thought of as a system for labelling Q-objects with 'compatible' elements of X; each 'label' is compatible with an isomorphism class of Q-objects. Then the action of T_Q assigns compatible labels to the source elements of Q-arrows in every way possible; the target is not affected. The resulting set of 'source-labelled Q-arrows' is itself made into a set of labels by regarding each arrow as a 'label' for its target.

We now give the formal definition of the functor $T_Q : \mathscr{E}_Q \to \mathscr{E}_Q$. For the action on object-categories, consider $(X, f) = (X \xrightarrow{f} S) \in \mathbf{Set}/S$. We have the following composite functor

$$\mathrm{elt}\, Q \xrightarrow{s} \mathscr{F}\mathbb{C}^{\mathrm{op}} \xrightarrow{\sim} \mathscr{F}S^{\mathrm{op}}$$

where $\mathscr{F}$ denotes the free symmetric strict monoidal category monad on **Cat**, and s and t the source and target functions, respectively. Consider the pullback $\mathrm{elt}\, Q \times_{\mathscr{F}S^{\mathrm{op}}} \mathscr{F}X^{\mathrm{op}}$. Since Q is tidy, $\mathrm{elt}\, Q$ is equivalent to a discrete category, so this pullback is equivalent to a discrete category X', say. Put $T_Q(X, f) = (X', f')$ where f' is the composite

$$X' \xrightarrow{\sim} \mathrm{elt}\, Q \times_{\mathscr{F}S^{\mathrm{op}}} \mathscr{F}X^{\mathrm{op}} \to \mathrm{elt}\, Q \xrightarrow{t} \mathbb{C} \xrightarrow{\sim} S.$$

We now define the action of T_Q on morphisms. A morphism $h : (X, f) \to (Y, g)$ in $\mathbf{Set}/S$ induces a functor $\mathrm{elt}\, Q \times_{\mathscr{F}S^{\mathrm{op}}} \mathscr{F}X^{\mathrm{op}} \to \mathrm{elt}\, Q \times_{\mathscr{F}S^{\mathrm{op}}} \mathscr{F}Y^{\mathrm{op}}$ which gives a morphism $h' : X' \to Y'$; by construction this is in fact a morphism in $\mathbf{Set}/S$. We define T_Q on morphisms by $T_Q(h) = h'$. This is clearly functorial; we now show that it inherits a cartesian monad structure from the identities and composition of Q. For convenience we write $\mathscr{E}_Q = \mathscr{E}$ and $T_Q = T$.

- unit

  Using the above notation, we define the unit by components

$$\eta_{(X,f)} : (X, f) \to (X', f').$$

Given $(X, f) \in \mathbf{Set}/S$, we have a functor $X \to \mathrm{elt}\, Q$ given by the composite

$$X \xrightarrow{f} S \xrightarrow{\sim} \mathbb{C} \xrightarrow{1_{-}} \mathrm{elt}\, Q.$$

We also have a functor $X \to \mathscr{F}X^{\mathrm{op}}$ given by the unit of the monad $\mathscr{F}$. These induce a functor

$$X \to \mathrm{elt}\, Q \times_{\mathscr{F}S^{\mathrm{op}}} \mathscr{F}X^{\mathrm{op}}$$

and we define the component $\eta_{(X,f)}$ to be the composite

$$X \to \mathrm{elt}\, Q \times_{\mathscr{F}S^{\mathrm{op}}} \mathscr{F}X^{\mathrm{op}} \xrightarrow{\sim} X'.$$

Explicitly, $\eta_{(X,f)}$ acts as follows. We have $\eta_{(x,f)}(x) = [(1_c, x)]$, the isomorphism class of $(1_c, x) \in \mathrm{elt}\, Q \times_{\mathscr{F}S^{\mathrm{op}}} \mathscr{F}X^{\mathrm{op}}$. So $(1_c, x)$ is an "identity labelled by x", where $c \in \mathbb{C}$ is any object in the isomorphism class fx.

