On the change of the Jordan form under the transition from the adjacency matrix of a vertex-transitive digraph to its principal submatrix of co-order one

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Abstract

Let \( J(\lambda; n_1, \ldots, n_k) \) be the set of matrices \( A \) such that \( \lambda \) is an eigenvalue of \( A \) and \( n_1 \leq \cdots \leq n_k \) are the sizes of the Jordan blocks associated with \( \lambda \). For a given index \( v \) of \( A \), denote by \( A - v \) the principal submatrix of co-order one obtained from \( A \) by deleting the \( v \)th row and column. In the present paper, all possible changes of the part of the Jordan form corresponding to \( \lambda \) under the transition from \( A \) to \( A - v \) are determined for matrices \( A \in J(\lambda; n_1, \ldots, n_k) \) such that for the eigenvalue \( \lambda \) of both \( A \) and \( A^\top \), there exists a Jordan chain of the largest length \( n_k \) whose eigenvector has nonzero \( v \)th entry. In particular, it is shown that for almost every matrix \( A \in J(\lambda; n_1, \ldots, n_k) \), \( n_1, \ldots, n_k - 1 \) are the sizes of Jordan blocks for \( \lambda \) considered as an eigenvalue of \( A - v \). Moreover, it is also proved that if \( A \) is the adjacency matrix of a vertex-transitive digraph and \( k \geq 2 \), then the change \( n_1, \ldots, n_k \rightarrow n_1, \ldots, n_{k-2}, 2n_{k-1} - 1 \) holds for the eigenvalue \( \lambda \) under the transition from \( A \) to \( A - v \). In the case of \( k = 1 \), \( \lambda \) is a simple eigenvalue of \( A \) and does not belong to the spectrum of \( A - v \).

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1. Introduction

Let $A$ be a square matrix and $v$ be one of its indices. Denote by $A - v$ the matrix obtained by deleting the $v$th column and row from $A$. In [1] such a matrix is called a principal submatrix of co-order one. It was also shown therein that the part of the Jordan form corresponding to $\lambda$ is changed under adding a generic column and row to an arbitrary matrix having $\lambda$ as an eigenvalue in the following way: one Jordan block of the largest size disappears and all the others remain unchanged. In the present paper, we show that the same holds under the transition from $A$ to $A - v$ for almost all matrices $A$ with a given sequence of Jordan blocks for $\lambda$. Of course, it is natural to compare this change with that when the group $\mathcal{C}(A)$ of permutation matrices $P$ commuting with the original matrix $A$ is sufficiently large. Here we shall consider the case of a $(0, 1)$-matrix $A$ such that for any two indices $v$ and $w$ there exists a permutation matrix $P \in \mathcal{C}(A)$ whose $(v, w)$th entry is equal to one. In other words, we shall assume that $A$ is the adjacency matrix of a vertex-transitive digraph. Many interesting facts concerning such digraphs and their spectra are contained in [2]. The reader will find the necessary information about them in Section 2 of our paper.

Without any doubt, it is interesting to compare the changes of the Jordan form when it contains nontrivial Jordan blocks. Different constructions allowing to obtain vertex-transitive and arc-transitive digraphs whose adjacency matrices are not diagonalizable were given in [3,4]. The simplest example of such a vertex-transitive digraph is the Cayley digraph on the dihedral group $D_4 = \langle a, b \mid a^4 = b^2 = e, (ab)^2 = e \rangle$ with respect to the standard generating set $S = \{a, b\}$. This example is due to Ch. Godsil and B.D. McKay (see [5]). In [6] it was proved that there exists a Cayley digraph on the dihedral group $D_n$ whose adjacency matrix is not diagonalizable if and only if $n$ is a composite number. This statement and the other results obtained in [6] show that groups with the “nondiagonalizable” property are not pathology. On the contrary, non-Abelian groups each of whose Cayley digraphs has diagonalizable adjacency matrix are rare. In [6,7] only two infinite series of such groups were found. One of them is $Q_8 \times Z_2^2$ and the other is $D_p$, where $n$ and $p$ run over all nonnegative integers and all primes, respectively. Thus, the subject of our work is not empty and its results can be interesting not only from the spectral point of view, but also in finite group theory and graph theory.

