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Galois Groups of Unramified Covers of Projective Curves in Characteristic p

Katherine F. Stevenson*

Department of Mathematics, Mathematics Building 084, University of Maryland, College Park, Maryland 20742-4015

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We show several results concerning the finite groups that occur as Galois groups of unramified covers of projective curves in characteristic p. In particular, we prove that every finite group with g generators occurs over some curve of genus g. This implies, for example, that every finite simple group occurs in genus 2. By similar methods, we obtain several other families of groups which occur in genus 2. In addition, we show that if a group G occurs over some curve of genus g, then it must occur over "almost all" curves of genus g or greater. The results are obtained using formal patching.

INTRODUCTION

This paper contains several results concerning the fundamental group of projective curves over algebraically closed fields of characteristic p. Although the fundamental group is known in characteristic 0, and in characteristic p for genus 0 and 1, little has been known for genus ≥ 2 in characteristic p. Here, we determine large classes of groups that are quotients of π_1 of curves of genus g in characteristic p, using the technique of formal patching. (See Section 3 and Propositions 5.5, 5.6.) Moreover, each such group occurs over a *generic* curve of genus g, though not necessarily over *all* curves of genus g. (See Section 4.)

More specifically, given a curve C over an algebraically closed field k, let $\pi_A(C)$ be the set of isomorphism classes of finite groups occurring as Galois groups of unramified Galois covers of C. Recall that this is equal to the set of continuous quotients of the algebraic fundamental group $\pi_1(C)$.

* E-mail: kfs@math.umd.edu.

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Since the same set of finite groups can form more than one inverse system, π_A does not in general determine π_1. But in the case of projective curves, π_A does indeed determine π_1; cf. the discussion below.

In the case where k is the complex numbers $\mathbb{C}$, the set π_A is well understood for both affine and projective curves. Namely, for non-negative integers g and r, let $F_{g,r}$ be the group generated by elements $a_1, \ldots, a_g$, $b_1, \ldots, b_g$, $c_1, \ldots, c_r$ subject to the single relation $\prod_{j=1}^{g}[a_j, b_j]\prod_{i=1}^{r}c_i = 1$, where $[a, b]$ denotes the commutator $aba^{-1}b^{-1}$. Then for a complex projective curve X (or equivalently Riemann surface) of genus g and for distinct closed points $\{\tau_1, \tau_2, \ldots, \tau_r\}$ on X, the group $F_{g,r}$ may be identified with the topological fundamental group of $X - \{\tau_1, \tau_2, \ldots, \tau_r\}$. This is done by letting a_j and b_j correspond to loops around the jth "hole," and c_i correspond to a loop around τ_i. With this identification, we get a complete description of π_A (and even π_1) for complex curves. Specifically, G lies in $\pi_A(X - \{\tau_1, \tau_2, \ldots, \tau_r\})$ if and only if G is a finite quotient of $F_{g,r}$. Moreover, for *any* algebraically closed field of characteristic 0, Grothendieck showed that the same result holds [Gr, XIII, Cor. 2.12].

In characteristic $p \neq 0$ the situation is quite different. For example, while the complex affine line is simply connected, the affine line over a field k of characteristic $p \neq 0$ is not. In particular, the Artin–Schreier equations (e.g., $y^p - y = x$) induce non-trivial covers of $\mathbb{P}_k^1$ which have Galois group $\mathbb{Z}/p\mathbb{Z}$ and are ramified only at infinity. Thus there are groups contained in $\pi_A(\mathbb{P}_k^1 - \{\tau_1\})$ which are not quotients of $F_{0,1} = \{e\}$. Abhyankar, in his 1957 paper [Ab], found many examples of groups lying in $\pi_A(\mathbb{P}_k^1 - \{\tau_1, \tau_2, \ldots, \tau_r\})$. From these, he conjectured that given a smooth connected projective k-curve X of genus g, a finite group G lies in $\pi_A(X - \{\tau_1, \tau_2, \ldots, \tau_r\})$ if and only if every prime-to-p quotient of G is a quotient of $F_{g,r}$. The forward implication was shown by Grothendieck in [Gr, XIII, Cor. 2.12], and recently the converse was proven by Harbater in [Ha2]. Harbater's proof uses formal patching (which is related to rigid analysis) and relies on Raynaud's proof [Ra] of the conjecture in the case where $X = \mathbb{P}_k^1$ and $r = 1$. From Abhyankar's conjecture we deduce two important facts about affine curves: First, that $\pi_A(X - \{\tau_1, \tau_2, \ldots, \tau_r\})$ depends only on the number $2g + r$, where g is genus of X and $r > 0$; and second, that π_A of an affine k-curve strictly contains π_A of the "analogous complex curve." However, Abhyankar's conjecture does *not* give us π_1 of affine curves. For example, an elliptic curve with one point removed has the same π_A as a projective curve with three points removed; however, their π_1's are different (for this and other such examples see [Bo, Ha3]). In fact, recent work of Tamagawa shows that for a k-curve U, the group $\pi_1(U)$ determines the genus and "p-rank" of the projective completion X of U, as well as the number of points in $X - U$, whereas these are not determined by π_A.

In this paper we consider unramified Galois covers of *projective* k-curves. For such curves, π_A is only known in genus zero ($\mathbb{P}^1_k$) and genus one (elliptic curves). Recall that $\pi_A(\mathbb{P}^1_k)$ is trivial and π_A of an elliptic k-curve X depends on the curve. Specifically, if X is ordinary then $\pi_A(X) = \{\mathbb{Z}/n \times \mathbb{Z}/m : m, n \in \mathbb{Z}_+, (p, n) = 1\}$, where (a, b) denotes the greatest common divisor of a and b, and if it is supersingular then $\pi_A(X) = \{\mathbb{Z}/n \times \mathbb{Z}/m : m, n \in \mathbb{Z}_+, (p, m) = (p, n) = 1\}$. Notice that in either case, $\pi_A(X)$ is contained in π_A of any complex elliptic curve, and that this inclusion is the opposite of the one arising in the situation of affine curves (of any genus). In fact, Grothendieck showed in [Gr, XIII, Cor. 2.12] that this containment holds for π_A of projective k-curves C of all possible genera, and that the containment is always strict if $g > 0$. (The strictness here holds because the p-rank of π_1 is less than or equal to g.) This implies that $\pi_1(C)$ is a quotient of π_1 of the "analogous complex curve" and hence is finitely generated. Since finitely generated profinite groups are determined by their continuous finite quotients [FJ, Prop. 15.4], we see that for projective curves π_A determines π_1.

Known results about $\pi_A(C)$ for C projective and of genus g include the following:

(1) [Gr, XIII, Cor. 2.12] If $G \in \pi_A(C)$ then G is in π_A of a complex curve of genus g (the latter π_A being known explicitly). The converse also holds if G has order prime-to-p.

(2) A p-group G cannot lie in $\pi_A(C)$ if its p-rank is greater than g. A given p-group G of rank g will lie in π_A of a *generic* curve of genus g. (This follows by combining results about the Hasse−Witt invariant with the Burnside Basis Theorem.)

(3) [Na1] (cf. Theorem 3.12 below) For G to lie in $\pi_A(C)$, it is necessary that G satisfy a certain condition concerning generators of group rings (although it is unknown whether this condition is a consequence of (1) and (2) above).

But for groups G that are neither of order prime-to-p nor a power of p, the above do not necessarily provide a condition that is sufficient to imply that G is in π_A of a curve of genus g.

The purpose of this paper is to provide conditions for G to lie in π_A of a curve of genus g. For example, we show that every finite group lies in π_A of some projective k-curve X (Corollary 3.7 below), and that in fact X may be chosen with its genus bounded by the number of generators of the group. Using the classification theorem of finite simple groups, this implies that any finite simple group lies in π_A of some smooth projective k-curve of genus 2 (Corollary 3.6 below). The drawback to the above results is that for a given group we find *one* curve X of genus g for which G lies in $\pi_A(X)$. However, in Section 4 we show that this in fact implies that G lies in π_A of almost all curves of genus g. Specifically, if G lies in π_A of some projective curve of genus g, then there exists a non-empty open subset V in the moduli space of curves of genus g such that G lies in π_A of any curve corresponding to a point in V (Proposition 4.2 below). With this result we show that if G lies in π_A of a curve of genus g then for all $g' > g$, the group G lies in π_A of almost all curves of genus g' (Corollary 3.9 and Proposition 4.2 below).

The main results of the paper, Theorems 3.1 and 5.4, construct G-Galois covers of projective k-curves of genus g by pasting together tamely ramified Galois covers of k-curves of lower genus in such a way that the ramification cancels. By induction, and Theorem 3.1 applied to unramified covers, we get Theorem 3.3, from which the above results follow. Theorem 5.4 allows us to use pasting arguments more complicated than those used in Section 3. In this way we obtain, for example, concrete descriptions of many groups which occur in π_A of projective k-curves of genus 2. (See Propositions 5.5 and 5.6 for a sample.) The proofs of all these results use formal patching and are similar to the methods used by Harbater to prove the Abhyankar conjecture. For a description of the use of formal patching (and rigid analysis) in Galois theory see the forthcoming books of Volklein [V] and Malle and Matzat [MM], as well as recent papers on the subject by Fried and Volklein [FV] and Haran and Volklein [HV].

The structure of this paper is as follows: Section 1 gives the necessary background. Section 2 gives various formal patching and deformation results; these can be viewed as corresponding to results in rigid analysis. Section 3 proves the main results and their corollaries using the preliminary work in Section 2. Section 4 proves the above result about open sets in the moduli space of curves of genus g. Finally, Section 5 gives some generalizations of the results in Section 3. This paper is adapted from the author's 1994 University of Pennsylvania Ph.D. thesis.

Many of the results in this paper also follow work done simultaneously by Mohamed Saidi in his 1994 Ph.D. thesis [Sa]. In his thesis M. Saidi uses rigid geometry instead of formal patching.

1. BACKGROUND

This section summarizes the necessary formal patching results and explains how they are applied. We begin by building the deformation, consisting of a scheme $\tilde{X}$ that is proper over a complete local ring D, with closed fibre X. (This construction can be thought of as a "very small" deformation of X.) Given $\tilde{X}$, Grothendieck's Existence Theorem [Gr, XIII, Cor. 2.12] says that there exists an equivalence of categories between the category of coherent sheaves on $\tilde{X}$, and the category of formal coherent sheaves on X. The result allows patching on covers which are defined over deformations of affine opens of X. However, here we will need a generalization of Grothendieck's result by Harbater which, loosely speaking, allows us to replace one of these affine opens of X by the spectrum of the complete local ring at a closed point on X (cf. [Ha1]). The gluing here is done by ''patching'' the corresponding modules.

For any ring R, let $\mathscr{M}(R)$ denote the category of finitely presented R-modules, and let $\mathscr{F}(R)$ and $\mathscr{P}(R)$ denote the subcategory of $\mathscr{M}(R)$ consisting of free and projective R-modules, respectively. Given categories $\mathscr{A}, \mathscr{B}, \mathscr{C}$ and functors $\mathscr{A} \to \mathscr{C}$ and $\mathscr{B} \to \mathscr{C}$, let $\mathscr{A} \times_{\mathscr{C}} \mathscr{B}$ denote the category of triples (a, b, c) of objects in $\mathscr{A}, \mathscr{B}, \mathscr{C}$ together with isomorphisms between c and the images of a and of b in $\mathscr{C}$. Let D be a ring; then for D-algebras A, B, and C define the *base change functor*

$$\mathscr{M}(D) \to \mathscr{M}(A) \times_{\mathscr{M}(C)} \mathscr{M}(B)$$

by

$$M \mapsto (M \otimes_D A, M \otimes_D B, M \otimes_D C).$$

Given a quadruple (A, B, C, D) of domains, we say that it satisfies the *patching property for modules* (cf. [Ha1]) if the base change functor is an equivalence of categories whose inverse (up to equivalence of functors) is given by the fiber product of modules

$$(M_A, M_B, M_C) \mapsto M_A \times_{M_C} M_B.$$

Similarly it satisfies the *patching property for free modules* (resp. for *projective modules*) if the corresponding assertion holds with $\mathscr{M}$ replaced by $\mathscr{F}$ (resp. by $\mathscr{P}$).

We can also define analogous categories for any scheme X. Let $\mathscr{P}(X)$ (resp. $\mathscr{AP}(X)$) denote the category of coherent sheaves of projective $\mathscr{O}_X$-modules (resp. projective $\mathscr{O}_X$-algebras). Also, let $\mathscr{SP}(X)$ denote the subcategory of $\mathscr{AP}(X)$ consisting of sheaves which are generically separable.

We will need only one case from the main formal patching result in [Ha1]. In order to state it, we introduce some notation which will be used throughout this paper. If X is a normal scheme and ξ is a point of X, then let $\hat{\mathscr{K}}_{X,\xi}$ be the total ring of fractions of the complete local ring $\hat{\mathscr{O}}_{X,\xi}$. Given another scheme Y, a morphism $Y \to X$ which is finite and generically separable (cf. [Ha, 2]) will be called a *cover*. If the cover $Y \to X$ is étale and if $G \subset \mathrm{Aut}(Y/X)$ is a finite group acting simply transitively on a geometric fibre, then $Y \to X$ together with this action will be called a

G-Galois cover. Now, for any finite group G let $G\mathscr{P}(X)$ be the subcategory of $\mathscr{S}\mathscr{P}(X)$ consisting of G-Galois sheaves. Let k be an algebraically closed field.

THEOREM 1.1 [Ha1, Thm. 1(3)]. *Let X be an irreducible regular projective curve over a field k, having function field $K(X)$. Let ξ be a closed point of X, and let* $\operatorname{Spec}(S) = X - \{\xi\}$. *Then the base change functors*

$$\mathscr{P}(X) \to \mathscr{P}(X) \times_{\mathscr{P}(\hat{\mathscr{K}}_{X,\xi})} \mathscr{P}\left(\hat{\mathscr{O}}_{X,\xi}\right)$$

$$\mathscr{P}(X \times_k \operatorname{Spec}(k[[t]])) \to \mathscr{P}(S[[t]]) \times_{\mathscr{P}(\hat{\mathscr{K}}_{X,\xi}[[t]])} \mathscr{P}\left(\hat{\mathscr{O}}_{X,\xi}[[t]]\right)$$

are equivalences of categories. Moreover, the result remains true if $\mathscr{P}$ is replaced by $\mathscr{A}\mathscr{P}$, $\mathscr{S}\mathscr{P}$, or $G\mathscr{P}$ (for any finite group G).

Since we will be looking at projective modules, the following lemma gives a condition for flatness, which implies projectivity in our situation.

LEMMA 1.2. *Let $R \subset S$ be noetherian rings of dimension 2. Suppose that R is regular and that S is normal, finitely generated R-module. Then S is flat over R.*

Proof. Let m be a maximal ideal of R, and take A to be the localization of R at m. Then, A is a noetherian local ring. Let $B = S \otimes_R A$. Since S is dimension 2 and normal, so is B, and hence B is Cohen–Macauley. By [Ma, Thm 18.H, p. 140; AB, p. 113, Ex. 4], B is a flat (in fact free) A-module, and this implies that S is a flat R-module. ∎

In addition to needing Theorem 1.1, we will also need a more specialized formal patching result [Ha2, Prop. 2.3] that relies on Theorem 1.1. Roughly, it is the following: Let G be a finite group generated by subgroups G_1 and G_2, and let $W_i \to X_i$ be connected G_i-Galois (possibly branched) covers of k-curves for $i = 1, 2$. By taking a disjoint union of copies of W_i, we obtain disconnected G-Galois covers of the X_i for $i = 1, 2$. These covers are then deformed to get disconnected G-Galois covers of curves over $k[[t]]$. The result shows that given some "patching data" for these deformations, there is an irreducible G-Galois cover defined over $k[[t]]$, whose closed fibre is a union of the original disconnected G-Galois covers which we deformed.

To understand the precise statement of the result we must recall the definition of induced covers. If H is a subgroup of a finite group G, and $Z \to Y$ is an H-Galois cover, then there is an induced G-Galois cover $\operatorname{Ind}_H^G Z \to Y$ which is the disjoint union of $(G : H)$ copies of Z, indexed by the left cosets of H in G. The stabilizer of the identity copy is $H \subset G$, and the stabilizers of the other copies are the conjugates of H in G.

As in [Ha3], let T^* be an irreducible $k[[t]]$-scheme of relative dimension 1, whose closed fibre is a union of two smooth irreducible curves X_1 and X_2. Because this result relies on Theorem 1.1, which only holds for smooth curves, we assume the existence of a smooth projective curve L and a flat covering $\phi^*\colon T^* \to L \times_k \operatorname{Spec}(k[[t]])$ over $k[[t]]$. For $i = 1, 2$ let $X_i' = X_i - \{\tau\} = \operatorname{Spec}(R_i)$ for some ring R_i. Let $X_i'^* = \operatorname{Spec}(R_i[[t]])$ and $\hat{X}_i'^* = \operatorname{Spec}(\hat{\mathscr{K}}_{X_{i,\tau}}[[t]])$, where τ is the unique common point of X_1 and X_2.

Consider a finite group G, together with subgroups G_1, G_2, and I that together generate G. For $i = 1, 2$ let $W_i'^* \to X_i'^*$ be an irreducible normal G_i-Galois cover, and let $\hat{W}_i'^*$ be an irreducible component of $W_i'^* \times_{X_i'^*} \hat{X}_i'^*$ such that $I_i = \operatorname{Gal}(\hat{W}_i'^*/\hat{X}_i'^*)$ is contained in I. Also, let $\hat{T}_\tau^* = \operatorname{Spec}(\hat{\mathscr{O}}_{T^*,\tau})$ and let $\hat{N}^* \to \hat{T}_\tau^*$ be an irreducible normal I-Galois cover, together with an isomorphisms $\hat{N}^* \times_{\hat{T}_\tau^*} \hat{X}_i'^* \to \operatorname{Ind}_{I_i}^I(\hat{W}_i'^*)$ of I-Galois covers of $\hat{X}_i'^*$, for $i = 1, 2$.

