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On wavelets related to the Walsh series

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Abstract

For any integers $p, n \geq 2$ necessary and sufficient conditions are given for scaling filters with p^n many terms to generate a p-multiresolution analysis in $L^2(\mathbb{R}_+)$. A method for constructing orthogonal compactly supported p-wavelets on $\mathbb{R}_+$ is described. Also, an adaptive p-wavelet approximation in $L^2(\mathbb{R}_+)$ is considered.

Keywords: Walsh–Fourier transform; Lacunary Walsh series; Orthogonal p-wavelets; Multifractals; Stability; Adapted wavelet analysis

1. Introduction

In the wavelet literature, there is some interest in the study of compactly supported orthonormal scaling functions and wavelets with an arbitrary dilation factor $p \in \mathbb{N}$, $p \geq 2$ (see, e.g., [3, Section 10.2], [21, Section 2.4], [4, and references therein]). Such wavelets can have very small support and multifractal structure, features which may be important in signal processing and numerical applications. In this paper we study compactly supported orthogonal p-wavelets related to the generalized Walsh functions $\{w_l\}$. There are two ways of considering these functions; either they may be defined on the positive half-line $\mathbb{R}_+ = [0, \infty)$, or, following Vilenkin [24], they may be identified with the characters of the locally compact Abelian group G_p which is a weak direct product of a countable set of the cyclic groups of order p. The classical Walsh functions correspond to the case $p = 2$, while the group G_2 is isomorphic to the Cantor

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dyadic group $\mathcal{C}$ (see [22,9]). Orthogonal compactly supported wavelets on the group $\mathcal{C}$ (and relevant wavelets on $\mathbb{R}_+$) are studied in [15–17,8]. Decimation by an integer different from 2 is discussed in [5,6], but construction for a general p is not completely treated. Here we review some of the elements of that construction on $\mathbb{R}_+$ and give an approach to the $p > 2$ case in a concrete fashion. An essential new element is the matrix extension in Section 4. Finally, in Section 5, we describe an adaptive p-wavelet approximation in $L^2(\mathbb{R}_+)$.

Let us consider the half-line $\mathbb{R}_+$ with the p-adic operations $\oplus$ and $\ominus$ (see Section 2 for the definitions). We say that a compactly supported function $\varphi \in L^2(\mathbb{R}_+)$ is a *p-refinable function* if it satisfies an equation of the type

$$\varphi(x) = p \sum_{\alpha=0}^{p^n-1} a_\alpha \varphi(px \ominus \alpha) \tag{1.1}$$

with complex coefficients a_α. Further, the generalized Walsh polynomial

$$m(\omega) = \sum_{\alpha=0}^{p^n-1} a_\alpha \overline{w_\alpha(\omega)} \tag{1.2}$$

is called the *mask* of Eq. (1.1) (or its solution φ).

An interval $I \subset \mathbb{R}_+$ is a *p-adic interval of range n* if $I = I_s^{(n)} = [sp^{-n}, (s+1)p^{-n})$ for some $s \in \mathbb{Z}_+$. Since w_α is constant on $I_s^{(n)}$ whenever $0 \le \alpha, s < p^n$, it is clear that the mask m is a p-adic step function. If $b_s = m(sp^{-n})$ are the values of m on p-adic intervals, i.e.,

$$b_s = \sum_{\alpha=0}^{p^n-1} a_\alpha \overline{w_\alpha(sp^{-n})}, \quad 0 \le s \le p^n - 1, \tag{1.3}$$

then

$$a_\alpha = \frac{1}{p^n} \sum_{s=0}^{p^n-1} b_s w_\alpha(s/p^n), \quad 0 \le \alpha \le p^n - 1, \tag{1.4}$$

and, conversely, equalities (1.3) follow from (1.4). These discrete transforms can be realized by the fast Vilenkin–Chrestenson algorithm (see, for instance, [22, p.463], [19]). Thus, an arbitrary choice of the values of the mask on p-adic intervals defines also the coefficients of Eq. (1.1).

