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Instantaneous Shrinking of the Support of Energy Solutions

Robert Kersner

Computer and Automation Institute, HAS, H-1518, Budapest, POB 63, Hungary

and

Andrey Shishkov

Institute of Applied Mathematics and Mechanics, UAS, R. Luxemburg St 74, 340114, Donetsk, Ukraine

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The Cauchy problem for a class of nonlinear diffusion-reaction equations is studied. The equations may be classified as being of degenerate parabolic type. It is shown that under certain conditions solutions of the problem exhibit instantaneous shrinking. This is to say, at any positive time the spatial support of the solution is bounded above, although the support of the initial data function is not. We also provide some estimates of the behavior of the free boundary. © 1996 Academic Press, Inc.

0. INTRODUCTION

This paper is devoted to the study of some inner properties of solutions of a large class of nonlinear partial differential equations. Methods used here can be called "energy methods"; they are based on getting integral estimates and have nothing in common with the maximum principle.

Our main results will be formulated in terms of the Cauchy problem (Problem P)

$$u_t = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(|\nabla u|^{p-1}\frac{\partial u}{\partial x_i}\right) - |u|^{\lambda-1}u; \qquad x \in \mathbb{R}^n, \quad t > 0 \tag{0.1}$$

$$u(x,0) = u_0(x), \qquad x \in \mathbb{R}^n, \tag{0.2}$$

where p and λ are positive real numbers, $\nabla u = \operatorname{grad}_x u$.

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We note that this is only for the sake of simplicity: analogously we can study more general equations, for instance the equation

$$u_t + (-1)^m \sum_{|\alpha|=m} D^\alpha\big(|D^\alpha u|^{p-1} D^\alpha u\big) + |u|^{\lambda-1}u = 0, \qquad (0.3)$$

where $p, \lambda > 0$ and

$$D^\alpha f = \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}, \qquad |\alpha| = \alpha_1 + \cdots + \alpha_n,$$

with the corresponding changes only in formulations but not in the essence of proofs.

We are interested in a phenomenon called "the instantaneous shrinking of the support of solution $u(x,t)$" (briefly, the (IS) property).

Let

$$\operatorname{supp} u(x,t) = \operatorname{clos}\{x \in \mathbb{R}^n : u(x,t) \neq 0\}$$

DEFINITION 1. The problems (0.1), (0.2) (or (0.3), (0.2)) have the (IS) property if for any $t > 0$ the support of solution $u(x,t)$ is bounded even if it is unbounded for $t = 0$.

Remark 1. The equality $u(x,t) = 0$ has to be understood in corresponding functional spaces. For the second order case ($m = 1$), in view of the well-known regularity results (see, e.g., [12]), $u = 0$ as a continuous function. As for the notion of solution, see Definition 2.

Remark 2. The paper [15] must be considered as the first one where the (IS) property was systematically investigated for the equation

$$u_t = \Delta u - g(x)\beta(u). \qquad (0.4)$$

In the case of $g(x) \equiv 1$ it has been shown that if $u_0(x)$ positive, continuous, bounded, uniformly goes to zero when $|x| \to \infty$ function, $\beta(s) \geq 0$ nondecreasing for $s \geq 0$, $\beta(0) = 0$ and $\int_0^\delta [s\beta(s)]^{-1/2}\, ds < 0$, $\delta > 0$, then (0.4), (0.2) has the (IS) property. As was mentioned in [15], for $\beta(u) = u^\lambda$, $\lambda \in (0,1)$, the same result was obtained earlier [28]. For variational inequalities the (IS) property was investigated in [11].

The method used in [15] was based on the construction of a special comparison function of the form $w = F(t) + G(x)$ and on the application of the maximum principle in the "far" cylinders $B_r(x_0) \times (0, t_0)$, $|x_0|$ large. Later this method was perfected and applied first to Eq. (1.4) with $n = 1$, $g(x) = (1 + x^2)^{-1/2}$, $\beta(u) = u^\lambda$, $\lambda \in (0,1)$ [17], then to (0.4) with a more general $g(x)$ [18]. See also [20], where for the arising free boundary two-sides estimates were given.

In the papers [21, 19] another method, still based on the comparison principle, was applied to one-dimensional equations such as

$$u_t = (u^m)_{xx} - g(x)u^p. \tag{0.5}$$

It was established for instance that if

$$u_0(x) \le c_0(1+|x|)^{-\gamma}, \quad g(x) \ge c_1(1+|x|)^{-\beta},$$
$$m \ge 1, \quad p \in (0,1), \quad \beta > 0, \quad \gamma > 0, \quad \text{and} \quad c_i > 0,$$

then the problem (0.5), (0.2) has the (IS) property iff $\beta < \gamma(1-p)$. In these articles global supersolutions have been used giving some information about free boundary $\zeta(t) = \sup\{x\colon u(x,t) > 0\}$ of type $\zeta(t) \le \text{const} \cdot t^{-\delta}$, $\delta > 0$. See also [10] for the case $0 < m < 1$.

This striking behavior of solutions in the above examples was the result of strong (with respect to diffusion) absorption ($0 < \lambda < 1$). We have to remark that analogous phenomenon can arise in other physically important models. Thus, in [16] for the equation

$$u_t = (u^m)_{xx} + (u^n)_x; \qquad 0 < n < 1, \quad m \ge 1,$$

the following theorem was proved: if $u_0(x) \sim cx^{-1/(1-n)}$ as $x \to \infty$, then $u(x,t) > 0$ for $t \in (0, (1/n)c^{1-n})$, $x \ge x_0 > 0$ and $u(x,t)$ has compact support in x ($\zeta(t) < \infty$) for $t > (1/n)c^{1-n}$. From this the (IS) property follows provided $u_0 = o(x^{-1/(1-n)})$.

Analogous results were obtained in [22] for the first order hyperbolic equation

$$u_t = (u^n)_x,$$

where $0 < n < 1$. Here, the (IS) property indicates also the instantaneous loss of continuity ($u_0(x) > 0$ was smooth).

Remark 3. All of the results mentioned have been obtained for second order equations, for non-negative solutions, and with the assumption that $u_0(x)$ is monotonous, goes to zero as $|x| \to \infty$, or has such a majorant. The main tool in getting the results was the maximum principle.

If the initial distribution $u_0(x)$ has no monotone majorant, for example $u_0(+k) = 1$, $k \in \mathbb{Z}$, $u_0(x) > 0$, $x \in \mathbb{R}$, then for the simplest equation

$$u_t = u_{xx} - u^p, \qquad 0 < p < 1, \tag{0.6}$$

we cannot tell anything about the solution's behavior, as the comparison principle here is inadequate. For higher order equations we have no such principles.

The method we will use in this paper is the result of a long evolution of ideas coming from the theory of linear elliptic and parabolic equations. It can be applied for different purposes and different equations.

The essence of this method consists in getting special (non-differential) inequalities linking different energy norms of solution. The analysis of these inequalities leads to the needed results. As to the origin of this method, first we have to mention the book [24] (growth lemmas) and the paper [25] (''method of parameter's introduction'').