It is not difficult to show that if $A$ is the adjacency matrix of a vertex-transitive (undirected) graph (so, $A$ is symmetric) and $\lambda$ is an eigenvalue of $A$, then its multiplicity decreases by one under the transition from $A$ to $A - v$. In Section 3 we prove that the same holds for the algebraic multiplicity in the directed case. If $\lambda$ is a semi-simple eigenvalue of $A$, this information is sufficient for determining the change of the Jordan form. In the opposite case we need some equality involving the (spectral) indices $\text{ind}_{A - v}(\lambda)$ and $\text{ind}_A(\lambda)$ of $\lambda$ considered as an eigenvalue of $A - v$ and $A$, respectively. In Section 4 it is shown that $\text{ind}_{A - v}(\lambda) = 2 \text{ind}_A(\lambda) - 1$ and the change of the Jordan form is completely determined.
2. The main definitions and notation

Let $D$ be a directed graph (digraph) without loops. Let $V(D)$ and $A(D)$ be the vertex-set and arc-set of $D$, respectively. The matrix $A$ such that $A(v, w) = 1$ if $(v, w)$ is an arc in $D$, and $A(v, w) = 0$ in the opposite case is called the adjacency matrix of $D$. Here we identify the set $V(A)$ of indices of $A$ with the vertex-set $V(D)$.

Let $T$ be any one-to-one map of $V(D)$. Consider the permutation matrix $P$ whose entries are defined in the following way: $P(v, w) = 1$ if $T(v) = w$, and $P(v, w) = 0$ in the opposite case. Since $T$ is a one-to-one correspondence of $V(D)$, there is only one nonzero entry in every column (row) of $P$. A one-to-one map $T$ of $V(D)$ is called an automorphism of $D$ if $(v, w) \in A(D)$ implies $(Tv, Tw) \in A(D)$ and vice versa. In the matrix language, this means that the corresponding permutation matrix $P$ commutes with the adjacency matrix $A$: $AP = PA$. We shall denote by $\mathcal{C}(A)$ the set of such permutation matrices.

By definition, $D$ is a vertex-transitive digraph if its automorphism group acts transitively on its vertex-set. This means that for any two vertices $v$ and $w$ there is an automorphism $T$ that moves $v$ to $w$. In other words, for any two indices $v$ and $w$ in $V(A)$, there is a permutation matrix $P \in \mathcal{C}(A)$ such that $P(v, w) = 1$.

3. The change of the algebraic multiplicity

Let $A$ be an arbitrary matrix and $v$ be any index of $A$. Denote by $A - v$ the principal submatrix of co-order one obtained by deleting the $v$th column and row from $A$. It is well known (see [2,8]) that

$$\frac{\partial}{\partial z} \det(A - zE) = - \sum_{v \in V(A)} \det((A - v) - zE),$$

where $E$ is the identity matrix of appropriate order.

Let $\lambda$ be an eigenvalue of $A$. Denote by $n_A(\lambda)$ the algebraic multiplicity of $\lambda$. By definition, $n_A(\lambda)$ coincides with the multiplicity of the point $\lambda$ as a root of the characteristic polynomial $\det(A - zE)$.

**Proposition 1.** Let $A$ be the adjacency matrix of a vertex-transitive digraph $D$ and $\lambda$ be any eigenvalue of $A$. Then for any index $v$ of $A$ we have

$$n_A(\lambda) = n_{A-v}(\lambda) + 1.$$

**Proof.** Take any two vertices $v$ and $w$ in $D$. Since $D$ is a vertex-transitive digraph, there exists an automorphism $T$ of $D$ that takes $v$ to $w$. Denote by $D - v$ the one-vertex-deleted subdigraph obtained by removing the vertex $v$ together with the incident arcs from $D$ (so, $A - v$ is the adjacency matrix of $D - v$). It is clear that $T$ also maps $D - v$ onto $D - w$ and therefore gives an isomorphism between them. This means that the submatrices $A - v$ and $A - w$ are similar via a permutation matrix.
In particular, their characteristic polynomials coincide with each other. By identity (1), we have
\[ \frac{\partial}{\partial z} \det(A - zE) = -\text{card} V(A) \det((A - v) - zE) \]
for any \( v \in V(A) \). So, if \( \lambda \) is a zero of order \( p \) of the polynomial \( \det(A - zE) \), then the function \( \frac{\partial}{\partial z} \det(A - zE) \) and therefore the polynomial \( \det((A - v) - zE) \) itself has a zero of order \( p - 1 \) at the point \( z = \lambda \). The proposition is proved. \( \square \)

Let \( (A - zE)^{-1}(v, v) \) be the \( (v, v) \)th entry of the resolvent matrix \( (A - zE)^{-1} \).

Using the cofactor formula for the inverse to the matrix \( A - zE \), we have
\[ (A - zE)^{-1}(v, v) = \frac{\det((A - v) - zE)}{\det(A - zE)}. \tag{2} \]

So, the following statement holds.