It was claimed in [6] that if a p-refinable function φ satisfies the condition $\widehat{\varphi}(0) = 1$ and the system $\{\varphi(\cdot \ominus k) \mid k \in \mathbb{Z}_+\}$ is orthonormal in $L^2(\mathbb{R}_+)$, then

$$m(0) = 1 \quad \text{and} \quad \sum_{l=0}^{p-1} |m(\omega + l/p)|^2 = 1 \quad \text{for all } \omega \in [0, 1/p).$$

From this it follows that the equalities

$$b_0 = 1, \quad |b_j|^2 + |b_{j+p^{n-1}}|^2 + \cdots + |b_{j+(p-1)p^{n-1}}|^2 = 1, \quad 0 \le j \le p^{n-1} - 1, \tag{1.5}$$

are necessary (but not sufficient, see Example 4) for the system $\{\varphi(\cdot \ominus k) | k \in \mathbb{Z}_+\}$ to be orthonormal in $L^2(\mathbb{R}_+)$.

Denote by $\mathbf{1}_E$ the characteristic function of a subset E of $\mathbb{R}_+$.

Example 1. If $a_0 = \cdots = a_{p-1} = 1/p$ and $a_\alpha = 0$ for all $\alpha \geq p$, then a solution of Eq. (1.1) is $\varphi = \mathbf{1}_{[0,\, p^{n-1})}$. Therefore the Haar function $\varphi = \mathbf{1}_{[0,1)}$ satisfies this equation for $n = 1$ (compare with [5, Remark 1.3] and [1, Section 5.1]).

Example 2. If we take $p = n = 2$ and put

$$b_0 = 1, b_1 = a, b_2 = 0, b_3 = b,$$

where $|a|^2 + |b|^2 = 1$, then by (1.4) we have

$$a_0 = (1 + a + b)/4, \qquad a_1 = (1 + a - b)/4,$$
$$a_2 = (1 - a - b)/4, \qquad a_3 = (1 - a + b)/4.$$

In particular, for $a = 1$ and $a = -1$ the Haar function: $\varphi(x) = \mathbf{1}_{[0,1)}(x)$ and the displaced Haar function: $\varphi(x) = \mathbf{1}_{[0,1)}(x \ominus 1)$ are obtained. If $0 < |a| < 1$, then

$$\varphi(x) = (1/2)\mathbf{1}_{[0,1)}(x/2) \left(1 + a \sum_{j=0}^{\infty} b^j w_{2^{j+1}-1}(x/2) \right)$$

and

$$\varphi(x) = \begin{cases} (1 + a - b)/2 + b\varphi(2x), & 0 \leq x < 1, \\ (1 - a + b)/2 - b\varphi(2x - 2), & 1 \leq x \leq 2 \end{cases}$$

(see [15,17]). Moreover, it was proved in [16] that, if $|b| < 1/2$, then the corresponding wavelet system $\{\psi_{jk}\}$ is an unconditional basis in all spaces $L^q(\mathbb{R}_+)$, $1 < q < \infty$. When $a = 0$ the system $\{\varphi(\cdot \ominus k) | k \in \mathbb{Z}_+\}$ is linear dependence (since $\varphi(x) = (1/2)\mathbf{1}_{[0,1)}(x/2)$ and $\varphi(x \ominus 1) = \varphi(x)$).