It is straightforward to check that this defines a cartesian natural transformation as required.

- multiplication

  We define multiplication by components $\mu_{(X,f)} : (X'', f'') \to (X', f')$.
  Now by definition

$$X' \simeq \mathrm{elt}\, Q \times_{\mathscr{F}S^{\mathrm{op}}} \mathscr{F}X^{\mathrm{op}} = A, \text{ say}$$

and

$$X'' \simeq \mathrm{elt}\, Q \times_{\mathscr{F}S^{\mathrm{op}}} \mathscr{F}X'^{\mathrm{op}} = B, \text{ say}.$$

We use the universal property of the pullback A to induce a morphism $B \to A$, and hence $X'' \to X'$; we do this via the morphisms

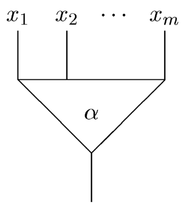
$$\mathrm{elt}\, Q \times \mathscr{F}X'^{\mathrm{op}} \xrightarrow{p_2} \mathscr{F}X'^{\mathrm{op}} \xrightarrow{\mathscr{F}p_2} \mathscr{F}\mathscr{F}X^{\mathrm{op}} \xrightarrow{\mu} \mathscr{F}X^{\mathrm{op}},$$

where p_1 and p_2 denote the first and second projections respectively, and

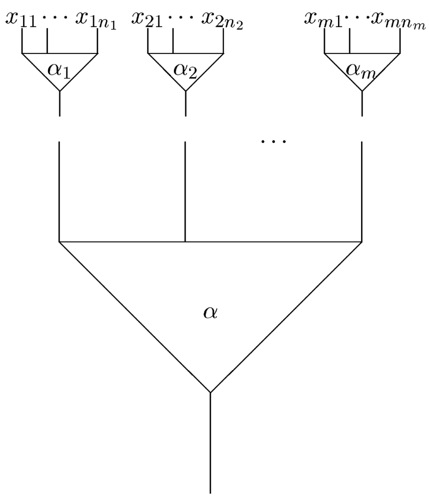
$$\mathrm{elt}\, Q \times \mathscr{F}X'^{\mathrm{op}} \xrightarrow{(1,\mathscr{F}p_1)} \mathrm{elt}\, Q \times \mathscr{F}(\mathrm{elt}\, Q)^{\mathrm{op}} \to \mathrm{elt}\, Q,$$

where the second morphism is composition in Q.

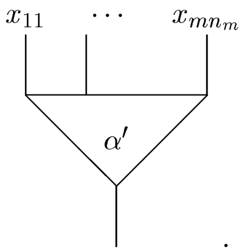
Informally, (X, f) is a system for labelling Q-objects, and $T(X, f) = (X', f')$ gives source-labelled Q-arrows. A typical element of X' may be thought of as the isomorphism class of



where $\alpha \in \mathrm{elt}\, Q$ and $s(\alpha) \cong (fx_1, \ldots, fx_n)$. Then we can draw θ as (the isomorphism class of)



where $\alpha, \alpha_1, \ldots, \alpha_m \in \mathrm{elt}\, Q$ and $s(\alpha) \cong (t(\alpha_1), \ldots t(\alpha_m))$. So, via the relevant object-isomorphisms, we may compose the underlying Q-arrows to give α', say, which is defined up to isomorphism. We then concatenate the X-labels (via the multiplication for $\mathscr{F}$) to give



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Finally, we take the isomorphism class of this to give $\mu_{(X,f)}(\theta) \in X'$, and $f''(\mu_{(X,f)}(\theta)) = [t(\alpha')] = [t(\alpha)] \in S$.

It follows that μ defined in this way is a cartesian natural transformation.