**Corollary 1.** Let \( A \) be the adjacency matrix of a vertex-transitive digraph and \( \lambda \) be any eigenvalue of \( A \). Then each diagonal entry of the function \( (A - zE)^{-1} \) has a simple pole at the point \( z = \lambda \).

### 4. The change of the spectral index

Let \( \lambda \) be an eigenvalue of \( A \). We say that \( \xi_1, \ldots, \xi_h \) is an \( A, \lambda \)-chain if
\[ \xi_{m-1} = (A - \lambda E)\xi_m \quad \text{for } m = 1, \ldots, h, \quad \text{where } \xi_0 = 0. \]

By definition, \( \xi \) is the generalized eigenvector associated with \( \lambda \) if \( (A - \lambda E)^m \xi = 0 \) for some \( m \). The smallest such \( m \) is called the height of \( \xi \) and is denoted by \( h \). In this case the vector \( \xi \) generates the \( A, \lambda \)-chain \( \xi_1, \ldots, \xi_h \) of length \( h \) by the rule:
\[ \xi_m = (A - \lambda E)^{h-m} \xi \quad \text{for } m = 1, \ldots, h. \]

All the generalized eigenvectors associated with \( \lambda \) form the generalized eigenspace \( L_A(\lambda) \) of the matrix \( A \) corresponding to \( \lambda \). Its dimension is equal to \( r_A(\lambda) \) (see [9]). The smallest number \( q \) such that \( (A - \lambda E)^q \xi = 0 \) for every \( \xi \in L_A(\lambda) \) is called the (spectral) index of the eigenvalue \( \lambda \). In our paper, we shall also denote it by \( \text{ind}_A(\lambda) \). It is clear that the index \( \text{ind}_A(\lambda) \) is equal to the size of the largest Jordan block in the part of the Jordan form of \( A \) corresponding to \( \lambda \).

We say that a vector \( \xi \in L_A(\lambda) \) has depth \( d \) if \( \xi \) can be represented as \( (A - \lambda E)^d \xi \) for some \( \xi \in L_A(\lambda) \) but there is no vector \( \theta \) such that \( \xi = (A - \lambda E)^{d+1} \theta \) (this definition is used, for instance, in [10]). It is not difficult to show that every vector in \( L_A(\lambda) \) with depth \( \text{ind}_A(\lambda) - 1 \) is an eigenvector. In the sequel, we shall assume that the depth of the zero vector \( 0 \) in \( L_A(\lambda) \) is equal to \( \text{ind}_A(\lambda) - 1 \). In this case all the vectors of depth \( \text{ind}_A(\lambda) - 1 \) in \( L_A(\lambda) \) form a subspace \( I_A(\lambda) \).
Lemma 1. Let $A$ be the adjacency matrix of a vertex-transitive digraph and $\lambda$ be any eigenvalue of $A$. Then for any $v \in V(A)$, there exists an eigenvector of depth \[ \text{ind}_A(\lambda) - 1 \] for $\lambda$ whose $v$th entry is nonzero.

Proof. Assume that every vector of $I_A(\lambda)$ has zero $v$th entry. Let $w$ be any index of $A$. Since $A$ is the adjacency matrix of a vertex-transitive digraph, there is a permutation matrix $P \in \mathbb{C}(A)$ such that $P(v, w) = 1$. It is clear that $P$ is invertible and commutes with $(A - \lambda E)^m$ for every natural $m$. So, for any $\xi \in I_A(\lambda)$, we have that $P \xi \in L_A(\lambda)$ and the vectors $\xi$ and $P \xi$ have the same depth. In particular, if $\xi \in I_A(\lambda)$, then $P \xi \in I_A(\lambda)$. It is also clear that the $v$th entry of $P \xi$ is equal to the $w$th entry of $\xi$. This implies that the $w$th entry of $\xi$ is equal to zero. We recall that $w$ and $\xi$ have been arbitrarily chosen in $V(A)$ and $I_A(\lambda)$, respectively. Thus, $I_A(\lambda) = \{0\}$ and therefore $L_A(\lambda) = \{0\}$. This contradicts the assumption that $\lambda$ is an eigenvalue of $A$. The lemma is proved. \[ \square \]

Let $\eta_v$ and $\xi_v$ be the vectors obtained from the $v$th column and $v$th row of $A$, respectively, by deleting the diagonal entry $A(v, v)$. In the sequel, we shall assume that $v$ is the first index of $A$. In this case the matrix $A$ has the following form:

\[
\begin{pmatrix}
A(v, v) & \xi_v \\
\eta_v & A - v
\end{pmatrix}.
\]