We recall that a collection of closed subspaces $V_j \subset L^2(\mathbb{R}_+)$, $j \in \mathbb{Z}$, is called a *p-multiresolution analysis* (*p-MRA*) in $L^2(\mathbb{R}_+)$ if the following hold:

(i) $V_j \subset V_{j+1}$ for all $j \in \mathbb{Z}$;
(ii) $\overline{\bigcup V_j} = L^2(\mathbb{R}_+)$ and $\bigcap V_j = \{0\}$;
(iii) $f(\cdot) \in V_j \Longleftrightarrow f(p\,\cdot) \in V_{j+1}$ for all $j \in \mathbb{Z}$;
(iv) $f(\cdot) \in V_0 \Longrightarrow f(\cdot \ominus k) \in V_0$ for all $k \in \mathbb{Z}_+$;
(v) there is a function $\varphi \in L^2(\mathbb{R}_+)$ such that the system $\{\varphi(\cdot \ominus k) | k \in \mathbb{Z}_+\}$ is an orthonormal basis of V_0.

The function φ in condition (v) is called a *scaling function* in $L^2(\mathbb{R}_+)$.

For any $\varphi \in L^2(\mathbb{R}_+)$, we set

$$\varphi_{j,k}(x) = p^{j/2}\varphi(p^j x \ominus k), \quad j \in \mathbb{Z}, k \in \mathbb{Z}_+.$$

We say that φ *generates a p-MRA in* $L^2(\mathbb{R}_+)$ if the system $\{\varphi(\cdot \ominus k) | k \in \mathbb{Z}_+\}$ is orthonormal in $L^2(\mathbb{R}_+)$ and, in addition, the family of subspaces

$$V_j = \mathrm{clos}_{L^2(\mathbb{R}_+)} \mathrm{span}\, \{\varphi_{j,k} |\, k \in \mathbb{Z}_+\}, \quad j \in \mathbb{Z}, \tag{1.6}$$

is a p-MRA in $L^2(\mathbb{R}_+)$. Any p-refinable function φ which generates a p-MRA in $L^2(\mathbb{R}_+)$ can be written as a sum of lacunary series by the generalized Walsh functions (see [5,6]).

The results of this paper are concerned mainly with the following two problems:

1. Find necessary and sufficient conditions in order that a p-refinable function φ generates a p-MRA in $L^2(\mathbb{R}_+)$.
2. Describe a method for constructing orthogonal compactly supported p-wavelets on $\mathbb{R}_+$.

Note that similar problems can be considered in framework of the biorthogonal p-wavelet theory (see [7] for the $p = 2$ case).

If a function φ generates a p-MRA, then it is a scaling function in $L^2(\mathbb{R}_+)$. In this case, the system $\{\varphi_{j,k} | \, k \in \mathbb{Z}_+\}$ is an orthonormal basis of V_j for each $j \in \mathbb{Z}$, and moreover, one can define *orthogonal p-wavelets* $\psi_1, \ldots, \psi_{p-1}$ in such a way that the functions

$$\psi_{l,j,k}(x) = p^{j/2}\psi_l(p^j x \ominus k), \quad 1 \le l \le p-1, \, j \in \mathbb{Z}, k \in \mathbb{Z}_+,$$

form an orthonormal basis of $L^2(\mathbb{R}_+)$. If $p = 2$, only one wavelet ψ is obtained and the system $\{2^{j/2}\psi(2^j \cdot \ominus k) | \, j \in \mathbb{Z}, k \in \mathbb{Z}_+\}$ is an orthonormal basis of $L^2(\mathbb{R}_+)$. In Section 4 we give a practical method to design orthogonal p-wavelets $\psi_1, \ldots, \psi_{p-1}$, which is based on an algorithm for matrix extension and on the following

Theorem. *Suppose that equation (1.1) possesses a compactly supported L^2-solution φ such that its mask m satisfies conditions (1.5) and $\widehat{\varphi}(0) = 1$. Then the following are equivalent:*

(a) *φ generates a p-MRA in $L^2(\mathbb{R}_+)$;*
(b) *m satisfies modified Cohen's condition;*
(c) *m has no blocked sets.*