Finally, it is easy to check that T preserves pullbacks, so $T_Q = (T, \eta, \mu)$ is a cartesian monad on $\mathscr{E}_Q = \mathscr{E}$. So we may define $\zeta(Q) = (\mathscr{E}_Q, T_Q)$; we define the action of ζ on morphisms in the obvious way.

We observe immediately that the construction of $(\mathscr{E}_Q, T_Q)$ uses only the isomorphism classes of objects and arrows of Q. So

$$(\mathscr{E}_{Q_1}, T_{Q_1}) \cong (\mathscr{E}_{Q_2}, T_{Q_2}) \Leftrightarrow Q_1 \simeq Q_2.$$

Recall [4] that we expect that a symmetric multicategory Q may be given as a monad in a certain bicategory, in which case the identities are given by the unit, and composition laws by multiplication. In this abstract framework there should be a morphism from the underlying bicategory to the 2-category **Cat**, taking the monad Q to the monad T_Q, but this is somewhat beyond the scope of this work.

2. The theory of opetopes

In this section we give the construction of Leinster opetopes, and show in what sense this is equivalent to the construction of opetopes given in [4]. That is, we show that the respective categories of k-opetopes are equivalent.

We first discuss the process by which $(k+1)$-cells are constructed from k-cells. Recall that, in [1], the 'slice' construction is used, giving for any symmetric multicategory Q the slice multicategory Q^+.

2.1. Slicing a $(\mathscr{E}, T)$-multicategory

In [10] the 'free $(\mathscr{E}, T)$-operad' construction is used to construct $(k+1)$-cells from k-cells; this gives, for any *suitable* monad $(\mathscr{E}, T)$, the 'free $(\mathscr{E}, T)$-operad' monad $(\mathscr{E}, T)' = (\mathscr{E}', T')$. In order to compare this construction with the Baez–Dolan slice, we will examine the monad $\zeta(Q)'$. However, we must first check that $\zeta(Q)'$ can actually be constructed, that is, that $\zeta(Q) = (\mathscr{E}_Q, T_Q)$ is a suitable monad.

Recall [10] that a cartesian monad $(\mathscr{E}, T)$ is *suitable* if it satisfies:

1. $\mathscr{E}$ has disjoint finite coproducts which are stable under pullback
2. $\mathscr{E}$ has colimits of nested sequences; these commute with pullbacks and have monic coprojections
3. T preserves colimits of nested sequences.

Here a nested sequence is a string of composable monics.

It is straightforward to check that if Q is a tidy symmetric multicategory, $(\mathscr{E}_Q, T_Q)$ is a suitable monad.

2.2. Comparison of slice

We are now ready to compare the slice constructions and make precise the sense in which they correspond to one another. We show that the functor

$$\zeta : \mathbf{TidySymMulticat} \to \mathbf{CartMonad}.$$

'commutes' with slicing, up to isomorphism, in the following sense.

Proposition 2.1. *Let Q be a tidy symmetric multicategory. Then*

$$\zeta(Q)' \cong \zeta(Q^+)$$

that is

$$(\mathscr{E}'_Q, T'_Q) \cong (\mathscr{E}_{Q^+}, T_{Q^+})$$

in the category **CartMonad**.

Note that Q^+ is tidy since Q is tidy (see [4]), so we can indeed form the monad $\zeta(Q^+)$.

This proof is somewhat technical and we defer it to the Appendix. Informally, the idea is as follows. T_{Q^+} takes a set A of 'labels for arrows of Q' and returns the set of configurations for composing labelled arrows according to their underlying arrows. On the other hand, T'_Q gives the set of all formal composites of arrows labelled in A according to the structure of T_Q, which is precisely the set of configurations as above.

Recall that

$$\zeta(Q_1) \cong \zeta(Q_2) \Leftrightarrow Q_1 \simeq Q_2.$$

We immediately deduce the following result, comparing all three processes of slicing (see [4]).

Corollary 2.2. *Let M be a generalised multicategory. Then*

$$\zeta\xi(M_+) \cong \zeta(\xi(M)^+) \cong \zeta\xi(M)'.$$

2.3. Opetopes

We are now ready to compare the different constructions of opetopes, applying the results we have already established.

