Multivalent functions with respect to n-ply points and symmetric conjugate points[☆]

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ABSTRACT

Multivalent analytic functions that are starlike, convex, close-to-convex, and quasi-convex with respect to n-ply points as well as multivalent functions with respect to symmetric points, conjugate points and symmetric conjugate points are considered. These classes of functions interestingly unify various known subclasses that have separately been introduced and studied. Inclusion and convolution properties of these generalized classes are investigated, and numerous known results could be obtained as special instances.

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1. Motivation and preliminaries

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disc and $\mathcal{A}_p$ be the class of analytic p-valent functions of the form

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n, \tag{1.1}$$

where $p \geq 1$, and let $\mathcal{A} := \mathcal{A}_1$. The Hadamard product or convolution of two p-valent functions f given by (1.1) and $g(z) = z^p + \sum_{n=p+1}^{\infty} b_n z^n$ is defined by

$$(f * g)(z) := z^p + \sum_{n=p+1}^{\infty} a_n b_n z^n.$$

The function f is subordinate to F in $\mathbb{D}$, written $f(z) \prec F(z)$, if there exists a Schwarz function w, analytic in $\mathbb{D}$ with $w(0) = 0$ and $|w(z)| < 1$, satisfying $f(z) = F(w(z))$. If the function F is univalent in $\mathbb{D}$, then $f(z) \prec F(z)$ is equivalent to $f(0) = F(0)$ and $f(\mathbb{D}) \subset F(\mathbb{D})$.

Two important subclasses of $\mathcal{A}$ are the classes of convex and starlike functions of order α, $0 \leq \alpha < 1$, denoted by $\mathcal{CV}(\alpha)$ and $\mathcal{ST}(\alpha)$ respectively. These classes and several other important subclasses in geometric function theory are characterized by either

$$\frac{zf'(z)}{f(z)} \prec h(z) \quad \text{or} \quad 1 + \frac{zf''(z)}{f'(z)} \prec h(z),$$

where h is a normalized univalent function with positive real part in $\mathbb{D}$ [1].

The well-known Alexander's relation states that a function f is convex if and only if the function zf' is starlike. Since $zf'(z) = f(z) * (z/(1-z)^2)$, it follows that f is convex if and only if $f * g$ is starlike for $g(z) = z/(1-z)^2$. Clearly $f(z) = f(z) * (z/(1-z))$, and so the investigation of the classes of convex and starlike functions can be given a unified treatment by considering the class of functions f satisfying

$$\frac{z(f * g)'(z)}{(f * g)(z)} \prec h(z),$$

where g is a given fixed function in $\mathcal{A}$, and h a convex function with a positive real part. Denote the class of all such functions by $\mathcal{ST}_g(h)$, and by $\mathcal{CV}_g(h)$ the corresponding class related to convex functions. Shanmugam [2] introduced these classes and other related classes, and investigated inclusion and convolution properties by using the convex hull method [3,4] and the method of differential subordination [5]. Ali et al. [6] investigated the subclasses of p-valent starlike and convex functions, and obtained several subordination and convolution properties, as well as sharp distortion, growth and rotation estimates. These works were recently extended by Supramaniam et al. [7]. Similar problems but for the class of meromorphic functions were also recently investigated by Mohd et al. [8]. Related results can also be found in [9].

Recall that a function $f \in \mathcal{A}$ is starlike with respect to symmetric points in $\mathbb{D}$ if

$$\text{Re}\left\{\frac{zf'(z)}{f(z) - f(-z)}\right\} > 0$$

for all $z \in \mathbb{D}$. This class was introduced and studied by Sakaguchi [10]. Further investigations into the class of starlike functions with respect to symmetric points can be found in [11–19]. El-Ashwah and Thomas [20] introduced and studied the classes consisting of starlike functions with respect to conjugate points, and starlike functions with respect to symmetric conjugate points defined respectively by the conditions

$$\text{Re}\left\{\frac{zf'(z)}{f(z) + \overline{f(\overline{z})}}\right\} > 0, \qquad \text{Re}\left\{\frac{zf'(z)}{f(z) - \overline{f(-\overline{z})}}\right\} > 0.$$

In 2004, Ravichandran [21] introduced the classes of starlike, convex and close-to-convex functions with respect to n-ply symmetric points, conjugate points and symmetric conjugate points, and obtained several convolution properties. Other investigations into the classes defined by using conjugate and symmetric conjugate points can be found in [22–29].

All these many investigations can be unified, and it is the aim of this paper to show such a unified method. For this purpose, we shall introduce general classes of p-valent starlike, convex, close-to-convex and quasi-convex functions with respect to n-ply points, as well as p-valent starlike and convex functions with respect to symmetric points, conjugate points and symmetric conjugate points respectively. Inclusion and convolution properties of these classes will be investigated, and it would be evident that previous earlier works are special instances of our present work.

For $\alpha < 1$, the class $\mathcal{R}_\alpha$ of prestarlike functions of order α is defined by

$$\mathcal{R}_\alpha := \left\{ f \in \mathcal{A} \,\middle|\, f * \frac{z}{(1-z)^{2-2\alpha}} \in \mathcal{ST}(\alpha) \right\},$$

while $\mathcal{R}_1$ consists of $f \in \mathcal{A}$ satisfying $\text{Re}\, f(z)/z > 1/2$.

The following theorems would be required.

Theorem 1.1 (*[4, Theorem 2.4, p. 54]*)**.** *Let $\alpha \leq 1$, $f \in \mathcal{R}_\alpha$ and $g \in \mathcal{ST}(\alpha)$. Then*

$$\frac{f * (Hg)}{f * g}(\mathbb{D}) \subset \overline{co}(H(\mathbb{D})),$$

for any analytic function H in $\mathbb{D}$, where $\overline{co}(H(\mathbb{D}))$ denote the closed convex hull of $H(\mathbb{D})$.

Theorem 1.1 due to Ruscheweyh [4] can easily be adapted to yield the following result.

Theorem 1.2. *If $f(z)/z^{p-1} \in \mathcal{R}_\alpha$ and $g(z)/z^{p-1} \in \mathcal{ST}(\alpha)$, then*

$$\frac{f * (Hg)}{f * g}(\mathbb{D}) \subset \overline{co}(H(\mathbb{D}))$$

for any analytic function H defined in $\mathbb{D}$.