We review some notation and terminology. Let $M \subset [0, 1)$ and let

$$T_p M = \bigcup_{l=0}^{p-1} \{l/p + \omega/p | \omega \in M\}.$$

The set M is said to be *blocked* (for the mask m) if it is a union of p-adic intervals of range $n-1$, does not contain the interval $[0, p^{-n+1})$, and satisfies the condition

$$T_p M \setminus M \subset \operatorname{Null} m,$$

where $\operatorname{Null} m := \{\omega \in [0, 1) | m(\omega) = 0\}$. It is clear that each mask can have only a finite number of blocked sets. In Section 3 we shall prove that if φ is a p-refinable function in $L^2(\mathbb{R}_+)$ such that $\widehat{\varphi}(0) = 1$, then the system $\{\varphi(\cdot \ominus k) | k \in \mathbb{Z}_+\}$ is linearly dependent if and only if its mask possesses a blocked set. The notion of blocked set (in the case $p = 2$) was introduced in the recent paper [8].

The family $\{[0, p^{-j}) | \, j \in \mathbb{Z}\}$ forms a fundamental system of the p-adic topology on $\mathbb{R}_+$. A subset E of $\mathbb{R}_+$ that is compact in the p-adic topology is said to be W-compact. It is easy to see that the union of a finite family of p-adic intervals is W-compact.

A W-compact set E is said to be *congruent to* $[0, 1)$ *modulo* $\mathbb{R}_+$ if its Lebesgue measure is 1 and, for each $x \in [0, 1)$, there is an element $k \in \mathbb{Z}_+$ such that $x \oplus k \in E$. As before, let m be the mask of refinable equation (1.1). We say that m satisfies the *modified Cohen condition* if there exists a W-compact subset E of $\mathbb{R}_+$ congruent to $[0, 1)$ modulo $\mathbb{Z}_+$ and containing a neighbourhood of zero such that

$$\inf_{j \in \mathbf{N}} \inf_{\omega \in E} |m(p^{-j}\omega)| > 0 \tag{1.7}$$

(cf. [3, Section 6.3], [16, Sect. 2]). Since E is W-compact, it is evident that if $m(0) = 1$ then there exists a number j_0 such that $m(p^{-j}\omega) = 1$ for all $j > j_0$, $\omega \in E$. Therefore (1.7) holds if m does not vanish on the sets $E/p, \ldots, E/p^{-j_0}$. Moreover, one can choose $j_0 \le p^n$ because m is 1-periodic and completely defined by the values (1.3).

Now we illustrate the theorem with the following two examples.

Example 3. Let $p = 3$, $n = 2$ and

$$b_0 = 1, b_1 = a, b_2 = \alpha, b_3 = 0, b_4 = b, b_5 = \beta, b_6 = 0, b_7 = c, b_8 = \gamma,$$

where

$$|a|^2 + |b|^2 + |c|^2 = |\alpha|^2 + |\beta|^2 + |\gamma|^2 = 1.$$

Then (1.4) implies precisely that

$$a_0 = \frac{1}{9}(1 + a + b + c + \alpha + \beta + \gamma),$$
$$a_1 = \frac{1}{9}(1 + a + \alpha + (b + \beta)\varepsilon_3^2 + (c + \gamma)\varepsilon_3),$$
$$a_2 = \frac{1}{9}(1 + a + \alpha + (b + \beta)\varepsilon_3 + (c + \gamma)\varepsilon_3^2),$$
$$a_3 = \frac{1}{9}(1 + (a + b + c)\varepsilon_3^2 + (\alpha + \beta + \gamma)\varepsilon_3),$$
$$a_4 = \frac{1}{9}(1 + c + \beta + (a + \gamma)\varepsilon_3^2 + (b + \alpha)\varepsilon_3),$$
$$a_5 = \frac{1}{9}(1 + b + \gamma + (a + \beta)\varepsilon_3^2 + (c + \alpha)\varepsilon_3),$$
$$a_6 = \frac{1}{9}(1 + (a + b + c)\varepsilon_3 + (\alpha + \beta + \gamma)\varepsilon_3^2),$$
$$a_7 = \frac{1}{9}(1 + b + \gamma + (a + \beta)\varepsilon_3 + (c + \alpha)\varepsilon_3^2),$$
$$a_8 = \frac{1}{9}(1 + c + \beta + (a + \gamma)\varepsilon_3 + (b + \alpha)\varepsilon_3^2),$$