PROPOSITION 1.3 [Ha3, Prop. 2.1, or Ha2, Prop. 2.3]. *In the above situation, there is an irreducible normal G-Galois cover $V^* \to T^*$ such that $V^* \times_{T^*} X_i'^* \cong \operatorname{Ind}_{G_i}^G(W_i'^*)$ as G-Galois covers of $X_i'^*$ for $i = 1, 2$, and $V^* \times_{T^*} \hat{T}_\tau^* \cong \operatorname{Ind}_I^G \hat{N}^*$ as G-Galois covers of $\hat{T}_\tau^*$.*

We will use this result in Section 2 to build an unramified connected G-Galois cover of a curve of genus g by deforming G_i-Galois covers of curves of lower genus. In Section 5, we generalize Proposition 1.3 to allow the closed fibre to have more than one double point. This gives us more flexibility in the construction of Galois covers.

2. FORMAL PATCHING RESULTS

In this section we prove some variants and applications of formal patching results in [Ha1, Ha2]. These results provide the deformation and patching methods needed to build G-Galois covers in Sections 3, 4, and 5.

Most of the patching arguments in this paper follow a similar pattern. We start with a degenerate curve T, consisting of a union of two projective k-curves crossing transversely at a point. Taking g to be the arithmetic genus of T, Lemma 2.3 deforms T to obtain T^* which is irreducible and projective of genus g. In Proposition 2.4, Galois covers of the components of the degenerate curve T are first pasted together and then deformed in order to construct a G-Galois cover of T^*. At this point, T^* and its cover are curves over $k[[t]]$, so in Proposition 2.5 we globalize the construction to obtain a family of covers parameterized by an actual k-variety of finite type over $k[t]$. In this way it is possible to choose a non-degenerate member of the family which will be a G-Galois cover of a smooth connected projective k-curve of genus g. These results roughly correspond to the steps needed to prove the main result, Theorem 3.1, as well as its generalizations in Section 5.

As a preliminary step, in Lemma 2.1 we use Theorem 1.1 to construct a $k[[t]]$-curve by patching together local deformations of a degenerate k-curve. As discussed in Section 1, Harbater's equivalence (Theorem 1.1) holds only for smooth curves, so we must first map our degenerate curve down to a projective k-line and then build its deformation locally over $\mathbb{P}^1_k \times_k \operatorname{Spec}(k[[t]])$.

Throughout this paper we will let L denote the projective k-line and identify L with its image in $L^* = L \times_k \operatorname{Spec}(k[[t]])$, via the inclusion $L \to L^*$ (induced by the natural ring morphism). Also, we let $K = k((t))$.

LEMMA 2.1. *Let* $\phi\colon T \to L$ *be a connected cover. Pick a closed point* λ *in* L, *and let* $L' = L - \lambda$, $\hat{L} = \operatorname{Spec}(\hat{\mathscr{O}}_{L,\lambda}) = \operatorname{Spec}(k[[x]])$, *and* $\hat{L}' = \operatorname{Spec}(\hat{\mathscr{K}}_{L,\lambda}) = \operatorname{Spec}(k((x)))$ (*where* x *is a local uniformizer on* L *at* λ). *Let* $\phi'\colon T'' \to L'$ *and* $\hat{\phi}\colon \hat{T} \to \hat{L}$ *be the pullbacks of* $\phi\colon T \to L$ *via the morphisms* $L' \to L$ *and* $\hat{L} \to L$, *respectively, and let* $L'^* = \operatorname{Spec}(\mathscr{O}_{L'}[[t]])$, $\hat{L}^* = \operatorname{Spec}(\hat{\mathscr{O}}_{L,\lambda}[[t]])$, *and* $\hat{L}'^* = \operatorname{Spec}(\hat{\mathscr{K}}_{L,\lambda}[[t]])$. *Suppose that* $\phi'^*\colon T''^* \to L'^*$ *and* $\hat{\phi}^*\colon \hat{T}^* \to \hat{L}^*$ *are flat covers for which the closed fibers are isomorphic to* $\phi'\colon T'' \to L'$ *and* $\hat{\phi}\colon \hat{T} \to \hat{L}$ (*respectively*) *and the pullbacks over* $\hat{L}'^*$ *are each étale. Then there exists a projective* $k[[t]]$-*curve* T^* *and a covering morphism* $\phi^*\colon T^* \to L^*$, *such that* $T^* \times_{L^*} L'^* \cong T''^*$ *as a cover of* L'^* *and* $T^* \times_{L^*} \hat{L}^* \cong \hat{T}^*$ *as a cover of* $\hat{L}^*$; *and the closed fibre of* ϕ^* *is isomorphic to* $\phi\colon T \to L$.

Proof. The cover ϕ is a flat, hence projective, morphism. Since it is also generically separable, the pair (T, ϕ) defines an object in $\mathscr{SP}(L)$, so by Theorem 1.1, it induces (via base change) $\mathscr{O}_{T'} \in \mathscr{SP}(\mathscr{O}_{L'})$ and $\mathscr{O}_{\hat{T}} \in \mathscr{SP}(\mathscr{O}_{\hat{L}})$ along with an isomorphism $\theta\colon \mathscr{O}_{T''} \otimes_{\mathscr{O}_{L'}} \hat{\mathscr{K}}_{L,\lambda} \to \mathscr{O}_{\hat{T}} \otimes_{\mathscr{O}_{\hat{L}}} \hat{\mathscr{K}}_{L,\lambda}$.

Consider the covers ϕ'^* and $\hat{\phi}^*$. These are flat by hypothesis, and consequently $\mathscr{O}_{T'^*}$ and $\mathscr{O}_{\hat{T}^*}$ are flat (hence projective) algebras over $\mathscr{O}_{L'^*}$ and $\mathscr{O}_{\hat{L}^*}$, respectively. Then because ϕ'^* and $\hat{\phi}^*$ are generically separable, $\mathscr{O}_{T'^*}$ and $\mathscr{O}_{\hat{T}^*}$ define objects in $\mathscr{SP}(\mathscr{O}_{L'^*})$ and $\mathscr{SP}(\mathscr{O}_{\hat{L}^*})$, respectively. Moreover, the closed fibre of $\phi'^*\colon T''^* \to L'^*$ is isomorphic to $\phi'\colon T'' \to L'$, so there is an isomorphism of $\hat{\mathscr{K}}_{L,\lambda}$-algebras

$$\alpha_1\colon \Big(\Big(\mathscr{O}_{T''^*} \otimes_{\mathscr{O}_{L'^*}} \hat{\mathscr{K}}_{L,\lambda}[[t]]\Big) \bmod t\Big) \to \mathscr{O}_{T''} \otimes_{\mathscr{O}_{L'}} \hat{\mathscr{K}}_{L,\lambda}.$$

Similarly, from the definition of $\hat{\phi}^*$, there is an isomorphism of $\hat{\mathscr{K}}_{L,\lambda}$-algebras

$$\alpha_2\colon \Big(\Big(\mathscr{O}_{\hat{T}^*} \otimes_{\mathscr{O}_{\hat{L}^*}} \hat{\mathscr{K}}_{L,\lambda}[[t]]\Big) \bmod t\Big) \to \mathscr{O}_{\hat{T}} \otimes_{\mathscr{O}_{\hat{L}}} \hat{\mathscr{K}}_{L,\lambda}.$$

By hypothesis the covers $T' \times_{L'} \hat{L}' \to \hat{L}'$, $\hat{T} \times_{\hat{L}} \hat{L}' \to \hat{L}'$, $T'^* \times_{L'^*} \hat{L}'^* \to \hat{L}'^*$, and $\hat{T}^* \times_{\hat{L}^*} \hat{L}'^* \to \hat{L}'^*$ are étale. Consider the category $Et(\hat{L}'^*)$ of étale covers of $\hat{L}'^*$ and the category $Et(\hat{L}')$ of étale covers of $\hat{L}'$. There is a natural functor between them defined by restricting to the closed fiber of the cover in $Et(\hat{L}'^*)$. By [Gr, I, Cor. 8.4], this is an equivalence of categories, so it must take homomorphisms to homomorphisms. In particular, it must take isomorphisms to isomorphisms. Therefore, the isomorphism θ: $\mathscr{O}_{T''} \otimes_{\mathscr{O}_{L'}} \hat{\mathscr{K}}_{L,\lambda} \to \mathscr{O}_{\hat{T}} \otimes_{\mathscr{O}_{\hat{L}}} \hat{\mathscr{K}}_{L,\lambda}$ induced (above) from ϕ: $T \to L$ is also induced by a unique isomorphism

$$\theta^*: \mathscr{O}_{T''^*} \otimes_{\mathscr{O}_{L'^*}} \hat{\mathscr{K}}_{L,\lambda}[[t]] \to \mathscr{O}_{\hat{T}^*} \otimes_{\mathscr{O}_{L^*}} \hat{\mathscr{K}}_{L,\lambda}[[t]].$$

With the isomorphism θ^*, the triple

$$\left(\mathscr{O}_{T''^*}, \mathscr{O}_{\hat{T}^*}, \left(\prod_{\eta \in \phi^{-1}(\lambda)} \hat{\mathscr{K}}_{T^*,\eta}\right)\right)$$

defines an object in

$$\mathscr{SP}(\mathscr{O}_{L'^*}) \times_{\mathscr{SP}(\mathscr{O}_{\hat{L}'^*})} \mathscr{SP}(\mathscr{O}_{\hat{L}^*}).$$

By Theorem 1.1 applied to L^*, the above triple combined with the isomorphism θ^* induces a covering ϕ^*: $T^* \to L^*$ such that $T^* \times_{L^*} L'^* \cong T''^*$ as a cover of L'^* and $T^* \times_{L^*} \hat{L}^* \cong \hat{T}^*$, as a cover of $\hat{L}^*$.

Notice that the closed fibre of ϕ^* gives an object in A in $\mathscr{SP}(L)$. We claim that A corresponds to the element $\mathscr{SP}(L)$ induced by ϕ: $T \to L$. This is because the base change functor in Theorem 1.1 commutes with the functor $\mathscr{SP}(L^*) \to \mathscr{SP}(L)$ defined by restriction. ∎

The above lemma allows us to patch deformed covers when we know the patching on the closed fibre. Next we prove a technical lemma which determines the genus of the deformed curves constructed in Lemma 2.3.

We will denote the arithmetic genus of a curve X by $p_A(X)$ and the Euler characteristic of sheaf $\mathscr{F}$ on X by $\chi(\mathscr{F}) = \sum(-1)^i \dim_k H^i(X, \mathscr{F})$. If $\mathscr{F} = \mathscr{O}_X$ then we write $\chi(X) = \chi(\mathscr{O}_X)$.

LEMMA 2.2. *Suppose that T is a connected reduced k-curve with irreducible components $X_1, \ldots, X_r$. Assume also that the singular locus Υ of T contains at most double points, and let $\nu_{ij} = \#(X_i \cap X_j)$ for $i \neq j$. Then $p_A(T) = (\sum_{i=1}^r p_A(X_i)) + (\sum_{i<j} \nu_{ij}) - r + 1$.*

Proof. Let $\tilde{T}$ be the normalization of T, and let f: $\tilde{T} \to T$ be the associated birational morphism. Similarly, let $\tilde{X}_i$ be the normalization of X_i for $i = 1, 2, \ldots, r$. For any point $\tau \in T$, denote by $\tilde{\mathscr{O}}_\tau$ the integral closure of $\mathscr{O}_\tau$. Then, the morphism f induces an exact sequence of sheaves on T, $0 \to \mathscr{O}_T \to f_* \mathscr{O}_{\tilde{T}} \to \sum_{\tau \in T} \tilde{\mathscr{O}}_\tau / \mathscr{O}_\tau \to 0$, which yields the equality

$$\chi(f_* \mathscr{O}_{\tilde{T}}) = \chi(\mathscr{O}_T) + \chi\left(\sum_{\tau \in T} \tilde{\mathscr{O}}_\tau / \mathscr{O}_\tau \right). \tag{1}$$

As $f: \tilde{T} \to T$ is an affine morphism of noetherian separated schemes, $H^i(\tilde{T}, \mathscr{O}_{\tilde{T}}) = H^i(T, f_* \mathscr{O}_{\tilde{T}})$ [Ht, III, Ex. 4.1]. Thus, $\chi(f_* \mathscr{O}_{\tilde{T}}) = \chi(\tilde{T})$. Moreover, the scheme $\tilde{T}$ is isomorphic to the disjoint union of $\tilde{X}_1, \dots, \tilde{X}_r$, so $\chi(\tilde{T}) = \sum_{i=1}^r \chi(\tilde{X}_i)$. To compute $\chi(\sum_{\tau \in T} \tilde{\mathscr{O}}_\tau / \mathscr{O}_\tau)$, we let δ_τ denote the length of $\tilde{\mathscr{O}}_\tau / \mathscr{O}_\tau$, so that $\dim_k H^0(T, \tilde{\mathscr{O}}_\tau / \mathscr{O}_\tau) = \delta_\tau$. Since the support of each $\tilde{\mathscr{O}}_\tau / \mathscr{O}_\tau$ is a point, $\chi(\tilde{\mathscr{O}}_\tau / \mathscr{O}_\tau) = \dim_k H^0(T, \tilde{\mathscr{O}}_\tau / \mathscr{O}_\tau)$, so $\chi(\sum_{\tau \in T} \tilde{\mathscr{O}}_\tau / \mathscr{O}_\tau) = \sum_{\tau \in T} \chi(\tilde{\mathscr{O}}_\tau / \mathscr{O}_\tau) = \sum_{\tau \in T} \delta_\tau$. If τ lies in $T - \Upsilon$, then T is smooth at τ, so $\delta_\tau = 0$. On the other hand, if $\tau \in \Upsilon$ then T has a normal crossing at τ so $\delta_\tau = 1$ [Ht, IV, Ex. 1.8]. Therefore, $\sum_{\tau \in T} \delta_\tau = \#\Upsilon$. For $i = 1, 2, \dots, r$, let Υ_i be the singular locus of X_i and let $\nu_{ii} = \#\Upsilon_i$. Then we have $\Upsilon = \bigcup_{i<j} (X_i \cap X_j) \cup \bigcup_{i=1}^r \Upsilon_i$, which implies that $\#\Upsilon = \sum_{i \le j} \nu_{ij}$. Therefore, $\sum_{\tau \in T} \delta_\tau = \sum_{i \le j} \nu_{ij}$. Rewriting (1), we get $\chi(T) = \sum_i^r \chi(\tilde{X}_i) - \sum_{i \le j} \nu_{ij}$. Since $p_A(X) = 1 - \chi(X)$ for any curve X, this implies

$$\begin{aligned} 1 - p_A(T) &= \sum_{i=1}^r \left(1 - p_A\left(\tilde{X}_i\right)\right) - \sum_{i \le j} \nu_{ij} \\ &= r - \sum_{i=1}^r \left(p_A\left(\tilde{X}_i\right) + \nu_{ii} \right) - \sum_{i<j} \nu_{ij}. \end{aligned} \tag{2}$$

Each component X_i of T is an irreducible k-curve, so again, $\chi(X_i) = \chi(\tilde{X}_i) - \sum_{\tau \in X_i} \delta_\tau$. From this we get $p_A(X_i) = p_A(\tilde{X}_i) + \sum_{\tau \in X_i} \delta_\tau$. Since every point $\tau \in \Upsilon_i$ is a node on X_i, $\delta_\tau = 1$ as above, so $\sum_{\tau \in X_i} \delta_\tau = \nu_{ii}$. Therefore $p_A(X_i) = p_A(\tilde{X}_i) + \nu_{ii}$. Substituting this into (2) yields $p_A(T) = (\sum_i^r p_A(X_i)) - r + 1 + \sum_{i<j} \nu_{ij}$, as desired. ∎

Next, in Lemma 2.3, we construct a degenerate k-curve T of genus g from two smooth k-curves of lower genus. Then, we deform T to get a smooth K-curve of genus g. The construction and local deformation of T are concrete, and the patching uses Lemmas 2.1 and 2.2. As stated in the introduction to this section, Lemma 2.3 corresponds to the first part of the proof of the main result (Theorem 3.1).

LEMMA 2.3. *Let L be the projective y-line over k, let n be a positive integer, and let X_1 and X_2 be smooth connected projective k-curves of genus g_1 and g_2, respectively. For $i = 1, 2$, suppose that ϕ_i: $X_i \to L$ is a covering morphism, with branch locus B_i, and suppose λ is a closed point in $L - (B_1 \cup B_2)$ with local parameter x. Choose $\tau_i \in \phi_i^{-1}(\lambda)$ and identify L with its* *image in* $L^* = L \times_k \operatorname{Spec}(k[[t]])$ (via the inclusion $L \to L^*$). *Then there exists a normal projective* $k[[t]]$*-curve* T^*, *a covering morphism* $\phi^*: T^* \to L^*$, *and a point* $\tau \in (\phi^*)^{-1}(\lambda)$ *such that*

(a) *on the closed fibre* $\phi: T \to L$ *of* $\phi^*: T^* \to L^*$, *the* k*-curve* T *is isomorphic to a union of* X_1 *and* X_2, *with* τ_1 *and* τ_2 *identified at a normal crossing* $\tau \in T$;

(b) *the* $k[[t]]$*-curve* T^* *is regular away from* τ, *and there is an isomorphism of* $k[[x,t]]$*-algebras between* $\hat{\mathscr{O}}_{T^*,\tau}$ *and* $k[[x_1, x_2, t]]/(x_1x_2 - t^n)$, *where* $x_1 + x_2 = x$;

(c) *on the generic fibre* $\phi^o: T^o \to L^o = L \times_k \operatorname{Spec}(K)$ *of* $\phi^*: T^* \to L^*$, *the scheme* T^o *is a regular irreducible projective* K*-curve of genus* $g_1 + g_2$.