2.4. Leinster opetopes

In [10], k-opetopes are defined by a sequence $(\mathbf{Set}/S_k, T_k)$ of cartesian monads given by iterating the slice as follows.

For any cartesian monad $(\mathscr{E}, T)$ write

$$(\mathscr{E}, T)^{k'} = \begin{cases} (\mathscr{E}, T), & k = 0, \\ ((\mathscr{E}, T)^{(k-1)'})', & k \geqslant 1. \end{cases}$$

Put $(\mathscr{E}_0, T_0) = (\mathbf{Set}, id)$ and for $k \geqslant 1$ put $(\mathscr{E}_k, T_k) = (\mathbf{Set}, id)^{k'}$. It follows that for each $k, (\mathscr{E}_k, T_k)$ is of the form $(\mathbf{Set}/S_k, T_k)$ where $S_0 = 1$ and S_{k+1} is given by

$$\begin{pmatrix} S_{k+1} \\ \downarrow \\ S_k \end{pmatrix} = T_k \begin{pmatrix} S_k \\ \downarrow 1 \\ S_k \end{pmatrix}.$$

Then *Leinster k-opetopes* are defined to be the members of S_k; as above, we will regard S_k as a discrete category.

2.5. Comparisons of opetopes

We now compare opetopes and Leinster opetopes.

Proposition 2.3. *For each* $k \geqslant 0$

$$\zeta(I^{k+}) \cong (\mathbf{Set}, \mathrm{id})^{k'} = (\mathbf{Set}/S_k, T_k).$$

Proof. By induction, using Proposition 2.1. □

Then on objects, the above equivalence gives the following result.

Corollary 2.4. *For each* $k \geqslant 0$

$$\mathbb{C}_k \simeq S_k.$$

Recall [4] that we also have for each k a (discrete) category P_k of 'multitopes', the analogous notion as defined in [6] (serialised in [7–9]); in [4] we prove that, for each $k \geqslant 0$, $P_k \simeq \mathbb{C}_k$. So we immediately have the following result, comparing all three theories:

Corollary 2.5. *For each $k \geqslant 0$*

$$P_k \simeq \mathbb{C}_k \simeq S_k.$$

This result shows that multitopes, opetopes, and Leinster opetopes are the same, up to isomorphism.

We eventually aim to define a category **Opetope** of opetopes of all dimensions, whose morphisms are 'face maps' of opetopes; this is the subject of [3].

Acknowledgements

This work was supported by a Ph.D. grant from EPSRC. I would like to thank Martin Hyland and Tom Leinster for their support and guidance.

Appendix A. Proof of Proposition 2.1

We now give the proof of Proposition 2.1 deferred from Section 2.2.

First we show that $\mathscr{E}'_Q \cong \mathscr{E}_{Q^+}$. Now $\mathscr{E}_{Q^+} = \mathbf{Set}/S_{Q^+}$ where $S_{Q^+} \cong o(Q^+) = \mathrm{elt}\, Q$, and $\mathscr{E}'_Q = \mathbf{Set}/S'_Q$ where

$$\begin{pmatrix} S'_Q \\ \downarrow \\ S_Q \end{pmatrix} = T_Q \begin{pmatrix} S_Q \\ \downarrow 1 \\ S_Q \end{pmatrix}.$$

So by definition S'_Q is equivalent to a pullback of $\mathrm{elt}\, Q$ over an identity morphism. Thus $S'_Q \simeq \mathrm{elt}\, Q$, giving $S'_Q \cong S_{Q^+}$. So we have $\mathscr{E}'_Q \cong \mathscr{E}_{Q^+}$. We write elements of both these categories as sets over S', since confusion is unlikely.