For the nonlinear degenerate parabolic equations in [2, 3] and later in [13, 14] results were obtained on the existence of free boundaries and their properties, by using some local energy estimates. For integral norms of solutions differential inequalities were obtained from which the results follow by integration.

Important contributions in developing the theory of energy solutions for elliptic and parabolic degenerate equations have been made in [5−9].

In [26, 27] some additional ideas have been introduced; we shall use some of them. From the above papers (see also references therein) one can form a good idea about the ''energy method'' and the results obtained by this method.

When this work was completed, we were informed of the paper [4], wherein the authors investigated similar questions with the help of the local energy method. We have also recently received the paper [10].

1. MAIN RESULTS

Let $u_0(x) \in L_2(\mathbb{R}^n)$. Define function $\tilde{h}(s)$ by

$$\tilde{h}(s) = \int_{|x|>s} u_0^2(x)\,dx. \tag{1.1}$$

Obviously, $\tilde{h}(s) \to 0$ as $s \to +\infty$.

Let $u(x,t)$ be any energy solution of the problme (0.1), (0.2).

THEOREM 1. *In both of the cases*

(i) $p \geq 1$, $0 < \lambda < 1$,

(ii) $0 < \lambda < p$, $1 > p > (n-2)/(n+2)$ *for* $n > 2$, $1 > p > 0$ *for* $n \leq 2$, *the support of* $u(x,t)$ *is bounded for* $t > 0$, *e.g., the problem* (0.1), (0.2) *has the* (IS) *property.*

The method we use gives information on diam{supp $u(x,t)$} in terms of the function $\tilde{h}(s)$.

Let $h(s) \geq \tilde{h}(s)$ be any nonincreasing continuous function such that $h(s) > 0$ as $s \to \infty$ and satisfies the condition

$$h\big(s + \Lambda h^{\nu(p-\lambda)/(1-\lambda)(p+1)}(s)\big) \geq \omega h(s) \tag{1.2}$$

for $s > s_0$, where Λ, $\omega < 1$ are positive numbers depending on p, n, λ;

$$\nu = \frac{(p+1)(1-\lambda)}{2(p+1) + n(p-\lambda)}.$$

We note that for $\tilde{h}(s)$ one can always choose $h(s)$ with (1.2). Actually, let

$$H(s) = h^{\alpha}(s),$$

where

$$\alpha = \frac{\nu(p-\lambda)}{(1-\lambda)(p+1)} = \frac{p-\lambda}{2(p+1) + n(p-\lambda)}.$$

Condition (1.2) is now equivalent to

$$H(s + \Lambda H(s)) \geq \omega_1 H(s), \qquad \omega_1 = \omega^{\alpha}. \tag{1.2a}$$

However, in order to ensure that any nonincreasing differentiable function $H(s)$ satisfies (1.2a) it is sufficient that $H'(s) \geq -\Lambda^{-1}(1-\omega_1)$ for $s > s_0$. This follows from the mean-value theorem

$$\begin{aligned} H(s + \Lambda H(s)) &= H(s) - \big(H(s) - H(s + \Lambda H(s))\big) \\ &= H(s)\big[1 - (-H'(\Theta s)\Lambda)\big] \geq \omega_1 H(s). \end{aligned}$$

A majorant of $\tilde{h}^{\alpha}$ can be chosen in the set of nonincreasing differentiable functions H such that $H'(s) \to 0$ as $s \to \infty$ and $H'(s) \geq -\Lambda^{-1}(1-\omega_1)$ for $s > s_0$. For example, if $\tilde{h}(s) = (1+s)^{-\gamma}$, $\gamma > 0$, then we can take $h = \tilde{h}$. Let $R(t) = \inf\{r\colon \operatorname{supp} u(x,t) \subseteq B_r = \{x\colon |x| < r\}\}$.

THEOREM 2. *Suppose that the conditions of Theorem* 1 *are fulfilled. Then, one has the following upper estimate for the free boundary*:

$$\begin{aligned} R(t) &\leq h^{-1}\big(D_1 t^{1/\nu}\big) + Q_1 t^{1/(p+1)} \qquad \text{for } p \geq 1, \\ R(t) &\leq h^{-1}\big(D_2 t^{(1-\lambda)/\nu(p-\lambda)}\big) + Q_2 t^{1/(p+1)} \qquad \text{for } p < 1; \end{aligned} \tag{1.3}$$

where $D_i > 0$ and $Q_i < \infty$ depend on known parameters only,

$$h^{-1}(t) := \inf\{s\colon h(s) < t\}.$$

Remark 4. If $h(s)$ is strictly decreasing, h^{-1} is the usual inverse function of h. For instance if $\tilde{h}(s) = (1+s)^{-\gamma}$, then the inequality (1.3) has the form (for $t \to 0$)

$$R(t) < Dt^{-1/\gamma\nu}, \qquad \text{for } p \geq 1.$$

The next step toward understanding the (IS) property is the case when $u_0(x) \in L_q(\mathbb{R}^n)$. If $u_0(x) \in L^\infty(\mathbb{R}^n)$ then Problem P has no (IS) property in general (take $u_0 \equiv 1$). In the case of $q = 1$, where we do not know the answer, it would be interesting to understand the situation for example for $n = 1$ and $u_0(x) = \sum_{i=-\infty}^{\infty} c_i \delta(x+i) + f(x)$, $\sum_{i=-\infty}^{\infty} |c_i| < \infty$, $f > 0$, $f \in L_1$. So, let $u_0(x) \in L_q(\mathbb{R}^n)$, $1 < q < \infty$. Analogously to (1.1), we introduce the function

$$\tilde{h}_q(s) := \int_{|x|>s} |u_0|^q \, dx$$

and $h_q(s) \geq \tilde{h}_q(s)$ satisfying condition (1.2) with

$$\nu = \nu_1 := \frac{(p+1)(1-\lambda)}{q(p+1) + n(p-\lambda)}.$$

THEOREM 3. *Let* $u_0 \in L_q(\mathbb{R}^n)$ *and assumptions* (i), (ii) *of Theorem* 1 *hold*; $u(x,t)$ *is a solution in the sense of Definition* 3. *Then*, *Problem* P *has the* (IS) *property*. *Moreover*,

$$R(t) \leq h_q^{-1}\left(D_3 t^{1/\nu_1}\right) + Q_3 t^{1/(p+1)} \qquad \text{for} \quad p > 1,$$

$$R(t) \leq h_q^{-1}\left(D_4 t^{(1-\lambda)/\nu_1(p-\lambda)}\right) + Q_4 t^{1/(p+1)} \quad \text{for } p < 1, \tag{1.4}$$

where $D_i > 0$ *and* $Q_i > 0$ *depend only on known parameters*.

If $\lambda \geq 1$ then Problem P has no (IS) property so the necessity of conditions (i), (ii) follows from our next result:

THEOREM 4. *Let* $0 < p < \lambda < 1$. *Then Problem* P *has no* (IS) *property in general*.

