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Stress concentration and surface instability of anisotropic solids with slightly wavy boundary

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ABSTRACT

This paper presents a first order perturbation analysis of stress concentration and surface morphology instability of elastically anisotropic solids. The boundary of the solids under consideration is periodic along two orthogonal directions. The magnitude of the undulation is sufficiently small so that a half-space model can be used for simplification. We derive expressions for the stress concentration factors and the critical wavelength of the perturbation in terms of the remote stresses, surface energy anisotropy and the elastic anisotropy of the solid. Numerical applications to cubic materials using Barnett–Lothe integrals are also given.

1. Introduction

A nominally flat material surface, in reality, always contains defects and inhomogeneities originating from the manufacturing process or environmental corrosion. When subject to mechanical loading, the presence of defects can magnify stresses in the material several times and eventually leads to the nucleation of plastic deformation or fracture. Stress concentration is one of major concerns in the reliability analysis of semiconductor thin films since residual stresses in these structures are usually very high.

Instability of the surface morphology is another problem of thin films mechanics which is also subject to this work. In presence of mechanical stresses, a perturbation in the film's shape can induce mass diffusion and render the free surface unstable (Asaro and Tiller, 1972; Srolovitz, 1989). The instability can be understood by considering the variation of the strain and surface energy in response to the perturbation (Freund and Suresh, 2003). Since perturbations are geometrically similar to surface defects, the study of stress concentration effect provides some insights into the surface instability problem.

In this paper, the surface defects or perturbations are both modeled as a sinusoidal fluctuation about the mean reference plane. The amplitude of the fluctuation is sufficiently small with respect to the wavelength so that the half space model can be used. To derive the solution which is first order accurate in perturbation, one considers a usual elasticity problem where the stress boundary conditions are determined from the zeroth-order stresses and the boundary profile. Based on this observation, Gao (1991a) used the Green's function for a half-space to solve directly the stress fields for surfaces with sinusoidally wavy profile and single wave profile. He concluded that for surfaces with relatively small amplitude a to wavelength λ ratio, say $a/\lambda = 0.1$, the perturbation can magnify the bulk stress by 2.25 times. In a separate work, Gao (1991b) made use of the complex potential function method to compute the complete stress field and the perturbation's critical wavelength λ_{cr} that induces the surface instability.

The works of Gao (1991a,b) are limited to a single material system where it is treated as a homogeneous elastic half space. In reality, thin film systems may be composed of several material layers that interact and have considerable impact on the global behavior. Freund and Jonsdottir (1993) formulated the problem concerning a film bonded to a substrate with bidimensional shape perturbation. They also determined the chemical potential that governs the diffusion process and computed the most unstable wavemode. Kim and Vlassak (2007) used the Airy stress function to investigate multilayer thin films, each of which can have its own residual stresses. In their works, all the considered materials are isotropic.

Anisotropic thin films are also studied in the past. Gao (1991c) used the Stroh formalism to analyze the instability of the thin films subject to perturbations along one direction. In Gao (1991c), the stress concentration factors were also obtained for some particular materials in closed form. Li et al. (2008) used a similar approach as Kim and Vlassak (2007) to study cubic film/substrate system. They found that anisotropic effect could enhance the surface stability if the anisotropy ratio AR less than 1. The works of (Gao, 1991c; Li et al., 2008) are limited to one dimensional surface perturbation so that two dimensional elasticity theory can be used.

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The motivation of the present work is based on the fact that thin film or imperfection profile can vary along any directions that lies on the material's mean plane. In the framework of anisotropic elasticity, the problem will be treated in a general way with the use of the Stroh formalism and the eigen solution method. The approach is then applied to several particular cases where closed form and numerical solutions are given. These results issued from the present work show that both the wavelength of the surface profile along two orthogonal directions and the complex stress state have significant influences on the stress concentration and morphology instability.

2. Problem formulation in anisotropic elasticity

2.1. Stress concentration in a wavy half space

In the Cartesian coordinate system $Ox_1x_2x_3$ associated with the orthonormal vector basis $(\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3)$, we consider a domain bounded by the inequality

$$x_2 - h(x_1,x_3) \geqslant 0, \tag{1}$$

where $h(x_1,x_3)$ is a biperiodic function of two variables x_1 and x_3, expressed in the form

$$h(x_1,x_3) = a\cos(\omega_1 x_1)\cos(\omega_3 x_3), \quad \omega_i = \frac{2\pi}{\lambda_i}, \quad i = 1,3. \tag{2}$$

In Eq. (2), ω_i, λ_i are respectively the wavenumber and the wavelength of $h(x_1,x_3)$ in direction i $(i = 1,3)$. The amplitude a of the wavy surface appearing in (2) is assumed to be very small with respect to the two wavelengths λ_1, λ_3, so that the dimensionless term $\varepsilon = a\omega$ satisfies

$$\varepsilon = a\omega \ll 1, \quad \omega = \sqrt{\omega_1^2 + \omega_3^2}. \tag{3}$$

The term ω introduced in (3) is called the equivalent wavenumber of the wavy surface. On the other hand, we denote λ as the equivalent wavelength that satisfies the relations

$$\lambda = 2\pi/\omega \quad \text{or} \quad \frac{1}{\lambda} = \sqrt{\frac{1}{\lambda_1^2} + \frac{1}{\lambda_3^2}}. \tag{4}$$

From (2) and (3), we remark that the case $\varepsilon = 0$ corresponds to a flat surface $h(x_1,x_3) = 0$ and a sufficiently small value of ε as in (3), can model a surface which is slightly deviated from the flat one. The parameter ε will be used in later perturbation analysis.

In terms of the boundary conditions, the surface $x_2 = h(x_1,x_3)$ is free of stress and at infinity, the halfspace is subject to uniform lateral stresses $\boldsymbol{\Sigma}^0$ with components $\Sigma^0_{2i} = 0$ $(i = 1,2,3)$. In the coordinate system $Ox_1x_2x_3$, the components of the remote stresses $\boldsymbol{\Sigma}^0$ are regrouped in the following matrix:

$$\boldsymbol{\Sigma}^0 = \begin{bmatrix} \Sigma^0_{11} & 0 & \Sigma^0_{13} \\ 0 & 0 & 0 \\ \Sigma^0_{13} & 0 & \Sigma^0_{33} \end{bmatrix} \quad \text{in } Ox_1x_2x_3. \tag{5}$$

Hence, the boundary conditions of the problem are the two equations

$$\boldsymbol{\Sigma}\cdot\mathbf{n} = \mathbf{0} \text{ if } x_2 = h(x_1,x_3), \quad \lim_{x_2\to\infty}\boldsymbol{\Sigma} = \boldsymbol{\Sigma}^0, \tag{6}$$

where $\boldsymbol{\Sigma}$ is the stress field in the solid and $\mathbf{n}$ is the normal vector to the surface $x_2 = h(x_1,x_3)$. Since $h(x_1,x_3)$ is given by (2), the normal vector $\mathbf{n}$ is expressed in the exact form as

$$\mathbf{n}(\mathbf{x}) = -\frac{\mathbf{e}_2 + a\omega_1\sin(\omega_1x_1)\cos(\omega_3x_3)\mathbf{e}_1 + a\omega_3\cos(\omega_1x_1)\sin(\omega_3x_3)\mathbf{e}_3}{\sqrt{1 + a^2\omega_1^2\sin^2(\omega_1x_1)\cos^2(\omega_3x_3) + a^2\omega_3^2\cos^2(\omega_1x_1)\sin^2(\omega_3x_3)}}. \tag{7}$$

Inside the considered solid, we assume that the displacement $\mathbf{u}$, strain $\mathbf{E}$ and stress $\boldsymbol{\Sigma}$ satisfy the fundamental equations of linear elasticity

$$\boldsymbol{\Sigma} = \mathbb{C} : \mathbf{E}, \quad \nabla_{\mathbf{x}}\cdot\boldsymbol{\Sigma} = \mathbf{0}, \quad \mathbf{E} = (\nabla_{\mathbf{x}}\mathbf{u} + \nabla^t_{\mathbf{x}}\mathbf{u})/2. \tag{8}$$

By writing $(8)_{1,2}$, we mean that the solid is elastically anisotropic and free of body force. The superscript t is used to designate the transpose. The fourth order tensor $\mathbb{C}$ in $(8)_1$ is called the elasticity tensor. When the surface is flat $(\varepsilon = 0)$, it is straightforward to verify that the trivial solution

$$\boldsymbol{\Sigma} = \boldsymbol{\Sigma}^0, \quad \mathbf{E} = \mathbf{E}^0 = \mathbb{C}^{-1} : \boldsymbol{\Sigma}^0, \quad \mathbf{u} = \mathbf{u}^0 = \mathbf{E}^0\cdot\mathbf{x} \tag{9}$$

satisfies all the elasticity Eq. (8) and the stress boundary conditions (6). When ε is nonzero and sufficiently small, i.e. $0 < \varepsilon \ll 1$, we treat the half space as if it has a flat boundary and account for the undulating effect of the surface by modifying the boundary equations at $x_2 = 0$ using (7). In the framework of perturbation analysis, we express the solution fields $\boldsymbol{\Sigma}$, $\mathbf{u}$, $\mathbf{E}$ and the normal vector $\mathbf{n}$ under the form

$$\begin{aligned} \boldsymbol{\Sigma}(\mathbf{x},\varepsilon) &= \boldsymbol{\Sigma}^0(\mathbf{x}) + \varepsilon\boldsymbol{\Sigma}^1(\mathbf{x}) + \varepsilon^2\boldsymbol{\Sigma}^2(\mathbf{x}) + \cdots \\ \mathbf{u}(\mathbf{x},\varepsilon) &= \mathbf{u}^0(\mathbf{x}) + \varepsilon\mathbf{u}^1(\mathbf{x}) + \varepsilon^2\mathbf{u}^2(\mathbf{x}) + \cdots \\ \mathbf{E}(\mathbf{x},\varepsilon) &= \mathbf{E}^0(\mathbf{x}) + \varepsilon\mathbf{E}^1(\mathbf{x}) + \varepsilon^2\mathbf{E}^2(\mathbf{x}) + \cdots \\ \mathbf{n}(\mathbf{x},\varepsilon) &= \mathbf{n}^0(\mathbf{x}) + \varepsilon\mathbf{n}^1(\mathbf{x}) + \varepsilon^2\mathbf{n}^2(\mathbf{x}) + \cdots \end{aligned} \tag{10}$$

Inserting (10) into (8) and carrying out the order analysis of ε, we can demonstrate that any group $(i \geqslant 1)$ composed of three elements $\boldsymbol{\Sigma}^i(\mathbf{x})$, $\mathbf{E}^i(\mathbf{x})$, $\mathbf{u}^i(\mathbf{x})$ satisfy (8) and the conditions

$$\begin{aligned} &\boldsymbol{\Sigma}^0\cdot\mathbf{n}^0 = \mathbf{0}, \quad \boldsymbol{\Sigma}^1\cdot\mathbf{n}^0 + \boldsymbol{\Sigma}^0\cdot\mathbf{n}^1 = \mathbf{0}, \quad \boldsymbol{\Sigma}^0\cdot\mathbf{n}^2 + \boldsymbol{\Sigma}^1\cdot\mathbf{n}^1 + \boldsymbol{\Sigma}^2\cdot\mathbf{n}^0 \\ &\quad = \mathbf{0},\ldots \quad \lim_{x_2\to\infty}\boldsymbol{\Sigma}^i = \mathbf{0}, \quad \forall i \geqslant 1. \end{aligned} \tag{11}$$