Consider $(A, f) = (A \xrightarrow{f} S') \in \mathscr{E}'_Q \cong \mathscr{E}_{Q^+}$. Write $T'_Q(A, f) = (A_1, f_1)$ and $T^+_Q(A, f) = (A_2, f_2)$. We show $(A_1, f_1) \cong (A_2, f_2)$.

Now $A_2 \simeq \mathrm{elt}\, Q^+ \times_{\mathscr{F}S'^{\mathrm{op}}} \mathscr{F}A^{\mathrm{op}}$, and f_2 is given by the composite

$$A_2 \simeq \mathrm{elt}\, Q^+ \times_{\mathscr{F}S'^{\mathrm{op}}} \mathscr{F}A^{\mathrm{op}} \to \mathrm{elt}\, Q^+ \xrightarrow{t_{Q^+}} \mathrm{elt}\, Q \xrightarrow{\sim} S'$$

where t_{Q^+} is the target map of Q^+.

Informally, since we are here considering $S' \simeq o(Q^+) = \mathrm{elt}\,(Q)$, the object $(A \xrightarrow{f} S')$ may be thought of as a set of labels for arrows of Q. Then A_2 is the set of all possible source-labelled arrows of Q^+.

Since an arrow of Q^+ is given by a tree with nodes corresponding to arrows of Q, an element of A_2 may be thought of as a configuration for composing labelled arrows of Q via object-isomorphisms, where composition is according to the underlying arrows only. f_2 acts by composing the underlying arrows of Q and then taking isomorphism classes.

We now turn our attention to the action of T'_Q. (For full details of the free multicategory construction we refer the reader to [10].) For convenience we write $T_Q = T$ and $S_Q = S$, so we need to form

$$(T, \mathbf{Set}/S)' = (T', S').$$

To construct A_1, we form the free multicategory on the following graph:

$$\begin{tikzcd}[column sep=small] & \begin{pmatrix} A \\ \downarrow \\ S \end{pmatrix} \arrow[dl, "f"'] \arrow[dr, "!"] & \\ T\begin{pmatrix} S \\ \downarrow 1 \\ S \end{pmatrix} & & \begin{pmatrix} S \\ \downarrow 1 \\ S \end{pmatrix} \end{tikzcd}$$

Recall we have $T\begin{pmatrix} S \\ \downarrow \\ S \end{pmatrix} = \begin{pmatrix} S' \\ \downarrow \\ S \end{pmatrix}$ and the map $A \to S$ is the composite $A \xrightarrow{f} S' \to S$. The graph underlying the free operad is then

$$\begin{tikzcd}[column sep=small] & \begin{pmatrix} A_1 \\ \downarrow \\ S \end{pmatrix} \arrow[dl, "f'"'] \arrow[dr] & \\ T\begin{pmatrix} S \\ \downarrow \\ S \end{pmatrix} & & \begin{pmatrix} S \\ \downarrow \\ S \end{pmatrix} \end{tikzcd}$$

The construction gives a sequence of graphs

$$\begin{tikzcd}[column sep=small] & \begin{pmatrix} C^{(k)} \\ \downarrow \\ S \end{pmatrix} \arrow[dl, "d_k"'] \arrow[dr, "c_k"] & \\ T\begin{pmatrix} S \\ \downarrow 1 \\ S \end{pmatrix} & & \begin{pmatrix} S \\ \downarrow 1 \\ S \end{pmatrix} \end{tikzcd}$$

where $C^{(0)} = S$, $d_0 = \eta_T$ and

$$\begin{pmatrix} C^{(k+1)} \\ \downarrow \\ S \end{pmatrix} = \begin{pmatrix} S \\ \downarrow 1 \\ S \end{pmatrix} + \begin{pmatrix} A \\ \downarrow \\ S \end{pmatrix} \circ \begin{pmatrix} C^{(k)} \\ \downarrow \\ S \end{pmatrix}.$$