When $m > 1$, Eq. (0.3) may be treated by the technique used in the proofs of Theorem 1 and 2. Here we state only the result; the proof will be given elsewhere.

THEOREM 5. *Let* $u(x,t)$ *be any energy solution of* (0.3), (0.2), $u_0 \in L_2(\mathbb{R}^n)$, $0 < \lambda < p$, $\lambda < 1$ *and* $p > (n-2m)/(n+2m)$ *for* $n > 2m$. *Then*, *this problem has the* (IS) *property. Moreover, one has the upper estimates*

$$R(t) \le h^{-1}\left(D_1 t^{1/\nu}\right) + Q_1 t^{1/m(p+1)} \qquad \text{for } p \ge 1,$$

$$R(t) \le h^{-1}\left(D_2 t^{1/\nu_1}\right) + Q_2 t^{1/m(p+1)} \qquad \text{for } p < 1,$$

for the free boundary, where

$$\nu = \frac{m(p+1)(1-\lambda)}{n(p-\lambda) + 2m(p+1)}, \qquad \nu_1 = \nu\frac{p-\lambda}{1-\lambda},$$

and $h(s)$ *is a monotonous majorant for* $\tilde{h}(s) = \int_{|x|>s} u_0^2(x)\,dx$.

2. (IS) PROPERTY FOR $u_0 \in L_2$

First we introduce some notations and definitions. For any given numbers $0 \le \tau_1 < \tau_2 \le T$, $0 < s_1 < s_2 < \infty$,

$$\Omega(s_1) = \left\{x \in \mathbb{R}^n : |x| > s_1\right\}$$

$$G_{\tau_1}^{\tau_2}(s_1) = \Omega(s_1) \times (\tau_1, \tau_2)$$

$$K_{\tau_1}^{\tau_2}(s_1, s_2 - s_1) = G_{\tau_1}^{\tau_2}(s_1) \setminus G_{\tau_1}^{\tau_2}(s_2).$$

Let us fix $\tau > 0$, $s > 0$, $\Delta\tau > 0$, and $\Delta s > 0$.

The cutoff functions $\eta(x,t)$ and $\eta_1(x)$ are such that $\eta, \eta_1 \ge 0$, $\eta = 1$ in $G_{\tau+\Delta\tau}^{T}(s+\Delta s)$, $\eta = 0$ in $\mathbb{R}^n \times (0,T) \setminus G_\tau^T(s)$, and $\eta_1 = 1$ in $\Omega(s+\Delta s)$, $\eta_1 = 0$ in $\mathbb{R}^n \setminus \Omega(s)$. We shall assume that

$$0 \le \eta_t \le \frac{c}{\Delta s}, \qquad |\eta_{x_i}| \le \frac{c}{\Delta s}, \qquad |\eta_{1x_i}| \le \frac{c}{\Delta s}$$

$$\eta_t = 0 \text{ if } \tau + \Delta\tau < t < T \qquad \text{and} \qquad \nabla\eta = 0 \quad \text{if } |x| > s + \Delta s. \quad (2.1)$$

Below we denote constants depending only on problem's parameters by c.

Definition 2. We call $u(x,t)$ the energy solution of (0.1) if $u(x,t) \in V_{p+1,2}^{1,0}(\mathbb{R}^n \times (0,T)) \cap L_{\lambda+1}(\mathbb{R}^n \times (0,T)) := C(0,T; L_2(\mathbb{R}^n)) \cap L_{1+p}(0,T; W_{p+1}^1(\mathbb{R}^n)) \cap L_{\lambda+1}(\mathbb{R}^n \times (0,T))$, $u(x,t)$ satisfies for $T_0 \le T$ integral identity

$$\int_{\mathbb{R}^n} u(x,T_0)v(x,T_0)\,dx - \int_0^{T_0}\int_{\mathbb{R}^n} u(x,t)v_t(x,t)\,dx\,dt$$
$$+ \int_0^{T_0}\int_{\mathbb{R}^n}\left[|\nabla u|^{p-1}u_{x_i}v_{x_i} + |u|^{\lambda-1}uv\right]dx\,dt = \int_{\mathbb{R}^n} u_0(x)v(x,0)\,dx, \tag{2.2}$$

when the test function v is from $L_{\lambda+1}(\mathbb{R}^n \times (0,T)) \cap W^{1,1}_{p+1,2}(\mathbb{R}^n \times (0,T))$; here $W^{1,1}_{p+1,2}(\mathbb{R}^n \times (0,T)) := \{w \in L_{1+p}(0,T; W^1_{p+1}(\mathbb{R}^n)), \;\; w_t \in L_2(\mathbb{R}^n \times (0,T))\}$.

Remark 5. The existence of solutions in the above sense is well known if $1 \le p$ or $0 < \lambda \le p$, see, e.g., [23, 8, 1].

In what follows we shall often use the Gagliardo–Nirenberg interpolation inequality

$$\|v\|_{\alpha,\Omega(s)} \le d_1 \|\nabla v\|^{\Theta}_{\beta,\Omega(s)} \|v\|^{1-\Theta}_{\gamma,\Omega(s)}, \tag{2.3}$$

where

$$v(x) \in W^1_\beta(\Omega(s)) \cap L_\gamma(\Omega(s)), \qquad \|v\|_{\alpha,\Omega} := \left(\int_\Omega |v|^\alpha \, dx\right)^{1/\alpha},$$

and

$$\frac{1}{\alpha} = \Theta\left(\frac{1}{\beta} - \frac{1}{n}\right) + (1-\Theta)\frac{1}{\gamma}, \qquad \gamma > 1, \quad \beta > 1,$$

and it is important that d_1 does not depend on $s > 0$.

Let $u(x,t)$ be any energy solution to (0.1). We set

$$E_T(\tau, s) = \int_{G^T_\tau(s)} u^2 \, dx\, dt, \qquad I_T(\tau, s) = \int_{G^T_\tau(s)} |u|^{p+1} \, dx\, dt.$$

If we show that for any $\tau > 0$ there exists $s(\tau) < \infty$ such that

$$H = H_T(\tau, s) := E_T(\tau, s) + I_T(\tau, s) = 0, \tag{2.4}$$

then Theorem 1 follows. In order to prove (2.4), as it follows from Lemma 1 of the Appendix, it is sufficient to show that

$$H_T(0, s) \to 0, \qquad \text{when } s \to \infty, \tag{2.5}$$

and

$$H(\tau + H^\alpha, s + H^\beta) \le \mu H, \tag{2.6}$$

where $\alpha, \beta > 0$, $0 < \mu < 1$.

First we prove (2.6). We obtain by substituting $v = u\eta^{p+1}$ into (2.2) and integrating by parts formally

$$2^{-1} \int_{\mathbb{R}^n} u^2(x,T)\eta^{p+1}(x,T)\, dx + \int_0^T \int_{\mathbb{R}^n} \left[|\nabla u|^{p+1} + |u|^{\lambda+1}\right] \eta^{p+1} \, dx\, dt$$
$$= (p+1) \int_0^T \int_{\mathbb{R}^n} \left(2^{-1} u^2 \eta_t + |\nabla u|^{p-1} u_{x_i} u \eta_{x_i}\right) \eta^p \, dx\, dt. \tag{2.7}$$

Remark 6. If u is not smooth in t, we can use approximation by smooth functions $\{u_k\}$, integrating by part and passing to the limit.