The final solution $\boldsymbol{\Sigma}$, $\mathbf{E}$, $\mathbf{u}$ can be constructed by solving successively problems related to order ε^i: $\boldsymbol{\Sigma}^i$, $\mathbf{E}^i$, $\mathbf{u}^i$. If the perturbation parameter ε is small, the consideration of up to the first order of ε can give satisfactory results and will be adopted in the following analysis. From (2), we can calculate the two leading terms $\mathbf{n}^0$ and $\mathbf{n}^1$ of $\mathbf{n}$

$$\begin{aligned} \mathbf{n}^0(\mathbf{x}) = -\mathbf{e}_2, \quad \mathbf{n}^1(\mathbf{x}) = &-\tilde{\omega}_1\sin(\omega_1x_1)\cos(\omega_3x_3)\mathbf{e}_1 \\ &-\tilde{\omega}_3\cos(\omega_1x_1)\sin(\omega_3x_3)\mathbf{e}_3 \end{aligned} \tag{12}$$

with $\tilde{\omega}_1$, $\tilde{\omega}_3$ being the normalized wavenumber

$$\tilde{\omega}_1 = \omega_1/\omega, \quad \tilde{\omega}_3 = \omega_3/\omega. \tag{13}$$

Inserting the formulae of $\mathbf{n}^0$ and $\mathbf{n}^1$ in (12) into the second equation of (11) yields

$$\begin{aligned} 2\mathbf{t}^1_2 + \left(\tilde{\omega}_1\mathbf{t}^0_1 + \tilde{\omega}_3\mathbf{t}^0_3\right)\sin(\omega_1x_1 + \omega_3x_3) + \left(\tilde{\omega}_1\mathbf{t}^0_1 - \tilde{\omega}_3\mathbf{t}^0_3\right)\sin(\omega_1x_1 \\ - \omega_3x_3) = \mathbf{0}. \end{aligned} \tag{14}$$

By writing $\mathbf{t}^i_j$ in (14), we mean the stress vector on the face normal to $\mathbf{e}_j$ associated to the stress state $\boldsymbol{\Sigma}^i$

$$\mathbf{t}^i_j = \boldsymbol{\Sigma}^i\cdot\mathbf{e}_j = \Sigma^i_{1j}\mathbf{e}_1 + \Sigma^i_{2j}\mathbf{e}_2 + \Sigma^i_{3j}\mathbf{e}_3, \quad j = 1,2,3;\ i = 1,2,\ldots \tag{15}$$

From (5) and (14), we can see that the distribution $\mathbf{t}^1_2$ is a sinusoidal function of x_1, x_3 and lies on the plane Ox_1x_3. The equation of $\mathbf{t}^1_2$ in (14) serves as boundary conditions to determine the solution corresponding to first order of ε: $\boldsymbol{\Sigma}^1$, $\mathbf{E}^1$, $\mathbf{u}^1$ from the previously known solution $\boldsymbol{\Sigma}^0$, $\mathbf{E}^0$, $\mathbf{u}^0$. In the framework of linear elasticity, $\boldsymbol{\Sigma}^1$ must be linearly proportional to $\boldsymbol{\Sigma}^0$ via a fourth order tensor $\mathbb{L}$, which allows us to write

$$\boldsymbol{\Sigma} = (\mathbb{I} + \varepsilon\mathbb{L}) : \boldsymbol{\Sigma}^0. \tag{16}$$

The tensor $\mathbb{I}$ is the fourth-order identity tensor and the term $\mathbb{I}+\varepsilon\mathbb{L}$ appears as the stress concentration tensor. Let us consider next a special case where the stress tensor $\mathbf{\Sigma}^0$ takes Ox_1 and Ox_3 as its principal axes, or

$$\mathbf{t}_1^0 = \Sigma_{11}^0\mathbf{e}_1, \quad \mathbf{t}_3^0 = \Sigma_{33}^0\mathbf{e}_3. \tag{17}$$

All the components of the stress tensor $\mathbf{\Sigma}$ are of order ε except Σ_{11} and Σ_{33}. The maximal value of these two components are related to Σ_{11}^0 and Σ_{33}^0 via the equations

$$\begin{aligned}\Sigma_{11\,\max} &= \Sigma_{11}^0 + k_{11}\varepsilon\Sigma_{11}^0 + k_{13}\varepsilon\Sigma_{33}^0,\\ \Sigma_{33\,\max} &= \Sigma_{33}^0 + k_{31}\varepsilon\Sigma_{11}^0 + k_{33}\varepsilon\Sigma_{33}^0.\end{aligned} \tag{18}$$

where k_{11}, k_{13}, k_{31}, k_{33} are four constant factors. These factors plays an important role in the stress concentration effect and their determination is the main objective of the stress concentration problem.

2.2. Surface morphology stability problem

Let us now consider another problem which is of practical interest and most of all, it is closely linked to the stress concentration problem discussed in the previous subsection. When we stress a halfspace, a small perturbation in shape can give rise to the instability. Typically, a perturbation can be a sinusoidal of surface coordinate (2) with increasing waviness ε and the instability correspond to an decrease in the free energy $\mathcal{F}$ in response to such perturbation. The instability criteria can be expressed as follows

$$\dot{\mathcal{F}} < 0 \quad \text{if } \dot{\varepsilon} > 0. \tag{19}$$

Due to the problem periodicity in x_1, x_3, we shall next restrict our study to one period only, say the region having the area $S_0 = \lambda_1\lambda_3$ defined by

$$-\lambda_1/2 \leqslant x_1 \leqslant \lambda_1/2, \quad -\lambda_3/2 \leqslant x_3 \leqslant \lambda_3/2. \tag{20}$$

The average free energy $\mathcal{F}$ (Freund and Suresh, 2003) per period is equal to

$$\mathcal{F} = U_e + U_s, \quad U_e = \frac{1}{S_0}\int_V w_e(\mathbf{E})dV, \quad U_s = \frac{1}{S_0}\int_S \gamma_s\, dS, \tag{21}$$

where U_e and U_s are the average bulk and the surface energy per period. These two energy terms can be evaluated by integrating the strain and surface energy density w_e and γ_s over the volume V and surface S per period of the considered system. If the distribution of the surface energy density γ_s is known, the average surface energy per period is given by the formula

$$U_s = \frac{1}{S_0}\int_{S_0} \gamma_s\sqrt{1+h_{,1}^2+h_{,3}^2}\,dS. \tag{22}$$

Eq. (22) is obtained from $(21)_3$ by transforming the integral over the curved surface S into the integral over S_0, the projection of S onto the Ox_1x_3 plane. In this work, the surface energy density γ_s is assumed to be a function of the local surface orientation (see Gao (1991c)), say

$$\gamma_s = \gamma_s(h_{,1}, h_{,3}). \tag{23}$$

We remark that the Taylor development of $\gamma_s(h_{,1},h_{,3})$ and $\sqrt{1+h_{,1}^2+h_{,3}^2}$ around the origin $(h_{,1},h_{,3}) = (0,0)$ admit

$$\begin{aligned}\gamma_s(h_{,1},h_{,3}) &= \gamma_s(0,0) + h_{,1}\frac{\partial\gamma_s}{\partial h_{,1}}(0,0) + h_{,3}\frac{\partial\gamma_s}{\partial h_{,3}}(0,0)\\ &\quad + \frac{h_{,1}^2}{2}\frac{\partial^2\gamma_s}{\partial h_{,1}^2}(0,0) + \frac{h_{,3}^2}{2}\frac{\partial^2\gamma_s}{\partial h_{,3}^2}(0,0) + h_{,1}h_{,3}\frac{\partial^2\gamma_s}{\partial h_{,1}\partial h_{,3}}(0,0) + \cdots\\ \sqrt{1+h_{,1}^2+h_{,3}^2} &= 1 + \frac{1}{2}\left(h_{,1}^2 + h_{,3}^2\right) + \cdots\end{aligned} \tag{24}$$

The surface anisotropy effect considered in this paper is due to its geometry only. The slope dependence of the surface energy density function γ_s is considered the same along both directions Ox_1 and Ox_3

$$\frac{\partial\gamma_s}{\partial h_{,1}}(0,0) = \frac{\partial\gamma_s}{\partial h_{,3}}(0,0), \quad \frac{\partial^2\gamma_s}{\partial h_{,1}^2}(0,0) = \frac{\partial^2\gamma_s}{\partial h_{,3}^2}(0,0). \tag{25}$$

As long as the perturbation amplitude is sufficiently small, the surface energy U_s can be approximated by the expression

$$U_s \simeq \gamma\left[\frac{\gamma_s(0,0)}{\gamma} + \frac{\varepsilon^2}{8}\right], \quad \gamma = \gamma_s(0,0) + \frac{\partial^2\gamma_s}{\partial h_{,1}^2}(0,0) \tag{26}$$

with γ being the reduced surface energy density. The temporal derivative of the surface energy $\dot{U}_s$ is given by

$$\dot{U}_s = \frac{1}{4}\gamma\varepsilon\dot{\varepsilon}. \tag{27}$$

The variation of the strain energy due to the film evolution is derived in a general way in Freund and Suresh (2003). In the absence of body force and work exchange between the materials and its surroundings, the formula for $\dot{U}_e$ is the following

$$\dot{U}_e = \frac{1}{S_0}\int_{S_0} w_e v_n\, dS. \tag{28}$$

where v_n is the normal velocity at the surface due to the perturbation. Given that the perturbation is described by (2), v_n takes the form as

$$v_n = -\frac{\dot{\varepsilon}}{\omega}\cos(\omega_1x_1)\cos(\omega_3x_3). \tag{29}$$

Next, assuming that we have solved the stress concentration problem described in the previous section and found the stress and strain field solution $\mathbf{\Sigma}$ and $\mathbf{E}$, the bulk strain energy density w_e is determined by the expression

$$2w_e = \mathbf{\Sigma} : \mathbf{E} \simeq \mathbf{\Sigma}^0 : \mathbf{E}^0 + 2\varepsilon\mathbf{\Sigma}^0 : \mathbf{E}^1. \tag{30}$$

The temporal derivative $\dot{U}_e$ becomes

$$\dot{U}_e = -\frac{\dot{\varepsilon}\varepsilon}{\omega}I, \tag{31}$$

in which I denotes the following integral

$$I = \frac{1}{S_0}\int_{S_0}\mathbf{\Sigma}^0 : \mathbf{E}^1\cos(\omega_1x_1)\cos(\omega_3x_3)dS. \tag{32}$$

The instability criteria (19) is now simplified into

$$\omega\gamma < 4I \quad \text{or} \quad \lambda > \lambda_{cr} = \frac{\pi\gamma}{2I}. \tag{33}$$

The quantity λ_{cr} in (33) is called the critical wavelength. In this paper, we also consider a special case where the waviness is in one direction only, say $\omega_3 = 0$ and $\omega = \omega_1 \neq 0$. The surface profile function h has the following form:

$$h(x_1, x_3) = a\cos\omega x_1. \tag{34}$$

The area S_0 to compute the average surface energy and average energy is reduced to a rectangular area having the length $\lambda = \lambda_1$ along the direction Ox_1 and unit length along the direction Ox_3. The surface energy is a scalar function of the slope $h_{,1}$, i.e. $\gamma_s = \gamma_s(h_{,1})$. In such situation, U_s and its time derivative $\dot{U}_s$ become

$$U_s = \gamma\left[\frac{\gamma_s(0)}{\gamma} + \frac{\varepsilon^2}{4}\right], \quad \dot{U}_s = \frac{1}{2}\gamma\varepsilon\dot{\varepsilon} \tag{35}$$

with $\gamma = \gamma_s(0) + \gamma_s''(0)$. On the other hand, the temporal derivative of the strain energy $\dot{U}_e$ is written in the form as

$$\dot{U}_e = -\frac{\dot{\varepsilon}\varepsilon}{\omega} I, \quad \text{with } I = \frac{1}{\lambda}\int_{-\lambda/2}^{+\lambda/2} \boldsymbol{\Sigma}^0 : \mathbf{E}^1 \cos(\omega x_1) dx_1. \tag{36}$$