Here $\circ$ is composition in the bicategory of spans, and d_{k+1} is given by the composite

$$\begin{pmatrix} A \\ \downarrow \\ S \end{pmatrix} \circ \begin{pmatrix} C^{(k)} \\ \downarrow \\ S \end{pmatrix} \to T \begin{pmatrix} C^{(k)} \\ \downarrow \\ S \end{pmatrix} \xrightarrow{Td_k} TT \begin{pmatrix} S \\ \downarrow \\ S \end{pmatrix} \xrightarrow{\mu_T} T \begin{pmatrix} S \\ \downarrow \\ S \end{pmatrix}.$$

This construction gives a nested sequence $(C^{(k)}, f^{(k)}) \in \mathbf{Set}/S$ with $(C^{(0)}, f^{(0)}) = (S, 1)$ and

$$C^{(k+1)} = S \coprod T(C^{(k)}) \times_{S'} A$$

where (by abuse of notation) we write

$$T \begin{pmatrix} C^{(k)} \\ \downarrow \\ S \end{pmatrix} = \begin{pmatrix} T(C^{(k)}) \\ \downarrow \\ S \end{pmatrix}.$$

$f^{(k+1)}$ is given by $1 \coprod (T(C^{(k)}) \times_{s'} A \xrightarrow{d_{k+1}} S' \to S)$ and $\begin{pmatrix} A_1 \\ \downarrow \\ S \end{pmatrix}$ is then the colimit of this nested sequence.

Informally, the sets $C^{(k)}$ may be thought of as formal composites of 'depth' at most k. The formula for $C^{(k)}$ says that a composite is either null or is a generating arrow composed with other composites. We aim to show that these formal composites correspond to the formal composites given by the source-labelled arrows of Q^+.

We show that $A_1 \cong A_2 \simeq \operatorname{elt} Q^+ \times_{\mathscr{F}S'^{\mathrm{op}}} \mathscr{F}A^{\mathrm{op}}$ as follows. For each k we exhibit an embedding

$$g_k : C^{(k)} \hookrightarrow A_2$$

which makes the following diagram commute:

$$\begin{tikzcd} C^{(k)} \arrow[rr, "g_k"] \arrow[dr, "d_k"'] & & A_2 \arrow[dl, "f_2"] \\ & S' & \end{tikzcd}$$

Then the colimit induces the map required.

We proceed by induction. Define $g_0 : S \to \operatorname{elt} Q^+ \times_{\mathscr{F}S'^{\mathrm{op}}} \mathscr{F}A^{\mathrm{op}}$ as follows. Let $[x] \in S$ denote the isomorphism class of $x \in \mathrm{o}(Q)$. Given any $[x] \in S$, we have a nullary arrow $\alpha_x \in Q^+(\cdot; 1_x)$. Recall that an arrow of Q^+ may be regarded as a tree with nodes corresponding to the source elements (which are themselves arrows of Q) and edges labelled by object-morphisms of Q. Then $\alpha_x \in Q^+(\cdot; 1_x)$ is given by a tree with no nodes, that is, a single edge labelled by 1_x. The source of α is empty, so we can define g_0 by

$$g_0([x]) = [(\alpha_x, \cdot)],$$

where $(\alpha_x, \cdot) \in \mathrm{elt}\, Q^+ \times_{\mathscr{F}S'^{\mathrm{op}}} \mathscr{F}A^{\mathrm{op}}$, and observe immediately that

$$x \cong x' \in \mathrm{o}(Q) \Leftrightarrow 1_x \cong 1_{x'} \in \mathrm{elt}\, Q.$$

Furthermore we have

$$d_0[x] = \mu_T[x] = [1_x] = f_2 g_0[x]$$

as required.