For the right-hand side of (2.7) we apply Young's inequality with ε and use (2.1)

$$\int_{\Omega(s)} u^2\eta^{p+1}\,dx + \int_{G_\tau^T(s)} \left(|\nabla u|^{p+1} + |u|^{\lambda+1}\right)\eta^{p+1}\,dx\,dt$$
$$\leq c\left[(\Delta s)^{-(p+1)}\int_{K_\tau^T(s,\Delta s)} |u|^{p+1}\,dx\,dt + (\Delta\tau)^{-1}\int_{G_\tau^{\tau+\Delta\tau}(s)} u^2\,dx\,dt\right]$$
$$:= cR_1 = cR_1(s,\Delta s,\tau,\Delta\tau). \tag{2.8}$$

On the right-hand side of (2.8) one can recognize an "H-like" function (cf. (2.4)). We would like to have H also on the left-hand side (cf. (2.6)). For this purpose, we write the Gagliardo–Nirenberg inequality (2.3) with $\alpha = 2$, $\beta = p+1$, $\gamma = \lambda + 1$ and use Young's inequality ((2.3) can be applied because $0 < \lambda < 1$ and $p > (n-2)/(n+2)$ for $n > 2$)

$$\left(\int_{\Omega(\bar{s})} u^2\,dx\right)^{1-\nu} \leq c\int_{\Omega(\bar{s})} \left(|\nabla u|^{p+1} + |u|^{\lambda+1}\right)dx, \qquad \bar{s} > s_0 > 0,$$

where $\nu = (p+1)(1-\lambda)/(2(p+1) + n(p-\lambda)) < 1$. Integration gives us

$$\Psi_{\bar{\tau},\bar{s}}^T(1-\nu) := \int_{\bar{\tau}}^T \left(\int_{\Omega(\bar{s})} u^2\,dx\right)^{1-\nu} dt \leq c\int_{G_{\bar{\tau}}^T(\bar{s})} \left(|\nabla u|^{p+1} + |u|^{\lambda+1}\right)dx\,dt. \tag{2.9}$$

In order to get new information about H, we return to the integral identity with test function

$$v = u\eta^{p+1}\chi_l(t), \qquad l > 0,$$

$$\chi_l(t) = \int_0^t \left(\int_{\Omega(s)} u^2\eta^{p+1}\,dx\right)^l dt, \qquad t > 0.$$

Substituting v into (2.2), we obtain

$$\chi_{l+1}(T) = \chi_l(T)\int_{\Omega(s)} u^2\eta^{p+1}\,dx + \int_{G_\tau^T(s)} \Big[2|\nabla u|^{p-1}u_{x_i}\left(u\eta^{p+1}\right)_{x_i}$$
$$+2|u|^{\lambda+1}\eta^{p+1} - u^2\left(\eta^{p+1}\right)_t\Big]\,\chi_l(t)\,dx\,dt,$$

from which and (2.8) we have

$$\chi_{l+1}(T) \leq c\chi_l(T)R_1. \tag{2.10}$$

Now we will estimate E_T in terms of $R_1^{1+\nu}$ (see (2.8)).

By the Hölder inequality, we have

$$\chi_l(T) \le \chi_{l_1}^{\mu}(T)\,\chi_{l_2}^{1-\mu}(T), \qquad 0 < l_1 < l < l_2 < \infty, \tag{2.11}$$

where $\mu = (l_2 - l)/(l_2 - l_1)$.

We iterate (2.10) and use (2.11) with suitable l_i

$$\chi_l(T) \le c\,\chi_\delta(T)\,R_1^{l-\delta}, \qquad \text{for any } l > \delta > 0. \tag{2.12}$$

By definition of $\eta(x,t)$ we have

$$\Psi^T_{\tau+\Delta\tau,\, s+\Delta s}(l) \le \chi_l(T) \le \Psi^T_{\tau,s}(l). \tag{2.13}$$

From (2.9) with $\bar{\tau} = \tau$ and $\bar{s} = s$ and (2.8) we have

$$\Psi^T_{\tau,s}(1-\nu) \le cR_1(s, \Delta s, \tau, \Delta\tau). \tag{2.14a}$$

From (2.12) with $l = 1$, $\delta = 1 - \nu$,

$$\chi_1(T) \le c\,\chi_{1-\nu}(T)\,R_1^{\nu}(s, \Delta s, \tau, \Delta\tau). \tag{2.14b}$$

From (2.13)

$$\chi_{1-\nu}(T) \le \Psi^T_{\tau,s}(1-\nu). \tag{2.14c}$$

Again from (2.13) and the definition of E_T

$$\Psi^T_{\tau+\Delta\tau,\, s+\Delta s}(1) := E_T(\tau + \Delta\tau, s + \Delta s) \le \chi_1(T). \tag{2.14d}$$

Inserting (2.14b) into (2.14d) and using (2.14c) and (2.14a) it follows that

$$E_T(\tau + \Delta\tau, s + \Delta s) \le cR_1^{1+\nu}(s, \Delta s, \tau, \Delta\tau). \tag{2.15}$$

Now we have to estimate the second term in H, i.e, I_T (see (2.4)). We separate the cases of $p > 1$ and $p < 1$. (For $p = 1$, $E_T = I_T$ and from (2.15) we can conclude the proof.)

Suppose first that $p > 1$. Take $\alpha = p + 1$, $\beta = p + 1$, $\gamma = 2$ in (2.3). After integration in t and using the Hölder inequality we obtain

$$I_T(\tau + \Delta\tau, s + \Delta s)$$
$$\le c\left(\int_{G^T_{\tau+\Delta\tau}(s+\Delta s)} |\nabla u|^{p+1}\,dx\,dt\right)^{\theta_1}\left(\Psi^T_{\tau+\Delta\tau,\, s+\Delta s}\left(\frac{p+1}{2}\right)\right)^{1-\theta_1}, \tag{2.16}$$

where

$$\theta_1 = \frac{n(p-1)}{2(p+1)+n(p-1)} < 1.$$

Inequality (2.12) with $l = (1+p)/2$ and $\delta = 1-\nu$ gives us

$$\Psi^T_{\tau+\Delta\tau,\, s+\Delta s}\left(\frac{p+1}{2}\right) \leq c\Psi^T_{\tau,s}(1-\nu)R_1^{(1+p)/2-1+\nu}.$$

Applying (2.15) one obtains

$$\Psi^T_{\tau+\Delta\tau,\, s+\Delta s}\left(\frac{p+1}{2}\right) \leq cR_1^{(1+p)/2+\nu}.$$

Using this in (2.16) we come to

$$I_T(\tau+\Delta\tau, s+\Delta s) \leq cR_1^{1+\nu_1},$$
$$\nu_1 = (1-\theta_1)\left(\frac{p-1}{2}+\nu\right) = \frac{\nu(p-\lambda)}{1-\lambda} > \nu. \quad (2.17)$$