The instability criteria for the one dimensionally undulating surface now reads

$$\omega\gamma < 2I \quad \text{or} \quad \lambda > \lambda_{cr} = \frac{\pi\gamma}{I}. \tag{37}$$

As we can see from the instability criteria (33) and (37), the most important work in the instability analysis is now the evaluation of the integral I specified by Eqs. (32) and (36). To facilitate the calculation of integral I, we rewrite the term $\boldsymbol{\Sigma}^0 : \mathbf{E}^1$ in another form

$$\boldsymbol{\Sigma}^0 : \mathbf{E}^1 = \mathbf{t}_1^0 \cdot \frac{\partial \mathbf{u}^1}{\partial x_1} + \mathbf{t}_2^0 \cdot \frac{\partial \mathbf{u}^1}{\partial x_2} + \mathbf{t}_3^0 \cdot \frac{\partial \mathbf{u}^1}{\partial x_3}. \tag{38}$$

Due to the form of $\boldsymbol{\Sigma}^0$ given by (5), one can deduce that $\mathbf{t}_2^0 = \mathbf{0}$ and simplify Eq. (38) into the expression

$$\boldsymbol{\Sigma}^0 : \mathbf{E}^1 = \mathbf{t}_1^0 \cdot \frac{\partial \mathbf{u}^1}{\partial x_1} + \mathbf{t}_3^0 \cdot \frac{\partial \mathbf{u}^1}{\partial x_3}. \tag{39}$$

2.3. The Stroh formalism

Let us introduce first the elements of the Stroh formalism, an important tool to solve our problems in anisotropic elasticity. We rewrite the Hooke's law (8) for anisotropic materials in the following form:

$$\mathbf{t}_i = \mathbf{C}_{ij} \frac{\partial \mathbf{u}}{\partial x_j}, \quad i,j = 1,2,3, \tag{40}$$

where $\mathbf{t}_i$, $\mathbf{u}$ are respectively the stress vector on the face normal to Ox_i and the displacement vector. The square matrices $\mathbf{C}_{ij}$ are made by rearranging the components of the fourth-order elasticity tensor $\mathbb{C}$ as follows:

$$\mathbf{C}_{ij} = \begin{bmatrix} C_{i11j} & C_{i12j} & C_{i13j} \\ C_{i21j} & C_{i22j} & C_{i23j} \\ C_{i31j} & C_{i32j} & C_{i33j} \end{bmatrix}, \quad i,j = 1,2,3. \tag{41}$$

Due to the symmetry of $\mathbb{C}$, we also must have the relation $\mathbf{C}_{ij} = \mathbf{C}_{ji}^t$. In the absence of body force, the Navier equations in anisotropic elasticity are then reduced to

$$\mathbf{C}_{ij} \frac{\partial^2 \mathbf{u}}{\partial x_i \partial x_j} = \mathbf{0}. \tag{42}$$

For generalized plane strain problems in the plane Ox_1x_2, Stroh (1958, 1962) investigated a special form of the displacement field

$$\mathbf{u} = \mathbf{a} f(z), \quad z = x_1 + \xi x_2, \tag{43}$$

where ξ is a constant to be determined. Substituting into (42), we obtain the matrix equation

$$(\mathbf{C}_{11} + \xi(\mathbf{C}_{12} + \mathbf{C}_{21}) + \xi^2 \mathbf{C}_{22})\mathbf{a} = \mathbf{0}. \tag{44}$$

The matrices $\mathbf{C}_{11}$, $\mathbf{C}_{12}$, $\mathbf{C}_{21}$, $\mathbf{C}_{22}$ are respectively equivalent to $\mathbf{Q}$,$\mathbf{R}$,$\mathbf{R}^t$,$\mathbf{T}$ in Ting (1996). To obtain a non trivial solution $\mathbf{a}$, ξ must be a root of the sextic equation

$$|\mathbf{C}_{11} + \xi(\mathbf{C}_{12} + \mathbf{C}_{21}) + \xi^2 \mathbf{C}_{22}| = 0. \tag{45}$$

Eq. (45) is called the characteristic equation and has no real roots. We denote by ξ_1, ξ_2, ξ_3 the three distinct complex roots of (45) with positive imaginary parts and $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$ the corresponding eigenvectors. A real general solution for the displacement is derived by superposing the eigensolutions as follows

$$\mathbf{u} = 2\Re\left\{\sum_{\alpha=1}^{3} \mathbf{a}_\alpha f_\alpha(z_\alpha)\right\}, \quad \text{or} \quad \mathbf{u} = 2\Re\{\mathbf{A}\mathbf{f}(z)\} \quad \text{for brevity} \tag{46}$$

in which

$$\mathbf{A} = [\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3], \quad \mathbf{f}(z) = [f_1(z_1), f_2(z_2), f_3(z_3)]^t, \quad z_i = x_1 + \xi_i x_2. \tag{47}$$

The two notations $\Re\{a\}$ and $\Im\{a\}$ denote respectively the real and imaginary part of the complex number a. Next, we define the two matrices $\mathbf{P}$ and $\mathbf{B}$ by the expressions

$$\mathbf{P} = \langle \xi_i \rangle = \mathrm{diag}[\xi_1, \xi_2, \xi_3], \quad \mathbf{B} = \mathbf{C}_{21}\mathbf{A} + \mathbf{C}_{22}\mathbf{A}\mathbf{P}. \tag{48}$$

In $(48)_1$, we adopt the notation $\langle \xi_i \rangle$ for diagonal matrix whose elements are ξ_i with i running from 1 to 3. Next, we introduce two Hermitian tensors in 2D anisotropic elasticity: the impedance tensor $\mathbf{M}$ and its inverse $\mathbf{M}^*$ which will be used in the later analysis. By definition, $\mathbf{M}$ and $\mathbf{M}^*$ are given by the expressions

$$\mathbf{M} = -i\mathbf{B}\mathbf{A}^{-1}, \quad \mathbf{M}^* = i\mathbf{A}\mathbf{B}^{-1}. \tag{49}$$

For monoclinic materials whose plane of symmetry coincides with the plane Ox_1x_2, the matrix $\mathbf{M}^*$ depends on two roots ξ_1 and ξ_2 only (see Ting (1996))

$$\mathbf{M}^* = \begin{bmatrix} s'_{11}\Im\{\xi_1 + \xi_2\} & -i(s'_{11}\xi_1\xi_2 - s'_{12}) & 0 \\ -i(s'_{12} - s'_{11}\bar{\xi}_1\bar{\xi}_2) & s'_{11}\Im\{\xi_1\xi_2(\bar{\xi}_1 + \bar{\xi}_2)\} & 0 \\ 0 & 0 & \sqrt{s'_{44}s'_{55} - s'_{45}s'_{45}} \end{bmatrix}. \tag{50}$$

The computation of ξ_1 and ξ_2 in (50) is now based on the quartic characteristic equation

$$s'_{11}\xi^4 - 2s'_{16}\xi^3 + (s'_{12} + s'_{66})\xi^2 - 2s'_{26}\xi + s'_{22} = 0, \tag{51}$$

and the remaining root ξ_3 is determined by a separate quadratic equation

$$s'_{55}\xi^2 - 2s'_{45}\xi + s'_{44} = 0. \tag{52}$$

The components $s'_{\alpha\beta}$, $\alpha, \beta = 1, 2, \ldots, 6$ in (50)–(52) are the reduced elastic compliances associated to a plane strain problem (Ting, 1996). They are related to the usual elastic compliances $s_{\alpha\beta}$ in Voigt notation by the following expression:

$$s'_{\alpha\beta} = s_{\alpha\beta} - \frac{s_{\alpha 3}s_{3\beta}}{s_{33}}. \tag{53}$$

Another matrix also needed for the later analysis is the matrix $\mathbf{N}$ defined by

$$\mathbf{N} = \mathbf{B}\mathbf{P}\mathbf{B}^{-1}. \tag{54}$$

For monoclinic materials, the matrix $\mathbf{B}$ takes the form as

$$\mathbf{B} = \begin{bmatrix} -k_1\xi_1 & -k_2\xi_2 & 0 \\ k_1 & k_2 & 0 \\ 0 & 0 & -k_3, \end{bmatrix}, \tag{55}$$

where k_1, k_2, k_3 are constants that depend on the normalization of matrix $\mathbf{B}$ (Ting, 1996). The matrix $\mathbf{N}$ is independent of k_1, k_2, k_3 and has a simple expression

$$\mathbf{N} = \mathbf{B}\mathbf{P}\mathbf{B}^{-1} = \begin{bmatrix} \xi_1 + \xi_2 & \xi_1\xi_2 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & \xi_3 \end{bmatrix}. \tag{56}$$

For general anisotropy, the matrices $\mathbf{M}^*$ and $\mathbf{N}$ can be determined via the Barnett–Lothe tensors $\mathbf{L}$, $\mathbf{S}$, $\mathbf{H}$ (see Barnett and Lothe, 1973; Ting, 1996)

$$\mathbf{M}^* = \mathbf{L}^{-1} - i\mathbf{S}\mathbf{L}^{-1}, \quad \mathbf{N} = -i\left[\mathbf{C}_{12}\mathbf{C}_{22}^{-1}\mathbf{C}_{21} - \mathbf{C}_{11}\right]\mathbf{M}^* - \mathbf{C}_{12}\mathbf{C}_{22}^{-1}, \tag{57}$$

The Barnett–Lothe tensors $\mathbf{L}$, $\mathbf{S}$, $\mathbf{H}$ can be computed numerically by the integrals

$$\mathbf{L} = -\frac{1}{\pi}\int_0^{\pi}\left[\mathbf{C}_{12}(\theta)\mathbf{C}_{22}^{-1}(\theta)\mathbf{C}_{21}(\theta) - \mathbf{C}_{11}(\theta)\right]d\theta,$$
$$\mathbf{S} = -\frac{1}{\pi}\int_0^{\pi}\mathbf{C}_{22}^{-1}(\theta)\mathbf{C}_{21}(\theta)d\theta, \quad \mathbf{H} = \frac{1}{\pi}\int_0^{\pi}\mathbf{C}_{22}^{-1}(\theta)d\theta, \tag{58}$$

in which the matrices $\mathbf{C}_{11}(\theta)$, $\mathbf{C}_{12}(\theta)$ and $\mathbf{C}_{22}(\theta)$ are defined by

$$\mathbf{C}_{11}(\theta) = \mathbf{C}_{11}\cos^2\theta + (\mathbf{C}_{12}+\mathbf{C}_{21})\sin\theta\cos\theta + \mathbf{C}_{22}\sin^2\theta,$$
$$\mathbf{C}_{12}(\theta) = \mathbf{C}_{12}\cos^2\theta + (\mathbf{C}_{22}-\mathbf{C}_{11})\sin\theta\cos\theta - \mathbf{C}_{21}\sin^2\theta, \tag{59}$$
$$\mathbf{C}_{22}(\theta) = \mathbf{C}_{22}\cos^2\theta - (\mathbf{C}_{12}+\mathbf{C}_{21})\sin\theta\cos\theta + \mathbf{C}_{11}\sin^2\theta.$$