Proof. We will proceed in three steps: Step 1 builds the closed fibre ϕ: $T \to L$; Step 2 deforms this cover locally and applies Lemma 2.1 to prove (a) and (b); then, Step 3 proves (c) using Lemma 2.2.

Step 1. Construction of the closed fibre T. Take $x_i \in \hat{\mathscr{O}}_{X_i, \tau_i}$, to be a local uniformizer at τ_i on X_i such that $\phi^*(x) = x_i$. Let $X = X_1 \times X_2$, and $T = (X_1 \times \{\tau_2\}) \cup (\{\tau_1\} \times X_2) \subset X$. Then the complete local ring $\hat{\mathscr{O}}_{T,\tau} \cong k[[x_1, x_2]]/(x_1x_2)$. Identifying X_i with its image in T, the k-curve T is a union of X_1 and X_2 with τ_1 identified with τ_2 at a normal crossing $\tau \in T$. The morphisms ϕ_i define rational functions s_i on T with $s_i(\gamma) = 0$ for all $\gamma \in X_j$ $(j \neq i)$. Consider the rational function $s = s_1 + s_2$. This restricts to a cover $\phi: T \to L$ which takes τ to the point λ on L. If we let d_i denote the degree of $\phi_i: X_i \to L$, then ϕ is a cover of degree $d_1 + d_2$.

Now we explicitly describe the local patches of ϕ which will be deformed in Step 2. Let $L' = L - \{\lambda\}$ and $\hat{L} = \operatorname{Spec}(\hat{\mathscr{O}}_{L,\lambda}) = \operatorname{Spec}(k[[x]])$. Pulling back $\phi: T \to L$ over the inclusions $L' \to L$ and $\hat{L} \to L$, we get covers $\phi': T'' \to L'$ and $\hat{\phi}: \hat{T} \to \hat{L}$, respectively (where $T'' = T - \phi^{-1}(\lambda)$, and $\hat{T}$ is the disjoint union over $\eta \in \phi^{-1}(\lambda)$ of $\hat{T}_\eta = \operatorname{Spec}(\hat{\mathscr{O}}_{T,\eta})$). For $\hat{\phi}$, if $\eta \neq \tau$, then there is a unique $i \in \{1, 2\}$ such that $\eta \in X_i$, and this implies that $\hat{T}_\eta = \operatorname{Spec}(\hat{\mathscr{O}}_{X_i,\eta})$. Otherwise, $\eta = \tau$, and we get $\hat{T}_\tau = \operatorname{Spec}(k[[x_1, x_2]]/(x_1x_2))$, where $\hat{\phi}$ restricted to $\hat{T}_\tau$ is defined by $x_1 + x_2 = x$. By hypothesis ϕ_1 and ϕ_2 are unramified at λ for $i = 1, 2$, and thus the cover $\phi': T'' \times_{L'} \hat{L}' \to \hat{L}'$ is étale and the cover $\hat{\phi}: \hat{T} \to \hat{L}$ is flat and étale away from $\tau \in T$.

Step 2. *Deformation of* $\phi: T \to L$. We define the deformation of ϕ: $T \to L$ locally over the patches described above in such a way that Lemma 2.1 applies.

To deform $\phi': T'' \to L'$, let $L'^* = \operatorname{Spec}(\mathscr{O}_{L'}[[t]])$ and define $\phi'^*: T'^* \to L'^*$ to be the pullback of ϕ' via the morphism $L'^* \to L'$ (from the natural ring morphism). The new cover ϕ'^* is a trivial deformation of its closed fibre ϕ', and as such is étale over $\hat{L}'^* = \operatorname{Spec}(\hat{\mathscr{K}}_{L,\lambda}[[t]])$.

To deform $\hat{\phi}$, we first deform each $\hat{\phi}|_{\hat{T}_\eta}$ for $\eta \in \phi^{-1}(\lambda)$, and then take their disjoint union. There are two cases: Either $\eta \in (\phi^{-1}(\lambda) - \{\tau\})$ or not. If $\eta \in (\phi^{-1}(\lambda) - \{\tau\})$, then let $\hat{\phi}^*_\eta \colon \hat{T}^*_\eta \to \hat{L}^*$ be the pullback of $\hat{\phi}|_{\hat{T}_\eta}$ via $\hat{L}^* \to \hat{L}$. Since the ϕ_i's are unramified over λ and $\eta \neq \tau$, $\hat{\phi}^*_\eta \colon \hat{T}^*_\eta \to \hat{L}^*$ is an étale cover. Otherwise, $\eta = \tau$, and we let $\hat{T}^*_\tau = \operatorname{Spec}(k[[x_1, x_2, t]]/(x_1 x_2 - t^n))$. Define $\hat{\phi}^*_\tau \colon \hat{T}^*_\tau \to \hat{L}^*$ via $x_1 + x_2 = x$. The Jacobian criterion shows that $\hat{T}^*_\tau$ is regular away from τ. Thus, $\hat{T}^*_\tau$ is normal over $\hat{L}^*$ and hence (by Lemma 1.2) flat. Moreover, the cover $\hat{\phi}^*_\tau$ is induced by the extension of $k[[x, t]]$ by $h(z) = z^2 - xz + t^n$, so ramification occurs over the locus B in $\hat{L}^*$ defined by the discriminant $x^2 - 4t^n$. Notice that in $\mathscr{O}_{\hat{L}'^*} = \mathscr{K}_{L,\lambda}[[t]] = k((x))[[t]]$ the power series $x^2 - 4t^n$ is invertible. Therefore, B does not lie in $\hat{L}'^*$, so $\hat{T}^*_\tau \times_{\hat{L}^*} \hat{L}'^* \to \hat{L}'^*$ is an étale cover of $\hat{L}'^*$. Let $\hat{T}^*$ be the disjoint union over $\eta \in \phi^{-1}(\lambda)$ of $\hat{T}^*_\eta$'s, and let $\hat{\phi}^* \colon \hat{T}^* \to \hat{L}^*$ be the flat cover induced by the $\hat{\phi}^*_\eta$'s. Since each $\hat{\phi}^*_\eta$ is étale over $\hat{L}'^*$, the cover $\hat{\phi}^*$ is also étale over $\hat{L}'^*$.

Now by Lemma 2.1 there exists a projective $k[[t]]$-curve T^*, a covering morphism $\phi^* \colon T^* \to L^*$, such that $T^* \times_{L^*} L'^* \cong T''^*$, $T^* \times_{L^*} \hat{L}^* \cong \hat{T}^*$, and $T^* \times_{L^*} L \cong T$ as covers of L'^*, $\hat{L}^*$, and L, respectively. So, (a) is satisfied. To check where T^* is regular, it is enough to look at T''^* and $\hat{T}^*$. The deformation T''^* is a trivial deformation of smooth curves so it is regular. Moreover, we showed above that $\hat{T}^*$ is regular away from τ, so (b) is now satisfied. And, since $\hat{T}^*$ is regular away from τ, we also have that T^* is normal. Thus by Lemma 1.2, T^* defines a flat family over L^* which is regular on the generic fiber.

Step 3. The genus of T^*. Let $\phi^o \colon T^o \to L^o$ be the generic fiber of $\phi^* \colon T^* \to L^*$; then T^o is a connected K-curve because T is. Since T^o is a smooth cover of $L^o \cong \mathbb{P}^1_K$, it must be projective. It remains to show that the genus of T^o is $g_1 + g_2$. The scheme T^* is a flat family of curves, and hence the arithmetic genus is constant on the fibres [Ht, III, Cor. 9.10]. Therefore, it suffices to show that $p_A(T) = g_1 + g_2$. This follows by applying Lemma 2.2 with $r = 2$ and $\#(X_1 \cap X_2) = 1$. ∎

Pick a compatible set of nth roots of unity $\zeta_n \in k$ for all positive integers n prime to p. That is, take a set of nth roots of unity so that $\zeta_{nm}^m = \zeta_n$ for all positive integers n and m. Let $\tau_1, \tau_2, \ldots, \tau_s$ be distinct closed points on a smooth connected projective k-curve X. Let $\pi_A^t(X - \{\tau_1, \tau_2, \ldots, \tau_s\})$ be the set of finite groups G which occur as Galois groups of regular Galois covers of X, at most tamely ramified over $\{\tau_1, \tau_2, \ldots, \tau_s\}$, and étale off this set. Let $Z \to X$ be such a cover and let n_i be the ramification index of a point σ_i over τ_i for each i. Pick a uniformizer x_i at τ_i on X, and take $z_i \in \hat{\mathscr{O}}_{Z,\sigma_i}$ to be a uniformizer at σ_i such that $z_i^{n_i} = x_i$. Then the *canonical generator* of the inertia group of σ_i over τ_i is the element c_i of the inertia group which takes z_i to $\zeta_{n_i} z_i$ (this is independent of the choice of x_i and z_i). Now, given $G \in \pi_A^t(X - \{\tau_1, \tau_2, \ldots, \tau_s\})$ and a corresponding G-Galois cover $Z \to X$, if for each i there is a point σ_i in Z over τ_i whose inertia group has canonical generator c_i, we say that G lies in $\pi_A^t(X - \{\tau_1, \tau_2, \ldots, \tau_s\})$ with *description* $(c_1, c_2, \ldots, c_s)$. Notice that this description is determined up to conjugation of the individual c_i's which corresponds to the choice of each point σ_i over each τ_i.

Recall from Section 1, that if H is a subgroup of a finite group G, and $Z \to Y$ is an H-Galois cover, then there is an induced G-Galois cover $\mathrm{Ind}_H^G Z \to Y$ which is the disjoint union of $(G : H)$ copies of Z, indexed by the left cosets of H in G. The stabilizer of the identity copy is $H \subset G$, and the stabilizers of the other copies are the conjugates of H in G.

PROPOSITION 2.4. *In the situation of Lemma* 2.3, *assume that* $(p, n) = 1$, *and let G be a finite group with subgroups H_1 and H_2 which together generate G. Suppose that for $i = 1, 2$, H_i lies in $\pi_A^t(X_i - \tau_i)$, with description (c_i) in H_i, such that $c_1 = c_2^{-1}$ and* $\mathrm{ord}(c_i) = n$. *Then over the $k[[t]]$-curve T^* there exists a normal connected G-Galois cover $\psi^*\colon C^* \to T^*$ such that the generic fibre $\psi^o\colon C^o \to T^o$ is an unramified connected G-Galois cover of the K-curve T^o.*

Proof. By hypothesis, for $i = 1, 2$ there exist H_i-Galois covers $\psi_i\colon W_i \to X_i$ ramified only over $\tau_i \in X_i$. Let ω_i be a point in $\psi_i^{-1}(\tau_i)$ with inertia group canonically generated by c_i. Restricting ψ_i gives H_i-Galois covers $\psi_i'\colon W_i' \to X_i' = X_i - \{\tau_i\}$, where X_i' is identified with its image in T (the closed fibre of T^* from Lemma 2.3). Let $X_i'^*$ and $\hat{X}_i'^*$ be the formal completions of T^* along X_i' and $\hat{X}_i' = \mathrm{Spec}(\hat{\mathscr{K}}_{X_i, \tau_i}) = \mathrm{Spec}(k((x_i)))$, respectively ($x_i$ is a local parameter at τ_i on X_i). Let $\hat{W}_i' = \mathrm{Spec}(\hat{\mathscr{K}}_{W_i, \omega_i})$. Take $w_i \in \hat{\mathscr{O}}_{W_i, \omega_i}$ to be a uniformizer so that $w_i^n = x_i$ and $c_i(w_i) = \zeta_n w_i$. Then $\hat{W}_i' = \mathrm{Spec}(k((w_i)))$. Pulling back W_i' and $\hat{W}_i'$ by $X_i'^* \to X_i'$ and $\hat{X}_i'^* \to \hat{X}_i'$, respectively, we obtain H_i-Galois unramified covers $\psi_i'^*\colon W_i'^* \to X_i'^*$ and $I = \langle c_i \rangle$-Galois covers $\hat{W}_i'^* \to \hat{X}_i'^*$.

Let $S^* = \mathscr{O}_{\hat{T}_\tau^*}$; here $S^* = k[[x_1, x_2, t]]/(x_1 x_2 - t^n))$. Now let

$$\hat{N}^* = \mathrm{Spec}\big(k[[z_1, z_2, t]]/(z_1 z_2 - t)\big),$$

and define an I-Galois cover $\hat{\psi}^*\colon \hat{N}^* \to \hat{T}_\tau^*$ by setting $z_1^n = x_1$ and $z_2^n = x_2$ and letting $I = \langle c_1 \rangle$ act by $c_1(z_1) = \zeta_n z_1$ and $c_1(z_2) = c_2^{-1}(z_2) = \zeta_n^{-1} z_2$. Consider the $I = \langle c_i \rangle$-Galois cover of $\hat{X}_i'^* = \mathrm{Spec}(k((x_i))[[t]])$ given by $\hat{N}_i'^* = \hat{N}^* \times_{\hat{T}_\tau^*} \hat{X}_i'^*$. Notice that for $(i, j) = (1, 2)$ or $(2, 1)$,

$$\begin{aligned}
\mathscr{O}_{\hat{N}_i'^*} &= k[[z_1, z_2, t]]/(z_1 z_2 - t) \otimes_{S^*} k((x_i))[[t]] \\
&= k((x_i))[[z_1, z_2, t]]/(z_1 z_2 - t, z_i^n - x_i) \\
&= k((z_i))[[z_j, t]]\Big/\left(z_j - \frac{t}{z_i}\right) = k((z_i))[[t]],
\end{aligned}$$

so the $\langle c_i \rangle$-cover $\hat{N}_i'^* \to \hat{X}_i'^*$ is defined via $z_i^n = x_i$ and $c_i(z_i) = \zeta_n z_i$. Now, sending z_i to w_i defines an isomorphism of I-covers $\hat{N}_i'^* \to \hat{W}_i'^*$.

By Proposition 1.3, applied with $H_i = G_i$, $i = 1, 2$, and $I = \langle c_1 \rangle = \langle c_2 \rangle$, there exists an irreducible normal G-Galois cover $\psi^*: C^* \to T^*$, such that $C^* \times_{T^*} X_i'^* \cong \mathrm{Ind}_{H_i}^G W_i'^*$ as G-Galois covers of $X_i'^*$, and $C^* \times_{T^*} \hat{T}_\tau^* \cong \mathrm{Ind}_I^G \hat{N}^*$ as G-Galois covers of $\hat{T}^*$.

If $\psi^o: C^o \to T^o$ denotes the generic fibre, then C^o is connected because C^* is irreducible. Since ψ_i' is unramified for $i = 1, 2$, the trivial deformation ψ'^* is also unramified. Therefore, ψ^* is ramified only on $\mathrm{Ind}_I^G \hat{N}^*$ over $\hat{T}_\tau^*$. The Jacobian criterion implies that $\mathrm{Ind}_I^G \hat{N}^* \to \hat{T}_\tau^*$ is branched only at the point defined by $(x_1 = x_2 = t = 0)$, which corresponds to τ on $\hat{T}_\tau^*$. Thus the cover ψ^o is étale, and hence C^o is regular and projective because T^o is. Consequently, $\psi^o: C^o \to T^o$, is an irreducible, unramified cover. ∎

The following result globalizes our construction.

PROPOSITION 2.5. *Let T^* be a normal (hence flat) projective $k[[t]]$-curve and let $\psi^*: C^* \to T^*$ be a G-Galois cover of $k[[t]]$-curves. Let $\psi^o: C^o \to T^o$ be its generic fibre and assume that C^o is geometrically connected over K, ψ^o is unramified, and T^o is a regular, projective K-curve of genus g. Then there exist a k-variety $E = \mathrm{Spec}(A)$, where $A \subset K$ and of finite type over $k[t]$; a flat projective E-curve T_E; and a regular G-Galois cover $\psi_E: C_E \to T_E$, such that*

(a) *$T_E \times_k \mathrm{Spec}(K)$ is isomorphic to T^o, and $C_E \times_k \mathrm{Spec}(K)$ is isomorphic to C^o as a G-Galois cover of T^o;*

(b) *for every closed point e in E, the fibre $\psi_e: C_e \to T_e$ is a connected unramified G-Galois cover of a smooth irreducible projective k-curve of genus g.*

Proof. Since the connected normal G-Galois cover $\psi^*: C^* \to T^*$ is of finite presentation, it descends to a regular $k[t]$-algebra $R \subset k[[t]]$ of finite type over $k[t]$. That is, for some such algebra R there is an irreducible normal flat projective R-scheme T_R and an irreducible normal projective R-scheme C_R, together with a G-Galois covering morphism $C_R \to T_R$ which induces $\psi^*: C^* \to T^*$ over $k[[t]]$. Moreover, if we let $A = R[t^{-1}]$ and $E = \mathrm{Spec}(A)$, then $\psi_E: C_E = (C_R \times_R E) \to T_E = (T_R \times_R E)$ is a regular (hence flat), projective cover which satisfies (a) and has unramified fibres.

Since C_R induces C^*, the fibre of C_R over $(t = 0)$ is connected and generically smooth. Moreover C^* is normal. If we apply [Ha1, Prop. 5] to $C_R \to \mathrm{Spec}(R)$, and let ϵ be the point $(t = 0)$, it follows that for all k-points e in a dense open subset of $\mathrm{Spec}(R)$ (and hence in a dense open subset E' of $E = \mathrm{Spec}(R) - (t = 0)$), the fibre C_e is geometrically irreducible. We may assume that E' is a basic open subset $\mathrm{Spec}(B)$ of E, where $B = R[t^{-1}, f^{-1}]$, for some non-zero $f \in R[t^{-1}]$.