Theorem 1.3 (*[5, Corollary 4.1h.1, p. 200]*)**.** *Let h be convex in $\mathbb{D}$, and S and T be analytic functions in $\mathbb{D}$ with $S(0) = T(0)$. If $\operatorname{Re}\left(zS'(z)/S(z)\right) > 0$, then*

$$\frac{T'(z)}{S'(z)} \prec h(z) \implies \frac{T(z)}{S(z)} \prec h(z).$$

2. Multivalent functions with respect to *n*-ply points

In the following sequel, the function $g \in \mathcal{A}_p$ is a fixed function and the function h is a convex univalent function with a positive real part satisfying $h(0) = 1$. On certain occasions, we would additionally require that $\operatorname{Re} h(z) > 1 - (1 - \alpha)/p$, where $0 \le \alpha < 1$. Multivalent functions starlike and convex with respect to n-ply points are given below:

Definition 2.1. Let $n \ge 1$ be an integer, $\epsilon^n = 1$, and $\epsilon \ne 1$. For $f \in \mathcal{A}_p$ of the form (1.1), let the function f_n be defined by

$$f_n(z) := \frac{1}{n} \sum_{k=0}^{n-1} \epsilon^{n-pk} f(\epsilon^k z) = z^p + a_{p+n} z^{p+n} + a_{p+2n} z^{p+2n} + \cdots.$$

The class $\mathcal{ST}_p^n(h)$ consists of functions $f \in \mathcal{A}_p$ satisfying $f_n(z)/z^p \ne 0$ in $\mathbb{D}$ and the subordination

$$\frac{1}{p} \frac{zf'(z)}{f_n(z)} \prec h(z). \tag{2.1}$$

For a fixed function $g \in \mathcal{A}_p$, denote by $\mathcal{ST}_{p,g}^n(h)$ the class

$$\mathcal{ST}_{p,g}^n(h) := \left\{ f \in \mathcal{A}_p : f * g \in \mathcal{ST}_p^n(h) \right\}.$$

Similarly, $\mathcal{CV}_p^n(h)$ consists of functions $f \in \mathcal{A}_p$ satisfying $f_n'(z)/z^{p-1} \ne 0$ in $\mathbb{D}$ and the subordination

$$\frac{1}{p} \frac{\left(zf'\right)'(z)}{f_n'(z)} \prec h(z),$$

and for a fixed function $g \in \mathcal{A}_p$, let

$$\mathcal{CV}_{p,g}^n(h) := \left\{ f \in \mathcal{A}_p : f * g \in \mathcal{CV}_p^n(h) \right\}.$$

Remark 2.2. If $n = 1$, the classes $\mathcal{ST}_{p,g}^1(h)$ and $\mathcal{CV}_{p,g}^1(h)$ were studied by Supramaniam et al. [7].

Evidently when $g(z) = z^p/(1-z)$, the classes $\mathcal{ST}_{p,g}^n(h)$ and $\mathcal{CV}_{p,g}^n(h)$ reduced respectively to the classes $\mathcal{ST}_p^n(h)$ and $\mathcal{CV}_p^n(h)$. Thus these new classes of p-valent starlike and convex functions with respect to n-ply points unify the classes $\mathcal{ST}_p^n(h)$ and $\mathcal{CV}_p^n(h)$. The notations $\mathcal{ST}_p^1(h) := \mathcal{ST}_p(h)$ and $\mathcal{CV}_p^1(h) := \mathcal{CV}_p(h)$ will be used for $n = 1$.

It is clear that $\mathcal{ST}_{p,zg'}^n(h) = \mathcal{CV}_{p,g}^n(h)$. Interestingly the property that every convex function is necessarily starlike remains valid even for multivalent functions with respect to n-ply points. Indeed the following result holds:

Lemma 2.3. *Let g be a fixed function in $\mathcal{A}_p$, and h be a convex univalent function having positive real part with $h(0) = 1$.*

(i) *If $f \in \mathcal{ST}_{p,g}^n(h)$, then $f_n \in \mathcal{ST}_{p,g}(h)$.*
(ii) *The function $f \in \mathcal{CV}_{p,g}^n(h)$ if and only if $\frac{1}{p} zf' \in \mathcal{ST}_{p,g}^n(h)$.*
(iii) *The inclusion $\mathcal{CV}_{p,g}^n(h) \subset \mathcal{ST}_{p,g}^n(h)$ holds.*

Proof. It is sufficient to prove the result for $g(z) = z^p/(1-z)$.

(i) Let $f \in \mathcal{ST}_p^n(h)$. For any fixed $z \in \mathbb{D}$,

$$\frac{1}{p} \frac{zf'(z)}{f_n(z)} \in h(\mathbb{D}). \tag{2.2}$$

Replacing z by $\epsilon^k z$ in (2.2), it follows that

$$\frac{1}{p} \frac{\epsilon^k zf'(\epsilon^k z)}{f_n(\epsilon^k z)} \in h(\mathbb{D}). \tag{2.3}$$

In light of the fact that

$$f_n(\epsilon^k z) = \epsilon^{pk} f_n(z), \tag{2.4}$$

the containment (2.3) becomes

$$\frac{1}{p}\frac{\epsilon^{k(1-p)}zf'(\epsilon^k z)}{f_n(z)} \in h(\mathbb{D}).$$

Since $h(\mathbb{D})$ is convex, it follows that

$$\frac{1}{n}\sum_{k=0}^{n-1}\frac{1}{p}\frac{\epsilon^{k(1-p)}zf'(\epsilon^k z)}{f_n(z)} \in h(\mathbb{D}). \tag{2.5}$$

Using the identity

$$f_n'(z) = \frac{1}{n}\sum_{k=0}^{n-1}\epsilon^{k(1-p)}f'(\epsilon^k z), \tag{2.6}$$

it is seen that (2.5) becomes

$$\frac{1}{p}\frac{zf_n'(z)}{f_n(z)} \in h(\mathbb{D}).$$

Thus

$$\frac{1}{p}\frac{zf_n'(z)}{f_n(z)} \prec h(z),$$

that is, $f_n \in \mathcal{ST}_p(h)$.

(ii) Since $\left(\frac{1}{p}zf'\right)_n(z) = \frac{1}{p}zf_n'(z)$, it is evident that

$$\frac{1}{p}\frac{(zf')'(z)}{f_n'(z)} = \frac{1}{p}\frac{z\left(\frac{1}{p}zf'\right)'(z)}{\left(\frac{1}{p}zf'\right)_n(z)}.$$

Thus $f \in \mathcal{CV}_p^n(h)$ if and only if $\frac{1}{p}zf' \in \mathcal{ST}_p^n(h)$.