3. One dimensional wavy surface

3.1. Stress concentration problem

In this section, we consider the case where the our surface is undulating along one direction only, say Ox_1. The values of the wavenumbers ω_1, ω_3 and ω are expressed as follows:

$$\omega_3 = 0, \quad \omega = \omega_1 \neq 0. \tag{60}$$

As discussed in Section 2, in order to determine the solution $\mathbf{u}^1$, $\mathbf{\Sigma}^1$, $\mathbf{E}^1$ corresponding to the first order of ε, we consider a half space subject to the stress boundary conditions at $x_2 = 0$ and at infinity:

$$\mathbf{\Sigma}^1\cdot\mathbf{e}_2 = \mathbf{t}_2^1 = -\sin(\omega x_1)\mathbf{t}_1^0 \quad \text{when } x_2 = 0, \quad \lim_{x_2\to\infty}\mathbf{\Sigma}^1 = \mathbf{0}, \tag{61}$$

where $\mathbf{t}_1^0$ is a known vector equal to

$$\mathbf{t}_1^0 = \mathbf{\Sigma}^0\cdot\mathbf{e}_1. \tag{62}$$

We note that the boundary conditions $(61)_1$ are derived from (14) using (60). With the Stroh formalism, one can verify that the vector function $\mathbf{f}(z)$ in the form

$$\mathbf{f}(z) = \frac{1}{2\omega}\langle e^{i\omega z_i}\rangle\mathbf{B}^{-1}\mathbf{t}_1^0 \tag{63}$$

correspond to a stress field satisfying the boundary conditions (61). Indeed, the stress components associated to the given vector function $\mathbf{f}(z)$ become

$$\mathbf{t}_1^1 = -\Re\{i\mathbf{BP}\langle e^{i\omega z_i}\rangle\mathbf{B}^{-1}\}\mathbf{t}_1^0,$$
$$\mathbf{t}_2^1 = \Re\{i\mathbf{B}\langle e^{i\omega z_i}\rangle\mathbf{B}^{-1}\}\mathbf{t}_1^0, \tag{64}$$
$$\mathbf{t}_3^1 = \Re\{i[\mathbf{C}_{31}\mathbf{A}+\mathbf{C}_{32}\mathbf{AP}]\langle e^{i\omega z_i}\rangle\mathbf{B}^{-1}\}\mathbf{t}_1^0.$$

Due to the fact that the imaginary parts of z_i are all positive, all stress components decay as $x_2\to\infty$, so that the second condition of (61) is fulfilled. Next, we also remark that the diagonal matrix $\langle\exp(i\omega z_i)\rangle$ at $x_2 = 0$ is reduced to a scalar matrix

$$\langle\exp(i\omega z_i)\rangle = \exp(i\omega x_1)\mathbf{I} \text{ at } x_2 = 0, \quad \mathbf{I} = \mathrm{diag}[1,1,1]. \tag{65}$$

After substituting (65) into (64), we recover the first condition of (61). To derive the stress field on the surface which is first order accurate in ε, we substitute $x_2 = 0$ in (64) and use (10) to obtain

$$\mathbf{t}_1 \simeq [\mathbf{I} - \varepsilon\Re\{i\mathbf{N}e^{i\omega x_1}\}]\mathbf{t}_1^0,$$
$$\mathbf{t}_2 \simeq -\varepsilon\sin(\omega x_1)\mathbf{t}_1^0, \tag{66}$$
$$\mathbf{t}_3 \simeq \mathbf{t}_3^0 + \varepsilon\Re\{[\mathbf{C}_{31}\mathbf{M}^* + \mathbf{C}_{32}\mathbf{M}^*\mathbf{N}]e^{i\omega x_1}\}\mathbf{t}_1^0,$$

In a particular case where the uniform remote stress field whose principal axes coincide with Ox_1 and Ox_3 as specified in Eq. (17). The maximal stress $\Sigma_{11\max}$ and $\Sigma_{33\max}$ on the faces normal to Ox_1 and Ox_3 become

$$\Sigma_{11\max} = \Sigma_{11}^0(1 + |N_{11}|\varepsilon),$$
$$\Sigma_{33\max} = \Sigma_{33}^0 + |\mathbf{e}_3^t(\mathbf{C}_{31}\mathbf{M}^* + \mathbf{C}_{32}\mathbf{M}^*\mathbf{N})\mathbf{e}_1|\Sigma_{11}^0\varepsilon. \tag{67}$$

In (67), $|a|$ denotes the modulus of the complex number a. The factors k_{ij} with i, $j = 1$, 3 according to the definition (18) are the followings

$$k_{11} = |N_{11}|, \quad k_{31} = |\mathbf{e}_3^t(\mathbf{C}_{31}\mathbf{M}^* + \mathbf{C}_{32}\mathbf{M}^*\mathbf{N})\mathbf{e}_1|,$$
$$k_{13} = k_{33} = 0. \tag{68}$$

Since the matrix $\mathbf{N}$ and $\mathbf{M}^*$ for general anisotropy can be computed numerically via the Barnett–Lothe integrals (57)–(59), the factors k_{ij} can also be calculated numerically. For monoclinic materials whose plane of symmetry coincide with Ox_1x_2, the form of the matrix $\mathbf{N}$ is given by (56). The coefficient k_{11} can now be expressed by an astonishingly simple equation

$$k_{11} = |\xi_1 + \xi_2|. \tag{69}$$

Thanks to the form of $\mathbf{M}^*$ in (50) and $\mathbf{N}$ in (56), the analytical formula of k_{31} can also be obtained in terms the roots ξ_i, the reduced elastic compliances s'_{ij} and the elasticity tensor components. We also remark that for orthotropic materials with planes of symmetry coinciding with Ox_1x_2, Ox_1x_3, Ox_2x_3, the characteristic Eq. (51) to find ξ_1, ξ_2 is reduced to a quadratic equation of ξ^2

$$s'_{11}\xi^4 + (2s'_{12} + s'_{66})\xi^2 + s'_{22} = 0. \tag{70}$$

Thus the roots ξ_1, ξ_2 with positive imaginary parts for this case can be determined explicitly from the relation

$$\xi_i^2 = \frac{2s'_{12}+s'_{66}}{2s'_{11}} \pm \sqrt{\left(\frac{2s'_{12}+s'_{66}}{2s'_{11}}\right)^2 - \frac{s'_{22}}{s'_{11}}}, \quad i = 1,2. \tag{71}$$

With ξ_3 given in (52), it is possible to express the matrices $\mathbf{M}^*$ and $\mathbf{N}$ purely in terms of elastic constants. For isotropic materials with the Young's modulus E and the Poisson ratio ν, the reduced elastic compliances become $s'_{11} = (1-\nu^2)/E, s'_{12} = -\nu(1+\nu)/E, s'_{16} = 2(1+\nu)/E$. In such situation, we can easily calculate the roots $\xi_1 = \xi_2 = \xi_3 = i$ and the stress concentration factors

$$k_{11} = 2, \quad k_{31} = 2\nu, \tag{72}$$

which is in agreement with the results of Gao (1991a).

Next, based on (68) and (69), we calculate the coefficients k_{11} and k_{31} for some cubic materials with elastic constants c_{11}, c_{12}, c_{44} in Voigt notation (see Table 1). The degree of departure from isotropy is characterized by the anisotropy ratio ARdefined as

$$AR = \frac{2c_{44}}{c_{11} - c_{12}}. \tag{73}$$

For the given materials in Table 1, AR ranges from 0.7 to 3.21 and $AR = 1$ for isotropic materials. As for the factor k_{11}, the numerical results show that the stress concentration for the materials with $AR < 1$ (the factor $k_{11} > 2$) is more critical than those with $AR > 1$ (the factor $k_{11} < 2$).

3.2. Surface stability problem

Having found $\mathbf{f}(z)$ in (63), we develop the displacement field $\mathbf{u}^1$ corresponding the first order of ε as follows:

$$\mathbf{u}^1 = \frac{1}{\omega}\Re\left\{\mathbf{A}\langle e^{i\omega z_i}\rangle\mathbf{B}^{-1}\right\}\mathbf{t}_1^0. \tag{74}$$

Differentiating $\mathbf{u}^1$ with respect to the spatial variable x_1, x_2, x_3 and substituting $x_2 = 0$, we obtain the value of $\mathbf{\Sigma}^0:\mathbf{E}^1$ in (39) on the surface, for example

$$\mathbf{\Sigma}^0:\mathbf{E}^1 = \mathbf{t}_1^0.\Re\{e^{i\omega x_1}\mathbf{M}^*\}\cdot\mathbf{t}_1^0. \tag{75}$$

Denoting $\mathbf{D}$ as the real part of $\mathbf{M}^*$, or $\mathbf{D} = \Re\{\mathbf{M}^*\}$, the integral I defined by (36) can be written as

Table 1
Factors k_{11} and k_{31} for some cubic materials ($\tilde{\omega}_1 = 1, \tilde{\omega}_3 = 0$).

Name	AR	c_{11} (GPa)	c_{44} (GPa)	c_{12} (GPa)	ξ_1	ξ_2	k_{11}	k_{31}
Cu	3.21	168.4	75.4	121.4	0.77 + 0.64i	−0.77 + 0.64i	1.28	0.54
Ag	3.01	124.0	46.1	93.4	0.77 + 0.64i	−0.77 + 0.64i	1.28	0.55
Pt	1.59	346.7	76.5	250.7	0.57 + 0.82i	−0.57 + 0.82i	1.66	0.69
Cr	0.70	339.8	99.0	58.6	0.60i	1.61i	2.21	0.33
Cb	0.78	240.2	28.2	125.6	0.45i	2.22i	2.67	0.92

$$I = \frac{\omega}{2\pi}\int_{-\lambda/2}^{+\lambda/2} \boldsymbol{\Sigma}^0 : \mathbf{E}^1 \cos(\omega x_1)dx_1 = \frac{1}{2}\mathbf{t}_1^0 \cdot \mathbf{D} \cdot \mathbf{t}_1^0. \tag{76}$$

The instability criteria (37) combined with (76) becomes

$$\lambda > \lambda_{cr} = \frac{2\pi\gamma}{\mathbf{t}_1^0 \cdot \mathbf{D} \cdot \mathbf{t}_1^0}. \tag{77}$$

which is in agreement with the results of Gao (1991c). In a particular case where $\boldsymbol{\Sigma}^0$ take the principal axes to be Ox_1 and Ox_3 (see Eq. (17)), the value of the critical wavelength λ_{cr} in (77) is simplified into the form

$$\lambda_{cr} = \frac{2\pi\gamma}{D_{11}\left(\Sigma_{11}^0\right)^2}. \tag{78}$$

Interestingly, the critical wavelength λ_{cr} according to (78) is independent of the lateral stress Σ_{33}^0 acting in the direction Ox_3. For general anisotropic materials, D_{11} can be computed numerically via the Barnett–Lothe integrals. For monoclinic materials whose plane of symmetry coincide with Ox_1x_2, from (50), D_{11} admits a simple analytic expression

$$D_{11} = s'_{11}\Im(\xi_1 + \xi_2). \tag{79}$$

For isotropic material, using the fact that $s'_{11} = (1-\nu^2)/E$ and $\xi_1 = \xi_2 = i$, the criteria (78) is in agreement with the results issued from Gao (1991b)

$$\lambda_{cr} = \frac{\pi E\gamma}{(1-\nu^2)\left(\Sigma_{11}^0\right)^2}. \tag{80}$$

For orthotropic materials taking both Ox_1x_2 and Ox_2x_3 as their planes of symmetry, the characteristic equation to determine the ξ_1, ξ_2 is reduced to a quadratic equation (70) of ξ^2. One can also demonstrate that if ξ_1, ξ_2 are roots with positive imaginary parts then the following relation must verify

$$\xi_1 + \xi_2 = \Im(\xi_1 + \xi_2). \tag{81}$$

Combined with (69), one can prove that $D_{11} = s'_{11}k_{11}$ and the instability criteria is directly related to the factor k_{11}

$$\lambda_{cr} = \frac{2\pi\gamma}{s'_{11}k_{11}\left(\Sigma_{11}^0\right)^2}. \tag{82}$$

As the consequences, the values of k_{11} for some cubic materials given in Table 1 can be used directly in the instability criteria.