This ν_1 is different from the ν_1 in Theorem 3. Now we add (2.15) and (2.17) and use the definition of R_1 (see (2.8)),

$$\begin{aligned} H_T(\tau+\Delta\tau, s+\Delta s) & \\ \leq c_0\Delta_\tau E_T(\tau,s)&\left[\frac{(\Delta_\tau E_T(\tau,s))^\nu}{(\Delta\tau)^{1+\nu}} + \frac{(\Delta_\tau E_T(\tau,s))^{\nu_1}}{(\Delta\tau)^{1+\nu_1}}\right] \\ + c_0\Delta_s I_T(\tau,s)&\left[\frac{(\Delta_s I_T(\tau,s))^\nu}{(\Delta s)^{(1+p)(1+\nu)}} + \frac{(\Delta_s I_T(\tau,s))^{\nu_1}}{(\Delta s)^{(1+p)(1+\nu_1)}}\right]. \quad (2.18) \end{aligned}$$

where $\Delta_\tau f(\tau,s) := f(\tau,s) - f(\tau+\Delta\tau, s)$, $\Delta_s f(t,s) := f(\tau,s) - f(\tau, s+\Delta s)$. Now—and this is one of the key points of our method—we fix Δs and $\Delta\tau$ in the following way:

$$\Delta s = (I_T(\tau,s))^{\nu/(p+1)(\nu+1)}, \qquad \Delta\tau = (E_T(\tau,s))^{\nu/(1+\nu)}.$$

In virtue of monotonicity of E and I one gets

$$H_T\left(\tau + E_T^{\nu/(1+\nu)}(\tau,s), s + I_T^{\nu/(1+p)(1+\nu)}(\tau,s)\right) \leq \mu_1 H_T(\tau,s) \quad (2.19)$$

for any $\tau > 0$, $s > s_0 > 0$ and

$$0 < \mu_1 = \frac{c_0\left[1 + \left(E_T(0, s_0) + I_T(0, s_0)\right)^{\alpha}\right]}{1 + c_0\left[1 + \left(E_T(0, s_0) + I_T(0, s_0)\right)^{\alpha}\right]} < 1,$$

$$\alpha = \frac{\nu_1 - \nu}{1 + \nu} > 0.$$

Increasing the arguments in the left-hand side of (2.19) we come to our final inequality

$$H_T\Big(\tau + H_T^{\nu/(1+\nu)}(\tau, s), s + H_T^{\nu/(1+p)(1+\nu)}(\tau, s) \le \mu_1 H_T(\tau, s)\Big). \quad (2.20)$$

In case of $0 < p < 1$ we can obtain an inequality analogous to (2.20) with $\nu_1 = \nu(p - \lambda)/(1 - \lambda) < \nu$ instead of ν. Only the starting point is different from the proof above: instead of the Gagliardo–Nirenberg inequality we use the Hölder inequality in the form

$$\int_{\Omega(s+\Delta s)} |u|^{p+1}\,dx \le c\left(\int_{\Omega(s+\Delta s)} u^2\,dx\right)^{(p-\lambda)/(1-\lambda)}\left(\int_{\Omega(s+\Delta s)} |u|^{1+\lambda}\,dx\right)^{(1-p)/(1-\lambda)}.$$

As we mentioned before, Theorem 1 follows from (2.20) and Lemma 1 of the Appendix provided (2.5), i.e., $H_T(0, s) \to 0$ when $s \to \infty$. This last result can be shown directly by using (2.2) with $v = u\eta_1(x)$, getting a (2.17)-type inequality. However, in the proof of our next theorem we show a much stronger estimation for $H_T(0, s)$.

Proof of Theorem 2. The equality

$$H_T(\tau, s) = 0, \qquad \forall(\tau, s)\colon \tau > \frac{1}{1 - \mu_1^{\nu/(1+\nu)}} H_T^{\nu/(1+\nu)}(0, s_0),$$

$$s \ge s_0 + \left(1 - \mu_1^{\nu/(1+p)(1+\nu)}\right)^{-1} H_T^{\nu/(1+p)(1+\nu)}(0, s_0), \quad (2.21)$$

for any $s_0 < \infty$, is a direct consequence of (2.20) and Lemma 1. Suppose for the moment that the inequality

$$H_T(0, s) \le B_1 h^{1+\nu}(s), \qquad \forall s > s_1, \quad (2.22)$$

holds, where $B_1 < \infty$, and recall that $h(s)$ is monotone majorant of

$\tilde{h}(s) = \int_{|x|>s} u_0^2(x)\,dx$. From (2.21) and (2.22) one has

$$H_T(\tau, s) = 0, \qquad \forall(\tau, s): \tau > \frac{B_1^{\nu/(1+\nu)}}{1 - \mu_1^{\nu/(1+\nu)}} h^{\nu}(s_0),$$

$$s \geq s_0 + \frac{B_1^{\nu/(p+1)(\nu+1)}}{1 - \mu_1^{\nu/(1+p)(1+\nu)}} h^{\nu/(1+p)}(s_0). \tag{2.23}$$

We define now s_0 by

$$s_0 = s_0(t) := h^{-1}\left(\frac{\left(1 - \mu_1^{\nu/(1+\nu)}\right) t^{1/\nu}}{B_1^{1/(1+\nu)}} \right).$$

From (2.23) one can see that for $t > 0$

$$H_T(\tau, s) = 0, \qquad \forall(\tau, s): \tau > t,\ s \geq s_0(t) + c_1 t^{1/(1+p)}, \tag{2.24}$$

and Theorem 2 is proved for $p > 1$. When $p < 1$, we have the same inequality (2.21) with $\nu_1 = ((p - \lambda)/(1 - \lambda))\nu$ instead of ν. With the same change in definition of $s_0(t)$, we can conclude the proof as above. Inequality (2.22) remains to be shown.

From integral identity (2.2), using $v = u\eta_1^{p+1}(x)$ as a test function, we can get in the same way as before (2.8) the inequality

$$\int_{\Omega(s)} u^2(x, T)\eta_1^{p+1}(x)\,dx + \int_{G_0^T(s)} \left(|\nabla u|^{p+1} + |u|^{\lambda+1}\right)\eta_1^{p+1}\,dx\,dt$$
$$\leq c\left((\Delta s)^{-(p+1)} \int_{K_0^T(s, \Delta s)} |u|^{p+1}\,dx\,dt + \tilde{h}(s) \right) := cR_2(s, \Delta s) = cR_2. \tag{2.25}$$

Similarly, as we derived (2.15), (2.17) from (2.8), from (2.25) we can obtain

$$E_T(0, s + \Delta s) := \int_{G_0^T(s+\Delta s)} u^2\,dx\,dt < cR_2^{1+\nu}, \qquad 0 < p < \infty, \tag{2.26}$$

$$I(0, s + \Delta s) := \int_{G_0^T(s+\Delta s)} |u|^{p+1}\,dx\,dt \leq cR_2^{1+\nu_1}, \qquad p > 1. \tag{2.27}$$