For the induction step, consider $y \in C^{(k+1)}$. If $y \in S$ then put $g_{k+1}(y) = g_0(y)$. Otherwise, we have $y = (\alpha, a) \in T(C^{(k)}) \times_{S'} A$. Here the map $T(C^{(k)}) \to S'$ is given by $Tf^{(k)}$. Recall that by definition, $T(C^{(k)})$ is equivalent to the pullback $\mathrm{elt}\, Q \times_{\mathscr{F}S^{\mathrm{op}}} \mathscr{F}(C^{(k)})^{\mathrm{op}}$ so an element of $T(C^{(k)})$ is an isomorphism class of arrows of Q source-labelled by compatible elements of $C^{(k)}$. We write the pullback as $\mathbb{C}^{(k)}$. Then $Tf^{(k)}$ is the map given by the composite

$$T(C^{(k)}) \xrightarrow{\sim} \mathbb{C}^{(k)} \to \mathrm{elt}\, Q \xrightarrow{\sim} S'.$$

Informally, $Tf^{(k)}$ removes the labels, leaving only the (isomorphism class of the) underlying arrow of Q.

Now we in fact exhibit a full and faithful functor

$$\mathbb{C}^{(k)} \times_{S'} A \to \mathrm{elt}\, Q^+ \times_{\mathscr{F}S'^{\mathrm{op}}} \mathscr{F}A^{\mathrm{op}}.$$

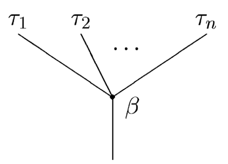
Let $((\beta, \underline{b}), a) \in \mathbb{C}^{(k)} \times_{S'} A$. So $\beta \in \mathrm{elt}\, Q$, $\underline{b} = b_1, \ldots, b_n \in \mathscr{F}(C^{(k)})^{\mathrm{op}}$ and $a \in A$ such that $[s_Q(\beta)] = (f^{(k)}(b_1), \ldots, f^{(k)}(b_n))$ and $f(a) = [\beta]$.

Informally, we have an arrow β of Q, source-labelled by the $b_i \in C^{(k)}$, and a compatible label $a \in A$. We seek a formal composite of labelled arrows, of depth up to $k+1$. By induction, we already have for each element of $C^{(k)}$ a formal composite of labelled arrows, of depth up to k. So we aim to form a formal composite of these together with β labelled by a.

By induction we have for each $1 \leqslant i \leqslant n$

$$g_k(b_i) = (\pi_i, p_i) \in \mathrm{elt}\, Q^+ \times_{\mathscr{F}S'^{\mathrm{op}}} \mathscr{F}A^{\mathrm{op}}, \text{ say.}$$

The commuting condition implies that for each i, $[s_Q(\beta)_i] = [t_Q t_{Q^+}(\pi_i)]$. This gives us a way of constructing a new element of $\mathrm{elt}\, Q^+$ from the data given, since each π_i can be composed with β at the ith place, via the appropriate object-isomorphism. That is, we form a tree by induction, as shown in the following diagram:



where τ_i is the tree for π_i. Each π_i has its nodes (that is, source elements) labelled by elements of A; to complete the definition it remains only to 'label' the node corresponding to β. But we have $f(a) = [\beta]$, that is, a is a compatible label for β. So we let a be the label for β.

So we have defined a full and faithful functor as intended, inducing, on isomorphism classes, an embedding $g_{k+1} : C^{(k)} \hookrightarrow A_2$ as required. It is straightforward to check the commuting condition; informally, d_k acts by ignoring the labels and composing the underlying arrows of Q, as does μ. Since μ is induced from composition in Q, and t_{Q^+} is constructed from composition of a formal composite of arrows of Q, we have $f_2 \circ g_{k+1} = d_{k+1}$ as required.

So we have for each $k \geqslant 0$ an embedding g_k, inducing a map $A_1 \to A_2$ in $\mathbf{Set}/S'$. It is straightforward to check that this is surjective and hence an isomorphism; it is also easy to check the axioms for a monad opfunctor. So we have

$$(\mathscr{E}_{Q^+}, T_{Q^+}) \cong (\mathscr{E}'_Q, T'_Q)$$

as required. □

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