4. Two dimensional wavy surface

We consider now the general case where the surface is undulating along two directions 1 and 3, e.g. $\omega_1 \neq 0$, $\omega_3 \neq 0$. In order to determine the solution $\mathbf{u}^1$, $\boldsymbol{\Sigma}^1$, $\mathbf{E}^1$ corresponding to the first order of ε, we consider a half space subject to the stress boundary conditions specified in (14). Before proceeding, we solve first the auxiliary problem presented in the following subsection.

4.1. Auxiliary problem and solution

The aim of this subsection is to find the solution $\mathbf{u}'^1$, $\boldsymbol{\Sigma}'^1$, $\mathbf{E}'^1$ that satisfies the following boundary conditions:

$$\boldsymbol{\Sigma}'^1 \cdot \mathbf{e}_2 = \mathbf{t}_2'^1 = -\frac{1}{2}\left(\tilde{\omega}_1\mathbf{t}_1^0 + \chi\tilde{\omega}_3\mathbf{t}_3^0\right)\sin(\omega_1x_1 + \chi\omega_3x_3)$$
$$\text{when } x_2 = 0,\ \lim_{x_2\to\infty}\boldsymbol{\Sigma}'^1 = \mathbf{0}, \tag{83}$$

The parameter χ in (83) can take either of the two values $\chi = +1$ or $\chi = -1$. To solve the problem with the boundary conditions (83), one can use directly the results of the surface wave theory: finding the displacement vector as a complex exponential function of the coordinates. The form of latter is a special form of surface wave displacement (see e.g.Ting, 1996; Tanuma, 2007) when the phase speed is equal to 0. On the other hand, one can reduce the problem to 2D case with a coordinate transformation and use the results of Section 3. Such methods have been used in the works of Gao and Suo (2003) and Gao (2003). They also showed that problems with periodic tractions can also be solved since any periodic function is equivalent to a Fourier series.

We study a special complex form of $\mathbf{u}'^1$ that satisfies the Navier equation for anisotropic elasticity (42)

$$\mathbf{u}'^1 = \mathbf{a}'e^{i\omega(\tilde{\omega}_1x_1 + \chi\tilde{\omega}_3x_3 + \xi'x_2)} \tag{84}$$

with $\mathbf{a}'$ being a constant complex vector. Next, we also define the matrix $\mathbf{C}'_{11}, \mathbf{C}'_{12}, \mathbf{C}'_{21}, \mathbf{C}'_{22}$ from $\mathbf{C}_{ij}$ as follows:

$$\begin{aligned}&\mathbf{C}'_{11} = \tilde{\omega}_1^2\mathbf{C}_{11} + \chi\tilde{\omega}_1\tilde{\omega}_3(\mathbf{C}_{13} + \mathbf{C}_{31}) + \tilde{\omega}_3^2\mathbf{C}_{33}, \quad \mathbf{C}'_{22} = \mathbf{C}_{22},\\ &\mathbf{C}'_{12} = \tilde{\omega}_1\mathbf{C}_{12} + \chi\tilde{\omega}_3\mathbf{C}_{32}, \quad \mathbf{C}'_{21} = \tilde{\omega}_1\mathbf{C}_{21} + \chi\tilde{\omega}_3\mathbf{C}_{23}.\end{aligned} \tag{85}$$

Inserting the formula (84) into (42) combined with (85), we obtain the equation

$$\left[\mathbf{C}'_{11} + \left(\mathbf{C}'_{12} + \mathbf{C}'_{21}\right)\xi' + \mathbf{C}'_{22}\xi'^2\right]\mathbf{a}' = \mathbf{0}. \tag{86}$$

To find a nonzero vector $\mathbf{a}'$, ξ must be a root of the equation

$$\left|\mathbf{C}'_{11} + \left(\mathbf{C}'_{12} + \mathbf{C}'_{21}\right)\xi' + \mathbf{C}'_{22}\xi'^2\right| = 0 \tag{87}$$

Here again, we encounter a characteristic sextic Eq. (87) to determine ξ' and its associated eigenvector $\mathbf{a}'$. Since this sextic equation has no real roots (Ting, 1996; Willis, 1966), we denote ξ'_1, ξ'_2, ξ'_3 as the three roots with positive imaginary parts and $\mathbf{a}'_1$, $\mathbf{a}'_2$, $\mathbf{a}'_3$ are the eigenvectors associated to them. We also define the matrix $\mathbf{A}'$, $\mathbf{P}'$ by

$$\mathbf{A}' = \left[\mathbf{a}'_1, \mathbf{a}'_2, \mathbf{a}'_3\right], \quad \mathbf{P}' = \langle\xi'_i\rangle = \mathrm{diag}\left[\xi'_1, \xi'_2, \xi'_3\right]. \tag{88}$$

Consequently, a complex solution of $\mathbf{u}'^1$ that satisfies the Navier's equation (42) can be constructed by a linear superposition. As we are interested in real solution, the following real form for the displacement field is chosen

$$\mathbf{u}'^1 = 2\Re\left\{\mathbf{A}'\langle e^{i\omega\xi'_i\cdot\mathbf{x}}\rangle\mathbf{h}\right\}. \tag{89}$$

Differentiating $\mathbf{u}'^1$ with respect to the coordinates x_1, x_2, x_3 and using the Hooke's law (40), we can find the stress vectors $\mathbf{t}_1'^1$, $\mathbf{t}_2'^1$, $\mathbf{t}_3'^1$ on the faces normal to $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$

$$\mathbf{t}_1'^1 = 2\omega\Re\left\{i\mathbf{F}'\langle e^{i\omega\xi_i'\cdot\mathbf{x}}\rangle\mathbf{h}\right\}, \quad \mathbf{t}_2'^1 = 2\omega\Re\left\{i\mathbf{B}'\langle e^{i\omega\xi_i'\cdot\mathbf{x}}\rangle\mathbf{h}\right\},$$
$$\mathbf{t}_3'^1 = 2\omega\Re\left\{i\mathbf{G}'\langle e^{i\omega\xi_i'\cdot\mathbf{x}}\rangle\mathbf{h}\right\}, \tag{90}$$

where the matrices $\mathbf{B}'$, $\mathbf{F}'$ and $\mathbf{G}'$ are defined by

$$\begin{aligned}\mathbf{F}' &= (\tilde{\omega}_1\mathbf{C}_{11} + \chi\tilde{\omega}_3\mathbf{C}_{13})\mathbf{A}' + \mathbf{C}_{12}\mathbf{A}'\mathbf{P}',\\ \mathbf{B}' &= (\tilde{\omega}_1\mathbf{C}_{21} + \chi\tilde{\omega}_3\mathbf{C}_{23})\mathbf{A}' + \mathbf{C}_{22}\mathbf{A}'\mathbf{P}',\\ \mathbf{G}' &= (\tilde{\omega}_1\mathbf{C}_{31} + \chi\tilde{\omega}_3\mathbf{C}_{33})\mathbf{A}' + \mathbf{C}_{32}\mathbf{A}'\mathbf{P}'.\end{aligned} \tag{91}$$

Due to the fact that all ξ_i' have positive imaginary parts, all stress components decay when $x_2 \to \infty$. At $x_2 = 0$, the traction $\mathbf{t}_2$ becomes

$$\mathbf{t}_2'^1 = 2\omega\Re\left\{i\mathbf{B}'\mathbf{h}e^{i(\omega_1 x_1 + \chi\omega_3 x_3)}\right\}. \tag{92}$$

As a result, by choosing conveniently the vector $\mathbf{h}$ as follows:

$$\mathbf{h} = \frac{1}{4\omega}\mathbf{B}'^{-1}\left(\tilde{\omega}_1\mathbf{t}_1^0 + \chi\tilde{\omega}_3\mathbf{t}_3^0\right), \tag{93}$$

the stress boundary conditions (83) are satisfied. The corresponding displacement field has now become

$$\mathbf{u}'^1 = \frac{1}{2\omega}\Re\left\{\mathbf{A}'\langle e^{i\omega\xi_i'\mathbf{x}}\rangle\mathbf{B}'^{-1}\right\}\left(\tilde{\omega}_1\mathbf{t}_1^0 + \chi\tilde{\omega}_3\mathbf{t}_3^0\right). \tag{94}$$

In another method, since the stress boundary condition $(83)_1$ is a function of $x_1 + \chi x_3$, the auxiliary problem can be reduced to a 2D problem by changing the frame of reference (see Appendix A).

4.2. Stress concentration problem

In the auxiliary problem, χ can take either of the two values: $\chi = +1$ and $\chi = -1$. In what follows, we label the associated notation with superscript + and −, for example

$$\begin{aligned}&\mathbf{u}'^{1+}, \boldsymbol{\Sigma}'^{1+}, \mathbf{E}'^{1+}, \mathbf{C}_{ij}'^{+}, \mathbf{A}'^{+}, \mathbf{P}'^{+}, \mathbf{B}'^{+}, \ldots \quad \text{for } \chi = +1,\\ &\mathbf{u}'^{1-}, \boldsymbol{\Sigma}'^{1-}, \mathbf{E}'^{1-}, \mathbf{C}_{ij}'^{-}, \mathbf{A}'^{-}, \mathbf{P}'^{-}, \mathbf{B}'^{-}, \ldots \quad \text{for } \chi = -1.\end{aligned} \tag{95}$$

Using the results (94) from the auxiliary problem and the linear superposition principle, one can obtain $\mathbf{u}^1$ in the form

$$\begin{aligned}\mathbf{u}^1 = \mathbf{u}'^{1+} + \mathbf{u}'^{1-} = \frac{1}{2\omega}\Re\Big\{&\mathbf{A}'^{+}\langle e^{i\omega\xi_i'^{+}\cdot\mathbf{x}}\rangle(\mathbf{B}'^{+})^{-1}\left(\tilde{\omega}_1\mathbf{t}_1^0 + \tilde{\omega}_3\mathbf{t}_3^0\right)\\ &+\mathbf{A}'^{-}\langle e^{i\omega\xi_i'^{-}\cdot\mathbf{x}}\rangle(\mathbf{B}'^{-})^{-1}\left(\tilde{\omega}_1\mathbf{t}_1^0 - \tilde{\omega}_3\mathbf{t}_3^0\right)\Big\}.\end{aligned} \tag{96}$$

The associated stress components grouped in vectors $\mathbf{t}_1^1, \mathbf{t}_2^1, \mathbf{t}_3^1$ are given below

$$\begin{aligned}\mathbf{t}_1^1 &= \frac{1}{2}\Re\{\mathbf{J}'^{+}(\tilde{\omega}_1\mathbf{t}_1^0 + \tilde{\omega}_3\mathbf{t}_3^0)e^{i(\omega_1x_1+\omega_3x_3)} + \mathbf{J}'^{-}(\tilde{\omega}_1\mathbf{t}_1^0 - \tilde{\omega}_3\mathbf{t}_3^0)e^{i(\omega_1x_1-\omega_3x_3)}\},\\ \mathbf{t}_2^1 &= -\frac{1}{2}(\tilde{\omega}_1\mathbf{t}_1^0 + \tilde{\omega}_3\mathbf{t}_3^0)\sin(\omega_1x_1+\omega_3x_3) - \frac{1}{2}(\tilde{\omega}_1\mathbf{t}_1^0 - \tilde{\omega}_3\mathbf{t}_3^0)\sin(\omega_1x_1-\omega_3x_3),\\ \mathbf{t}_3^1 &= \frac{1}{2}\Re\{\mathbf{K}'^{+}(\tilde{\omega}_1\mathbf{t}_1^0 + \tilde{\omega}_3\mathbf{t}_3^0)e^{i(\omega_1x_1+\omega_3x_3)} + \mathbf{K}'^{-}(\tilde{\omega}_1\mathbf{t}_1^0 - \tilde{\omega}_3\mathbf{t}_3^0)e^{i(\omega_1x_1-\omega_3x_3)}\}.\end{aligned} \tag{97}$$