Let $\hat{\xi}_1, \ldots, \hat{\xi}_h$ be an $A - v, \lambda$-chain. By definition, the sequence

\[
\begin{pmatrix}
0 \\
\hat{\xi}_1 \\
\vdots \\
0
\end{pmatrix}
\]

is its $v$-extension. It is clear that the $v$-extension of the $A - v, \lambda$-chain $\hat{\xi}_1, \ldots, \hat{\xi}_h$ is an $A, \lambda$-chain iff $(\xi_v, \hat{\xi}_1) = \cdots = (\xi_v, \hat{\xi}_h) = 0$ (in other words, the vectors $\hat{\xi}_1, \ldots, \hat{\xi}_h$ are orthogonal to $\xi_v$ with respect to the scalar product $(\cdot, \cdot)$ without the complex conjugation of the entries of the second vector in it).

Lemma 2. Let $A$ be an arbitrary matrix, $\lambda$ be an eigenvalue of $A$, and $q$ be the spectral index of $\lambda$. Assume that the $v$th entry of some eigenvector $\xi$ of depth $q - 1$ in $L_A(\lambda)$ is nonzero. Then there exists a vector $\hat{\eta}_q$ such that $(A - v - \lambda E)^q \hat{\eta}_q = \eta_v$.

Proof. Let $\xi$ be a vector such that $\xi = (A - \lambda E)^q - 1 \xi$. Consider the $A, \lambda$-chain $\hat{\xi}_1, \ldots, \hat{\xi}_q$ generated by the vector $\xi$ (so, $\hat{\xi}_1 = \xi$ and $\hat{\xi}_q = \xi$). Without loss of generality, we can assume that this chain has the following form:

\[
\begin{pmatrix}
-1 \\
\hat{\eta}_1 \\
0 \\
\hat{\eta}_2 \\
\vdots \\
0
\end{pmatrix}, \begin{pmatrix}
0 \\
\hat{\eta}_2 \\
\hat{\xi}_2 \\
0 \\
\vdots \\
0
\end{pmatrix}, \ldots, \begin{pmatrix}
0 \\
\hat{\eta}_q \\
\hat{\xi}_q \\
0 \\
\vdots \\
0
\end{pmatrix}.
\]

Indeed, multiplying the vector $\xi$ (and therefore the vector $\hat{\xi}_i$) by a nonzero constant $\alpha$, we can assume that the $v$th entry of $\hat{\xi}$ is equal to $-1$. Define the numbers $\alpha_1, \ldots, \alpha_{q-1}$ recurrently by the condition that the $v$th entry of the vector
is equal to zero for \( m = 2, \ldots, q \). Then the vector \( \xi_q + \sum_{i=1}^{q-1} \alpha_{q-i} \xi_i \) generates an \( A, \lambda \)-chain of length \( q \) with the desired properties.

The fact that \( \xi_1 \) is an eigenvector of \( A \) for \( \lambda \) can be rewritten in the following form:

\[
\begin{pmatrix}
A(v, v) & \xi_v \\
\eta_v & A - v
\end{pmatrix}
\begin{pmatrix}
-1 \\
\hat{\eta}_1
\end{pmatrix}
= \begin{pmatrix}
-\lambda & \xi_v + (\xi_v, \hat{\eta}_1) \\
-\eta_v + (A - v)\hat{\eta}_1 & -1
\end{pmatrix}
\begin{pmatrix}
\hat{\eta}_1
\end{pmatrix}.
\]

In particular, this implies that \( (A - v)\hat{\eta}_1 = \eta_v \).

Moreover,

\[
\begin{pmatrix}
A(v, v) & \xi_v \\
\eta_v & A - v
\end{pmatrix}
\begin{pmatrix}
0 \\
\hat{\eta}_2
\end{pmatrix}
= \begin{pmatrix}
(\xi_v, \hat{\eta}_2) \\
(A - v)\hat{\eta}_2
\end{pmatrix}
= \lambda \begin{pmatrix}
0 \\
\hat{\eta}_2
\end{pmatrix} + \begin{pmatrix}
-1 \\
\hat{\eta}_1
\end{pmatrix}
\]

and for \( n = 3, \ldots, q \) we have

\[
\begin{pmatrix}
A(v, v) & \xi_v \\
\eta_v & A - v
\end{pmatrix}
\begin{pmatrix}
0 \\
\hat{\eta}_n
\end{pmatrix}
= \begin{pmatrix}
(\xi_v, \hat{\eta}_n) \\
(A - v)\hat{\eta}_n
\end{pmatrix}
= \lambda \begin{pmatrix}
0 \\
\hat{\eta}_n
\end{pmatrix} + \begin{pmatrix}
0 \\
\hat{\eta}_{n-1}
\end{pmatrix}.
\]