(iii) Let $f \in \mathcal{CV}_p^n(h)$. Then part (ii) shows that $\frac{1}{p}zf' \in \mathcal{ST}_p^n(h)$. We deduce from part (i) that $\left(\frac{1}{p}zf'\right)_n \in \mathcal{ST}_p(h)$. From $\left(\frac{1}{p}zf'\right)_n = \frac{1}{p}zf_n'$, part (ii) now shows that $f_n \in \mathcal{CV}_p(h)$. Since $\mathcal{CV}_p(h)$ is subset of $\mathcal{ST}_p(h)$ [7, Theorem 2.1], it follows that $f_n \in \mathcal{ST}_p(h)$, and because h is a function with positive real part, the function f_n is starlike.

Define the functions T and S by

$$T(z) := \frac{1}{p}zf'(z) \quad \text{and} \quad S(z) := f_n(z).$$

Since the function S is starlike and

$$\frac{T'(z)}{S'(z)} = \frac{1}{p}\frac{(zf')'(z)}{f_n'(z)} \prec h(z),$$

Theorem 1.3 implies that

$$\frac{1}{p}\frac{zf'(z)}{f_n(z)} = \frac{T(z)}{S(z)} \prec h(z),$$

whence $f \in \mathcal{ST}_p^n(h)$. $\quad\square$

Ruscheweyh and Sheil-Small [3] proved the Polya–Schoenberg conjecture that the classes of convex functions, starlike functions and close-to-convex functions are closed under convolution with convex functions. In the following theorem, this result is extended for the convolution between prestarlike functions and multivalent functions with respect to n-ply points.

Theorem 2.4. *Let h be a convex univalent function satisfying the condition*

$$\operatorname{Re} h(z) > 1 - \frac{1-\alpha}{p} \quad (0 \le \alpha < 1),$$

and $\phi \in \mathcal{A}_p$ with $\phi/z^{p-1} \in \mathcal{R}_\alpha$.

(i) *If $f \in \mathcal{ST}_{p,g}^n(h)$, then $\phi * f \in \mathcal{ST}_{p,g}^n(h)$. Equivalently, $\mathcal{ST}_{p,g}^n(h) \subset \mathcal{ST}_{p,g*\phi}^n(h)$.*
(ii) *If $f \in \mathcal{CV}_{p,g}^n(h)$, then $\phi * f \in \mathcal{CV}_{p,g}^n(h)$. Equivalently, $\mathcal{CV}_{p,g}^n(h) \subset \mathcal{CV}_{p,g*\phi}^n(h)$.*

Proof. (i) Let $f \in \mathcal{ST}_p^n(h)$. From Lemma 2.3(i), it follows that $f_n \in \mathcal{ST}_p(h)$. The function ψ_n defined by

$$\psi_n(z) := \frac{f_n(z)}{z^{p-1}}$$

is analytic and satisfies

$$\frac{z\psi_n'(z)}{\psi_n(z)} = \frac{zf_n'(z)}{f_n(z)} - (p-1) \prec p\, h(z) - (p-1).$$

Since $\operatorname{Re} h(z) > 1 - (1-\alpha)/p$, it follows that

$$\operatorname{Re} \frac{z\psi_n'(z)}{\psi_n(z)} > \alpha,$$

and hence $\psi_n \in \mathcal{ST}(\alpha)$. Define the function H by

$$H(z) := \frac{1}{p} \frac{zf'(z)}{f_n(z)}.$$

Since $H \prec h$ and h is convex, an application of Theorem 1.2 shows that

$$\frac{1}{p} \frac{z(\phi * f)'(z)}{(\phi * f)_n(z)} = \frac{\phi(z) * \frac{1}{p}zf'(z)}{\phi(z) * f_n(z)} = \frac{(\phi * Hf_n)(z)}{(\phi * f_n)(z)} \prec h(z),$$

and thus $\phi * f \in \mathcal{ST}_p^n(h)$.

The general result for $f \in \mathcal{ST}_{p,g}^n(h)$ follows from the fact that

$$f \in \mathcal{ST}_{p,g}^n(h) \Leftrightarrow f * g \in \mathcal{ST}_p^n(h).$$

(ii) Now let $f \in \mathcal{CV}_{p,g}^n(h)$ so that $\frac{1}{p}zf' \in \mathcal{ST}_{p,g}^n(h)$. The result of part (i) yields $\left(\frac{1}{p}zf'\right) * \phi = \frac{1}{p}z(f * \phi)' \in \mathcal{ST}_{p,g}^n(h)$, and thus $\phi * f \in \mathcal{CV}_{p,g}^n(h)$. $\square$

Close-to-convex and quasi-convex multivalent functions with respect to n-ply points are defined as follows:

Definition 2.5. The class $\mathcal{CC}_p^n(h)$ consists of functions $f \in \mathcal{A}_p$ satisfying the subordination

$$\frac{1}{p} \frac{zf'(z)}{\phi_n(z)} \prec h(z)$$

for some $\phi \in \mathcal{ST}_p^n(h)$. The general class $\mathcal{CC}_{p,g}^n(h)$ then consists of functions $f \in \mathcal{A}_p$ satisfying the subordination

$$\frac{1}{p} \frac{z(g * f)'(z)}{(g * \phi)_n(z)} \prec h(z)$$

for some $\phi \in \mathcal{ST}_{p,g}^n(h)$. The class $\mathcal{QC}_p^n(h)$ consists of functions $f \in \mathcal{A}_p$ satisfying the subordination

$$\frac{1}{p} \frac{\left(zf'\right)'(z)}{\phi_n'(z)} \prec h(z)$$

for some $\phi \in \mathcal{CV}_p^n(h)$, while the class $\mathcal{QC}_{p,g}^n(h)$ consists of $f \in \mathcal{A}_p$ such that

$$\frac{1}{p} \frac{\left(z(g * f)'\right)'(z)}{(g * \phi)_n'(z)} \prec h(z)$$

for some $\phi \in \mathcal{CV}_{p,g}^n(h)$.