The matrices $\mathbf{J}'^{\pm}$ and $\mathbf{K}'^{\pm}$ in (97) are equal to the matrix products

$$\mathbf{J}'^{\pm} = i\mathbf{F}'^{\pm}(\mathbf{B}'^{\pm})^{-1}, \quad \mathbf{K}'^{\pm} = i\mathbf{G}'^{\pm}(\mathbf{B}'^{\pm})^{-1}. \tag{98}$$

From the relations (91) and (98), the matrices $\mathbf{J}'^{\pm}$ and $\mathbf{K}'^{\pm}$ are given explicitly by the expressions

$$\begin{aligned}\mathbf{J}'^{\pm} &= (\tilde{\omega}_1\mathbf{C}_{11} \pm \tilde{\omega}_3\mathbf{C}_{13})\mathbf{M}'^{*\pm} + \mathbf{C}_{12}\mathbf{M}'^{*\pm}\mathbf{N}'^{\pm},\\ \mathbf{K}'^{\pm} &= (\tilde{\omega}_1\mathbf{C}_{31} \pm \tilde{\omega}_3\mathbf{C}_{33})\mathbf{M}'^{*\pm} + \mathbf{C}_{32}\mathbf{M}'^{*\pm}\mathbf{N}'^{\pm}\end{aligned} \tag{99}$$

with $\mathbf{N}'^{\pm}$ and $\mathbf{M}'^{*\pm}$ being the matrices

$$\mathbf{N}'^{\pm} = \mathbf{B}'^{\pm}\mathbf{P}'^{\pm}(\mathbf{B}'^{\pm})^{-1}, \quad \mathbf{M}'^{*\pm} = i\mathbf{A}'^{\pm}(\mathbf{B}'^{\pm})^{-1}. \tag{100}$$

We consider next a special case where the principal axes of the stress tensor coincide with Ox_1 and Ox_3 (see Eq. (17)) and the considered material has two planes of symmetry Ox_1x_2, Ox_2x_3. Due to the symmetry of the problem (see Appendix B), the stress components Σ_{11}^1, Σ_{33}^1 can be evaluated by the expression

$$\begin{aligned}\Sigma_{11}^1 &= \left[\Sigma_{11}^0\tilde{\omega}_1 J_{11}'^{+} + \Sigma_{33}^0\tilde{\omega}_3 J_{13}'^{+}\right]\cos(\omega_1x_1)\cos(\omega_3x_3),\\ \Sigma_{33}^1 &= \left[\Sigma_{11}^0\tilde{\omega}_1 K_{31}'^{+} + \Sigma_{33}^0\tilde{\omega}_3 K_{33}'^{+}\right]\cos(\omega_1x_1)\cos(\omega_3x_3),\end{aligned} \tag{101}$$

where $J_{11}'^{+}$, $J_{13}'^{+}$, $K_{31}'^{+}$ and $K_{33}'^{+}$ are the real elements of the matrices $\mathbf{J}'^{+}$ and $\mathbf{K}'^{+}$. The factor k_{ij} with $i, j = 1, 3$ according to definition (18) can be computed as

$$k_{11} = \tilde{\omega}_1 J_{11}'^{+}, \quad k_{33} = \tilde{\omega}_3 K_{33}'^{+}, \quad k_{13} = \tilde{\omega}_3 J_{13}'^{+}, \quad k_{31} = \tilde{\omega}_3 K_{31}'^{+}. \tag{102}$$

For isotropic materials with Young's modulus E and Poisson ratio ν, the matrices $\mathbf{J}'^{+}$ and $\mathbf{K}'^{+}$ can be determined explicitly (see Appendix C), which allows us to derive the factors k_{ij}. The factors k_{ij} for isotropic materials are independent of the Young's modulus E and written as follows:

$$\begin{aligned}&k_{11} = 2\tilde{\omega}_1^2(1 - \nu\tilde{\omega}_3^2), \quad k_{33} = 2\tilde{\omega}_3^2(1 - \nu\tilde{\omega}_1^2),\\ &k_{13} = 2\nu\tilde{\omega}_3^4, \quad k_{31} = 2\nu\tilde{\omega}_1^4.\end{aligned} \tag{103}$$

For general anisotropic materials, it is possible to calculate $\mathbf{J}'^{+}$ and $\mathbf{K}'^{+}$ numerically with the Barnett–Lothe integrals. As an example, we consider a special system where the material is cubic ones and the perturbation wavelength is the same for both two directions Ox_1, Ox_3 or

$$h = a\cos(\omega x_1)\cos(\omega x_3), \quad \tilde{\omega}_1 = \tilde{\omega}_3 = 1/\sqrt{2}. \tag{104}$$

The mechanical properties of the considered materials are taken to be the same as Table 1 in Section 3 and the calculated values of k_{ij} are given in Table 2. As in the one dimensional wavy surface case, we observe the different stress concentration features for materials with $AR > 1$ and $AR < 1$. The stress factors k_{ij} for materials with $AR < 1$ is more critical than those with $AR > 1$.

4.3. Surface stability problem

The integral I specified by (32) becomes

$$I = \frac{A}{S_0}\int_{S_0}\cos^2(\omega_1x_1)\cos^2(\omega_3x_3)dS = A/4, \tag{105}$$

in which the constant A is defined by the expression

$$\begin{aligned}A = \frac{1}{2}&(\tilde{\omega}_1\mathbf{t}_1^0 + \tilde{\omega}_3\mathbf{t}_3^0)\mathbf{D}'^{+}(\tilde{\omega}_1\mathbf{t}_1^0 + \tilde{\omega}_3\mathbf{t}_3^0)\\ &+\frac{1}{2}(\tilde{\omega}_1\mathbf{t}_1^0 - \tilde{\omega}_3\mathbf{t}_3^0)\mathbf{D}'^{-}(\tilde{\omega}_1\mathbf{t}_1^0 - \tilde{\omega}_3\mathbf{t}_3^0).\end{aligned} \tag{106}$$

The matrices $\mathbf{D}'^{\pm}$ are the real parts of the matrices $\mathbf{M}'^{*\pm}$. The instability criteria (33) now becomes

$$\begin{aligned}\frac{4\pi\gamma}{\lambda_{cr}} = &(\tilde{\omega}_1\mathbf{t}_1^0 + \tilde{\omega}_3\mathbf{t}_3^0)\mathbf{D}'^{+}(\tilde{\omega}_1\mathbf{t}_1^0 + \tilde{\omega}_3\mathbf{t}_3^0)\\ &+ (\tilde{\omega}_1\mathbf{t}_1^0 - \tilde{\omega}_3\mathbf{t}_3^0)\mathbf{D}'^{-}(\tilde{\omega}_1\mathbf{t}_1^0 - \tilde{\omega}_3\mathbf{t}_3^0).\end{aligned} \tag{107}$$

We consider next a special case where the principal axes of the stress tensor coincide with Ox_1 and Ox_3 and the material possesses two planes of symmetry as Ox_1x_2, Ox_2x_3. The symmetry of the prob-

Table 2
Factors k_{ij} for some cubic materials ($\tilde{\omega}_1 = \tilde{\omega}_3 = 1/\sqrt{2}$).

Name	AR	c_{11} (GPa)	c_{44} (GPa)	c_{12} (GPa)	$k_{11} = k_{33}$	$k_{13} = k_{31}$
Cu	3.21	168.4	75.4	121.4	0.63	0.07
Ag	3.01	124.0	46.1	93.4	0.66	0.08
Pt	1.59	346.7	76.5	250.7	0.94	0.17
Cr	0.70	339.8	99.0	58.6	1.29	0.10
Cb	0.78	240.2	28.2	125.6	1.70	0.28

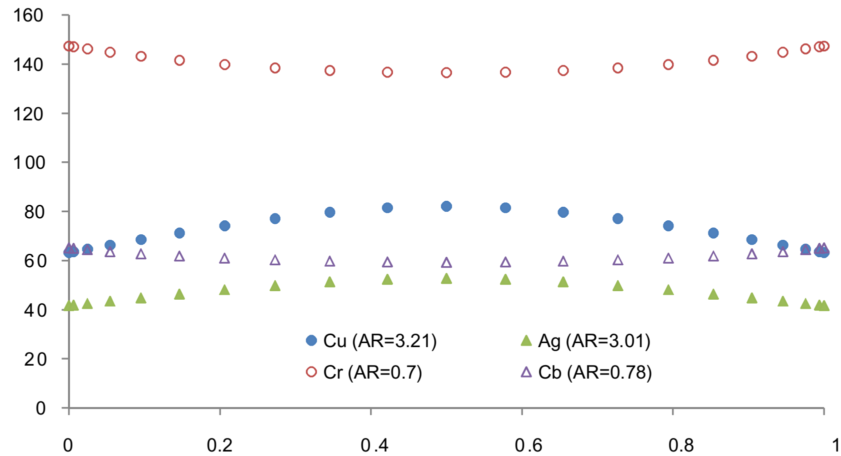

Fig. 1. The variation of $\lambda_{cr}(\Sigma^0)^2/2\pi\gamma$ (the vertical axis) in terms of $\tilde{\omega}_1^2$ (the horizontal axis) for different cubic materials in unit GPa. Due to the material symmetry, all curves are symmetric with respect to the value $\tilde{\omega}_1^2 = 1/2$.

Table 3
Summary of results.

	Generally anisotropic	Monoclinic	Isotropic
Stress factors (1D)	Eq. (68)	Eq. (69)	Eq. (72)
Stability criteria (1D)	Eqs. (77) and (78)	Eq. (79)	Eq. (80)
	Generally anisotropic	Orthotropic	Isotropic
Stress factors (2D)	Eq. (97)	Eq. (102)	Eq. (103)
Stability criteria (2D)	Eq. (107)	Eq. (108)	Eq. (110)

lem (see Appendix C) allows us to rewrite the instability criteria (33) as

$$\frac{2\pi\gamma}{\lambda_{cr}} = \tilde{\omega}_1^2 D'^{+}_{11}\left(\Sigma^0_{11}\right)^2 + (D'^{+}_{13} + D'^{+}_{31})\tilde{\omega}_1\tilde{\omega}_3\Sigma^0_{11}\Sigma^0_{33} + \tilde{\omega}_3^2 D'^{+}_{33}\left(\Sigma^0_{33}\right)^2, \tag{108}$$

where D'^{+}_{11}, D'^{+}_{13}, D'^{+}_{31}, D'^{+}_{33} denotes the elements of the matrix $\mathbf{D}'^{+}$. For isotropic materials, the matrix $\mathbf{M}'^{*\pm}$ can be determined explicitly (see Appendix C) and we can obtain immediately the required elements of the matrix $\mathbf{D}'^{\pm}$

$$D'^{+}_{11} = \frac{2(1+\nu)}{E}(1-\nu\tilde{\omega}_1^2), \quad D'^{+}_{33} = \frac{2(1+\nu)}{E}(1-\nu\tilde{\omega}_3^2),$$
$$D'^{+}_{13} = D'^{+}_{31} = -\frac{2\nu(1+\nu)}{E}\tilde{\omega}_1\tilde{\omega}_3. \tag{109}$$

With (109), the instability criteria (107) now takes the new form as

$$\lambda_{cr} = \frac{\pi E\gamma}{1+\nu}\left[\tilde{\omega}_1^2\left(\Sigma^0_{11}\right)^2 + \tilde{\omega}_3^2\left(\Sigma^0_{33}\right)^2 - \nu\left(\tilde{\omega}_1^2\Sigma^0_{11} + \tilde{\omega}_3^2\Sigma^0_{33}\right)^2\right]^{-1}. \tag{110}$$