So, \( (A - v)\hat{\eta}_n = \hat{\eta}_{n-1} \) for \( n = 2, \ldots, q \). In particular, \( (A - v) - \lambda E)\hat{\eta}_q = \eta_v \). The lemma is proved. □

The following lemma is a direct consequence of the results from [11,12]. Nevertheless, we give our own proof based on a very simple and natural orthogonalization procedure.

**Lemma 3.** Let \( A \) be an arbitrary matrix, \( \lambda \) be an eigenvalue of \( A \), and \( q \) be the spectral index of \( \lambda \). Assume that \( \lambda \) is also an eigenvalue of \( A - v \) and \( q < \text{ind}_{A-v}(\lambda) \).

Then for any Jordan basis of \( A - v \), there exists exactly one Jordan chain of length strictly greater than \( q \) corresponding to \( \lambda \).

**Proof.** Assume the converse. Then there exist two \( A - v, \lambda \)-chains \( \hat{\xi}_1, \ldots, \hat{\xi}_{q+1} \) and \( \hat{\zeta}_1, \ldots, \hat{\zeta}_{q+1} \) of length \( q + 1 \) whose eigenvectors are linearly independent (they can be obtained by cutting any two Jordan chains of length strictly greater than \( q \) corresponding to the eigenvalue \( \lambda \) of \( A - v \)). Let \( m \) be the smallest number such that \( (\hat{\xi}_v, \hat{\xi}_m) \neq 0 \) and \( n \) be the smallest number such that \( (\hat{\xi}_v, \hat{\zeta}_n) \neq 0 \). Without loss of generality, we can assume that \( m \geq n \). Take the unique solution \( \alpha \) to the equation \( (\hat{\xi}_v, \hat{\xi}_m - \alpha \hat{\zeta}_n) = 0 \) and consider the \( A - v, \lambda \)-chain of length \( q + 1 \) generated by the vector \( \hat{\xi}_{q+1} - \alpha \hat{\zeta}_{q+1} \). Its \( m \)th member coincides with \( \hat{\xi}_m = \alpha \hat{\zeta}_n \). By the choice of \( \alpha \), this vector is orthogonal to \( \hat{\xi}_v \). Moreover, the same holds for the \( i \)th member of the considered chain if \( i < m \).

Repeating the above orthogonalization procedure, we obtain an \( A - v, \lambda \)-chain of length \( q + 1 \) all of whose vectors are orthogonal to \( \hat{\xi}_v \). In this case its \( v \)-extension
is an $A, \lambda$-chain of length $q + 1$. But the existence of such a chain contradicts the definition of the spectral index $q$. The lemma is proved. □

Remark 1. For an arbitrary matrix $A$, the numbers
$$\sigma_m(A, \lambda) = \dim \ker(A - \lambda E)^m - \dim \ker(A - \lambda E)^{m-1},$$
where $m = 1, \ldots, \text{ind}_A(\lambda)$, form a sequence which is called the Weyr characteristic of the eigenvalue $\lambda$ of the matrix $A$ (see [13]). It is clear that $\sigma_m(A, \lambda) = 0$ if $m > \text{ind}_A(\lambda)$. The proof of Lemma 3 allows us to prove the inequality
$$\sigma_m(A - v, \lambda) - 1 \leq \sigma_m(A, \lambda) \leq \sigma_m(A - v, \lambda) + 1$$
for every natural number $m$. It is not difficult to show that Lemma 3 itself is a direct consequence of these inequalities. Moreover, they imply that if $n_A(\lambda) = n_A - v(\lambda) - \text{ind}_A - v(\lambda)$, then in the part of the Jordan form corresponding to $\lambda$, one Jordan block of size $\text{ind}_A - v(\lambda)$ disappears and the others remain unchanged under the transition from $A - v$ to $A$. This fact was first proved in [1] by quite another way.

It is not difficult to show that if $A$ is the adjacency matrix of a digraph $D$, then $A^\top$ is the adjacency matrix of the digraph obtained from $D$ by reversing the direction of every arc in it. Since every permutation matrix $P$ is an unitary matrix, $C(A^\top) = C(A)$ for every matrix $A$. In particular, if $A$ is the adjacency matrix of a vertex-transitive digraph, then $A^\top$ is the adjacency matrix of such a digraph, too. In the sequel, we shall use the fact that the matrix $A$ and its transpose $A^\top$ have the same Jordan form. Moreover, now Lemma 1 can be applied not only to the matrix $A$ but also to $A^\top$.