where $\varepsilon_3 = \exp(2\pi \mathrm{i}/3)$. Further, if

$$\gamma(1, 0) = a, \gamma(2, 0) = \alpha, \gamma(1, 1) = b, \gamma(2, 1) = \beta, \gamma(1, 2) = c, \gamma(2, 2) = \gamma$$

and $\nu_j \in \{1, 2\}$, then we let

$$c_l = \gamma(\nu_0, 0) \quad \text{for } l = \nu_0;$$
$$c_l = \gamma(\nu_1, 0)\gamma(\nu_0, \nu_1) \quad \text{for } l = \nu_0 + 3\nu_1;$$
$$\ldots$$
$$c_l = \gamma(\nu_k, 0)\gamma(\nu_{k-1}, \nu_k) \ldots \gamma(\nu_0, \nu_1) \quad \text{for } l = \sum_{j=0}^{k} \nu_j 3^j, k \ge 2.$$

The solution of Eq. (1.1) can be decomposed (see [6]) as follows:

$$\varphi(x) = (1/3)\mathbf{1}_{[0,1)}(x/3) \left(1 + \sum_l c_l w_l(x/3)\right).$$

The blocked sets are: (1) $[1/3, 2/3)$ for $a = c = 0$, (2) $[2/3, 1)$ for $\alpha = \beta = 0$, (3) $[1/3, 1)$ for $a = \alpha = 0$. Hence, φ generates a MRA in $L^2(\mathbb{R}_+)$ in the following cases: (1) $a \neq 0, \alpha \neq 0$, (2) $a = 0, \alpha \neq 0, c \neq 0$, (3) $\alpha = 0, a \neq 0, \beta \neq 0$.

Example 4. Suppose that for some numbers b_s, $0 \leq s \leq p^n - 1$, equalities (1.5) are true. Using (1.4), we find the mask

$$m(\omega) = \sum_{\alpha=0}^{p^n-1} a_\alpha \overline{w_\alpha(\omega)},$$

which takes the values b_s on the intervals $I_s^{(n)}$, $0 \leq s \leq p^n - 1$. When $b_j \neq 0$ for $1 \leq j \leq p^{n-1} - 1$ Eq. (1.1) has a solution, which generates a p-MRA in $L^2(\mathbb{R}_+)$ (the modified Cohen condition is fulfilled for $E = [0, 1)$). The expansion of this solution in a lacunary series by generalized Walsh functions is contained in [6].

2. Preliminaries

For the integer and the fractional parts of a number x we are using the standard notations, $[x]$ and $\{x\}$, respectively. For any $s \in \mathbb{Z}$ let us denote by $\langle s \rangle_p$ the remainder upon dividing s by p. Then for $x \in \mathbb{R}_+$ we set

$$x_j = \langle [p^j x] \rangle_p, \qquad x_{-j} = \langle [p^{1-j} x] \rangle_p, \quad j \in \mathbb{N}. \tag{2.1}$$

For each $x \in \mathbb{R}_+$, these numbers are the digits of the p-ary expansion

$$x = \sum_{j<0} x_j p^{-j-1} + \sum_{j>0} x_j p^{-j}$$

(for a p-adic rational x we obtain an expansion with finitely many nonzero terms). It is clear that

$$[x] = \sum_{j=1}^{\infty} x_{-j} p^{j-1}, \qquad \{x\} = \sum_{j=1}^{\infty} x_j p^{-j},$$

and there exists $k = k(x)$ in $\mathbb{N}$ such that $x_{-j} = 0$ for all $j > k$.