In the case where $\Sigma^0_{33} = \Sigma^0_{11} = \Sigma^0$, we recover the instability criteria for isotropic materials (Freund and Suresh, 2003)

$$\lambda_{cr} = \frac{\pi E\gamma}{(1-\nu^2)(\Sigma^0)^2}. \tag{111}$$

For general anisotropic materials and arbitrary values of $\tilde{\omega}_1$ and $\tilde{\omega}_3$, explicit formulae for $\mathbf{D}'^{\pm}$ in terms of elastic constants are not available. However, the computation of $\mathbf{D}'^{\pm}$ can always be done numerically by the Barnett–Lothe integrals. As an examples, we consider a special case where the materials is cubic and the principal lateral stresses are the same in both directions Ox_1 and Ox_3, i.e. $\Sigma^0_{33} = \Sigma^0_{11} = \Sigma^0$. The instability criteria can be written as

$$\lambda_{cr} = \frac{2\pi\gamma}{D(\tilde{\omega}_1,\tilde{\omega}_3)(\Sigma^0)^2} \quad \text{with}$$
$$D(\tilde{\omega}_1,\tilde{\omega}_3) = \tilde{\omega}_1^2 D'^{+}_{11} + (D'^{+}_{13} + D'^{+}_{31})\tilde{\omega}_1\tilde{\omega}_3 + \tilde{\omega}_3^2 D'^{+}_{33}. \tag{112}$$

The values of the term $\lambda_{cr}(\Sigma^0)^2/2\pi\gamma$ for different materials and values of $\tilde{\omega}_1$ and $\tilde{\omega}_3$ are plotted in Fig. 1. From the numerical results, we remark that for anisotropic materials, the critical wavelength λ_{cr} is also dependent on the wavelength ratio λ_1/λ_3 (via normalized wavenumbers $\tilde{\omega}_1, \tilde{\omega}_3$), what is in contrast with the isotropic materials (see Eq. (111)). In terms of variation trend, we remark that materials with $AR > 1$ tends to have maximal critical wavelength λ_{cr} when the waviness is in one direction only while those with $AR < 1$ tends to have maximal critical wavelength λ_{cr} when the waviness is the same in both direction $\tilde{\omega}_1 = \tilde{\omega}_3 = 1/\sqrt{2}$.

5. Conclusions and perspectives

In this paper, we have studied the two problems associated to the thin film systems: the stress concentration and the surface morphology instability. The considered thin film is elastically anisotropic bounded by a bidimensional undulating free surface. The surface anisotropy effect is also captured via a surface energy function that depends on the perturbation slopes. The stress concentration factors and the critical wavelength of the shape perturbation are expressed in terms of matrices which can be computed easily with Barnett–Lothe integrals. Analytical solutions are also obtained for some particular cases.

The results issued from this work have shown that the material anisotropy have a considerable effect on both stress concentration factors and the critical wavelength of the thin films. Interestingly, the enhancement of the stress factors is observed in the cubic materials with anisotropy ratio less than unity $AR < 1$. Although the numerical examples concern bidimensional perturbations whose directions coincide with the stress directions and the material's plane of symmetry, the general case can be treated without difficulties using the same approach.

We also demonstrate that the thin film instability criteria written in terms of critical equivalent wavelength λ_{cr} is now dependent on the wavelength ratio λ_1/λ_3, $\lambda_{cr} = \lambda_{cr}(\lambda_1/\lambda_3)$. Numerical examples on some cubic materials with $AR > 1$ and $AR < 1$ show two completely different variation trends of λ_{cr} in terms of the ratio λ_1/λ_3. Remarking that the cases $\lambda_1/\lambda_3 = 0$ or ∞ correspond to the one dimensional perturbation problem. We conclude that the ratio $\sqrt{2}$ given by Gao (1994) between the biaxial and the plane strain critical wavelength for isotropic materials is no longer applicable in this situation.

Based on the Stroh formalism, the solutions derived by the author are subject to the limitations of the method: the displacement constraints. In the Stroh formalism, the displacement components are invariant in one direction, say Ox_3. However, in practice, there are also problems with stress constraints, e.g. free stress conditions as in isotropic elastic plane stress problem, $\sigma_{i3} = 0$, $i = 1, 2, 3$. Isotropic elastic film systems with free lateral stress boundary conditions were studied by Gao (1994). Similar results can also be obtained in the framework of anisotropic elasticity if the material has a plane of symmetry that coincides with Ox_1x_2. In this case, the solutions to the plane stress problem can be obtained from the solutions in the present paper (plane strain problem) by replacing the reduced elastic compliances s'_{ij} with the usual elastic compliances s_{ij} (Ting, 1996).

The thin film studied in this paper is assumed to be sufficiently thick so that it can be treated as a halfspace. As a result, the present model can not account for the substrate stiffness. Another interesting problem, the evolution of the shape perturbation is also lacking. These problems can be solved using the same approach and shall be addressed in the future work.

Finally, the main results of this paper are summarized in Table 3. Some numerical examples of the derived equations are also presented in Tables 1 and 2.

Appendix A

We consider the transformation matrix $\mathbf{\Omega}$ in the following form

$$\mathbf{\Omega} = \begin{bmatrix} \tilde{\omega}_1 & 0 & \chi\tilde{\omega}_3 \\ 0 & 1 & 0 \\ -\chi\tilde{\omega}_3 & 0 & \tilde{\omega}_1 \end{bmatrix}. \tag{A.1}$$

Under the change of frame, a point with coordinates $\mathbf{x}(x_1,x_2,x_3)$ will have coordinates $\hat{\mathbf{x}}(\hat{x}_1,\hat{x}_2,\hat{x}_3)$ that satisfy the relations

$$\hat{\mathbf{x}} = \mathbf{\Omega}\mathbf{x}, \quad \hat{x}_1 = \tilde{\omega}_1 x_1 + \chi\tilde{\omega}_3 x_3, \quad \hat{x}_2 = x_2, \quad \hat{x}_3 = -\chi\tilde{\omega}_3 x_1 + \tilde{\omega}_1 x_3 \tag{A.2}$$

and the boundary condition $(83)_1$ becomes

$$\hat{\mathbf{t}}_2^1 = -\frac{1}{2}\mathbf{\Omega}(\tilde{\omega}_1\mathbf{t}_1^0 + \chi\tilde{\omega}_3\mathbf{t}_3^0)\sin(\omega\hat{x}_1) \quad \text{when } \hat{x}_2 = 0 \tag{A.3}$$

with $\hat{\mathbf{t}}_2^1 =$ being the traction vector in the new system. The boundary conditions (A.3) are similar to $(61)_1$ in Section 3 and the solutions of Section 3 can be used directly. However, we must also recalculate the Stroh's matrices: the matrices of elastic constants $\widehat{\mathbf{C}}_{ij}$, the matrix of eigenvalues $\widehat{\mathbf{P}}$, the matrices of eigen vectors $\widehat{\mathbf{A}}, \widehat{\mathbf{B}}$. The matrices $\widehat{\mathbf{C}}_{ij}$ are defined from $\mathbf{C}_{ij}$ via the transformation rules and thus are related to the matrices $\mathbf{C}'_{ij}$ in (85) via the expression

$$\widehat{\mathbf{C}}_{11} = \mathbf{\Omega}\mathbf{C}'_{11}\mathbf{\Omega}^t, \quad \widehat{\mathbf{C}}_{12} = \mathbf{\Omega}\mathbf{C}'_{12}\mathbf{\Omega}^t, \quad \widehat{\mathbf{C}}_{21} = \mathbf{\Omega}\mathbf{C}'_{21}\mathbf{\Omega}^t, \quad \widehat{\mathbf{C}}_{22} = \mathbf{\Omega}\mathbf{C}'_{22}\mathbf{\Omega}^t. \tag{A.4}$$

The matrices $\widehat{\mathbf{A}}$, $\widehat{\mathbf{B}}$, $\widehat{\mathbf{P}}$ are determined using the Stroh's procedure and the matrices $\widehat{\mathbf{C}}_{ij}$. One can also find the following connections between $\widehat{\mathbf{A}}, \widehat{\mathbf{B}}, \widehat{\mathbf{P}}$ and the matrices $\mathbf{A}', \mathbf{B}', \mathbf{P}'$

$$\widehat{\mathbf{P}} = \mathbf{P}', \quad \widehat{\mathbf{A}} = \mathbf{\Omega}\mathbf{A}', \quad \widehat{\mathbf{B}} = \mathbf{\Omega}\mathbf{B}'. \tag{A.5}$$

Appendix B

The material is assumed to be orthotropic with two planes of symmetry Ox_1x_2 and Ox_2x_3 and the stress state $\mathbf{\Sigma}^0$ takes Ox_1 and Ox_3 as its principal axes. Under these circumstances, the two set of solutions $\mathbf{\Sigma}'^{1+}$, $\mathbf{E}'^{1+}$, $\mathbf{u}'^{1+}$ and $\mathbf{\Sigma}'^{1-}$, $\mathbf{E}'^{1-}$, $\mathbf{u}'^{1-}$ correspond to two symmetric loadings with respect to the material symmetry planes. We shall verify that the solutions with + sign are symmetric to those with − sign, or mathematically

$$\mathbf{u}'^{1+}(\mathbf{x}) = \mathbf{Q}\mathbf{u}'^{1-}(\mathbf{x}^*), \quad \mathbf{E}'^{1+}(\mathbf{x}) = \mathbf{Q}\mathbf{E}'^{1-}(\mathbf{x}^*)\mathbf{Q}^t, \quad \mathbf{\Sigma}'^{1+}(\mathbf{x}) = \mathbf{Q}\mathbf{\Sigma}'^{1-}(\mathbf{x}^*)\mathbf{Q}^t, \quad \mathbf{x}^* = \mathbf{Q}\mathbf{x} \tag{B.1}$$

with $\mathbf{Q}$ being either of the two matrices $\mathbf{Q}_1$ and $\mathbf{Q}_3$

$$\mathbf{Q}_1 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{Q}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}. \tag{B.2}$$

We also remark that with matrices $\mathbf{Q}_1$ and $\mathbf{Q}_3$ given by (B.2), the point $\mathbf{x}^*$ is symmetric to $\mathbf{x}$ with respect to the plane Ox_2x_3 and Ox_1x_2.

Since the elasticity tensor $\mathbb{C}$ is invariant with respect to the change of frame defined by $\mathbf{Q}$, if the solution set $\mathbf{\Sigma}'^{1-}(\mathbf{x})$, $\mathbf{E}'^{1-}(\mathbf{x})$, $\mathbf{u}'^{1-}(\mathbf{x})$ satisfy the elasticity Eq. (8), $\mathbf{\Sigma}'^{1+}(\mathbf{x})$, $\mathbf{E}'^{1+}(\mathbf{x})$, $\mathbf{u}'^{1+}(\mathbf{x})$ defined by (B.1) also satisfy (8). Next, as the principal directions of the stress state $\mathbf{\Sigma}^0$ coincide with $\mathbf{e}_1$ and $\mathbf{e}_3$ as in (17), using (83), we can prove that the stress boundary conditions associated to $\mathbf{\Sigma}'^{1+}$, $\mathbf{E}'^{1+}$,$\mathbf{u}'^{1+}$ are symmetric to those associated to $\mathbf{\Sigma}'^{1-}$, $\mathbf{E}'^{1-}$, $\mathbf{u}'^{1-}$ with respect to both planes Ox_1x_2 and Ox_2x_3

$$\mathbf{\Sigma}'^{1+}(\mathbf{x})\mathbf{e}_2 = \mathbf{Q}\mathbf{\Sigma}'^{1-}(\mathbf{x}^*)\mathbf{e}_2. \tag{B.3}$$