Proposition 2. Let $A$ be an arbitrary matrix, $\lambda$ be an eigenvalue of $A$, $q$ be the spectral index of $\lambda$, and $n_1 \leq \cdots \leq n_k$ be the sizes of the Jordan blocks associated with $\lambda$. Assume that for both $L_A(\lambda)$ and $L_A - v(\lambda)$ there exists an eigenvector of depth $q - 1$ for $\lambda$ whose $v$th entry is nonzero. Then
$$n_1, \ldots, n_{k-1}, \text{ind}_A - v(\lambda)$$
are the sizes of Jordan blocks for $\lambda$ considered as an eigenvalue of $A - v$. Moreover,

(1) if $\text{ind}_A - v(\lambda) \leq q$, then $\text{ind}_A - v(\lambda) = n_k - 1$;
(2) if $\text{ind}_A - v(\lambda) > q$, then $n_k - 1 = n_k = q$ and $\text{ind}_A - v(\lambda) = 2q - (n_A(\lambda) - n_A - v(\lambda)).$

Proof. The matrix $A^\top$ can be represented in the following form:
$$\begin{pmatrix} A(v, v) & \eta_v \\ \xi_v & (A - v)^\top \end{pmatrix}.$$
From this representation it follows that the vector $\xi_v$ plays the same role for $A^\top$ as $\eta_v$ plays for the original matrix $A$. By Lemma 2 applied to the matrix $A^\top$, there exists a vector $\xi_q$ such that $( (A - v)^\top - \lambda E )^q \xi_q = \xi_v$. 


Let $\hat{\xi}$ be any vector in $L_{A,v}(\lambda)$ such that $((A - v) - \lambda E)^q \hat{\xi} = 0$. Then

$$
(\hat{\xi}, \xi) = ((A - v)^T - \lambda E)^q \hat{\xi}, \xi_q \rangle = ((A - v) - \lambda E)^q \hat{\xi}, \xi_q \rangle = 0.
$$

This means that the $v$-extension of $\text{Ker}((A - v) - \lambda E)^q$ belongs to the generalized eigenspace $L_A(\lambda)$.

By assumption, there is an $A, \lambda$-chain of length $q$ whose eigenvector has nonzero $v$th entry. Without loss of generality, we can assume that it has the form (3). Take now any vector $\xi \in L_A(\lambda)$. Adding a linear combination of the vectors of the $A, \lambda$-chain (3) to $\xi$ if it is necessary, we can assume that the $v$th entry of every vector of the $A, \lambda$-chain generated by $\xi$ is equal to zero. In this case, $\xi$ is the $v$-extension of a vector $\hat{\xi}$ such that $((A - v) - \lambda E)^q \hat{\xi} = 0$. This means that $L_A(\lambda)$ coincides with the direct sum of the $v$-extension of the subspace $\text{Ker}((A - v) - \lambda E)^q$ and the linear hull of the vectors of the $A, \lambda$-chain (3). In particular, the geometric multiplicity of the eigenvalue $\lambda$ of $A - v$ is equal to $k - 1$.

Let $m_1 \leq \cdots \leq m_{k-1}$ be the sizes of Jordan blocks for $\lambda$ considered as an eigenvalue of $A - v$. Since the $v$-extension of any Jordan basis of $\text{Ker}((A - v) - \lambda E)^q$ combined with the $A, \lambda$-chain (3) of length $q$ forms a Jordan basis of $L_A(\lambda)$, the proposition is proved.

\[ n_A(\lambda) = n_1 + \cdots + n_{k-1} = 2q - m_{k-1}. \]

Moreover, $n_1 + \cdots + n_{k-2} + m_{k-1} = n_{A,v}(\lambda)$. Thus, $n_A(\lambda) - n_{A,v}(\lambda) = 2q - m_{k-1}$. Recalling now that $m_{k-1} = \text{ind}_{A,v}(\lambda)$, we obtain (1) and (2) in the statement of the proposition. The proposition is proved. \( \square \)