It remains to show that each fibre T_e has genus g. The scheme T_E is a flat projective algebraic family of k-curves parameterized by the variety E, so by [Ht, Cor. 9.3], the Hilbert polynomial, and hence the arithmetic genus, is constant on the fibres. Consider the fibre $T_E \times_k \mathrm{Spec}(K)$, which is isomorphic to T^o. Since T^o is a projective regular K-curve of (geometric) genus g, its arithmetic genus is also g. Thus, the arithmetic genus of the fibre $T_E \times_k \mathrm{Spec}(K)$ is g, which implies that the arithmetic genus of every fibre of T_E is g. The fibres T_e are smooth and projective, so in fact their geometric genus is g. ∎

3. CONSTRUCTION OF GALOIS COVERS

The goal of this section is to prove that certain classes of groups are Galois groups over projective k-curves of genus g. This is done by using the results of Section 2 to construct Galois covers of certain projective curves of genus g. In the next section, we show that each of these groups must therefore also arise as Galois groups over generic curves of genus g.

We begin with tamely ramified Galois covers of two k-curves of genus g_1 and g_2 (where $g_1 + g_2 = g$). In Theorem 3.1, these covers are patched together in such a way that the ramification cancels. This can be interpreted as an analog to Van Kampen's theorem. Using this result and generalizations of it, we gain some insight into which groups occur in the set π_A of a curve of genus g. For example, Corollary 3.2 implies that when $g = 2$ there are groups generated by a minimum of three elements, which occur in π_A (see Remark (1) below). This is compatible with Grothendieck's result, which shows that any group occurring in π_A of a curve of genus g must be generated by $2g$ generators subject to the commutator relation [Gr, XIII, Cor. 2.12]. Also, Corollary 3.5 shows that every finite group with g generators occurs in π_A of some curve of genus g.

Notation. Let $\pi_A(g)$ denote the set of groups G for which there exists a smooth connected projective k-curve X of genus g with $G \in \pi_A(X)$.

THEOREM 3.1. *Let G be a finite group, let H_1 and H_2 be subgroups which together generate G, and let g_1 and g_2 be positive integers. Suppose that for $i = 1, 2$, there exists a smooth connected projective k-curve X_i of genus g_i such that H_i lies in $\pi_A^t(X_i - \{\tau_i\})$ with description (c_i), where $c_1 = c_2^{-1}$ in G. Then G lies in $\pi_A(g_1 + g_2)$.*

Proof. Let L be the projective k-line. There exists a branched cover ϕ_i: $X_i \to L$, for $i = 1, 2$. Let B_i be the branch locus of ϕ_i and choose a closed point λ on L so that $\lambda \notin B_1 \cup B_2$. Now pick $\tau_i \in \phi_i^{-1}(\lambda)$ and identify L with its image in $L^* = L \times_k \operatorname{Spec}(k[[t]])$. Applying Lemma 2.3 (with $n = \operatorname{ord}(c_i)$) we obtain a normal projective $k[[t]]$-curve T^*, a covering morphism $\phi^*: T^* \to L^*$, and a point $\tau \in (\phi^*)^{-1}(\lambda)$ satisfying (a), (b), and (c) of that result. Then the curve T^o in the generic fiber $\phi^0: T^o \to L^o$ of $\phi^*: T^* \to L^*$ is a regular connected projective K-curve of genus $g_1 + g_2$.

By Proposition 2.4 with $n = \operatorname{ord}(c_1)$, there exists a normal connected G-Galois cover $\psi^*: C^* \to T^*$ such that the generic fibre $\psi^o: C^o \to T^o$ is an unramified regular connected G-Galois cover of K-curves. Moreover, the genus (arithmetic and geometric) of T^o is $g_1 + g_2$. Proposition 2.5 allows us to descend this cover to one defined over to a ring A of finite type over $k[t]$ satisfying (a) and (b) of that result. Therefore by specializing to a k-point of E, we are done. ∎

COROLLARY 3.2. *Suppose that G is a finite group which is generated by elements a_1, b_1, a_2, b_2. Assume also that $[a_1, b_1][a_2, b_2] = 1$ and that* $\operatorname{char}(k) = p$ *is prime to the orders of the groups $\langle a_i, b_i \rangle$ for $i = 1, 2$. Then $G \in \pi_A(2)$.*

Proof. Let $H_i = \langle a_i, b_i \rangle$ and let $[a_i, b_i] = c_i$ for $i = 1, 2$. Let X_i be an elliptic curve, and pick a closed point τ_i on X_i. Now because p is prime to the order of the group H_i, we know that H_i lies in $\pi_A^t(X_i - \{\tau_i\})$ with description (c_i). In the group G, we have $c_1 = c_2^{-1}$ because $c_1 c_2 = [a_1, b_1][a_2, b_2] = 1$. Since G is generated by H_1 and H_2, we can apply Theorem 3.1. ∎

Remarks. (1) Using the computer program GAP one can find a group G of order 288 with the above description in terms of generators and relations. This G is neither solvable nor generated by any two elements (cf. Cor. 3.5). (2) Notice that in Corollary 3.2 the condition that p does not divide the order of the group $H_i = \langle a_i, b_i \rangle$ is just to ensure that H_i lies in $\pi_A^t(X_i - \{\tau_i\})$. This is probably a much stronger condition than is necessary, but very little is currently understood about the groups arising as Galois groups of tame coverings of affine curves. Finally: (3) Taking the genus equal to 2 was not important in Corollary 3.2. A similar argument shows that a group with $2g_1 + 2g_2$ generators (subject to a similar commutator relation), will occur in $\pi_A(g_1 + g_2)$.

If we allow only unramified covers of the components of the degenerate curve, then by induction on the number of components, we obtain the following result.

THEOREM 3.3. *Let G be a finite group and let $H_1, H_2, \dots, H_m$ be subgroups of G which together generate G. Let $g_1, g_2, \dots, g_m$ be non-negative integers with $g = \sum_{i=1}^m g_i$. Suppose that for every $i = 1, 2, \dots, m$ there* *exists a smooth connected projective curve* X_i *of genus* g_i *with* $H_i \in \pi_A(X_i)$. *Then* $G \in \pi_A(g)$.

Proof. Using induction on m, it suffices to show the result for $m = 2$. Now the theorem follows from Theorem 3.1 in the special case where c_i is the identity. ∎

Notice that combining results 3.2 and 3.3 gives another family of groups with $4g$ generators that occur in $\pi_A(2g)$: Let G be a group by elements $\{c_{i1}, d_{i1}, c_{i2}, d_{i2} : i = 1, 2, \ldots, g\}$ subject to the relations $[c_{i1}, d_{i1}][c_{i2}, d_{i2}] = 1$ for all i, and the restriction that p does not divide the orders of the subgroups generated by each $\{c_{i1}, d_{i1}, c_{i2}, d_{i2}\}$; then 3.2 and 3.3 together imply that G lies in $\pi_A(2g)$.

Note: In the rest of this paper, all projective curves are required to be smooth connected k-curves.

COROLLARY 3.4. *Let* X *be a projective curve of genus* g, *and let* G *be a finite group lying in* $\pi_A^t(X - \{\tau\})$. *Then* $G \in \pi_A(2g)$.

Proof. Let $Z \to X$ be the G-Galois cover tamely ramified at τ. Then by [Gr, XIII, Cor. 2.12], G must be generated by elements $\{a_1, b_1, \ldots, a_g, b_g, c\}$, where $[a_1, b_1]\ldots[a_g, b_g] = c$. Therefore G may be generated by $2g$ elements, and so by Theorem 3.3 there exists a projective curve of genus $2g$ with $G \in \pi_A(X)$. ∎

COROLLARY 3.5. *Given any finite group* G *with* g *generators*, $G \in \pi_A(g)$.

Proof. Suppose that $\{a_1, a_2, \ldots, a_g\}$ generate G. In the hypothesis of Theorem 3.3 let $m = g$. For each $i = 1, 2, \ldots, g$ let $H_i = \langle a_i \rangle$ and let X_i be an ordinary elliptic curve. Then the H_i's together generate G, and since H_i is cyclic, H_i lies in $\pi_A(X_i)$. Apply Theorem 3.3. ∎

As an example, consider $G = S_n$ the symmetric group on n objects. The group S_n is generated by (12) and $(1234\ldots n)$, so by Corollary 3.5 the group S_n lies in $\pi_A(2)$. Similarly, we have the following result for any finite simple group.

COROLLARY 3.6. *If* G *is any finite simple group, then* $G \in \pi_A(2)$.

Proof. By the Classification Theorem of finite simple groups [Go], every finite simple group is generated by two elements. Now use Corollary 3.5. ∎

Corollary 3.5 also gives us the following result, which was originally shown by Serre in 1956 [Se].

COROLLARY 3.7. *If* G *is any finite group, then* $G \in \pi_A(g)$ *for some positive integer* g.

Proof. Clear from Corollary 3.5. ∎

For any real number s let $\lfloor s \rfloor$ denote the greatest integer less than or equal to s.

COROLLARY 3.8. *If G is a finite group of order n, then the group G lies in $\pi_A(g)$ for $g = \lfloor \log_2(n) \rfloor$.*

Proof. We use induction on n to show that G can be generated by $\lfloor \log_2(n) \rfloor$ elements. If G has only two elements, then clearly it is generated by $1 = \lfloor \log_2(2) \rfloor$ element. Suppose that for all groups of order less than n the statement holds. Then, let H be a maximal proper subgroup of G, so it has at most $n/2$ elements. By the induction hypothesis, H must have a generating set consisting of $\lfloor \log_2(n/2) \rfloor$ elements, and hence G has a generating set of $\lfloor \log_2(n/2) \rfloor + 1 = \lfloor \log_2(n) \rfloor$ elements. To finish the proof, apply Corollary 3.5. ∎

The next result relates π_A's of curves of genus g to π_A's of curves of higher genus.

COROLLARY 3.9. *Let G be a finite group and g be a positive integer. Then $G \in \pi_A(g)$ implies that for every integer $g' > g$, $G \in \pi_A(g')$.*

Proof. It suffices to show this for $g' = g + 1$. Let X_1 be a projective curve of genus g such that $G \in \pi_A(X_1)$ and let X_2 be an elliptic curve. Then let $H_1 = G$ and $H_2 = \langle e \rangle$, where e is the identity element of G. Now apply Theorem 3.3 with $m = 2$. ∎

COROLLARY 3.10. *Given a finite group $G = \langle a_1, b_1, a_2, b_2 \rangle$, if $[a_1, b_1] = 1$, $[a_2, b_2] = 1$, and if $p = \mathrm{char}(k)$ does not divide orders of a_1 and a_2, then $G \in \pi_A(2)$.*

Proof. For $i = 1, 2$ let H_i be generated by elements a_i, b_i in G, and let X_i be an ordinary elliptic curve. Then since $[a_i, b_i] = 1$ and p does not divide orders of a_i, $H_i \in \pi_A(X_i)$. Apply Theorem 3.3. ∎

The above results all relate to the following question.

QUESTION 3.11. For a fixed genus, which groups in π_A in characteristic 0 also occur in characteristic $p > 0$?

Since it is believed that the answer to Question 3.11 will depend on more than just the genus of the k-curve, we may ask instead which finite groups occur over most curves of a fixed genus. More specifically, for fixed genus g, let M_g denote the moduli space of curves of genus g, and let X_g be the generic curve of genus g (i.e., the curve corresponding to the geometric generic point of M_g). Then for any finite group G, let U_G denote the subset of M_g consisting of the points for which G lies in π_A of the corresponding curve. Both here and in Section 5 we find examples of groups G for which $U_G \subset M_g$ is not empty. Consider now $\pi_A(X_g)$, which is equal to the set of finite groups for which U_G is *dense* in M_g. In the next section we prove that in fact $\pi_A(g) = \pi_A(X_g)$. Therefore, our modified question is asking how $\pi_A(g)$ in characteristic 0 compares with that in characteristic p.

One approach to this question is to look at Hasse–Witt invariants of curves and their n-cyclic covers (n is prime to p) [cf. Bo, Kt, Na1, Na2]. This is interesting because the Hasse–Witt invariant of a curve indicates how many p-cyclic covers of that curve exist. In [Na1], Nakajima found a family of groups occurring in $\pi_A(g)$, and related these groups to the "p-rank" (Hasse–Witt invariant) of n-cyclic covers of curves of genus g. The following is an example of one of these groups.

EXAMPLE. Let k be an algebraically closed field of characteristic $p = 2$ and X be the smooth connected projective k-curve of genus 2 defined by the equation $y^2 + y = x^5 + Ax^3$. Nakajima shows in [Na1, Sect. 6] that there are 40 non-isomorphic connected étale A_4-Galois covers of X. So, A_4 lies in $\pi_A(2)$. Moreover, he uses the fact that 40 is maximal for the number of non-isomorphic A_4-covers of any genus two curve, to show that X is "3-ordinary" (a statement concerning the "p-rank" of 3-cyclic Galois covers of X).

Recall from the Introduction that Grothendieck's result [Gr, XIII, Cor. 2.12] implies that $\pi_A(g)$ is contained in the finite quotients of $F_{g,0}$ (or equivalently, of $\hat{F}_{g,0}$, the profinite completion of $F_{g,0}$). Moreover, this containment is strict because the p-rank is bounded by the genus in characteristic p. For genus one, $\pi_A(1)$ is equal to the set of finite quotients of the inverse limit $\hat{F}_1$ of the finite groups generated by elements a, b such that $[a, b] = 1$ and p does not divide the order of a. Thus $\hat{F}_1$ is a quotient of $\hat{F}_{0,1}$. In genus ≥ 1, consider the following two quotients of $\hat{F}_{g,0}$:

(1) the inverse limit $\hat{F}'_g$ of the finite groups generated by elements $a_1, b_1, a_2, b_2, \ldots, a_g, b_g$ such that $[a_1, b_1][a_2, b_2]\ldots[a_g, b_g] = 1$ and p does not divide the order of a_i for all $i = 1, 2, \ldots, g$;

(2) the inverse limit $\hat{F}''_g$ of the finite groups generated by elements $a_1, b_1, a_2, b_2, \ldots, a_g, b_g$ such that $[a_1, b_1][a_2, b_2]\ldots[a_g, b_g] = 1$ and p does not divide the order of the subgroup generated by $\{a_1, a_2, \ldots, a_n\}$.

Notice that for genus one, $\hat{F}_1 = \hat{F}'_1 = \hat{F}''_1$, whereas for all $g > 1$ the groups $\hat{F}'_g$ and $\hat{F}''_g$ are not equal. However, in either case the set of finite quotients is *strictly* contained within the set of quotients of $\hat{F}_{g,0}$. Moreover, all the examples (both here and in Section 5) of groups occurring over genus g curves also occur as quotients of $\hat{F}'_g$ and $\hat{F}''_g$. This suggests the question of whether $\pi_A(g)$ is the set of finite quotients of either of these groups.

For the first of these groups, this is not in general the case because of a result of Nakajima.

THEOREM 3.12 [Na1, Theorem 5]. *If a finite group* G *lies in* $\pi_A(g)$*, then there exists a surjective* $k[G]$*-homomorphism* $k[G]^g \to I_G$.

Using this we obtain the following proposition. I thank Bob Guralnick for his help with this, and particularly for the example in the proof below.

PROPOSITION 3.13. *For* $g > 2$*, the set* $\pi_A(g)$ *is not equal to the set of finite quotients of* $\hat{F}'_g$.

Proof. Let p be an odd prime. Let G be the semi-direct product of $V = (\mathbb{Z}/p\mathbb{Z})^{2g-2}$ and $C = C_q$ the cyclic group of order q, where $1 < q | p-1$, and C acts on V as a scalar of order q. Let x be a generator for C and choose generators a_i, b_i for G as follows: $a_i = xv_i$ for some v_i in V with $v_g = 1$, and $b_i = w_i$ for some $w_i \in V$, such that $v_1, \dots, v_{g-1}, w_1, \dots, w_{g-1}$ are independent in V and $b_g \in V$ with $[x, b_g]^{-1} = [a_1, b_1] \dots [a_{g-1}, b_{g-1}]$ (this is possible because every element in V is of the form $[x, v]$ for some $v \in V$). On the other hand, if we take the 1-dimensional $k[G]$-module S where V acts trivially and x acts via the character corresponding to the qth root of 1 corresponding to the action on V, then $\dim H^1(G, S) = 2g - 2$. This in turn implies that the augmentation ideal needs at least $2g - 2$ generators, so with Nakajima's result, for $g > 2$ this is a counterexample. ∎

In light of Proposition 3.13, we are left with the question of whether the set of quotients of $\hat{F}''_g$, or some variant of it, is equal to $\pi_A(g)$. However, we could also ask the corresponding question about covers of affine k-curves and finite quotients of $\hat{F}_{g,r}$. Namely, let $\pi_A^t(g, r)$ be the set of Galois groups over a genus g curve tamely branched at r given points and unramified elsewhere. Since we are not allowing wild ramification, this set is strictly contained in the finite quotients of $\hat{F}_{g,r}$ [Gr, XIII, Cor. 2.12]. How much smaller is it? The following example gives a family of quotients of $\hat{F}_{g,r}$, the members of which are *not* Galois groups of tame covers of a genus g curves tamely branched at r points.

EXAMPLE. For positive integers n, a, and m with $p = \operatorname{char} k$ not dividing na, let G_m be the semi-direct product of the elementary abelian p-group $E = \langle \tau_1, \dots, \tau_s \rangle$ of rank m by the cyclic group $h = \langle \sigma \rangle$ of order n, where the action of H on E is given by $\tau_i^\sigma = \tau_i^a$, $1 \le i \le m$. Then all G_m with $r \ge g + m$ and $a \ne 1 \bmod p$, are quotients of $F_{g,r}$ but not elements of π_A^t of a curve of genus g with r puncturs. This example is due to Kani [Ka, p. 204].

In the patching results of Sections 3 and 5 (viz. patching at tame branch points), knowing that certain groups are Galois groups of tame covers of curves of low genus can be used inductively to show that certain larger groups are Galois groups of unramified covers of projective curves of higher genus. Therefore, the set $\pi_A^t(g, r)$ is useful for constructing covers of projective curves.