In the case of $p > 1$, we set $\Delta s = I_T^{\nu_1/(1+p)(1+\nu_1)}(0, s)$ in (2.27) and after simple calculation we have

$$A_T(s + A_T(s)) \leq \xi_1 A_T(s) + c_1 \tilde{h}^{\nu_1/(1+p)}(s), \tag{2.28}$$

where

$$A_T(s) := I_T^{\nu_1/(1+p)(1+\nu_1)}(0,s) \qquad \text{and} \qquad \xi_1 = \left(\frac{c}{1+c}\right)^{\nu_1/(1+p)(1+\nu_1)} < 1.$$

Now we apply Lemma 2 (Appendix): for any k such that $0 < k < 1 - \xi_1$ there exists a sequence $s_i \to \infty$ such that

$$I_T(0,s_i) \le \left(\frac{c_1}{k}\right)^{(p+1)(\nu_1+\nu)/\nu_1} \tilde{h}^{1+\nu_1}(s_i) := N_1 \tilde{h}^{1+\nu_1}(s_i) \qquad (2.29)$$

with

$$s_{i+1} - s_i \le \frac{c_1}{k(1-\xi_1-k)}\left[\tilde{h}(s_i)\right]^{\nu_1/(1+p)} := K\tilde{h}^{\nu_1/(1+p)}(s_i).$$

From the monotonicity of $\tilde{h}(s)$ and Lemma 3 of the Appendix it follows that from $\{s_i\}$ one can choose a subsequence $\{s_i\}$ for which

$$\frac{K}{2}\tilde{h}^{\nu_1/(1+p)}(s_i) \le \Delta s_i := s_{i+1} - s_i < \frac{3}{2}K\tilde{h}^{\nu_1/(1+p)}(s_i). \qquad (2.30)$$

Setting $\Delta s = \Delta s_i$ in (2.26), using the left-hand side inequality from (2.30) and (2.29), we obtain

$$E_T(0,s_{i+1}) \le N_2 \tilde{h}^{1+\nu}(s_i). \qquad (2.31)$$

Adding (2.29) and (2.31) we obtain

$$H_T(0,s_{i+1}) \le (N_1+N_2)\tilde{h}^{1+\nu_2}(s_i) \qquad \text{for any } i \in \mathbb{N}, \qquad (2.32)$$

where $\nu_2 = \min(\nu, \nu_1)$. Consider first the case $p \ge 1$. Plainly, $\nu_2 = \nu$. Set in (1.2), $\Lambda = \frac{3}{2}K$. Then from (2.32) for any $s \in [s_i, s_{i+1}]$ we obtain ($\tilde{h} < h$)

$$\begin{aligned} H_T(0,s) &\le H_T(0,s_i) \le (N_1+N_2)\tilde{h}^{1+\nu_2}(s_i) \le (N_1+N_2)h^{1+\nu_2}(s_i) \\ &\le \frac{N_1+N_2}{\omega^{1+\nu_2}} h^{1+\nu_2}\big(s_i + \Lambda h^{\nu_1/(p+1)}(s_i)\big) \le \frac{N_1+N_2}{\omega^{1+\nu}} h^{1+\nu}(s_{i+1}) \\ &\le \frac{N_1+N_2}{\omega^{1+\nu}} h^{1+\nu}(s), \qquad \text{for any } i \in \mathbb{N}, \end{aligned} \qquad (2.33)$$

from which (2.22) follows. For $p < 1$ we proceed in the same way, obtaining inequalities analogous to (2.29), (2.31). Theorem 2 is proved. ∎

3. (IS) PROPERTY FOR $u_0 \in L_q(\mathbb{R}^n)$ AND COUNTEREXAMPLE

DEFINITION 3. In the case of $u_0(x) \in L_q(\mathbb{R}^n)$, $q > 1$ we call the function $u(x,t) \in C(0,T; L_q(\mathbb{R}^n)) \cap L_{q-1+\lambda}(\mathbb{R}^n \times (0,T))$ an energy solution of (0.1), (0.2), if the hypotheses

$$|u|^{(q-2)/(p+1)} u_{x_i} \in L_{p+1}(\mathbb{R}^n \times (0,T)) \tag{3.1}$$

and

$$\begin{aligned}\int_{\mathbb{R}^n} |u|^q \varphi(x,T)\,dx - \int_{\mathbb{R}^n} |u_0|^q \varphi(x,0)\,dx \\ = \int_{\mathbb{R}^n \times (0,T)} \Big[|u|^q \varphi_t(x,t) - q|u|^{q-1+\lambda}\varphi \\ - q|\nabla u|^{p-1} U_{x_i} \big(|u|^{q-2} u\,\varphi(x,t)\big)_{x_i} \Big]\,dx\,dt\end{aligned} \tag{3.2}$$

are satisfied for any $\varphi(x,t) \in C^1(\mathbb{R}^n \times (0,T))$, such that φ and φ_t are bounded and $\nabla\varphi$ has compact support.

Remark 7. With regard to existence theorems in the sense of Definition 3, we cannot give exact references. However, let $u_0^{(i)}(x) \in L_\infty(\mathbb{R}^n) \cap L_2(\mathbb{R}^n)$, $u_0(x) \in L_q(\mathbb{R}^n)$, $u_i^{(i)} \to u_0(x)$ in $L_q(\mathbb{R}^n)$. If $u^{(i)}(x,t)$ is a solution from Theorem 1, then Theorem 3 gives a uniform in i estimate for the (IS) property. We think that on the basis of corresponding integral estimates for $u^{(i)}(x,t)$ it is possible to show the existence of a solution in the sense of Definition 3.

We start proving Theorem 3 by setting $\varphi(x,t) = \eta^{p+1}(x,t)$ in integral identity (3.2), where η is the same as the cutoff function in Section 2:

$$\begin{aligned}\int_{\Omega(s)} |u|^q \eta^{p+1}(x,T)\,dx\,dt + \frac{q(q-1)(p+1)^{p+1}}{(q+p-1)^{p+1}} \\ \times \int_{G_\tau^T(s)} \Big|\nabla\big(u|u|^{(q-2)/(p+1)}\big)\Big|^{p+1} \eta^{p+1}(x,t)\,dx\,dt \\ + q\int_{G_\tau^T(s)} |u|^{q+\lambda-1}\eta^{p+1}\,dx\,dt \\ = -\frac{q(p+1)^{2+p}}{(p+q-1)^{p+1}} \int_{K_\tau^T(s,As)} \Big|\nabla\big(u|u|^{(q-2)/p+1}\big)\Big|^{p+1}\end{aligned}$$

$$\cdot\left(u|u|^{(q-2)/(p+1)}\right)_{x_i} u|u|^{(q-2)/(p+1)} \eta^p \eta_{x_i}\, dx\, dt + (p+1)$$
$$\times \int_G |u|^q \eta^p \eta_t \, dx\, dt. \tag{3.3}$$