Let $A$ be an arbitrary matrix, $d$ be the number of Jordan blocks in the Jordan form of $A$, and $n_1, \ldots, n_d$ be their sizes. Let $\{\xi_{n_1}^{(p)}\}_{n_1=1}^{\cdots} \subseteq \text{id}$ be an arbitrary Jordan basis of $A$ and $\{\eta_{n_1}^{(p)}\}_{n_1=1}^{\cdots} \subseteq \text{id}$ be the dual Jordan basis for its transpose $A^T$. Here we assume that the $p$th Jordan chain corresponds to the eigenvalue $\lambda_p, (A - \lambda_p E)\xi_{n_p}^{(p)} = \xi_{n_{p-1}}^{(p)}$ and $\langle A^T - \lambda_p E \rangle \eta_{n_p}^{(p)} = \eta_{n_{p-1}}^{(p)}$ for $n = 1, \ldots, n_p$ and $p = 1, \ldots, d$ (by definition, $\xi_{0}^{(p)} = \eta_{0}^{(p)} = 0$). The bases are dual in the sense that $\langle \xi_{n_p}^{(p)} | \eta_{n_{p+1-n_p}}^{(p)} \rangle = \delta_{n_p,n_{p+1-n_p}}$, where $\delta$ is the Kronecker symbol.

Consider any eigenvalue $\lambda$ of $A$. Let $q$ be the spectral index of $\lambda$ and $k$ be the geometric multiplicity of $\lambda$. Without loss of generality, we can assume that $\lambda_1 = \cdots = \lambda_k = \lambda$ and $n_1 \leq \cdots \leq n_k$. Denote by $s$ the number defined by the condition $n_1 \leq \cdots \leq n_{s-1} < n_s = \cdots = n_k$ (if $n_1 = \cdots = n_k = q$, then we assume that $s = 1$). Then the leading coefficient $c_{-q}$ of the Laurent expansion

$$c_{-q} (z - \lambda)^{-q} + \cdots + c_{-1} (z - \lambda)^{-1} + \cdots
$$

of the function $(z E - A)^{-1}(v, v)$ at $z = \lambda$ has the following form:

$$c_{-q} = \xi_1^{(s)}(v)\eta_1^{(s)}(v) + \cdots + \xi_1^{(k)}(v)\eta_1^{(k)}(v).$$
By identity (2), the inequality \( c_{-q} \neq 0 \) holds iff \( n_{A-v}(\lambda) = n_A(\lambda) - \text{ind}_A(\lambda) \).

One can also see that the condition \( c_{-q} \neq 0 \) implies the inequalities \( \xi_1^{(p)}(v) \neq 0 \) and \( n_1^{(m)}(v) \neq 0 \) for some \( p,m \in \{s, \ldots, k\} \). From this fact and Proposition 2 it follows that \( n_1, \ldots, n_{k-1} \) are the sizes of the Jordan blocks associated with the eigenvalue \( \lambda \) of \( A - v \) if \( n_{A-v}(\lambda) = n_A(\lambda) - \text{ind}_A(\lambda) \). This result can be also obtained with the use of inequalities (4).

The condition \( c_{-q} \neq 0 \) is generic (see Remark 2 below). This means that it holds for almost all matrices with eigenvalue \( \lambda \) whose spectral index is equal to \( q \). Thus, for almost every matrix \( A \) such that \( \lambda \) is an eigenvalue of \( A \) and \( n_1, \ldots, n_k \Rightarrow n_1, \ldots, n_{k-1} \) is the change of the part of the Jordan form corresponding to \( \lambda \) under the transition from \( A \) to \( A - v \).

Remark 2. If there exists an eigenvector of depth \( q - 1 \) for \( \lambda \) whose \( v \)-th entry is nonzero (this condition is generic), then we can choose a Jordan basis \( \{\xi_1^{(p)}\}_{n=1}^{d} \) such that \( \xi_1^{(k)}(v) = 1, \xi_n^{(k)}(v) = 0 \) for \( n = 2, \ldots, n_k \), and \( \xi_n^{(p)}(v) = 0 \) for \( n = 1, \ldots, n_p \) and \( p = 1, \ldots, k-1 \). Let \( \{\eta_1^{(p)}\}_{n=1}^{d} \) be the dual Jordan basis for the transpose \( A^\top \). Then the coefficients \( c_{-q}, \ldots, c_{-1} \) of the leading part of the Laurent expansion of the function \((zE - A)^{-1}(v,v)\) at \( z = \lambda \) can be expressed in the following very simple form:

\[
c_{-q} = \xi_1^{(k)}(v)\eta_1^{(k)}(v), \ldots, c_{-1} = \xi_1^{(k)}(v)\eta_q^{(k)}(v).
\]

In particular, the condition \( c_{-q} \neq 0 \) is equivalent to the inequality \( \eta_1^{(k)}(v) \neq 0 \) and therefore is also generic.