Lemma 2.6. *Let g be a fixed function in $\mathcal{A}_p$, and h be a convex univalent function with positive real part satisfying $h(0) = 1$. Then*

(i) $\mathcal{CV}_{p,g}^n(h) \subset \mathcal{QC}_{p,g}^n(h) \subset \mathcal{CC}_{p,g}^n(h)$,
(ii) $f \in \mathcal{QC}_{p,g}^n(h)$ *if and only if* $\frac{1}{p}zf' \in \mathcal{CC}_{p,g}^n(h)$.

Proof. (i) By taking $\phi = f$, it is evident from the definition that $\mathcal{CV}_{p,g}^n(h) \subset \mathcal{QC}_{p,g}^n(h)$. To prove the second inclusion, suppose that $f \in \mathcal{QC}_{p,g}^n(h)$. Then there exists $\phi \in \mathcal{CV}_{p,g}^n(h)$ such that

$$\frac{1}{p} \frac{\left(z(g * f)'\right)'(z)}{(g * \phi)_n'(z)} \prec h(z).$$

Since $\phi \in \mathcal{CV}_{p,g}^n(h)$, it follows that $(g * \phi)_n \in \mathcal{CV}_p(h)$ which is a subset of $\mathcal{ST}_p(h)$ [7, Theorem 2.1]. Thus $(g * \phi)_n \in \mathcal{ST}_p(h)$. The result now follows from Theorem 1.3 with

$$T(z) = \frac{1}{p} z(g * f)'(z) \quad \text{and} \quad S(z) = (g * \phi)_n(z).$$

(ii) Here the proof follows from the identity

$$\frac{1}{p} \frac{\left(z(g*f)'\right)'(z)}{(g*\phi)'_n(z)} = \frac{1}{p} \frac{z\left[\left(g * \frac{1}{p}zf'\right)\right]'(z)}{\left(g * \frac{1}{p}z\phi'\right)_n(z)},$$

and Lemma 2.3(ii). □

Theorem 2.7. *Let h and ϕ satisfy the conditions of Theorem 2.4.*

(i) *If $f \in \mathcal{CC}_{p,g}^n(h)$ with respect to a function $f_1 \in \mathcal{ST}_{p,g}^n(h)$, then $\phi * f \in \mathcal{CC}_{p,g}^n(h)$ with respect to the function $\phi * f_1 \in \mathcal{ST}_{p,g}^n(h)$. Also $\mathcal{CC}_{p,g}^n(h) \subset \mathcal{CC}_{p,g*\phi}^n(h)$.*

(ii) *If $f \in \mathcal{QC}_{p,g}^n(h)$ with respect to $f_1 \in \mathcal{CV}_{p,g}^n(h)$, then $\phi * f \in \mathcal{QC}_{p,g}^n(h)$ with respect to $\phi * f_1 \in \mathcal{CV}_{p,g}^n(h)$. Also $\mathcal{QC}_{p,g}^n(h) \subset \mathcal{QC}_{p,g*\phi}^n(h)$.*

Proof. (i) It is sufficient to prove the result for the case $g(z) = z^p/(1-z)$. Let $f \in \mathcal{CC}_p^n(h)$ with respect to a function $f_1 \in \mathcal{ST}_p^n(h)$. Theorem 2.4 yields $\phi * f_1 \in \mathcal{ST}_p^n(h)$, and Lemma 2.3(i) gives $(f_1)_n$ is in $\mathcal{ST}_p(h)$. Also it is easy to see that the function $(f_1)_n/z^{p-1} \in \mathcal{ST}(\alpha)$. Now define the analytic function H by

$$H(z) := \frac{1}{p} \frac{zf'(z)}{(f_1)_n(z)}.$$

Since $H(z) \prec h(z)$, an application of Theorem 1.2 shows that

$$\frac{1}{p} \frac{z(\phi*f)'(z)}{(\phi*f_1)_n(z)} = \frac{\phi(z) * \frac{1}{p}zf'(z)}{\phi(z) * (f_1)_n(z)} = \frac{(\phi * H(f_1)_n)(z)}{(\phi * (f_1)_n)(z)} \prec h(z).$$

This completes the proof of part (i).

(ii) If $f \in \mathcal{QC}_{p,g}^n(h)$, then Lemma 2.6(ii) gives $\frac{1}{p}zf' \in \mathcal{CC}_{p,g}^n(h)$. Since

$$\frac{1}{p}z(\phi*f)'(z) = \phi(z) * \frac{1}{p}zf'(z),$$

the result of part (i) shows that $\frac{1}{p}z(\phi*f)' \in \mathcal{CC}_{p,g}^n(h)$. From Lemma 2.6(ii), $\phi * f \in \mathcal{QC}_{p,g}^n(h)$. □

3. Multivalent functions with respect to *n*-ply symmetric, conjugate and symmetric conjugate points

In this section, it is assumed that *p is an odd number*. Also, the function $g \in \mathcal{A}_p$ is a fixed function and the function h is convex univalent with a positive real part satisfying $h(0) = 1$. Here the classes of multivalent functions that are p-valent starlike and p-valent convex with respect to n-ply symmetric, conjugate, and symmetric conjugate points are introduced, and their convolution properties will be discussed. These classes are respectively defined below:

$$\mathcal{STS}_p^n(h) := \left\{ f \in \mathcal{A}_p : \frac{1}{p} \frac{2zf'(z)}{f_n(z) - f_n(-z)} \prec h(z), \frac{f_n(z) - f_n(-z)}{z^p} \neq 0 \right\},$$

$$\mathcal{CVS}_p^n(h) := \left\{ f \in \mathcal{A}_p : \frac{1}{p} \frac{2\left(zf'\right)'(z)}{f_n'(z) + f_n'(-z)} \prec h(z), \frac{f_n'(z) + f_n'(-z)}{z^{p-1}} \neq 0 \right\},$$

$$\mathcal{STC}_p^n(h) := \left\{ f \in \mathcal{A}_p : \frac{1}{p} \frac{2zf'(z)}{f_n(z) + \overline{f_n(\overline{z})}} \prec h(z), \frac{f_n(z) + \overline{f_n(\overline{z})}}{z^p} \neq 0 \right\},$$

$$\mathcal{CVC}_p^n(h) := \left\{ f \in \mathcal{A}_p : \frac{1}{p} \frac{2\left(zf'\right)'(z)}{f_n'(z) + \overline{f_n'(\overline{z})}} \prec h(z), \frac{f_n'(z) + \overline{f_n'(\overline{z})}}{z^{p-1}} \neq 0 \right\},$$

$$\mathcal{STSC}_p^n(h) := \left\{ f \in \mathcal{A}_p : \frac{1}{p} \frac{2zf'(z)}{f_n(z) - \overline{f_n(-\overline{z})}} \prec h(z), \frac{f_n(z) - \overline{f_n(-\overline{z})}}{z^p} \neq 0 \right\},$$

$$\mathcal{CVSC}_p^n(h) := \left\{ f \in \mathcal{A}_p : \frac{1}{p} \frac{2\left(zf'\right)'(z)}{f_n'(z) + \overline{f_n'(-\overline{z})}} \prec h(z), \frac{f_n'(z) + \overline{f_n'(-\overline{z})}}{z^{p-1}} \neq 0 \right\}.$$