To the first term on the right-hand side of (3.3) we apply Young's inequality and after some standard calculation for the function

$$w(x,t) = u|u|^{(q-2)/(p+1)},$$

we obtain

$$\int_{\Omega(s)} |w|^{\alpha} \eta^{p+1}(x,T)\, dx + \int_{G_\tau^T(s)} \left(|\nabla w|^{p+1} + |w|^{\alpha-\delta}\right) \eta^{p+1}\, dx\, dt$$
$$\leq c\left(\frac{1}{(\Delta s)^{p+1}} \int_{K_\tau^T(s,\Delta s)} |w|^{p+1}\, dx\, dt + \frac{1}{\Delta\tau} \int_{G_\tau^{\tau+\Delta\tau}(s)} |w|^{\alpha}\, dx\, dt\right)$$
$$:= cR_3 = cR_3(s, \Delta s, \tau, \Delta\tau), \tag{3.4}$$

where

$$\alpha = \frac{q(p+1)}{p+q-1}, \qquad \delta = \frac{(p+1)(1-\lambda)}{p+q-1}, \qquad 0 < \lambda < 1.$$

Now, we can proceed with the proof as in Theorem 1 by taking the new values of parameters into account. Let us denote

$$E_T(\tau, s) := \int_{G_\tau^T(s)} |w|^{\alpha}\, dx\, dt, \qquad I_T(\tau, s) := \int_{G_\tau^T(s)} |w|^{p+1}\, dx\, dt.$$

Interpolation inequality (2.3) is to be applied to w with $\beta = p+1$, $\gamma = \alpha - \delta$, using Young's inequality and integrating in t,

$$\Psi_{\tau,s}^T(1-\nu_3) := \int_\tau^T \left(\int_{\Omega(s)} |w|^{\alpha}\, dx\right)^{1-\nu_3} dt$$
$$\leq c \int_{G_\tau^T(s)} \left(|\nabla w|^{p+1} + |w|^{\alpha-\delta}\right) dx\, dt, \tag{3.5}$$

where

$$\nu_3 = \frac{(p+1)(1-\lambda)}{(p+1)q + n(p-\lambda)}.$$

Notice that ν_3 is ν_1 of Theorem 3. Substitute into (3.2) the function $\varphi = \eta^{p+1}\chi_l(t)$ with

$$\chi_l(t) = \int_0^t \left(\int_{\Omega(s)} |w(x,\theta)|^\alpha \eta^{p+1}(x,\theta)\, dx \right)^l d\theta.$$

Repeating practically identical calculations which led to (2.15), using (3.4), we obtain

$$\Psi^T_{\tau+\Delta\tau,\, s+\Delta s}(1) \equiv E_T(\tau+\Delta\tau, s+\Delta s) \le cR_3^{1+\nu_3}. \tag{3.6}$$

In the case of $p > 1$, we write Gagliardo–Nirenberg for w with $\alpha = p+1 = \beta$, $\gamma = q(p+1)/(p+q-1)$, use Hölder, and integrate in t,

$$\begin{aligned} &I_T(\tau+\Delta\tau, s+\Delta s) \\ &\quad \le c\left(\int_{G^T_{\tau+\Delta\tau}(s+\Delta s)} |\nabla w|^{p+1}\, dx\, dt \right)^\Theta \left(\Psi^T_{\tau+\Delta\tau,\, s+\Delta s}\left(\frac{p+q-1}{q} \right) \right)^{1-\Theta}, \end{aligned} \tag{3.7}$$

where

$$\Theta = \frac{n(p-1)}{q(p+1) + n(p-1)} < 1.$$

Acting as in (2.17), we will have

$$I_T(\tau+\Delta\tau, s+\Delta s) \le cR_3^{1+\nu_4}, \tag{3.8}$$

where

$$\nu_4 = (1-\Theta)\left(\frac{p+1}{\alpha} - 1 + \nu_3 \right) = \frac{(p+1)(p-\lambda)}{(p+1)q + n(p-\lambda)} = \nu_3 \frac{p-\lambda}{1-\lambda}.$$

We add (3.6) and (3.8):

$$\begin{gathered} H_T(\tau+\Delta\tau, s+\Delta s) \le c(N^{1+\nu_3} + N^{1+\nu_4}) \\ N = \left(\frac{\Delta_s I_T(\tau,s)}{(\Delta s)^{p+1}} + \frac{\Delta_\tau E_T(\tau,s)}{\Delta\tau} \right) \qquad (\text{cf. } (2.18)). \end{gathered} \tag{3.9}$$

In the case of $p < 1$ we start applying Hölder in the form

$$\begin{aligned} \int_{\Omega(s+\Delta s)} |w|^{p+1}\, dx \le &\left(\int_{\Omega(s+\Delta s)} |w|^\alpha\, dx \right)^{(p-\lambda)/(1-\lambda)} \\ &\cdot \left(\int_{\Omega(s+\Delta s)} |w|^{\alpha-\delta}\, dx \right)^{(1-p)/(1-\lambda)}. \end{aligned}$$

Integration in t gives

$$I_T(\tau + \Delta\tau, s + \Delta s) \leq \left(E_T(\tau + \Delta\tau, s + \Delta s)\right)^{(p-\lambda)/(1-\lambda)} \cdot \left(\int_{G^T_{\tau+\Delta\tau}(s+\Delta s)} |w|^{\alpha-\delta}\, dx\, dt \right)^{(1-p)/(1-\lambda)}.$$

If we estimate the first term on the right-hand side by (3.6) and the second term by (3.4), we obtain

$$I_T(\tau + \Delta\tau, s + \Delta s) \leq cR_1^{1+\nu_4}. \tag{3.10}$$

The addition of (3.6) and (3.10) gives (3.9).

Now, we can conclude the proof of Theorem 3 exactly as in Theorems 1 and 2. In order to prove Theorem 4 it is sufficient to give a counterexample. Consider the equation

$$Lu := u_t - \left(|u_x|^{p-1}u_x\right)_x + u^\lambda = 0$$

in the half-strip $S = \{(x,t)\colon x > 0,\ 0 < t < \varepsilon\}$, $\varepsilon > 0$.

Define (in S) the function v by

$$v(x,t) = \frac{(\varepsilon - t)}{(x+1)^\gamma}.$$

First, we calculate Lv. We have

$$\begin{aligned} Lv = &-(x+1)^{-\gamma} - \gamma^p p(\gamma+1)(x+1)^{-p(\gamma+1)-1}(\varepsilon - t)^p \\ &+ (\varepsilon - t)^\gamma (x+1)^{-\gamma\lambda} \\ &< \gamma^p(\varepsilon - t)^p (x+1)^{-p(\gamma+1)-1} \\ &\times \left[-p(\gamma+1) + \gamma^{-p}(\varepsilon - t)^{\lambda - p}(x+1)^{\gamma(p-\lambda)+p+1}\right]. \end{aligned}$$

The quantity enclosed in square brackets is negative when

$$\lambda > p, \qquad \gamma \geq \frac{p+1}{\lambda - p}, \qquad \text{and} \qquad \varepsilon^{\lambda - p} < p(\gamma+1)\gamma^p. \tag{3.11}$$

If $u_0(x) > 0$ for $x \geq 0$, we choose, by continuity of $u(x,t)$, $\varepsilon > 0$ so small that we have $v(0,t) = (\varepsilon - t) \leq \varepsilon \leq u(0,t)$ for $t \in (0, \varepsilon)$. If we choose γ and ε in correspondence with (3.11) and $u_0(x) \geq \varepsilon(x+1)^{-\gamma}$ for $x > 0$, then, by the comparison principle, we have $u(x,t) \geq v(x,t) > 0$ in S.