The following proposition shows that the change of the part of the Jordan form corresponding to a given eigenvalue \( \lambda \) under the transition from \( A \) to \( A - v \) when \( A \) is the adjacency matrix of a vertex-transitive digraph is different from the typical change described above if \( \text{ind}_A(\lambda) > 1 \).

**Theorem 1.** Let \( A \) be the adjacency matrix of a vertex-transitive digraph, \( \lambda \) be an eigenvalue of \( A \), and \( n_1, \ldots, n_k \) be the sizes of the Jordan blocks associated with \( \lambda \). Assume that \( k \geq 2 \). Then for any index \( v \) of \( A \),

\[
n_1, \ldots, n_{k-2}, 2n_{k-1} - 1
\]

are the sizes of Jordan blocks for \( \lambda \) considered as an eigenvalue of \( A - v \). Moreover, if \( k = 1 \), then \( \lambda \) is a simple eigenvalue of \( A \) and does not belong to the spectrum of \( A - v \).

**Proof.** Let \( q \) be the spectral index of \( \lambda \). By Lemma 1 applied to \( A \) and \( A^\top \), for both of these matrices there exists an eigenvector of depth \( q - 1 \) for \( \lambda \) whose \( v \)-th entry is nonzero. First of all, consider the case of \( q = 1 \) and \( k \geq 2 \). Assume that \( q < \text{ind}_{A-v}(\lambda) \). Then Part (2) of Proposition 2 and the equality \( n_A(\lambda) =
n_{A-v}(\lambda) + 1 \text{ (see Proposition 1) imply the equality } \text{ind}_{A-v}(\lambda) = 1. \text{ This contradiction to the assumption } q < \text{ind}_{A-v}(\lambda) \text{ shows that } \text{ind}_{A-v}(\lambda) = 1 \text{ and therefore the change } n_1, \ldots, n_k \rightarrow n_1, \ldots, n_{k-1} \text{ holds under the transition from } A \text{ to } A - v \text{ (see Part (1) of Proposition 2). Since } 2n_{k-1} - 1 = n_{k-1} \text{ for } n_{k-1} = 1, \text{ the change of the Jordan form described in the statement of the theorem is correct for the considered case.}

Assume now that } q \geq 2. \text{ If } q \geq \text{ind}_{A-v}(\lambda), \text{ then } n_A(\lambda) = n_{A-v}(\lambda) + q \text{ (see Part (1) of Proposition 2)} \text{ and therefore } n_A(\lambda) \geq n_{A-v}(\lambda) + 2. \text{ This inequality contradicts Proposition 1. Thus, the inequality } q < \text{ind}_{A-v}(\lambda) \text{ holds. Now the change of the Jordan form described in the statement of the theorem follows from Part (2) of Proposition 2 and the equality } n_A(\lambda) = n_{A-v}(\lambda) + 1.

It is easy to see that if the conditions of Proposition 2 hold, then the geometric multiplicity of the eigenvalue } \lambda \text{ always decreases by one under the transition from } A \text{ to } A - v. \text{ So, if } k = 1, \text{ then } \lambda \text{ does not belong to the spectrum of } A - v. \text{ Since } n_A(\lambda) = n_{A-v}(\lambda) + 1, \text{ we have that } \lambda \text{ must be a simple eigenvalue of } A. \text{ The theorem is proved.} \quad \square

Remark 3. \text{ The sequence } n_1, \ldots, n_k \text{ in the statement of Theorem 1 cannot be arbitrary. Indeed, it is not difficult to show that the number of copies of the number } n_p \text{ in this sequence is not less than } n_p. \text{ For the adjacency matrix } A \text{ of the Cayley digraph on a group } G \text{ with respect to a Cayley set } S, \text{ this observation directly follows from the equality }

A^T = \sum_{s \in S} R(s),

\text{where } R \text{ is the regular representation of } G, \text{ and the fact that the direct sum decomposition of } R \text{ contains } n_{\Phi} \text{ copies of any irreducible representation } \Phi \text{ of degree } n_{\Phi} \text{ (we note that for any such representation } \Phi, \text{ the sum } \sum_{s \in S} \Phi(s) \text{ can only give Jordan block of size not greater than } n_{\Phi}. \text{ Other restrictions on the Jordan form are less evident. In particular, we do not know whether the Jordan form of } A \text{ for } \lambda \text{ can consist only of } q \text{ Jordan blocks of size } q \text{ or not when } q > 1.}

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References