In particular, we obtain the following relations for displacements

$$\begin{aligned} u_1'^{1+}(x_1,x_2,x_3) &= u_1'^{1-}(x_1,x_2,-x_3), \\ u_1'^{1+}(x_1,x_2,x_3) &= -u_1'^{1-}(-x_1,x_2,x_3), \\ u_3'^{1+}(x_1,x_2,x_3) &= u_3'^{1-}(-x_1,x_2,x_3), \\ u_3'^{1+}(x_1,x_2,x_3) &= -u_3'^{1-}(x_1,x_2,-x_3). \end{aligned} \tag{B.4}$$

and other relations for stresses

$$\begin{aligned} &\Sigma_{11}'^{1+}(x_1,x_2,x_3) = \Sigma_{11}'^{1-}(-x_1,x_2,x_3) = \Sigma_{11}'^{1+}(-x_1,x_2,x_3) = \Sigma_{11}'^{1-}(x_1,x_2,-x_3), \\ &\Sigma_{33}'^{1+}(x_1,x_2,x_3) = \Sigma_{33}'^{1-}(x_1,x_2,-x_3) = \Sigma_{33}'^{1+}(-x_1,x_2,x_3) = \Sigma_{33}'^{1-}(-x_1,x_2,x_3), \\ &\Sigma_{13}'^{1+}(x_1,x_2,x_3) = -\Sigma_{13}'^{1-}(-x_1,x_2,x_3), \quad \Sigma_{/13}^{1+}(x_1,x_2,x_3) = -\Sigma_{13}'^{1-}(x_1,x_2,-x_3). \end{aligned} \tag{B.5}$$

The relations (B.4) and (B.5) are valid for any $\Sigma_{11}^0, \Sigma_{33}^0$. Now we study the components $\Sigma_{11}'^{1+}$ and $\Sigma_{11}'^{1-}$ which can be computed from (97) using (17)

$$\begin{aligned} \Sigma_{11}'^{1+} &= \frac{1}{2}\Re\left\{\left(J_{11}'^{+}\tilde{\omega}_1\Sigma_{11}^0 + J_{13}'^{+}\tilde{\omega}_3\Sigma_{33}^0\right)e^{i(\omega_1x_1+\omega_3x_3)}\right\}, \\ \Sigma_{11}'^{1-} &= \frac{1}{2}\Re\left\{\left(J_{11}'^{-}\tilde{\omega}_1\Sigma_{11}^0 - J_{13}'^{-}\tilde{\omega}_3\Sigma_{33}^0\right)e^{i(\omega_1x_1-\omega_3x_3)}\right\}. \end{aligned} \tag{B.6}$$

Making use of $(B.5)_{1,2}$ and (B.6) which are valid for any $\Sigma_{11}^0,\ \Sigma_{33}^0$, we must have

$$\begin{aligned} J_{11}'^{+} &= J_{11}'^{-}, \quad \Im\{J_{11}'^{+}\} = \Im\{J_{11}'^{-}\} = 0, \\ J_{13}'^{+} &= -J_{13}'^{-}, \quad \Im\{J_{13}'^{+}\} = -\Im\{J_{13}'^{-}\} = 0. \end{aligned} \tag{B.7}$$

Similarly, other relations concerning Σ_{33}^{1+} and Σ_{33}^{1-} can also be obtained

$$\begin{aligned} K_{33}'^{+} &= -K_{33}'^{-}, \quad \Im\{K_{33}'^{+}\} = -\Im\{K_{33}'^{-}\} = 0, \\ K_{31}'^{+} &= K_{31}'^{-}, \quad \Im\{K_{31}'^{+}\} = \Im\{K_{31}'^{-}\} = 0. \end{aligned} \tag{B.8}$$

Concerning the symmetry of displacement, we rewrite the displacement on the surface $\mathbf{u}'^{1+}$ and $\mathbf{u}'^{1-}$

$$\begin{aligned} \mathbf{u}'^{1+} &= \frac{1}{2\omega}\Re\left\{-i\mathbf{M}'^{*+}\left(\tilde{\omega}_1\Sigma_{11}^0\mathbf{e}_1 + \tilde{\omega}_3\Sigma_{33}^0\mathbf{e}_3\right)e^{i(\omega_1x_1+\omega_3x_3)}\right\}, \\ \mathbf{u}'^{1-} &= \frac{1}{2\omega}\Re\left\{-i\mathbf{M}'^{*+}\left(\tilde{\omega}_1\Sigma_{11}^0\mathbf{e}_1 - \tilde{\omega}_3\Sigma_{33}^0\mathbf{e}_3\right)e^{i(\omega_1x_1-\omega_3x_3)}\right\}. \end{aligned} \tag{B.9}$$

Making use of (B.4) and (B.9), we obtain

$$M'^{*+}_{11} = M'^{*-}_{11}, \quad M'^{*+}_{33} = M'^{*-}_{33}, \quad M'^{*+}_{31} = -M'^{*-}_{31}, \quad M'^{*+}_{13} = -M'^{*-}_{13}. \tag{B.10}$$

If the matrices $\mathbf{D}'^{\pm}$ are the real part of $\mathbf{M}'^{*\pm}$, then we must have

$$D'^{+}_{11} = D'^{-}_{11}, \quad D'^{+}_{33} = D'^{-}_{33}, \quad D'^{+}_{31} = -D'^{-}_{31}, \quad D'^{+}_{13} = -D'^{-}_{13}. \tag{B.11}$$

Appendix C

In this appendix, we present the calculation of the matrices $\mathbf{J}'^{\pm}$, $\mathbf{K}'^{\pm}$, $\mathbf{M}'^{*\pm}$, $\mathbf{N}'^{\pm}$. Since the computation procedure is identical both cases $\chi = \pm 1$, in what follows, we use notations without + or − sign, i.e. the matrices $\mathbf{J}'$, $\mathbf{K}'$, $\mathbf{M}'^{*}$, $\mathbf{N}'$ that satisfy the relations

$$\begin{aligned}
\mathbf{J}' &= i\mathbf{F}'\mathbf{B}'^{-1} = (\tilde{\omega}_1\mathbf{C}_{11} + \chi\tilde{\omega}_3\mathbf{C}_{13})\mathbf{M}'^{*} + \mathbf{C}_{12}\mathbf{M}'^{*}\mathbf{N}', \\
\mathbf{K}' &= i\mathbf{G}'\mathbf{B}'^{-1} = (\tilde{\omega}_1\mathbf{C}_{31} + \chi\tilde{\omega}_3\mathbf{C}_{33})\mathbf{M}'^{*} + \mathbf{C}_{32}\mathbf{M}'^{*}\mathbf{N}', \\
\mathbf{N}' &= \mathbf{B}'\mathbf{P}'\mathbf{B}'^{-1}, \quad \mathbf{M}'^{*} = i\mathbf{A}'\mathbf{B}'^{-1}.
\end{aligned} \tag{C.1}$$

In the frame $O\hat{x}_1\hat{x}_2\hat{x}_3$ defined in Appendix A, we denote $\widehat{\mathbf{M}}^{*}, \widehat{\mathbf{N}}$ as the matrices with similar meanings as $\mathbf{M}^{*}$, $\mathbf{N}$ in the Stroh formalism. They are derived from the matrices $\widehat{\mathbf{A}}$, $\widehat{\mathbf{B}}$, $\widehat{\mathbf{P}}$ by the expressions

$$\widehat{\mathbf{N}} = \widehat{\mathbf{B}}\widehat{\mathbf{P}}\widehat{\mathbf{B}}^{-1}, \quad \widehat{\mathbf{M}}^{*} = i\widehat{\mathbf{A}}\widehat{\mathbf{B}}^{-1}. \tag{C.2}$$

Thus, the matrices $\mathbf{M}'^{*}$,$\mathbf{N}'$ and $\mathbf{M}'^{*}\mathbf{N}'$ are connected to $\widehat{\mathbf{M}}^{*}, \widehat{\mathbf{N}}$ and $\widehat{\mathbf{M}}^{*}\widehat{\mathbf{N}}$. Comparing the Eqs. (C.2) and (100), these relations read

$$\mathbf{M}'^{*} = \mathbf{\Omega}^t\widehat{\mathbf{M}}^{*}\mathbf{\Omega}, \quad \mathbf{N}' = \mathbf{\Omega}^t\widehat{\mathbf{N}}\mathbf{\Omega}, \quad \mathbf{M}'^{*}\mathbf{N}' = \mathbf{\Omega}^t\widehat{\mathbf{M}}^{*}\widehat{\mathbf{N}}\mathbf{\Omega}. \tag{C.3}$$

For isotropic materials, $\widehat{\mathbf{M}}^{*}$ and $\widehat{\mathbf{N}}$ is independent of frame, or

$$\mathbf{M}^{*} = \widehat{\mathbf{M}}^{*}, \quad \mathbf{N} = \widehat{\mathbf{N}}. \tag{C.4}$$

Making use of (50) and (56) and the fact that $\xi_1 = \xi_2 = \xi_3 = i$, one can determine explicitly $\mathbf{M}^{*}$ and $\mathbf{N}$ in terms of E and ν

$$\begin{aligned}
\mathbf{M}^{*} &= \widehat{\mathbf{M}}^{*} = \frac{1+\nu}{E}\begin{bmatrix} 2(1-\nu) & (1-2\nu)i & 0 \\ -(1-2\nu)i & 2(1-\nu) & 0 \\ 0 & 0 & 2 \end{bmatrix}, \\
\mathbf{N} &= \widehat{\mathbf{N}} = \begin{bmatrix} 2i & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & i \end{bmatrix}.
\end{aligned} \tag{C.5}$$

With (C.3, C.5) and (A.1), one can determine explicitly the matrices $\mathbf{M}'^{*}$, $\mathbf{J}'$, $\mathbf{K}'$

$$\mathbf{M}'^{*} = \frac{1+\nu}{E}\begin{bmatrix} 2(1-\nu\tilde{\omega}_1^2) & \tilde{\omega}_1(1-2\nu)i & -2\chi\nu\tilde{\omega}_1\tilde{\omega}_3 \\ -\tilde{\omega}_1(1-2\nu)i & 2(1-\nu) & -\chi\tilde{\omega}_3(1-2\nu)i \\ -2\chi\tilde{\omega}_1\tilde{\omega}_3\nu & \chi\tilde{\omega}_3(1-2\nu)i & 2(1-\nu\tilde{\omega}_3^2) \end{bmatrix}, \tag{C.6}$$

$$\mathbf{J}' = \begin{bmatrix} 2\tilde{\omega}_1(1-\nu\tilde{\omega}_3^2) & (2\nu\tilde{\omega}_3^2 + \tilde{\omega}_1^2)i & 2\chi\nu\tilde{\omega}_3^3 \\ i & 0 & 0 \\ \chi\tilde{\omega}_3(1-2\nu\tilde{\omega}_1^2) & 0 & \tilde{\omega}_1(1-2\nu\tilde{\omega}_3^2) \end{bmatrix}, \tag{C.7}$$

$$\mathbf{K}' = \begin{bmatrix} \chi\tilde{\omega}_3(1-2\nu\tilde{\omega}_1^2) & \chi\tilde{\omega}_1\tilde{\omega}_3(1-2\nu)i & \tilde{\omega}_1(1-2\nu\tilde{\omega}_3^2) \\ 0 & 0 & i \\ 2\tilde{\omega}_1^3 & (\tilde{\omega}_3^2 + 2\nu\tilde{\omega}_1^2)i & 2\chi\tilde{\omega}_3(1+\nu\tilde{\omega}_1^2) \end{bmatrix}. \tag{C.8}$$

By setting $\chi = +1$ or $\chi = -1$ in ((C.6)–(C.8)), we obtain the matrices $\mathbf{J}'^{\pm}$, $\mathbf{K}'^{\pm}$, $\mathbf{M}'^{*\pm}$. For general anisotropy, since the matrix $\widehat{\mathbf{M}}^{*}$ in the frame $O\hat{x}_1\hat{x}_2\hat{x}_3$ can be computed via the Barnett–Lothe integrals (57)–(59), $\mathbf{J}'^{\pm}$, $\mathbf{K}'^{\pm}$, $\mathbf{M}'^{*\pm}$ can also be calculated numerically.

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