4. GALOIS GROUPS OVER GENERIC CURVES

In this section we show that if a finite group G lies in π_A of a projective k-curve of genus g, then G lies in π_A of k-curves of genus g generically. More precisely, let X be a smooth connected projective curve of genus g corresponding to a point u in the moduli space M_g of curves of genus g. Suppose that G lies in $\pi_A(X)$. We show that there exists a dense open neighborhood $U \subset M_g$ of u, such that for every point in U, G lies in π_A of the corresponding curve. This implies (in the notation of Section 3) that $\pi_A(g) = \pi_A(X_g)$.

To do this, one might try building a cover over a neighborhood of u in M_g for which a versal family exists. However, M_g is not a fine moduli space, and if X is a curve with too many automorphisms then a neighborhood with versal family does not exist. Therefore, we must build the family some other way. Recall that a *curve of genus g with level n-structure* is a pair, (X, δ), where X is a curve of genus g and δ is a symplectic monomorphism $H^1_{et}(X, \mathbb{Z}/n\mathbb{Z}) \to (\mathbb{Z}/n\mathbb{Z})^{2g}$, where $(\mathbb{Z}/n\mathbb{Z})^{2g}$ has the standard symplectic structure and $H^1_{et}(X, \mathbb{Z}/n\mathbb{Z})$ has the symplectic structure defined by the cup product (cf. [Po, Lecture 10]). By [Po, p. 135] if we pick a positive integer n that is prime to p and greater than or equal to 3, then there exists a scheme $H^{(n)}$ which parameterizes 3-canonical smooth curves of genus g with level n-structure. Moreover, this $H^{(n)}$ has a universal family $X^{(n)} \to H^{(n)}$, and $X^{(n)}$ is also a scheme. Over a point v in $H^{(n)}$ the fibre X_v of $X^{(n)}$ is a (3-canonically embedded) curve equipped with a level n-structure δ_v. By [Po, p. 137] there is a covering morphism κ: $H^{(n)} \to M_g$ taking v to the point in M_g corresponding to the isomorphism class $[X_v]$ of X_v.

Given $u \in M_g$, let v be a point of $H^{(n)}$ lying over u, and consider $\mathrm{Spec}(\hat{\mathscr{O}}_{H^{(n)},v}) \subset H^{(n)}$. Pulling $X^{(n)}$ back to $\mathrm{Spec}(\hat{\mathscr{O}}_{H^{(n)},v})$ we get a local family $\hat{X}$ of (3-canonical) smooth curves of genus g. However, $\hat{X}$ is also a genus g curve over the complete local ring $\hat{\mathscr{O}}_{H^{(n)},v}$ (equipped with a level n-structure $\hat{\delta}$), so it corresponds via κ to a point in M_g. We will build a family of G-Galois covers over $\hat{X}$, and then, the following lemma globalizes this local family to one in which the base is an affine variety mapping to M_g.

LEMMA 4.1. *Let v be a point in $H^{(n)}$, $S = \hat{\mathscr{O}}_{H^{(n)},v}$, and $\hat{V} = \mathrm{Spec}(S)$ with inclusion map $\hat{\alpha}$: $\hat{V} \to H^{(n)}$. Define $\hat{X}$ to be the pullback of the universal* *family* $X^{(n)} \to H^{(n)}$ *via* $\hat{\alpha}$. *Let* $\hat{\beta}\colon \hat{W} \to \hat{X}$ *be an étale Galois cover with Galois group* G, *and let* $\beta\colon W \to X$ *be its closed fibre* (*the fibre over* v). *Then there exist a* k-*variety* $E = \mathrm{Spec}(A)$, *where* $A \subset S$ *and* A *is of finite type over* k; *a morphism* $\alpha_E\colon E \to H^{(n)}$; *the pullback* $X_E \to E$ *of* $X^{(n)} \to H^{(n)}$ *via* α_E; *and a* G-*Galois cover* $\beta_E\colon W_E \to X_E$ *such that*

(a) *the pullback* $X_E \times_E \hat{V}$ *is isomorphic to* $\hat{X}$ *as a* 3-*canonical curve over* $\hat{V}$, *and the pullback* $W_E \times_E \hat{V}$ *is isomorphic to* $\hat{W}$ *as a* G-*Galois cover of* $\hat{X}$;

(b) *for every closed point* e *in* E, *the fibre* $\beta_e\colon W_e \to X_e$ *is a connected unramified* G-*Galois cover of a smooth irreducible projective* k-*curve of genus* g.

Proof. Since both the morphism $\hat{\alpha}\colon \hat{V} \to H^{(n)}$ and the connected G-Galois cover $\hat{\beta}\colon \hat{W} \to \hat{X}$ are of finite presentation, this construction descends to an algebra $R \subset S$ such that R is of finite type over k. That is, for some such algebra R, if we let $E = \mathrm{Spec}(R)$ there is a morphism $\alpha_E\colon E \to H^{(n)}$ the pullback X_E of $X^{(n)}$ via α_E; and moreover, there is an irreducible normal projective R-scheme W_E, together with a G-Galois étale covering morphism $\beta_E\colon W_E \to X_E$ which induces $\hat{\beta}\colon \hat{W} \to \hat{X}$ over S.

Let $\varepsilon\colon \hat{V} \to E$ be the morphism induced by $R \subset S$. Then $v' = \varepsilon(v)$ is a closed point of E, and since W_E induces $\hat{W}$, the fibre of W_E over v' is connected and smooth. Moreover $\hat{W}$ is smooth. Applying [Ha1, Proposition 5] to $W_E \to E$, and letting $\epsilon = v'$, it follows that for all k-points e in a dense open subset E' of E, the fibre W_e is (geometrically) irreducible. We may assume that E' is a basic open subset $\mathrm{Spec}(A)$ of E, where $A = R[f^{-1}]$, for some non-zero $f \in R$.

With E replaced by E', it remains to show that each fibre X_e is a smooth irreducible projective k-curve of genus g. Since the fibre X_E is the pullback of the universal family $X^{(n)}$ over $H^{(n)}$, each of the fibres is a smooth irreducible projective curve of genus g with level n-structure. Dropping the level n-structure, we have the desired curve. ▮

PROPOSITION 4.2. *Let* u *be a closed point in* M_g, *and let* X *be the corresponding curve of genus* g. *Suppose that* G *is a finite group lying in* $\pi_A(X)$. *Then there exists an open neighborhood* U *of* u *in* M_g, *such that for all* $w \in U$, *the group* G *lies in* π_A *of the corresponding curve of genus* g.

Proof. Since G lies in $\pi_A(X)$, there exists a G-Galois cover $\beta\colon W \to X$. Let v be a point in $H^{(n)}$ lying over u. Let $\hat{V} = \mathrm{Spec}(\hat{\mathscr{O}}_{H^{(n)},v})$. Pulling back the universal family $X^{(n)}$ via the inclusion $\hat{\alpha}\colon \hat{V} \to H^{(n)}$, we get a family $\hat{X}$ of (3-canonical) curves, such that the fibre over $v \in \hat{V}$ is X. By the equivalence of categories between the $Et(\hat{X})$ and $Et(X)$ [Gr, I, Cor. 8.4] there exists a Galois cover $\hat{W} \to \hat{X}$ over $\hat{V}$ with Galois group G, such that the v-fibre is $W \to X$.

Let $S = \hat{\mathscr{O}}_{H^{(n)},v}$. By Lemma 4.1, there exist a k-variety $E = \operatorname{Spec}(A)$, where $A \subset S$ and of finite type over k; a morphism $\alpha_E\colon E \to H^{(n)}$ and the pullback X_E of $X^{(n)} \to H^{(n)}$ via α_E; and a G-Galois cover $\beta_E\colon W_E \to X_E$, satisfying (a) and (b) of that result. Let $\varepsilon\colon \hat{V} \to E$ be the morphism induced by the inclusion $A \subset S$. Recall that the fibre of $X^{(n)} \to H^{(n)}$ over a point $w \in H^{(n)}$ is a 3-canonical smooth curve X_w with a level n-structure δ_w, and that $\kappa\colon H^{(n)} \to M_g$ is the map taking w to the point of M_g corresponding to the isomorphism class $[X_w]$. Similarly, the pullback $\kappa \circ \hat{\alpha}\colon \hat{V} \to M_g$ of κ takes a point $w \in \hat{V}$ to the point of M_g corresponding to $[\hat{X} \times_{\hat{V}} \{w\}]$.

Since X_E is the pullback of $X^{(n)} \to H^{(n)}$ via α_E, the map $(\alpha_E \circ \varepsilon \circ \mathrm{pr}_2)\colon X_E \times_E \hat{V} \to H^{(n)}$ is equal to the following composition of the projection maps $X_E \times_E \hat{X} \to X_E \to X^{(n)} \to H^{(n)}$. Composing with κ tells us that $\kappa \circ \alpha_E \circ \varepsilon$ is the morphism taking a point $w \in \hat{V}$, with fiber X_w in $X_E \times_E \hat{V}$, to the point of M_g corresponding to $[X_w]$. By part (a) of Lemma 4.1, $[X_E \times_E \hat{V}] = [\hat{X}]$. Therefore for any $w \in \hat{V}$, $[(X_E \times_E \hat{V}) \times \{w\}] = [\hat{X} \times_{\hat{V}} \{w\}]$, and this implies that $\kappa \circ \hat{\alpha} = \kappa \circ \alpha_E \circ \varepsilon$.

Let f be a non-zero function on an affine open neighborhood of u in M_g. To show that the image of E in M_g is dense, it suffices to show that the pullback of f to E is also non-zero. Well, the pullback of f to $H^{(n)}$ is non-zero since $\kappa\colon H^{(n)} \to M_g$ is a covering morphism, and hence its pullback to $\hat{V} = \operatorname{Spec}(\hat{\mathscr{O}}_{H^{(n)},v})$ is also non-zero. Since $\hat{V} \to M_g$ factors through E (because $\kappa \circ \hat{\alpha} = \kappa \circ \alpha_E \circ \varepsilon$), the pullback of f to E is non-zero, as desired.

Let $Z = \operatorname{Im}(\kappa \circ \alpha_E)$. Now E is a variety, hence constructible, so Z, as the image of a constructible, must be a constructible subset of M_g. By [Ht, Chap. II, Ex. 3.8], Z contains an open neighborhood U of u in $U \subset M_g$. Conclude by part (b) of Lemma 4.1. ∎

This result says that for any g and G, either G occurs over no curve of genus g, or G occurs over almost every such curve. More precisely, the subset U_G of points in M_g whose corresponding curves have a G-Galois cover contains a dense open subset. However, coming back to the question of what $\pi_A(g)$ is, it is unclear how U_G varies with G, and it is also unclear what the intersection of all the U_G's is.

QUESTION 4.3. Is the intersection of all the U_G's dense? Does it contain any $\overline{\mathbb{F}}_p$-points?

5. GENERALIZATIONS

This section generalizes the results of Sections 2 and 3 in order to realize additional groups as Galois groups over projective curves. Specifically, Theorem 5.4 permits more general constructions of G-Galois covers over curves of arbitrary genus. Applying this in the case $g = 2$, we deduce that certain additional classes of groups must lie in $\pi_A(2)$ (Props. 5.5 and 5.6). This construction involves the deformation of configurations more complicated than those in the earlier sections of the paper, and in order to do this we will first generalize some of the results in Section 2 as well as two results (Cor. 2.2 and Prop. 2.3) from [Ha2].

For convenience, we state here the result in [Ha2] from which the original [Ha2, Cor. 2.2] follows.

PROPOSITION 5.1 [Ha2, Prop. 2.1]. *Let L be a regular connected projective k-curve let λ be a closed point of L, and let* $\operatorname{Spec}(S) = L - \{\lambda\}$. *Let* $L^* = L \times_k \operatorname{Spec}(k[[v]])$, *let* $\phi\colon T^* \to L^*$ *be a cover, and assume that ϕ is flat (e.g., if T^* is normal). Let $\mathscr{B}$ be the category*

$$\mathscr{P}(\phi^*(S[[v]])) \times_{\mathscr{P}(\phi^*(\hat{\mathscr{K}}_{L,\lambda}[[v]]))} \mathscr{P}\Big(\phi^*\Big(\hat{\mathscr{O}}_{L,\lambda}[[v]]\Big)\Big).$$

Then the base change functor $\mathscr{P}(T^*) \to \mathscr{B}$ is an equivalence of categories. Moreover this remains true if $\mathscr{P}$ is replaced by $\mathscr{AP}$, $\mathscr{SP}$, or $G\mathscr{P}$ for any finite group G.

The following is the necessary generalization of [Ha2, Cor. 2.2]. The proof given below is just a modification of the proof of the original result.

COROLLARY 5.2 [cf. Ha2, Cor. 2.2]. *Under the hypotheses of Proposition* 5.1, *assume that the closed fibre of T of T^* is connected and has irreducible components $X_1, X_2, \ldots, X_r$, each of which is a k-curve; that the singular locus Λ of T consists of a finite collection of nodes; and that $\Lambda \subset \phi^{-1}(\lambda) = D$. For $i = 1, 2, \ldots, r$, let $\Lambda_i = X_i \cap \Lambda$, and let R_i be the ring of functions on the affine curve $X_i' = X_i - \Lambda_i$; let* $X_i'^* = \operatorname{Spec}(R_i[[t]])$, *and for $\tau \in \Lambda_i$ let* $\hat{X}_{i\tau}'^* = \operatorname{Spec}(\hat{\mathscr{K}}_{X_i,\tau}[[t]])$. *Also, for $\tau \in \Lambda$ let* $\hat{T}_\tau^* = \operatorname{Spec}(\hat{\mathscr{O}}_{T^*,\tau})$. *Let $\mathscr{C}$ be the category*

$$\mathscr{P}\left(\bigcup_{i=1}^{r} X_i'^*\right) \times_{\mathscr{P}(\bigcup_{i=1}^{r}\bigcup_{\tau\in\Lambda_i}(\hat{X}_{i\tau}'^*))} \mathscr{P}\left(\bigcup_{\tau\in\Lambda} \hat{T}_\tau^*\right).$$

Then the base change functor $\mathscr{P}(T^*) \to \mathscr{C}$ is an equivalence of categories. Moreover this remains true if $\mathscr{P}$ is replaced by $\mathscr{AP}$, $\mathscr{SP}$, or $G\mathscr{P}$ for any finite group G.

Proof. Viewing $L \subset L^*$, let $D_i = D \cap X_i$, $D_i' = D \cap X_i'$, and $L' = L - \{\lambda\} = \operatorname{Spec}(S)$. For $i = 1, 2, \ldots, r$, let $T_i' = X_i - D_i$ and let S_i be the ring of functions on the affine curve T_i'. Let Y^* be the pullback of $\phi^*\colon T^* \to L^*$ over $\hat{L}^* = \operatorname{Spec}(\hat{\mathscr{O}}_{L,\lambda}[[t]])$, so $Y^* = \bigcup_{\delta\in D}\operatorname{Spec}(\mathscr{O}_{\hat{T}^*,\delta})$, and let $Y'^* = \bigcup_{i=1}^{r}\bigcup_{\delta\in D_i}\operatorname{Spec}(\hat{\mathscr{K}}_{X_i,\delta}[[t]])$. For $i = 1, 2, \ldots, r$, let $Y_i^* =$

$\bigcup_{\delta \in D'_i} \operatorname{Spec}(\mathscr{O}_{\hat{T}^*,\delta}[[t]])$, $Y_i'^* = \bigcup_{i=1}^r \bigcup_{\delta \in D'_i} \operatorname{Spec}(\hat{\mathscr{K}}_{X_i,\delta}[[t]])$, and $T_i'^* = \operatorname{Spec}(S_i[[t]])$. Let $\mathscr{B}$ be as in Proposition 5.1. Thus $\mathscr{B} = \mathscr{P}(\bigcup_{i=1}^r T_i'^*) \times_{\mathscr{P}(Y'^*)} \mathscr{P}(Y^*)$, and the base change functor $\mathscr{P}(T^*) \to \mathscr{B}$ is an equivalence of categories.

By [Ha1, Prop. 3] (which is the affine analog of Theorem 1.1) and induction on $\#(D'_i)$, base change induces an equivalence of categories

$$\mathscr{P}(X_i'^*) \xrightarrow{\sim} \mathscr{P}(T_i'^*) \times_{\mathscr{P}(Y_i'^*)} \mathscr{P}(Y_i^*)$$

for $i = 1, 2, \ldots, r$. So base change induces an equivalence of categories $\mathscr{P}(\bigcup_{i=1}^r X_i'^*) \xrightarrow{\sim} \mathscr{D}$, where $\mathscr{D} = \mathscr{P}(\bigcup_{i=1}^r T_i'^*) \times_{\mathscr{P}(\bigcup_{i=1}^r Y_i'^*)} \mathscr{P}(\bigcup_{i=1}^r Y_i^*)$, and hence also induces an equivalence $\mathscr{C} \xrightarrow{\sim} \mathscr{D} \times_{\mathscr{P}(\bigcup_{i=1}^r \bigcup_{\tau \in \Lambda_i}(\hat{X}_{i\tau}'^*))} \mathscr{P}(\bigcup_{\tau \in \Lambda} \hat{T}_\tau^*)$. The latter category is canonically equivalent to $\mathscr{B}$, because of the disjoint unions $Y^* = (\bigcup_{i=1}^r Y_i^*) \cup (\bigcup_{\tau \in \Lambda} \hat{T}_\tau^*)$ and $Y'^* = (\bigcup_{i=1}^r Y_i'^*) \cup (\bigcup_{i=1}^r \bigcup_{\tau \in \Lambda_i} \hat{X}_{i\tau}'^*)$. Thus the base change functors $\mathscr{P}(T^*) \to \mathscr{B}$ and $\mathscr{C} \to \mathscr{B}$ are equivalences of categories, and hence so is the base change functor $\mathscr{P}(T^*) \to \mathscr{C}$. This proves the result for $\mathscr{P}$. Replacing $\mathscr{P}$ throughout by $\mathscr{AP}$, $\mathscr{SP}$, or $G\mathscr{P}$ yields the proofs in those cases. ∎

The next step is to use Galois covers of k-curves $X_1, \ldots, X_r$ to build a G-Galois cover of a degenerate curve T. Proposition 5.3 generalizes Harbater's Proposition 2.3 [Ha2], by not only allowing more intersections between the X_i's but also allowing the X_i's to have self-intersections. The changes here are somewhat more involved than those needed to generalize [Ha2, Prop. 2.2]. We begin with some notation.