The classes $\mathscr{ST}\mathscr{S}^n_{p,g}(h)$, $\mathscr{CV}\mathscr{S}^n_{p,g}(h)$, $\mathscr{ST}\mathscr{C}^n_{p,g}(h)$, $\mathscr{CV}\mathscr{C}^n_{p,g}(h)$, $\mathscr{ST}\mathscr{SC}^n_{p,g}(h)$ and $\mathscr{CV}\mathscr{SC}^n_{p,g}(h)$ consist of functions $f \in \mathscr{A}_p$ for which $f * g$ belongs to the classes $\mathscr{ST}\mathscr{S}^n_p(h)$, $\mathscr{CV}\mathscr{S}^n_p(h)$, $\mathscr{ST}\mathscr{C}^n_p(h)$, $\mathscr{CV}\mathscr{C}^n_p(h)$, $\mathscr{ST}\mathscr{SC}^n_p(h)$ and $\mathscr{CV}\mathscr{SC}^n_p(h)$ respectively.

In the special case $n = 1$, we shall adopt the following usual notations: $\mathscr{ST}\mathscr{S}^1_{p,g}(h) =: \mathscr{ST}\mathscr{S}_{p,g}(h)$, $\mathscr{CV}\mathscr{S}^1_{p,g}(h) =: \mathscr{CV}\mathscr{S}_{p,g}(h)$, $\mathscr{ST}\mathscr{C}^1_{p,g}(h) =: \mathscr{ST}\mathscr{C}_{p,g}$, $\mathscr{CV}\mathscr{C}^1_{p,g}(h) =: \mathscr{CV}\mathscr{C}_{p,g}(h)$, $\mathscr{ST}\mathscr{SC}^1_{p,g}(h) =: \mathscr{ST}\mathscr{SC}_{p,g}(h)$ and $\mathscr{CV}\mathscr{SC}^1_{p,g}(h) =: \mathscr{CV}\mathscr{SC}_{p,g}(h)$.

Remark 3.1. When $p = 1$, these classes were investigated by Ravichandran [21]. We also took note that these classes reduced to the classes studied in [30] when $n = 1$ and $g(z) = z/(1 - z)$.

Lemma 3.2. *Let g be a fixed function in $\mathscr{A}_p$, and h be a convex univalent function with a positive real part satisfying $h(0) = 1$.*

(i) *If $f \in \mathscr{ST}\mathscr{S}^n_{p,g}(h)$ and $F(z) := \frac{1}{2}[f(z) - f(-z)]$, then $F_n \in \mathscr{ST}_{p,g}(h)$.*
(ii) *If $f \in \mathscr{ST}\mathscr{S}^n_{p,g}(h)$, then $f_n \in \mathscr{ST}_{p,g}(h)$.*
(iii) *The function $f \in \mathscr{CV}\mathscr{S}^n_{p,g}(h)$ if and only if $\frac{1}{p}zf' \in \mathscr{ST}\mathscr{S}^n_{p,g}(h)$.*
(iv) *The inclusion $\mathscr{CV}\mathscr{S}^n_{p,g}(h) \subset \mathscr{ST}\mathscr{S}^n_{p,g}(h)$ holds.*

Proof. Again it is enough to prove the results for $g(z) = z^p/(1 - z)$.

(i) Let $f \in \mathscr{ST}\mathscr{S}^n_p(h)$. For any fixed $z \in \mathbb{D}$,

$$\frac{1}{p}\frac{zf'(z)}{F_n(z)} \in h(\mathbb{D}).$$

Replacing z by $-z$ and taking the convex combination of these two expressions, it readily follows that

$$\frac{1}{2p}\left[\frac{zf'(z)}{F_n(z)} + \frac{(-z)f'(-z)}{F_n(-z)}\right] = \frac{1}{p}\frac{zF'(z)}{F_n(z)} \in h(\mathbb{D}).$$

This shows that the function $F \in \mathscr{ST}^n_p(h)$ and Lemma 2.3(i) now yields $F_n \in \mathscr{ST}_p(h)$.

(ii) Replacing z by $\epsilon^k z$ in

$$\frac{1}{p}\frac{2zf'(z)}{f_n(z) - f_n(-z)} \in h(\mathbb{D}),$$

and using (2.4) and (2.6), it follows from the convexity of $h(\mathbb{D})$ that

$$\frac{1}{n}\sum_{k=0}^{n-1}\frac{1}{p}\frac{2\epsilon^k zf'(\epsilon^k z)}{f_n(\epsilon^k z) - f_n(-\epsilon^k z)} = \frac{2}{p}\frac{\frac{1}{n}\sum_{k=0}^{n-1}\epsilon^{k(1-p)}zf'(\epsilon^k z)}{f_n(z) - f_n(-z)} = \frac{2}{p}\frac{zf'_n(z)}{f_n(z) - f_n(-z)} \in h(\mathbb{D}).$$

Thus $f_n \in \mathscr{ST}_p(h)$.

(iii) Since $\left(\frac{1}{p}zf'\right)_n(-z) = -\frac{1}{p}zf'_n(-z)$, it is clear that

$$\frac{1}{p}\frac{2\left(zf'\right)'(z)}{f'_n(z) + f'_n(-z)} = \frac{1}{p}\frac{2z\left(\frac{1}{p}zf'\right)'(z)}{\left(\frac{1}{p}zf'\right)_n(z) - \left(\frac{1}{p}zf'\right)_n(-z)}.$$

Thus $f \in \mathscr{CV}\mathscr{S}^n_p(h)$ if and only if $\frac{1}{p}zf' \in \mathscr{ST}\mathscr{S}^n_p(h)$.