APPENDIX

The three lemmas below were often used in the article. They appeared in different forms also in [26, 27].

LEMMA 1. *If* $f(\tau, s)$ *is a non-negative, nonincreasing or* $\tau > \tau_0$, $s > s_0$, *function satisfying*

$$f\big(\tau + f^{\alpha}(\tau, s), s + f^{\beta}(\tau, s)\big) \le \delta f(\tau, s) \tag{4.1}$$

for each $\tau > \tau_0$, $s > s_0$; $\delta < 1$, $\alpha > 0$, $\beta > 0$, *then,*

$$f(\tau, s) \equiv 0 \qquad \text{for every } (\tau, s) \text{ such that}$$

$$\tau > \tau_0 + \frac{1}{1 - \delta^{\alpha}} f^{\alpha}(\tau_0, s_0),\ s > s_0 + \frac{1}{1 - \delta^{\beta}} f^{\beta}(\tau_0, s_0).$$

Proof. Define the sequences $\{\tau_i\}$ and $\{s_i\}$ recursively by

$$\tau_{i+1} = \tau_i + f^{\alpha}(\tau_i, s_i),\ s_{i+1} = s_i + f^{\beta}(\tau_i, s_i), \qquad i = 1, 2, \dots .$$

From (4.1), one has

$$f(\tau_{i+1}, s_{i+1}) \le \delta f(\tau_i, s_i).$$

After iteration, we obtain

$$f(\tau_{j+1}, s_{j+1}) \le \delta^{j} f(\tau_0, s_0) \qquad \text{for each } j \in \mathbb{N}.$$

Now,

$$\begin{aligned} \tau_{j+1} &= \tau_j + f^{\alpha}(\tau_j, s_j) = \tau_{j-1} + f^{\alpha}(\tau_{j-1}, s_{j-1}) = f^{\alpha}(\tau_j, s_j) \\ &= \cdots = \tau_0 + \sum_{i=0}^{j} f^{\alpha}(\tau_i, s_i) \le \tau_0 + f^{\alpha}(\tau_0, s_0) \cdot \sum_{i=0}^{j} \delta^{i\alpha} \\ &\le \tau_0 + f^{\alpha}(\tau_0, s_0) \frac{1}{1 - \delta^{\alpha}}. \end{aligned}$$

In an analogous way,

$$s_{j+1} \le s_0 + f^{\beta}(\tau_0, s_0) \frac{1}{1 - \delta^{\beta}}.$$

Because $\lim_{j \to \infty} f(\tau_j, s_j) = 0$ and our sequences are uniformly bounded, the lemma is proved. ∎

LEMMA 2. *Let $f(s)$, $g(s)$ be monotone non-negative nonincreasing functions, satisfying*

$$f(s+f(s)) \leq \delta f(s) + g(s) \tag{4.2}$$

for all $s > s_0 > 0$; $0 < \delta < 1$.

Then, for any number $K > 1/(1-\delta)$ there exists a sequence $s_i \to \infty$ such that

$$f(s_i) \leq K g(s_i), \qquad i = 1, 2,, \dots, \tag{4.3}$$

and

$$s_{i+1} - s_i < \frac{K}{1-\delta-K^{-1}} g(s_i), \qquad i = 1, 2, \dots . \tag{4.4}$$

Proof. If there is no sequence with (4.3), then

$$f(s) > K g(s), \qquad \text{for all } s > s_0. \tag{4.5}$$

From (4.2) it follows that

$$f(s+f(s)) \leq f(s)\left(\delta + \frac{g(s)}{f(s)}\right) < (\delta + K^{-1}) f(s) := \delta_1 f(s). \tag{4.6}$$

By Lemma 1, $f(s) = 0$ for all $s > s_0 + (1/(1-\delta_1))f(s_0)$, which contradicts (4.5).

If (4.4) is not true, then there exists $i \in \mathbb{N}$ such that

$$f(s) > K g(s), \qquad \forall s \in (s_i, s_{i+1}). \tag{4.7}$$

Moreover $f(s_i) = K g(s_i)$, $f(s_{i+1}) = K g(s_{i+1})$, and

$$s_{i+1} - s_i > \frac{K}{1-\delta-K^{-1}} g(s_i). \tag{4.8}$$

In this case $(s_i, s_i + Kg(s_i)/(1-\delta-K^{-1})) \subset (s_i, s_{i+1})$; thus, on the interval $(s_i, s_i + Kg(s_i)/(1-\delta-K^{-1})) \equiv (s_i, s_i + f(s_i)/(1-\delta-K^{-1}))$ inequality (4.6) is satisfied. By Lemma 1, $f(s_i + f(s_i)/(1-\delta-K^{-1})) = f(s_i + Kg(s_i)/(1-\delta-K^{-1})) = 0$ in contradiction with (4.7) (see (4.8)). Lemma 2 is proved. ∎

LEMMA 3. *From* $\{s_i\}$ *in* (2.29) *one can choose a subsequence* $\{\tilde{s}_i\}$ *for which*

$$\frac{K}{2}\tilde{h}^{\nu_1/(1+p)}(\tilde{s}_i) \le \Delta\tilde{s}_i := \tilde{s}_{i+1} - \tilde{s}_i < \frac{3}{2}K\tilde{h}^{\nu_1/(1+p)}(\tilde{s}_i).$$

Proof. Let

$$\frac{K}{2}\tilde{h}^{\nu_1/(p+1)} := g(s).$$

Arguing by contradiction, let $i < \infty$ such that

$$s_{i+1} - s_i < g(s_i).$$

We have, using the monotonicity of $\tilde{h}(s)$,

$$s_{i+2} - s_i = s_{i+2} - s_{i+1} + s_{i+1} - s_i \le 2g(s_{i+1}) + g(s_i) \le 3g(s_i).$$

If now $s_{i+2} - s_i > g(s_i)$, then we take $\tilde{s}_{i+1} = s_{i+2}$.

If not, then $s_{i+3} - s_i = s_{i+3} - s_{i+2} + s_{i+2} - s_i \le 2g(s_{i+2}) + g(s_i) \le 3g(s_i)$ and we can take $\tilde{s}_i = s_{i+3}$.

If $s_{i+3} - s_i < g(s_i)$, we pass to s_{i+1}. Because $s_k \to \infty$ when $k \to \infty$, there exists $j < \infty$ such that $g(s_i) < s_{i+j} - s_i \le 3g(s_i)$. This s_{i+j} we take for $\tilde{s}_{i+1}$ in the subsequence $\{\tilde{s}_j\}$. ∎

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