Notation. Given a connected k-curve T with singular locus Λ consisting of at most finitely many nodes (e.g., T in Cor. 5.2), let $X_1, \ldots, X_r$ be its irreducible components, and let $f\colon \tilde{T} \to T$ be its normalization. Thus $\tilde{T}$ is equal to the disjoint union of the normalizations $\tilde{X}_i$ of the X_i's. If $i \neq j$ then let $\Lambda_{ij} = X_i \cap X_j$, and let Λ_{ii} be the singular locus of X_i. If $\tau \in \Lambda_{i_1 i_2}$ and $\tau_k \in f^{-1}(\tau) \cap \tilde{X}_{i_k}$, then $x_{i_k \tau_k}$ will denote the local parameter of $\tilde{X}_{i_k}$ at τ_k. Let $X'_{i_k\tau_k} = \operatorname{Spec}(\hat{\mathscr{K}}_{\tilde{X}_{i_k}, \tau_{i_k}}) = \operatorname{Spec}(k((x_{i_k\tau_k})))$. If $\tau \in \Lambda_{i_1 i_2}$, where $i_1 \neq i_2$ (i.e., τ is a smooth point of X_{i_k}) then $\hat{X}'_{i_k\tau_k}$ is isomorphic to $\hat{X}'_{i_k\tau} = \operatorname{Spec}(\hat{\mathscr{K}}_{X_{i_k},\tau})$. Otherwise, $\tau \in \Lambda_{i_1 i_2}$, where $i_1 = i_2$ (i.e., τ is a node on X_{i_k}, and $\hat{X}'_{i_k\tau_k}$ is canonically isomorphic to one of the two irreducible components of $\hat{X}'_{i_k\tau}$). Thus for any $\tau \in X_{i_k}$ we identify $\hat{X}'_{i_k\tau_k}$ with its image in X_{i_k}. Then, for $\tau \in \Lambda_{i_1 i_2} \hat{T}_r \cong \operatorname{Spec}(\hat{\mathscr{O}}_{T,\tau}) = \operatorname{Spec}(k[[x_{i_1\tau_1}, x_{i_2\tau_2}]]/(x_{i_1\tau_1} x_{i_2\tau_2}))$, and $\hat{X}_{i_k\tau_k}'^* = \operatorname{Spec}(\hat{\mathscr{K}}_{\tilde{X}_{i_k},\tau i_k}[[t]])$ is identified with the formal completion of T along $\hat{X}'_{i_k\tau_k}$.

Proposition 5.3 [cf. Ha2, Prop. 2.3]. *Assume the hypotheses of Corollary* 5.2 *and the notation above. Given a finite group G and subgroups*

$G_1, G_2, \ldots, G_r$, *suppose that there exist*:

(1) *a connected G-Galois cover* $\psi\colon W \to T$, *and irreducible normal G_i-Galois covers* $\psi_i^*\colon W_i'^* \to X_i'^*$ *such that for each i the closed fibre of* $\psi_i^*\colon W_i'^* \to X_i'^*$ *is smooth, irreducible, and isomorphic to the restriction of ψ to* $W \times_T X_i'$;

(2) *for each τ in Λ, a subgroup I_τ of G and an irreducible normal I_τ-Galois cover* $\hat{N}_\tau^* \to \hat{T}_\tau^*$;

(3) *for each $\tau \in \Lambda_i$ and $\gamma \in f^{-1}(\tau) \cap \tilde{X}_i$ (where f is defined above), an irreducible component $\hat{W}_{i\gamma}'^*$ of $W_i'^* \times_{X_i'^*} \hat{X}_{i\gamma}'^*$ and a subgroup I_γ of $I_\tau \cap G_i$ such that* $\mathrm{Gal}(\hat{W}_{i\gamma}'^*/\hat{X}_{i\gamma}'^*) = I_\gamma$; *and*

(4) *for each $\tau \in \Lambda_i$ and $\gamma \in f^{-1}(\tau) \cap \tilde{X}_i$, an isomorphism* $\hat{N}_\tau^* \times_{\hat{T}_\tau^*} \hat{X}_{i\gamma}'^* \to \mathrm{Ind}_{I_\gamma}^{I_\tau} \hat{W}_{i\gamma}'^*$ *of I_τ-Galois covers of* $\hat{X}_{i\gamma}'^*$.

Then there is an irreducible normal G-Galois cover $V^* \to T^*$ such that $V^* \times_{T^*} X_i'^* \approx \mathrm{Ind}_{G_i}^G W_i'^*$ *as G-Galois covers of $X_i'^*$ for $i = 1, 2, \ldots, r$, and* $V^* \times_{T^*} \hat{T}_\tau^* \approx \mathrm{Ind}_{I_\tau}^G \hat{N}_\tau^*$ *as G-Galois covers of $\hat{T}_\tau^*$ for each $\tau \in \Lambda$.*

Proof. We preserve the notation of the statements of 5.1 and 5.2. The covers $W_i'^* \to X_i'^*$ and $\hat{N}^* \to \hat{T}^*$ are flat and hence define projective modules, since the total spaces are normal surfaces. So for $\tau \in \Lambda$, $\mathscr{V}_{2\tau} = \mathrm{Ind}_{I_\tau}^G \mathscr{O}_{\hat{N}_\tau^*}$ is an object in $G\mathscr{P}(\hat{T}_\tau^*)$ and $\mathrm{Ind}_{G_i}^G \mathscr{O}_{W_i'^*}$ is an object in $G\mathscr{P}(X_i'^*)$, and so for all $i_1 \le i_2$ in $\{1, 2, \ldots, r\}$, $\mathscr{V}_{i_1 i_2} = \mathrm{Ind}_{G_{i_1}}^G \mathscr{O}_{W_{i_1}'^*} \times \mathrm{Ind}_{G_{i_2}}^G \mathscr{O}_{W_{i_2}'^*}$ is an object in $G\mathscr{P}(X_{i_1}'^* \cup X_{i_2}'^*)$. Since each $\tau \in \Lambda$ is assumed to be a normal crossing, $f^{-1}(\tau) \subset \tilde{T}$ consists of a pair of points $\{\tau_1, \tau_2\}$, where τ_k lies in a unique $\tilde{X}_{i_k}$ for some $i_k \in \{1, 2, \ldots, r\}$. Moreover, we may assume that $i_1 \le i_2$, so that $\tau \in \Lambda_{i_1 i_2}$ (see notation above). In this way, we associate to each $\tau \in \Lambda_{i_1 i_2}$ a unique pair of points $\{\tau_1, \tau_2\} \subset \tilde{T}$. As above $\mathscr{V}_{0\tau} = \mathrm{Ind}_{I_{\tau_1}}^G \mathscr{O}_{\hat{W}_{i_1\tau_1}'^*} \times \mathrm{Ind}_{I_{\tau_2}}^G \mathscr{O}_{\hat{W}_{i_2\tau_2}'^*}$ is an object in $G\mathscr{P}(\hat{X}_{i_1\tau_1}'^* \cup \hat{X}_{i_2\tau_2}'^*)$. For $l = 0, 2$, let $V_{l\tau} = \mathrm{Spec}(\mathscr{V}_{l\tau})$ and for $i_1 \le i_2 \in \{1, 2, \ldots, r\}$ let $V_{i_1 i_2} = \mathrm{Spec}(\mathscr{V}_{i_1 i_2})$.

By definition of induced modules, if γ lies in $\tilde{X}_i \cap f^{-1}(\Lambda)$ then we have an isomorphism

$$\mathscr{O}_{W_i'^*} \otimes_{R_i[[t]]} \mathscr{O}_{\hat{X}_{i\gamma}'^*} \xrightarrow{\sim} \mathrm{Ind}_{I_\gamma}^{G_i} \mathscr{O}_{\hat{W}_{i\gamma}'^*}$$

of modules over $\hat{X}_{i\gamma}'^*$. For each $\tau \in \Lambda_{i_1 i_2}$, we have the associated pair of points $\{\tau_1, \tau_2\} = f^{-1}(\tau)$, where $\tau_k \in \tilde{X}_{i_k}$, so using $\mathscr{O}_{\hat{X}_{i_k\tau_k}'^*} \approx \hat{\mathscr{K}}_{\tilde{X}_{i_k}, \tau_k}[[t]]$, the above induces an isomorphism

$$\begin{aligned}\mathscr{V}_{i_1 i_2} &\otimes_{R_{i_1}[[t]] \times R_{i_2}[[t]]} \left(\hat{\mathscr{K}}_{\tilde{X}_{i_1}, \tau_1}[[t]] \times \hat{\mathscr{K}}_{\tilde{X}_{i_2}, \tau_2}[[t]]\right) \\ &= \left(\mathrm{Ind}_{G_{i_1}}^G \mathscr{O}_{W_{i_1}'^*} \otimes_{R_{i_1}[[t]]} \mathscr{O}_{\hat{X}_{i_1\tau_1}'^*}\right) \times \left(\mathrm{Ind}_{G_{i_2}}^G \mathscr{O}_{W_{i_2}'^*} \otimes_{R_{i_2}[[t]]} \mathscr{O}_{\hat{X}_{i_2\tau_2}'^*}\right) \xrightarrow{\sim} \mathscr{V}_{0\tau}\end{aligned}$$

in $G\mathscr{P}(\hat{X}'_{i_1\tau_1^*}\cup \hat{X}'_{i_2\tau_2}{}^*)$. Here the left hand side is the object in $G\mathscr{P}(\hat{X}'_{i_1\tau_1}{}^* \cup \hat{X}'_{i_2\tau_2}{}^*)$ induced by $\mathscr{V}_{i_1i_2}$. Then because the $V_{0\tau}$ are disjoint, we get an isomorphism

$$\mathscr{V}_{i_1i_2}\otimes_{R_{i_1}[[t]]\times R_{i_2}[[t]]}\left(\prod_{\tau\in\Lambda_{i_1i_2}}\left(\hat{\mathscr{K}}_{\tilde{X}_{i_1,\tau_1}}[[t]]\times\hat{\mathscr{K}}_{\tilde{X}_{i_2,\tau_2}}[[t]]\right)\right)\tilde{\to}\prod_{\tau\in\Lambda_{i_1i_2}}\mathscr{V}_{0\tau}$$

in $G\mathscr{P}(\bigcup_{\tau\in\Lambda_{i_1i_2}}(\hat{X}'_{i_1\tau_1^*}\cup\hat{X}'^*_{i_2\tau_2}))$. Here the left hand side is the object in the category $G\mathscr{P}(\bigcup_{\tau\in\Lambda_{i_1i_2}}(\hat{X}'_{i_1\tau_1}{}^*\cup\hat{X}'_{i_2\tau_2^*}))$ induced by $\mathscr{V}_{i_1i_2}$.

Let $\mathscr{V}_0=\prod_{\tau\in\Lambda}\mathscr{V}_{0\tau}$. Notice that if $\tau\in\Lambda_{ii}$ then $\hat{X}'_{i\tau_1}{}^*\cup\hat{X}'_{i\tau_2}{}^*\approx\hat{X}'^*_{i\tau}$, and if $\tau\in\Lambda_{i_1i_2}$ for $i_1<i_2$ then $\hat{X}'_{i_k\tau_k}{}^*=\hat{X}'_{i_k\tau}{}^*$ for $k=1,2$. Thus, $\mathscr{V}_0$ is an object in $G\mathscr{P}(\bigcup_{i=1}^r\bigcup_{\tau\in\Lambda_i}(\hat{X}'^*_{i\tau}))$. Similarly, let $\mathscr{V}_1=\prod_{i_1\le i_2}\mathscr{V}_{i_1i_2}$. Again, because the $V_{0\tau}$ are disjoint, we get an isomorphism

$$\mathscr{V}_1\otimes_{(\prod_{i_1\le i_2}R_{i_1}[[t]]\times R_{i_2}[[t]])}\left(\prod_{i_1\le i_2}\prod_{\tau\in\Lambda_{i_1i_2}}\left(\hat{\mathscr{K}}_{\tilde{X}_{i_1,\tau_1}}[[t]]\times\hat{\mathscr{K}}_{\tilde{X}_{i_2,\tau_2}}[[t]]\right)\right)\tilde{\to}\mathscr{V}_0$$

in $G\mathscr{P}(\bigcup_{i_1\le i_2}\bigcup_{\tau\in\Lambda_{i_1i_2}}(\hat{X}'_{i_1\tau_1}{}^*\cup\hat{X}'_{i_2\tau_2}{}^*))$. Here the left hand side is the object in the category $G\mathscr{P}(\bigcup_{i_1\le i_2}\bigcup_{\tau\in\Lambda_{i_1i_2}}(\hat{X}'_{i_1\tau_1}{}^*\cup\hat{X}'_{i_2\tau_2}{}^*))$ induced by $\mathscr{V}_1$.

Meanwhile, if $\tau\in\Lambda_{i_1i_2}$, then the associated τ_1 and τ_2 lie in $\tilde{X}_{i_1}$ and $\tilde{X}_{i_2}$ respectively, and the given isomorphisms $\hat{N}^*_\tau\times_{\hat{T}^*_\tau}\hat{X}'_{i_k\tau_k}{}^*\tilde{\to}\operatorname{Ind}^{I_\tau}_{I_{\tau_k}}\hat{W}'_{i_k\tau_k}{}^*$ $(k=1,2)$ induce an isomorphism

$$\begin{aligned}\mathscr{V}_{2\tau}\otimes_{\hat{\mathscr{O}}_{T*\tau}}&\left(\hat{\mathscr{K}}_{\tilde{X}_{i_1,\tau_1}}[[t]]\times\hat{\mathscr{K}}_{\tilde{X}_{i_2,\tau_2}}[[t]]\right)\\ &=\prod_{k=1}^2\left(\operatorname{Ind}^G_{I_\tau}\mathscr{O}_{\hat{N}^*_\tau}\otimes_{\hat{\mathscr{O}}_{T^*_\tau}}\mathscr{O}_{\hat{X}'^*_{i_k\tau_k}}\right)\tilde{\to}\mathscr{V}_{0\tau}\end{aligned}$$

in $G\mathscr{P}(\hat{X}'_{i_1\tau_1}{}^*\cup\hat{X}'_{i_2\tau_2}{}^*)$. Here, the left hand side is the object in $G\mathscr{P}(\hat{X}'_{i_1\tau_1}{}^* \cup\hat{X}'_{i_2\tau_2}{}^*)$ induced by $\mathscr{V}_{2\tau}$. Let $\mathscr{V}_2=\prod_{\tau\in\Lambda}\mathscr{V}_{2\tau}$. Then because the $V_{2\tau}$ are disjoint, we get an isomorphism

$$\mathscr{V}_2\otimes_{\prod_{\tau\in\Lambda}(\hat{\mathscr{O}}_{T*,\tau})}\left(\prod_{i_1\le i_2}\prod_{\tau\in\Lambda_{i_1i_2}}\left(\hat{\mathscr{K}}_{\tilde{X}_{i_1,\tau_1}}[[t]]\times\hat{\mathscr{K}}_{\tilde{X}_{i_2,\tau_2}}[[t]]\right)\right)\tilde{\to}\mathscr{V}_0$$

in $G\mathscr{P}(\bigcup_{i_1\le i_2}\bigcup_{\tau\in\Lambda_{i_1i_2}}(G\mathscr{P}(\hat{X}'_{i_1\tau_1}{}^*\cup\hat{X}'_{i_2\tau_2}{}^*)))$. Here, the left hand side is the object in the category $G\mathscr{P}(\bigcup_{i_1\le i_2}\bigcup_{\tau\in\Lambda_{i_1i_2}}(G\mathscr{P}(\hat{X}'_{i_1\tau_1}{}^*\cup\hat{X}'_{i_2\tau_2}{}^*)))$ induced by $\mathscr{V}_2$.

Let $G\mathscr{C}$ be the category which is the analog of the category $\mathscr{C}$ in Corollary 5.2, but with $\mathscr{P}$ replaced by $G\mathscr{P}$. Then the triple $(\mathscr{V}_1,\mathscr{V}_2,\mathscr{V}_0)$, together with the above isomorphisms, defines an object in $G\mathscr{C}$. So by Corollary 5.2 this object is induced (up to isomorphism) by an object $\mathscr{V}$ in

$G\mathscr{P}(T^*)$. Thus $V^* = \mathrm{Spec}(\mathscr{V})$ is a G-Galois cover of T^* which induces $\mathrm{Ind}_{G_i}^G W_i'^* \to X_i'^*$ and $\mathrm{Ind}_{I_\tau}^G \hat{N}_\tau^* \to \hat{T}_\tau^*$, as G-Galois covers. Moreover, the closed fibre is isomorphic to $\psi: W \to T$. Because this is a flat family whose closed fibre is connected, the cover V^* is connected.