(iv) Let $f \in \mathscr{CV}\mathscr{S}^n_p(h)$ and $F(z) := \frac{1}{2}[f(z) - f(-z)]$. The result in part (iii) shows that $\frac{1}{p}zf' \in \mathscr{ST}\mathscr{S}^n_p(h)$. Hence, by part (i), $\left(\frac{1}{p}zF'\right)_n \in \mathscr{ST}_p(h)$. Since $\left(\frac{1}{p}zF'\right)_n = \frac{1}{p}zF'_n$, Lemma 2.3(ii) shows that $F_n \in \mathscr{CV}_p(h)$. So it follows from Lemma 2.3(iii) that $F_n \in \mathscr{ST}_p(h)$. Since h is a function with a positive real part, we deduce that the function F_n is starlike.

Now let $T(z) := \frac{1}{p}zf'(z)$ and $S(z) := [f_n(z) - f_n(-z)]/2 = F_n(z)$. Since $f \in \mathscr{CV}\mathscr{S}^n_p(h)$,

$$\frac{T'(z)}{S'(z)} = \frac{1}{p}\frac{2\left(zf'\right)'(z)}{f'_n(z) + f'_n(-z)} \prec h(z).$$

Since S is starlike, the above subordination together with Theorem 1.3 implies that

$$\frac{1}{p}\frac{2zf'(z)}{f_n(z) - f_n(-z)} = \frac{T(z)}{S(z)} \prec h(z),$$

and hence $f \in \mathscr{ST}\mathscr{S}^n_p(h)$. $\square$

Theorem 3.3. *Let h and ϕ satisfy the conditions of Theorem 2.4.*

(i) *If $f \in \mathcal{ST}\mathcal{S}_{p,g}^n(h)$, then $\phi * f \in \mathcal{ST}\mathcal{S}_{p,g}^n(h)$. Equivalently, $\mathcal{ST}\mathcal{S}_{p,g}^n(h) \subset \mathcal{ST}\mathcal{S}_{p,g*\phi}^n(h)$.*

(ii) *If $f \in \mathcal{CV}\mathcal{S}_{p,g}^n(h)$, then $\phi * f \in \mathcal{CV}\mathcal{S}_{p,g}^n(h)$. Equivalently, $\mathcal{CV}\mathcal{S}_{p,g}^n(h) \subset \mathcal{CV}\mathcal{S}_{p,g*\phi}^n(h)$.*

Proof. It is enough to prove the results when $g(z) = z^p/(1-z)$.

(i) Define the functions F and H by

$$F(z) := \frac{1}{2}[f(z) - f(-z)] \quad \text{and} \quad H(z) := \frac{1}{p}\frac{zf'(z)}{F_n(z)}.$$

Lemma 3.2(i) shows that $F_n \in \mathcal{ST}_p(h)$. Since h is a convex function with $\operatorname{Re} h(z) > 1 - (1-\alpha)/p$, it follows that

$$\operatorname{Re}\frac{zF_n'(z)}{F_n(z)} > p - 1 + \alpha,$$

and whence the function $F_n(z)/z^{p-1}$ is starlike of order α. Since $H(z) \prec h(z)$, Theorem 1.2 yields

$$\frac{1}{p}\frac{2z(\phi*f)'(z)}{(\phi*f)_n(z) - (\phi*f)_n(-z)} = \frac{\phi(z) * \frac{1}{p}zf'(z)}{\phi(z) * [f_n(z) - f_n(-z)]/2} = \frac{(\phi * HF_n)(z)}{(\phi * F_n)(z)} \prec h(z),$$

and thus $\phi * f \in \mathcal{ST}\mathcal{S}_p^n(h)$.

(ii) If $f \in \mathcal{CV}\mathcal{S}_p^n(h)$, Lemma 3.2(iii) and the result of part (i) above yield

$$\phi * \frac{1}{p}zf' = \frac{1}{p}z(\phi*f)' \in \mathcal{ST}\mathcal{S}_p^n(h).$$

Hence $\phi * f \in \mathcal{CV}\mathcal{S}_p^n(h)$. $\quad\square$

Lemma 3.4. *Let g be a fixed function in $\mathcal{A}_p$, and h be a convex univalent function with positive real part satisfying $h(0) = 1$.*

(i) *If $f \in \mathcal{ST}\mathcal{C}_{p,g}^n(h)$ and $F(z) := \frac{1}{2}[f(z) + \overline{f(\overline{z})}]$, then $F_n \in \mathcal{ST}_{p,g}(h)$.*

(ii) *If $f \in \mathcal{ST}\mathcal{C}_{p,g}^n(h)$, then $f_n \in \mathcal{ST}\mathcal{C}_{p,g}(h)$.*

(iii) *The function $f \in \mathcal{CV}\mathcal{C}_{p,g}^n(h)$ if and only if $\frac{1}{p}zf' \in \mathcal{ST}\mathcal{C}_{p,g}^n(h)$.*

(iv) *The inclusion $\mathcal{CV}\mathcal{C}_{p,g}^n(h) \subset \mathcal{ST}\mathcal{C}_{p,g}^n(h)$ holds.*

Proof. Again it is enough to prove the results when $g(z) = z^p/(1-z)$.

(i) Since $F_n(z) = [f_n(z) + \overline{f_n(\overline{z})}]/2$, if $f \in \mathcal{ST}\mathcal{C}_p^n(h)$, then

$$\frac{1}{p}\frac{zf'(z)}{F_n(z)} \in h(\mathbb{D})$$

for any fixed $z \in \mathbb{D}$. Thus

$$\frac{1}{2p}\left[\frac{zf'(z)}{F_n(z)} + \overline{\left(\frac{\overline{z}f'(\overline{z})}{F_n(\overline{z})}\right)}\right] = \frac{1}{p}\frac{zF'(z)}{F_n(z)} \in h(\mathbb{D}).$$

This shows that the function $F \in \mathcal{ST}_p^n(h)$ and by Lemma 2.3(i) it follows that $F_n \in \mathcal{ST}_p(h)$.