To verify that V^* is normal, it suffices to show that for every closed point σ in the closed fibre of T^*, $\hat{V}_\sigma^* = V^* \times_{T^*} \mathrm{Spec}(\hat{\mathscr{O}}_{T^*,\sigma})$ is normal. If $\sigma = \tau \in \Lambda$ then $\hat{V}_\sigma^* = V^* \times_{T^*} \hat{T}_\tau^* \cong \mathrm{Ind}_{I_\tau}^G \hat{N}_\tau^*$, which is normal. Otherwise, we may identify σ with some other point on the closed fibre of T^*, i.e., a point on the closed fibre X_i' of some $X_i'^*$, for some $i \in \{1, 2, \ldots, r\}$. Choosing $\tilde{\sigma} \in W_i^*$ lying over $\sigma \in X_i'^*$, we have that $\hat{V}_\sigma^* = \mathrm{Ind}_{G_i}^G W_i^* \times_{X_i'^*} \mathrm{Spec}(\hat{\mathscr{O}}_{X_i'^*,\sigma})$ is a union of copies of $\mathrm{Spec}(\hat{\mathscr{O}}_{W_i^*,\tilde{\sigma}})$; this is normal since W_i^* is. ∎

The next proposition essentially reduces our construction of smooth G-Galois covers to the problem of constructing the degenerate cover. We add the following new notation to that which precedes Proposition 5.3.

Notation. In what follows let T be a connected k-curve with singular locus Λ consisting of at most finitely many nodes; let $X_1, \ldots, X_r$ be its irreducible components; and let L be the projective k-line with a closed point λ such that the local parameter at λ on L is x. Then let $\phi: T \to L$ be a cover and $\psi: W \to T$ be a connected G-Galois cover for some finite group G. Now we define the following special situation:

DEFINITION. Given T, L, G, and W as above, the pair (ϕ, ψ) is called *admissible* if the following conditions are satisfied:

(a) The set Λ is contained in $\phi^{-1}(\lambda)$.

(b) For every $\tau \in \Lambda_{i_1 i_2} \subset \Lambda$ the complete local ring $\hat{\mathscr{O}}_{T,\tau} \cong k[[x_{i_1\tau_1}, x_{i_2\tau_2}]]/(x_{i_1\tau_1}x_{i_2\tau_2})$ is an $\hat{\mathscr{O}}_{L,\lambda}$-algebra defined by $x_{i_1\tau_1} + x_{i_2\tau_2} = x$.

(c) For each $i = 1, 2, \ldots, r$ the restriction $W_i = W \times_T X_i \to X_i$ is a tame G_i-Galois cover for some subgroup G_i of G, and this cover is branched only at $\tau \in \bigcup_{i \neq j} \Lambda_{ij}$.

(d) For any point $\omega \in W$ over $\tau \in \Lambda_{i_1 i_2}$ where $i_1 \neq i_2$, assume that $g_{i_1\omega} = g_{i_2\omega}^{-1}$, where $g_{i_k\omega}$ is the canonical generator of inertia of ω on W_{i_k}.

Let $\pi_A^{\mathrm{adm}}(T)$ denote the set of finite groups G such that there exists a pair (ϕ, ψ) as above which is admissible.

Notice that in terms of building the desired G-Galois cover, condition (a) is just a matter of building the cover $\phi: T \to L$ so that the formal patching results apply, and condition (b) is simply notational. Thus conditions (c) and (d) are the essential part of admissibility, yet in practice it is convenient to have all four conditions together.

THEOREM 5.4. *Given T as above, if G lies in $\pi_A^{\mathrm{adm}}(T)$ and $g = p_A(T)$, then G lies in $\pi_A(g)$.*

Proof. Since G lies in $\pi_A^{\mathrm{adm}}(T)$, there exist curves L and W as above along with a pair of covers (ϕ, ψ) which is admissible. We proceed in three steps: First we deform $\phi: T \to L$; second we locally deform $\psi: W \to T$; and third we use Proposition 5.3 to patch the local deformations of ψ.

Step 1. *Construction of $\phi^*: T^* \to L^*$.* As in Lemma 2.3, we will deform $\phi_i: X_i \to L$ locally in such a way that Lemma 2.1 applies. Let $L' = L - \{\lambda\}$, $T' = T - \phi^{-1}(\lambda)$, and $L'^* = \mathrm{Spec}(\mathscr{O}_{L'}[[t]])$. Then take $\phi'^*: T'^* \to L'^*$ to be the pullback of ϕ with respect to the morphism $L'^* \to L$ (induced by the natural ring morphism). Let $\hat{L} = \mathrm{Spec}(\hat{\mathscr{O}}_{L,\lambda})$, $\hat{L}^* = \mathrm{Spec}(\hat{\mathscr{O}}_{L,\lambda}[[t]])$, and for every $\tau \in \phi^{-1}(\lambda)$ let $\hat{T}_\tau = \mathrm{Spec}(\hat{\mathscr{O}}_{T,\tau})$. For $\tau \in \Lambda_{i_1 i_2}$, by hypothesis the restriction $\phi|_{\hat{T}_\tau}: \hat{T}_\tau \to \hat{L}$ is defined by $x_{i_1\tau_1} + x_{i_2\tau_2} = x$. Let $n_\tau = \mathrm{ord}(g_{i_1\omega}) = \mathrm{ord}(g_{i_2\omega})$, where ω lies over τ. Notice that if ω is unramified then $n_\tau = 1$, and moreover since the cover ψ is Galois this definition of n_τ is independent of the choice of the point ω over τ. Now let $\hat{T}_\tau^* = \mathrm{Spec}(k[[x_{i_1\tau_1}, x_{i_2\tau_2}, t]]/(x_{i_1\tau_1}x_{i_2\tau_2} - t^{n_\tau}))$, and define $\hat{\phi}_\tau^*: \hat{T}_\tau^* \to \hat{L}^*$ by $x_{i_1\tau_1} + x_{i_2\tau_2} = x$. Meanwhile, for $\tau \in \phi^{-1}(\lambda) - \Lambda$, let $\hat{\phi}_\tau^*: \hat{T}_\tau^* \to \hat{L}^*$ be the pullback of $\phi|_{\hat{T}_\tau}$ via $\hat{L}^* \to \hat{L}$ (induced by the natural ring inclusion). These covers $\hat{\phi}_\tau^*$ induce $\hat{\phi}^*: \hat{T}^* = (\bigcup_{\tau \in \phi^{-1}(\lambda)} \hat{T}_\tau^*) \to \hat{L}^*$.

Let $L^* = L \times_k \mathrm{Spec}(k[[t]])$. Applying Lemma 2.1, we obtain a normal projective $k[[t]]$-curve T^* and a covering morphism $\phi^*: T^* \to L^*$, such that $T^* \times_{L^*} L'^* \cong T'^*$ as a cover of L'^* and $T^* \times_{L^*} \hat{L}^* \cong \hat{T}^*$ as a cover of $\hat{L}^*$; and the closed fibre of ϕ^* is isomorphic to $\phi: T \to L$. This cover satisfies all the hypotheses of Propositions 5.3.

Step 2. *Local construction of $\psi^*: W^* \to T^*$.* We now deform $\psi: W \to T$ in such a way that Proposition 5.3 applies. Let $\Lambda_i = \Lambda \cap X_i = \bigcup_{j=1}^r \Lambda_{ij}$, and let $X_i' = X_i - \Lambda_i = \mathrm{Spec}(R_i)$, and $X_i'^* = \mathrm{Spec}(R_i[[t]])$. Denote the pullback of ψ_i via $X_i'^* \to X_i$ by $\psi_i'^*: W_i'^* \to X_i'^*$. This is an irreducible normal G_i-Galois cover such that the closed fibre is smooth, irreducible, and isomorphic to the restriction of ψ to $W \times_T X_i'$. Hence it satisfies (1) of Proposition 5.3.

Next for $\tau \in \Lambda_{i_1 \neq i_2}$, let $\hat{X}_{i_k\tau_k}'^* = \mathrm{Spec}(\mathscr{O}_{\hat{X}_{i_k\tau_k}'}[[t]])$. (Recall from the above notation that under the identification of $\hat{X}_{i_k\tau_k}' = \mathrm{Spec}(\mathscr{K}_{\tilde{X}_{i_k,\tau_k}}) = k((x_{i_k\tau_k}))$ with its isomorphic image in X_{i_k}, the scheme $\tilde{X}_{i_k\tau_k}'^*$ is identified with the formal completion of T along $\hat{X}_{i_k\tau}'$). Let ω be a point over τ. By hypothesis we denote its canonical generator of inertia on W_{i_k} by $g_{i_k\omega}$. Let $I_\tau = \langle g_{i_1\omega} \rangle = \langle g_{i_2\omega} \rangle$ and define

$$\hat{N}_\tau^* = \mathrm{Spec}\Big(k\big[\big[z_{i_1\tau_1}, z_{i_2\tau_2}, t\big]\big]/\big(z_{i_1\tau_1}z_{i_2\tau_2} - t\big)\Big) \to \hat{T}_\tau^*$$

to be the I_τ-Galois cover defined by setting $z_{i_k\tau_k}^{n_\tau} = x_{i_k\tau_k}$ and letting I_τ act by $g_{i_k\omega}(z_{i_k\tau_k}) = \zeta_{n_\tau} z_{i_k\tau_k}$.

Consider now the cover ψ. For $\tau \in \Lambda_{i_1 i_2}$ where $i_1 = i_2$, the cover ψ is unramified over τ, so there exist local parameters $w_{i_k\tau_k}$ $(k = 1, 2)$ on W (hence on W_{i_k}) such that $w_{i_k\tau_k} = x_{i_k\tau_k}$ for $k = 1, 2$; and $\hat{W}_{i_k,\omega} = \mathrm{Spec}(\hat{\mathscr{O}}_{W_{i_k,\omega}}) = \mathrm{Spec}(k[[w_{i_1\tau_1}, w_{i_2\tau_2}]]/(w_{i_1\tau_1} w_{i_2\tau_2})$. Let $\hat{W}'^{\,*}_{i_k,\tau_k} = \mathrm{Spec}(k((w_{i_k\tau_k}))[[t]])$, and define a trivial cover $\hat{W}'^{\,*}_{i_k,\tau_k} \to \hat{X}'^{\,*}_{i_k,\tau_k}$ by letting $w_{i_k\tau_k} = x_{i_k\tau_k}$. Then $\hat{W}'^{\,*}_{i_k,\tau_k}$ is naturally identified with an irreducible component of the restriction $W'^{*}_{i_k} \times_{X'^{*}_{i_k}} \hat{X}'^{\,*}_{i_k,\tau_k}$.

Similarly for $\tau \in \Lambda_{i_1 i_2}$ where $i_1 \neq i_2$, W_{i_k} is smooth at ω, so we choose a local parameter $w_{i_k\tau_k}$ such that $\hat{W}_{i_k,\omega} = \mathrm{Spec}(\hat{\mathscr{O}}_{W_{i_k,\omega}}) = \mathrm{Spec}(k[[w_{i_k\tau_k}]])$; $w_{i_k\tau_k}^{n_\tau} = x_{i_k\tau_k}$; and $g_{i_k\omega}(w_{i_k\tau_k}) = \zeta_{n_\tau} w_{i_k\tau_k}$. Again, let $\hat{W}'^{\,*}_{i_k,\tau_k} = \mathrm{Spec}(k((w_{i_k\tau_k}))[[t]])$, and define an I_τ-Galois cover $\hat{W}'^{\,*}_{i_k\tau_k} \to \hat{X}'^{\,*}_{i_k\tau_k}$ by $w_{i_k\tau_k}^{n_\tau} = x_{i_k\tau_k}$ and $g_{i_k\omega}(w_{i_k\tau_k}) = \zeta_{n_\tau} w_{i_k\tau_k}$. As above, $\hat{W}'^{\,*}_{i_k,\tau_k}$ is naturally identified with an irreducible component of $W'^{*}_{i_k} \times_{X'^{*}_{i_k}} \hat{X}'^{\,*}_{i_k,\tau_k}$.

The I_τ-Galois covers $\hat{N}^*_\tau \to \hat{T}^*_\tau$ and $\hat{W}'^{\,*}_{i_k\tau_k} \to \hat{X}'^{\,*}_{i_k\tau_k}$ satisfy hypotheses (2) and (3) of Proposition 5.3 (respectively). Thus to apply Proposition 5.3, it remains to show that hypothesis (4) of that result holds. Let $\hat{N}'^{*}_{\tau_k} = \hat{N}^*_\tau \times_{\hat{T}^*_\tau} \hat{X}'^{*}_{i_k\tau_k}$. Then

$$\begin{aligned}\mathscr{O}_{\hat{N}'^{*}_{\tau_k}} &= k\big((x_{i_k\tau_k})\big)\big[\big[z_{i_1\tau_1}, z_{i_2\tau_2}, t\big]\big]/\big(z_{i_1\tau_1} z_{i_2\tau_2} - t^{n_\tau}, z_{i_k\tau_k}^{n_\tau} - x_{i_k\tau_k}\big)\\ &= k\big((z_{i_k\tau_k})\big)[[t]].\end{aligned}$$

We deduce that the I_τ-cover $\hat{N}'^{*}_{\tau_k} = \hat{N}^*_\tau \times_{\hat{T}^*_\tau} \hat{X}'^{*}_{i_k\tau_k} \to \hat{X}'^{*}_{i_k\tau_k}$ is defined via the extension $k((z_{i_k\tau_k}))[[t]]$ of $k((x_{i_k\tau_k}))[[t]]$ with the I_τ action induced from $\hat{N}^*_\tau \to \hat{T}^*_\tau$. So we define an isomorphism of I_τ-Galois covers $\hat{N}'^{*}_{\tau_k} \to \hat{W}'^{*}_{i\tau_k}$ by sending z_{τ_k} to w_{τ_k}, and this satisfies hypothesis (4) of Proposition 5.3.

Step 3. *Patching.* Applying Proposition 5.3, we get an irreducible normal G-Galois cover $V^* \to T^*$ over $k[[t]]$ such that $V^* \times_{T^*} X'^{*}_i \cong \mathrm{Ind}_{I_\tau}^G \hat{N}^*_\tau$ as G-Galois covers of $\hat{T}^*_\tau$ for each $\tau \in \Lambda$. The deformation $T''^{*} = \bigcup_{i=1}^r X'^{*}_i$ is a trivial deformation of a regular k-curve, so it is regular. Similarly, the Jacobian criterion implies that $\hat{T}^*$ is regular away from $\tau \in \Lambda$ (see Lemma 2.3, Step 2). Thus, T^* is regular away from Λ, which implies that its generic fibre is regular and also that T^* defines a flat family over L^* (Lemma 1.2). Let $V^o \to T^o$ be the generic fibre of $V^* \to T^*$. Then, T^o is a connected K-curve because T is connected and T^* is flat, and also T^o is projective because it is smooth over $\mathbb{P}^1_K$. Since T^* is a flat family over L^*, the arithmetic genus is constant on the fibres [Ht, III, Cor. 9.10], and therefore the genus of T^o is equal to $p_A(T)$.

The K-curve V^o is connected because V^* is irreducible. Since ψ restricted to $W_i \times X'_i$ is unramified for $i = 1, 2, \ldots, r$, the trivial deformation ψ'^* is also unramified. Therefore, ψ^* is ramified only on $\mathrm{Ind}_{I_\tau}^G \hat{N}_\tau^*$ over $\hat{T}_\tau^*$ for $\tau \in \Lambda$. The Jacobian criterion implies that $\mathrm{Ind}_{I_\tau}^G \hat{N}_\tau^* \to \hat{T}_\tau^*$ is branched only at τ. Thus the cover ψ^o is étale, and hence V^o is regular and projective because T^o is. Consequently, ψ^o: $V^o \to T^o$, is an irreducible, unramified G-Galois cover. Proposition 2.5 allows us to descend the G-Galois cover $V^* \to T^*$ to a ring A of finite type over $k[t]$ satisfying (a) and (b) of that result. Therefore, by specializing to a k-point of $E = \mathrm{Spec}(A)$, we are done. ∎

Remark. Now, consider the semi-stable compactification $\overline{M}_g$ of M_g. Theorem 5.4 shows that for any degenerate curve T corresponding to a point on the boundary, $\pi_A^{\mathrm{adm}}(T)$ is contained in $\pi_A(g)$. For example, Proposition 5.5 below examines a family of groups lying in $\pi_A^{\mathrm{adm}}(T_0)$, where T_0 is the semi-stable curve of genus 2 constructed by taking two $\mathbb{P}_k^1$'s which meet one another in three points. One can also show that $\pi_A(T_0)$ is in fact strictly contained in $\pi_A(2)$ by examining the p-rank of l-cyclic covers of both T_0 and X_2 (cf. [Bo]).

Before constructing the final examples, we must recall that for any G-Galois cover θ: $C \to D$, if $\tau \in D$ and $\sigma \in \theta^{-1}(\tau)$ are fixed, then for any $\sigma' \in \theta^{-1}(\tau)$ there exists an element $h \in G$ taking σ to σ' under the action of G on C. We denote such a σ' by σ^h. Moreover, if τ is a branch point of θ and g is the canonical generator of inertia at σ on D, then hgh^{-1} is the canonical generator of inertia at σ^h on D. Given any induced cover, $\mathrm{Ind}_H^G C' \to D$ the connected components of $\mathrm{Ind}_H^G C'$ are indexed by the cosets of H in G, so we denote by $(C')_g$ the component of $\mathrm{Ind}_H^G C'$ corresponding to the coset gH.

The constructions in 5.5 and 5.6 have only smooth irreducible components, and therefore do not need the full strength of the previous results. We build G-Galois covers of a curve of genus 2 by deforming a G-Galois cover of degenerate curve T. As mentioned above, the next proposition builds a G-Galois cover of some genus two curve by deforming two $\mathbb{P}^1$'s which meet in three points.

PROPOSITION 5.5. *Given finite groups H_1 and H_2 generated by elements $\{c_1, d_1\}$ and $\{c_2, d_2\}$ (resp.), and take G to be a finite group generated by H_1 together with H_2, where $c_1 = gc_2g^{-1}$ and $d_1 = hd_2h^{-1}$ for some $g, h \in G$; and $c_1d_1 = c_2d_2$. Suppose that for $i = 1, 2$, $H_i \in \pi_A^t(\mathbb{P}^1 - \{0, 1, \infty\})$ with descriptions $(c_1^{-1}, c_1d_1, d_1^{-1})$ and $(d_2, d_2^{-1}c_2^{-1}, c_2)$, respectively. Then G lies in $\pi_A(2)$.*

Proof. We proceed in two steps which then allow us to apply Theorem 5.4: First, we build a degenerate k-curve T of genus 2 and a covering ϕ: $T \to L$; second, we build a G-Galois cover ψ: $W \to T$ such that the pair (ϕ, ψ) is admissible.