(ii) Replacing z by $\epsilon^k z$ in

$$\frac{1}{p}\frac{2zf'(z)}{f_n(z) + \overline{f_n(\overline{z})}} \in h(\mathbb{D}),$$

and using (2.4) and (2.6), it follows from the convexity of $h(\mathbb{D})$ that

$$\frac{1}{p}\frac{2zf_n'(z)}{f_n(z) + \overline{f_n(\overline{z})}} = \frac{1}{n}\sum_{k=0}^{n-1}\frac{1}{p}\frac{2\epsilon^k zf'(\epsilon^k z)}{f_n(\epsilon^k z) + \overline{f_n(\epsilon^{-k}\overline{z})}} \in h(\mathbb{D}).$$

Thus $f_n \in \mathcal{ST}\mathcal{C}_p(h)$.

(iii) Since

$$\overline{\left(\frac{1}{p}zf'\right)_n}(\overline{z}) = \frac{1}{p}z\overline{f_n'(\overline{z})},$$

it follows that

$$\frac{1}{p}\frac{2\left(zf'\right)'(z)}{f_n'(z) + \overline{f_n'(\overline{z})}} = \frac{1}{p}\frac{2z\left(\frac{1}{p}zf'\right)'(z)}{\left(\frac{1}{p}zf'\right)_n(z) + \overline{\left(\frac{1}{p}zf'\right)_n}(\overline{z})}.$$

Thus $f \in \mathcal{CVC}_{p,g}^n(h)$ if and only if $\frac{1}{p}zf' \in \mathcal{STC}_{p,g}^n(h)$.

(iv) If $f \in \mathcal{CVC}_p^n(h)$, then part (iii) gives $\frac{1}{p}zf' \in \mathcal{STC}_p^n(h)$. With $F(z) = [f(z) + \overline{f(\overline{z})}]/2$, it follows from part (i) that

$$\left(\frac{1}{p}zF'\right)_n = \frac{\left(\frac{1}{p}zf'\right)_n + \left(\overline{\left(\frac{1}{p}zf'\right)(\overline{z})}\right)_n}{2}$$

$$= \frac{\frac{1}{p}zf_n' + \frac{1}{p}z\overline{f_n'(\overline{z})}}{2}$$

$$= \frac{1}{p}zF_n' \in \mathcal{ST}_p(h).$$

Lemma 2.3(ii) now gives $F_n \in \mathcal{CV}_p(h)$, and so $F_n \in \mathcal{ST}_p(h)$. Thus F_n is starlike.

Next let $T(z) := \frac{1}{p}zf'(z)$ and $S(z) := [f_n(z) + \overline{f_n(\overline{z})}]/2 = F_n(z)$. Since

$$\frac{T'(z)}{S'(z)} = \frac{1}{p}\frac{2\left(zf'\right)'(z)}{f_n'(z) + \overline{f_n'(\overline{z})}} \prec h(z),$$

and S is starlike, Theorem 1.3 shows that

$$\frac{1}{p}\frac{2zf'(z)}{f_n(z) + \overline{f_n'(\overline{z})}} = \frac{T(z)}{S(z)} \prec h(z),$$

whence $f \in \mathcal{STC}_{p,g}^n(h)$. □

Theorem 3.5. *Let h and ϕ satisfy the conditions of Theorem 2.4 and ϕ has real coefficients.*

(i) *If $f \in \mathcal{STC}_{p,g}^n$, then $\phi * f \in \mathcal{STC}_{p,g}^n(h)$. Equivalently, $\mathcal{STC}_{p,g}^n(h) \subset \mathcal{STC}_{p,g*\phi}^n(h)$.*

(ii) *If $f \in \mathcal{CVC}_{p,g}^n(h)$, then $\phi * f \in \mathcal{CVC}_{p,g}^n(h)$, and $\mathcal{CVC}_{p,g}^n(h) \subset \mathcal{CVC}_{p,g*\phi}^n(h)$.*

Proof. (i) Let $f \in \mathcal{STC}_p^n(h)$. Define the functions $F(z)$ and $H(z)$ by

$$F(z) = \frac{f(z) + \overline{f(\overline{z})}}{2} \quad \text{and} \quad H(z) = \frac{1}{p}\frac{zf'(z)}{F_n(z)}.$$

Using Lemma 3.4, and proceeding similarly as in the proof of Theorem 3.3, it can be shown that the function $F_n(z)/z^{p-1}$ is starlike of order α, where $F_n(z) = [f_n(z) + \overline{f_n(\overline{z})}]/2$.

Since $H(z) \prec h(z)$ and because ϕ has real coefficients, Theorem 1.2 yields

$$\frac{1}{p}\frac{2z(\phi * f)'(z)}{(\phi * f)_n(z) + \overline{(\phi * f)_n(\overline{z})}} = \frac{\phi(z) * \frac{1}{p}zf'(z)}{\phi(z) * \left[f_n(z) + \overline{f_n(\overline{z})}\right]/2} = \frac{(\phi * HF_n)(z)}{(\phi * F_n)(z)} \prec h(z),$$

or $\phi * f \in \mathcal{STC}_p^n(h)$.

(ii) If $f \in \mathcal{CVC}_{p,g}^n(h)$, it follows from Lemma 3.4(iii) that $\frac{1}{p}zf' \in \mathcal{STC}_{p,g}^n(h)$. By part (i), it is now evident that

$$\phi * \frac{1}{p}zf' = \frac{1}{p}z(\phi * f)' \in \mathcal{STC}_{p,g}^n(h),$$

and thus we deduce that $\phi * f \in \mathcal{CVC}_{p,g}^n(h)$ from Lemma 3.4(iii). □

The following two results can readily be established by proceeding analogously as in the proofs of Lemmas 3.2 and 3.4, and Theorems 3.3 and 3.5. We omit these proofs.