Step 1. *Construction of* $T \to L$. Let X_1 denote the projective k-line and let $\Lambda_1 = \{\alpha_{11}, \alpha_{12}, \alpha_{13}\}$ be an ordered set of closed points on X_1. Similarly, let X_2 denote the projective k-line and let $\Lambda_2 = \{\alpha_{23}, \alpha_{22}, \alpha_{21}\}$ be an ordered set of closed points on X_2. Then by hypothesis, H_i lies in $\pi_A(X_i - \Lambda_i)$. As before, let L be the projective k-line and take a closed point λ with local parameter x. Then for each $i = 1, 2$, there exists a covering morphism $\phi_i \colon X_i \to L$, unramified at λ, such that $\phi_i^{-1}(\lambda) = \Lambda_i$ (because the X_i's are all $\mathbb{P}_k^1$'s). Take $x_{ij} \in \hat{\mathscr{O}}_{X_i, \alpha_{ij}}$ to be a local uniformizer at α_{ij} on X_i such that $\phi^*(x) = x_{ij}$, and let T be a union of X_1 and X_2 such that X_1 and X_2 cross transversely at closed points α_1, α_2, and α_3 in T (where $\alpha_{1j} \in X_1$ is identified with $\alpha_{2j} \in X_2$ for $j = 1, 2, 3$). Then, the complete local ring $\hat{\mathscr{O}}_{T, \alpha_j} \cong k[[x_{1j}, x_{2j}]]/(x_{1j}x_{2j})$ for $j = 1, 2, 3$. Also, T is constructed so that it is smooth outside of $\Lambda = \{\alpha_1, \alpha_2, \alpha_3\}$. Identifying X_i with its image in T, we equate α_{ij} with α_j for all i, j. The covers ϕ_1 and ϕ_2 induce a cover $\phi \colon T \to L$ for which $\phi^{-1}(\lambda) = \Lambda$. Moreover, letting $\hat{L} = \operatorname{Spec}(\hat{\mathscr{O}}_{L, \lambda})$ and $\hat{T}_{\alpha_j} = \operatorname{Spec}(\hat{\mathscr{O}}_{T, \alpha_j})$, we get $\hat{\phi} = \phi|_{\hat{T}_{\alpha_j}} \colon \hat{T}_{\alpha_j} \to \hat{L}$ is defined by $x_{1j} + x_{2j} = x$. Thus $\phi \colon T \to L$ satisfies conditions (a) and (b) of admissibility. In the notation of Lemma 2.2, we have $r = 2$ and $\nu_{12} = 3$, so $p_A(T) = 0 + 3 - 2 + 1 = 2$.

Step 2. *Construction of G-Galois cover of T.* Since $H_1 \in \pi_A^t(X_1 - \{\alpha_{11}, \alpha_{12}, \alpha_{13}\})$ with description $(g_1, g_2, g_3) = (c_1^{-1}, c_1 d_1, d_1^{-1})$, there exists a smooth connected H_1-Galois cover $W_1 \to X_1$ such that over each α_{1j} there is a ramified point ω_{1j} with canonical generator of inertia g_j. Similarly, as $H_2 \in \pi_A^t(X_2 - \{\alpha_{23}, \alpha_{22}, \alpha_{21}\})$ with description $(d_2, d_2^{-1}c_2^{-1}, c_2)$, there exists a smooth connected H_2-Galois cover $W_2 \to X_2$ such that over each α_{2j} there is a ramified point ω_{2j} with canonical generator of inertia h_j, where the $h_1 = c_2$, $h_2 = d_2^{-1}c_2^{-1}$, and $h_3 = d_2$. Since $c_1 = g c_2 g^{-1}$, $d_1 = h d_2 h^{-1}$, and $c_1 d_1 = c_2 d_2$, we have that $\operatorname{ord}(h_j) = \operatorname{ord}(g_j) = n_j$.

Let $\psi_i \colon \operatorname{Ind}_{H_i}^G W_i \to X_i$ be the induced G-Galois covers, and for any $a \in G$ let $(W_i)_a$ denote the component of $\operatorname{Ind}_{H_i}^G W_i$ corresponding to the coset aH_i. Let W be a union of $\operatorname{Ind}_{H_1}^G W_1$ and $\operatorname{Ind}_{H_2}^G W_2$ such that for every $a \in G$, $(W_1)_a$ and $(W_2)_{ga}$ cross transversely at a closed point ω_1^a (where $\omega_{11}^a \in (W_1)_a$ is identified with $\omega_{21}^{ga} \in (W_2)_{ga}$); $(W_1)_a$ and $(W_2)_a$ cross transversely at a closed point ω_2^a in W (where $\omega_{12}^a \in (W_1)_a$ is identified with $\omega_{22}^a \in (W_2)_a$); and $(W_1)_a$ and $(W_2)_{ha}$ cross transversely at a closed point ω_3^a in W (where $\omega_{13}^a \in (W_1)_a$ is identified with $\omega_{23}^{ha} \in (W_2)_{ha}$). Also, W may be constructed so that it is smooth outside of $\Omega = \{\omega_j^a : a \in G,\ j = 1, 2, 3, 4\}$. Identify $(W_i)_a$ with its image in W.

Let $(W)_a$ be the connected component of $(W_1)_a$ on W. Since the ψ_i's agree on the points of Ω, they induce a G-Galois cover $\psi \colon W \to T$. This cover is connected because H_1 and H_2 together generate G. Thus ψ: $W \to T$ satisfies conditions (c) and (d) of admissibility.

We have now defined covers $\phi: T \to L$ and $\psi: W \to T$ satisfying all the conditions of admissibility. Hence by Theorem 5.4 we obtain the desired smooth connected projective k-curve Y of genus $p_a(T) = 2$ such that G lies in $\pi_A(Y)$. ▮

PROPOSITION 5.6. *Let G be a finite group generated by elements a_1, b_1, a_2, b_2, where $[a_1, b_1][a_2, b_2] = 1$ and p is prime to the orders of a_1 and a_2. Let $H_0 = \langle a_1, a_2, [a_1, b_1]\rangle$, and suppose that $H_0 \in \pi_A^t(\mathbb{P}_k^1 - \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\})$ with description*

$$\left(a_1, b_1 a_1^{-1} b_1^{-1}, a_2, b_2 a_2^{-1} b_2^{-1}\right).$$

Then G lies in $\pi_A(2)$.

Proof. We follow the same two steps as those in Proposition 5.5.

Step 1. *Construction of $T \to L$.* Let $X_0 = \mathbb{P}_k^1$ and rename $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ by $\Lambda_0 = \{\alpha_{01}, \alpha_{02}, \alpha_{03}, \alpha_{04}\}$. By hypothesis, $H_0 \in \pi_A^t(X_0 - \Lambda_0)$. For $i = 1, 2$, let $H_i = \langle a_i \rangle$ and X_i be another projective k-line. Given $\Lambda_1 = \{\alpha_{11}, \alpha_{12}\}$, $\Lambda_2 = \{\alpha_{23}, \alpha_{24}\}$ closed points in X_1 and X_2, respectively, we know that $H_1 \in \pi_A^t(X_1 - \Lambda_1)$ with description (a_1^{-1}, a_1), and $H_2 \in \pi_A^t(X_2 - \Lambda_2)$ with description (a_2^{-1}, a_2). Let L be the projective k-line and take a closed point λ with local parameter x. Then for each $i = 0, 1, 2$ there exists a covering morphism $\phi_i: X_i \to L$, unramified at λ, such that $\phi_i^{-1}(\lambda) = \Lambda_i$ (because the X_i's are all $\mathbb{P}_k^1$'s). Take $x_{ij} \in \hat{\mathscr{O}}_{X_i, \alpha_{ij}}$ to be a local uniformizer at α_{ij} on X_i such that $\phi^*(x) = x_{ij}$. Let T be a union of X_0, X_1, and X_2 such that X_0 and X_1 cross transversely at closed points α_1 and α_2 in T (where $\alpha_{0j} \in X_0$ is identified with $\alpha_{1j} \in X_1$ for $j = 1, 2$); and X_0 and X_2 cross transversely at closed points α_3 and α_4 in T (where $\alpha_{0j} \in X_0$ is identified with $\alpha_{2j} \in X_2$ for $j = 3, 4$). We identify X_i with its image in T, and then equate α_{ij} and α_j. For $i = 1, 2$, if $\alpha_j \in X_i$ then the complete local ring $\hat{\mathscr{O}}_{T, \alpha_j} \cong k[[x_{0j}, x_{ij}]]/(x_{0j} x_{ij})$. We construct T so that it is smooth outside of $\Lambda = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$. Since $\phi_i(\Lambda_i) = \lambda \in L$, the covers ϕ_0, ϕ_1, ϕ_2 induce a cover $\phi: T \to L$ for which $\phi^{-1}(\lambda) = \Lambda$. Moreover, letting $\hat{L} = \mathrm{Spec}(\hat{\mathscr{O}}_{L, \lambda})$ and $\hat{T}_{\alpha_j} = \mathrm{Spec}(\hat{\mathscr{O}}_{T, \alpha_j})$, we get $\hat{\phi}_j = \phi|_{\hat{T}_{\alpha_j}}: \hat{T}_{\alpha_j} \to \hat{L}$ is defined by $x_{0j} + y_{ij} = x$. Thus, $\phi: T \to L$ satisfies conditions (a) and (b) of admissibility. In the notation of Lemma 2.2, we have $r = 3$ and $\nu_{01} = \nu_{02} = 2$ and $\nu_{12} = 0$, so $p_A(T) = 0 + 4 - 3 + 1 = 2$.

Step 2. *Construction of G-Galois cover of T.* Since $H_1 \in \pi_A^t(X_1 - \{\alpha_{11}, \alpha_{12}\})$ with description (a_1^{-1}, a_1), there exists a smooth connected H_1-Galois cover $W_1 \to X_1$ such that over each α_{1j} $(j = 1, 2)$ there is a ramified point ω_{1j} with canonical generator of inertia h_j, where $h_1 = a_1^{-1}$

and $h_2 = a_1$. For $j = 3, 4$, do the same for H_2 and X_2 with $h_3 = a_2^{-1}$ and $h_4 = a_2$.

Meanwhile, by hypothesis $H_0 \in \pi_A^t(X_0 - \{\alpha_{01}, \alpha_{02}, \alpha_{03}, \alpha_{04}\})$ with description

$$\left(a_1, b_1 a_1^{-1} b_1^{-1}, a_2, b_2 a_2^{-1} b_2^{-1}\right).$$

So, there exists a smooth connected H_0-Galois cover $W_0 \to X_0$ such that over each α_{0j} there is a ramified point ω_{0j} with canonical generator of inertia g_j, where $g_1 = a_1$, $g_2 = b_1 a_1^{-1} b_1^{-1}$, $g_3 = a_2$, and $g_4 = b_2 a_2^{-1} b_2^{-1}$.

Let ψ_i: $\mathrm{Ind}_{H_i}^G W_i \to X_i$ be the induced G-Galois covers, and as above for any $g \in G$ let $(W_i)_g$ denote the component of $\mathrm{Ind}_{H_i}^G W_i$ corresponding to the coset gH_i. Let W be a union of $\mathrm{Ind}_{H_0}^G W_0$, $\mathrm{Ind}_{H_1}^G W_1$, and $\mathrm{Ind}_{H_2}^G W_2$ such that for every $h \in G$, $(W_0)_h$ and $(W_1)_h$ cross transversely at a closed point ω_1^h in W (where $\omega_{01}^h \in (W_0)_h$ is identified with $\omega_{11}^h \in (W_1)_h$); $(W_0)_h$ and $(W_1)_{b_1 h}$ cross transversely at a closed point ω_2^h (where $\omega_{02}^h \in (W_0)_h$ is identified with $\omega_{12}^{b_1 h} \in (W_1)_{b_1 h}$); $(W_0)_h$ and $(W_2)_h$ cross transversely at a closed point ω_3^h in W (where $\omega_{03}^h \in (W_0)_h$ is identified with $\omega_{23}^h \in (W_2)_h$); and $(W_0)_h$ and $(W_2)_{b_2 h}$ cross transversely at a closed point ω_4^h in W (where $\omega_{04}^h \in (W_0)_h$ is identified with $\omega_{24}^{b_2 h} \in (W_2)_{b_2 h}$). Also, we construct W so that it is smooth outside of $\Omega = \{\omega_j^h : h \in G,\ j = 1, 2, 3, 4\}$ and then we identify $(W_i)_g$ with its image in W.

Let $(W)_g$ be the connected component of $(W_0)_g$ on W. Since the ψ_i's agree on the points of Ω, they induce a G-Galois cover ψ: $W \to T$. Notice that by construction, $(W_i)_e \subset (W)_e$ and $(W_i)_{b_i} \subset (W)_e$, where e is the identity element of the group G. To show that W is connected it suffices to show that for each $g \in G$ the component $(W)_g$ is contained in $(W)_e$. Recall that g is a word in a_1, b_1, a_2, b_2, so by induction on the length of that word it suffices to show that each $(W)_{a_i}$ and $(W)_{b_i}$ is contained in $(W)_e$. Well, $(W_0)_e$ is contained in $(W)_e$, so H_0 is contained in the stabilizer of $(W)_e$. Thus, as a_i is contained in H_0, $(W)_{a_i}$ must be contained in $(W)_e$. For b_i, we know $(W_i)_{b_i} \subset (W)_{b_i}$ and $(W_i)_{b_i} \subset (W)_e$, so $(W)_{b_i}$ is contained in $(W)_e$. Thus ψ: $W \to T$ satisfies conditions (c) and (d) of admissibility.

We have now defined covers ϕ: $T \to L$ and ψ: $W \to T$ satisfying all the conditions of admissibility. Hence by Theorem 5.4, we obtain the desired smooth connected projective k-curve Y of genus $p_A(T) = 2$ such that G lies in $\pi_A(Y)$. ∎

Remark. In Proposition 5.6, the hypothesis that $H_0 \in \pi_A^t(\mathbb{P}^1 - \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\})$ with description $(a_1, b_1 a_1^{-1} b_1^{-1}, a_2, b_2 a_2^{-1} b_2^{-1})$ is satisfied if p is prime to the order of the group H_0.

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REFERENCES

[Ab] S. Abhyankar, Coverings of algebraic curves, *Amer. J. Math.* **79** (1957), 825–856.

[AB] M. Auslander and D. Buchsbaum, Homological dimension in local rings, *Trans. Amer. Math. Soc.* **85** (1957), 390–405.

[Bo] I. Bouw, Ph.D. thesis, University of Utrecht, forthcoming.

[FJ] M. Fried and M. Jarden, "Field Arithmetic," Ergeb. Math. Grenzgeb., Vol. 11, Springer-Verlag, Berlin/New York, 1986.

[FV] M. Fried and H. Volklein, The Inverse Galois problem and rational points on moduli space, *Math. Ann.*, **209** (1991), 771–800.

[Go] D. Gorenstein, "The Classification of Finite Simple Groups," Plenum Press, New York, 1983.

[Gr] A. Grothendieck, "Revêtements étales et groupe fondamental (SGA 1)," Lecture Notes in Mathematics, Vol. 224, Springer-Verlag, Berlin/Heidelberg/New York, 1971.

[Ha1] D. Harbater, Formal patching and adding branch points, *Amer. J. Math.* **115** (1993), 487–508.

[Ha2] D. Harbater, Abhyankar's conjecture on Galois groups over curves, *Invent. Math.* **117** (1993), 1–25.

[Ha3] D. Harbater, Fundamental groups and embedding problems in characteristic p, *in* "Recent Developments in the Inverse Galois Problem" (M. Fried et al., Eds.), pp. 353–369, Contemp. Math. Ser., Vol. 186, Amer. Math. Soc., Providence, RI, 1995.

[Ht] R. Hartshorne, "Algebraic Geometry," Springer-Verlag, New York/Berlin, 1977.

[HV] D. Haran and H. Volklein, Galois groups over complete valued fields, preprint, 1994.

[Ka] E. Kani, The Galois-module structure of the space of holomorphic differentials of a curve, *Crelle's J.* **367** (1986), 187–206.

[Kt] H. Katsurada, Generalized Hasse–Witt invariants over families of varieties defined over finite fields, *J. Math. Soc. Japan* **31** (1979), 119–218.

[Ma] H. Matsumara, "Commutative Algebra," Benjamin, New York, 1970.

[MM] G. Malle and B. H. Matzat, "Inverse Galois Theory," in preparation.

[Na1] S. Nakajima, On Galois module structure of the cohomology groups of an algebraic variety, *Invent. Math.* **75** (1984), 1–8.

[Na2] S. Nakajima, "On Generalized Hasse–Witt Invariants of an Algebraic Curve," pp. 69–88, Adv. Stud. Pure Math., Vol. 2, Academic Press, Boston, 1983.

[Po] P. Popp, "Moduli Theory and Classification Theory of Algebraic Varieties," Springer-Verlag, Berlin, 1977.

[Ra] M. Raynaud, Revêtements de la droite affine en caractéristique $p > 0$ et conjecture d'Abhyankar, 1992 preprint; *Invent. Math.* **116** (1994), 425–462.

[Sa] M. Saidi, "Revêtements étales et réducation semi-stable des courbes," Thèse, Université Bordeaux I, Proposition, 1994.

[Se] J. P. Serre, Sur la topologie des variétiés algébriques en caractéristique p, *in* "Proceedings, Symposium de topologia algebrica, Mexico, 1956," pp. 24–53.

[V] H. Volklein, "Groups as Galois Groups—An Introduction," in preparation.