Lemma 3.6. *Let g be a fixed function in $\mathcal{A}_p$, and h be a convex univalent function with positive real part satisfying $h(0) = 1$.*

(i) *If $f \in \mathcal{STSC}^n_{p,g}(h)$ and $F(z) := \frac{1}{2}[f(z) - \overline{f(-\overline{z})}]$, then $F_n \in \mathcal{ST}_{p,g}(h)$.*

(ii) *If $f \in \mathcal{STSC}^n_{p,g}(h)$, then $f_n \in \mathcal{STSC}_{p,g}(h)$.*

(iii) *The function $f \in \mathcal{CVSC}^n_{p,g}(h)$ if and only if $\frac{1}{p}zf' \in \mathcal{STSC}^n_{p,g}(h)$.*

(iv) *The inclusion $\mathcal{CVSC}^n_{p,g}(h) \subset \mathcal{STSC}^n_{p,g}(h)$ holds.*

Theorem 3.7. *Let h and ϕ satisfy the conditions of Theorem 2.4 and ϕ has real coefficients.*

(i) *If $f \in \mathcal{STSC}^n_{p,g}$, then $\phi * f \in \mathcal{STSC}^n_{p,g}(h)$.*

(ii) *If $f \in \mathcal{CVSC}^n_{p,g}(h)$, then $\phi * f \in \mathcal{CVSC}^n_{p,g}(h)$.*

References

[1] W.C. Ma, D. Minda, A unified treatment of some special classes of univalent functions, in: Proceedings of the Conference on Complex Analysis (Tianjin, 1992), Int. Press, Cambridge, MA, 1994, pp. 157–169.
[2] T.N. Shanmugam, Convolution and differential subordination, Int. J. Math. Math. Sci. 12 (2) (1989) 333–340.
[3] St. Ruscheweyh, T. Sheil-Small, Hadamard products of Schlicht functions and the Polya–Schoenberg conjecture, Comment. Math. Helv. 48 (1973) 119–135.
[4] St. Ruscheweyh, Convolutions in Geometric Function Theory, in: Sem. Math. Sup, vol. 83, Presses Univ. de Montreal, 1982.
[5] S.S. Miller, P.T. Mocanu, Differential Subordinations, Dekker, New York, 2000.
[6] R.M. Ali, V. Ravichandran, S.K. Lee, Subclasses of multivalent starlike and convex functions, Bull. Belg. Math. Soc. Simon Stevin 16 (2009) 385–394.
[7] S. Supramaniam, R.M. Ali, S.K. Lee, V. Ravichandran, Convolution and differential subordination for multivalent functions, Bull. Malays. Math. Sci. Soc. (2) 32 (3) (2009) 351–360.
[8] M.H. Mohd, R.M. Ali, S.K. Lee, V. Ravichandran, Subclasses of meromorphic functions associated with convolution, J. Inequal. Appl. 2009 (2009) Art. ID 190291, 10pp.
[9] R.M. Ali, M.N. Mahnaz, V. Ravichandran, K.G. Subramanian, Convolution properties of classes of analytic and meromorphic functions, J. Inequal. Appl. 2010 (2010) Article ID 385728, 14 pp.
[10] K. Sakaguchi, On a certain univalent mapping, J. Math. Soc. Japan 11 (1959) 72–75.
[11] R. Chand, P. Singh, On certain Schlicht mappings, Indian J. Pure Appl. Math. 10 (9) (1979) 1167–1174.
[12] I.R. Nezhmetdinov, A sharp lower bound in the distortion theorem for the Sakaguchi class, J. Math. Anal. Appl. 242 (2) (2000) 129–134.
[13] K.S. Padmanabhan, J. Thangamani, The effect of certain integral operators on some classes of starlike functions with respect to symmetric points, Bull. Math. Soc. Sci. Math. Roumanie (N.S.) 26 (74) (1982) 355–360. 4.
[14] T.N. Shanmugam, C. Ramachandran, V. Ravichandran, Fekete–Szegő problem for subclasses of starlike functions with respect to symmetric points, Bull. Korean Math. Soc. 43 (3) (2006) 589–598.
[15] J. Thangamani, On starlike functions with respect to symmetric points, Indian J. Pure Appl. Math. 11 (3) (1980) 392–405.
[16] Z.-G. Wang, A new subclass of quasi-convex functions with respect to k-symmetric points, Lobachevskii J. Math. 19 (2005) 41–50. (electronic).
[17] Z.-G. Wang, Certain subclasses of close-to-convex and quasi-convex functions with respect to K-symmetric points, Lobachevskii J. Math. 21 (2006) 73–83. (electronic).
[18] Z.-G. Wang, C.-Y. Gao, S.-M. Yuan, On certain subclasses of close-to-convex and quasi-convex functions with respect to k-symmetric points, J. Math. Anal. Appl. 322 (1) (2006) 97–106.
[19] Z.-G. Wang, Some subclasses of close-to-convex and quasi-convex functions, Mat. Vesnik 59 (1–2) (2007) 65–73.
[20] R.Md. El-Ashwah, D.K. Thomas, Some subclasses of close-to-convex functions, J. Ramanujan Math. Soc. 2 (1) (1987) 85–100.
[21] V. Ravichandran, Functions starlike with respect to n-ply symmetric, conjugate and symmetric conjugate points, J. Indian Acad. Math. 26 (1) (2004) 35–45.
[22] H. Al-Amiri, D. Coman, P.T. Mocanu, Some properties of starlike functions with respect to symmetric-conjugate points, Int. J. Math. Math. Sci. 18 (3) (1995) 469–474.
[23] M.-P. Chen, Z.-R. Wu, Z.-Z. Zou, On functions α-starlike with respect to symmetric conjugate points, J. Math. Anal. Appl. 201 (1) (1996) 25–34.
[24] M.S. Kasi, V. Ravichandran, On starlike functions with respect to n-ply conjugate and symmetric conjugate points, J. Math. Phys. Sci. 30 (6) (1996) 307–316. (1999).
[25] Z.-G. Wang, Y.-P. Jiang, On certain subclasses of close-to-convex and quasi-convex functions with respect to $2k$-symmetric conjugate points, J. Math. Appl. 29 (2007) 167–179.
[26] Z.-G. Wang, D.-Z. Chen, On subclasses of close-to-convex and quasi-convex functions with respect to $2k$-symmetric conjugate points, Lobachevskii J. Math. 26 (2007) 125–135.
[27] Z.-G. Wang, C.-Y. Gao, S.-M. Yuan, Some coefficient inequalities for certain subclasses of analytic functions with respect to k-symmetric points, Soochow J. Math. 33 (2) (2007) 223–227.
[28] Z.-G. Wang, C.-Y. Gao, On starlike and convex functions with respect to $2k$-symmetric conjugate points, Tamsui Oxf. J. Math. Sci. 24 (3) (2008) 277–287.
[29] Z. Wu, S. Owa, Convolution theorems for functions of Sakaguchi type, Mathematica (Cluj) 32(55) (1) (1990) 93–97.
[30] V. Ravichandran, Starlike and convex functions with respect to conjugate points, Acta Math. Acad. Paedagog. Nyházi. (N.S.) 20 (1) (2004) 31–37. (electronic).