Consider the p-adic addition defined on $\mathbb{R}_+$ as follows: if $z = x \oplus y$, then

$$z = \sum_{j<0} \langle x_j + y_j \rangle_p p^{-j-1} + \sum_{j>0} \langle x_j + y_j \rangle_p p^{-j}.$$

As usual, the equality $z = x \ominus y$ means that $z \oplus y = x$. According to our notation

$$[x \oplus y] = [x] \oplus [y] \quad \text{and} \quad \{x \oplus y\} = \{x\} \oplus \{y\}.$$

Note that for $p = 2$ we have

$$x \oplus y = \sum_{j<0} |x_j - y_j| 2^{-j-1} + \sum_{j>0} |x_j - y_j| 2^{-j}.$$

Letting $\varepsilon_p = \exp(2\pi \mathrm{i}/p)$, we define a function w_1 on $[0, 1)$ by

$$w_1(x) = \begin{cases} 1, & x \in [0, 1/p), \\ \varepsilon_p^l, & x \in [lp^{-1}, (l+1)p^{-1}), l \in \{1, \ldots, p-1\}, \end{cases}$$

and extend it to $\mathbb{R}_+$ by periodicity: $w_1(x+1) = w_1(x)$ for all $x \in \mathbb{R}_+$. Then *the generalized Walsh system* $\{w_l | l \in \mathbb{Z}_+\}$ is defined by

$$w_0(x) \equiv 1, \qquad w_l(x) = \prod_{j=1}^{k} (w_1(p^{j-1}x))^{l_{-j}}, \quad l \in \mathbb{N}, x \in \mathbb{R}_+,$$

where the l_{-j} are the digits of the p-ary expansion of l:

$$l = \sum_{j=1}^{k} l_{-j} p^{j-1}, \quad l_{-j} \in \{0, 1, \dots, p-1\}, l_{-k} \neq 0, k = k(l).$$

For any $x, y \in \mathbb{R}_+$, let

$$\chi(x, y) = \varepsilon_p^{t(x,y)}, \quad t(x, y) = \sum_{j=1}^{\infty} (x_j y_{-j} + x_{-j} y_j), \tag{2.2}$$

where x_j, y_j are given by (2.1). Note that

$$\chi(x, p^{-s}l) = \chi(p^{-s}x, l) = w_l(p^{-s}x), \quad l, s \in \mathbb{Z}_+, x \in [0, p^s),$$

and

$$\chi(x, z)\chi(y, z) = \chi(x \oplus y, z), \qquad \chi(x, z)\overline{\chi(y, z)} = \chi(x \ominus y, z), \tag{2.3}$$

if $x, y, z \in \mathbb{R}_+$ and $x \oplus y$ is p-adic irrational. Thus, for fixed x and z, equalities (2.3) hold for all $y \in \mathbb{R}_+$ except countably many of them (see [9, Section 1.5]).

It is known also that Lebesgue measure is translation invariant on $\mathbb{R}_+$ with respect to p-adic addition, and so we can write

$$\int_{\mathbb{R}_+} f(x \oplus y)\,\mathrm{d}x = \int_{\mathbb{R}_+} f(x)\,\mathrm{d}x, \quad f \in L^1(\mathbb{R}_+),$$

for all $y \in \mathbb{R}_+$ (see [22, Section 1.3], [9, Section 6.1]).

The *Walsh–Fourier transform* of a function $f \in L^1(\mathbb{R}_+)$ is defined by

$$\widehat{f}(\omega) = \int_{\mathbb{R}_+} f(x)\overline{\chi(x, \omega)}\,\mathrm{d}x,$$

where $\chi(x, \omega)$ is given by (2.2). If $f \in L^2(\mathbb{R}_+)$ and

$$J_a f(\omega) = \int_0^a f(x)\overline{\chi(x, \omega)}\,\mathrm{d}x, \quad a > 0,$$

then $\widehat{f}$ is the limit of $J_a f$ in $L^2(\mathbb{R}_+)$ as $a \to \infty$. We say that a function $f : \mathbb{R}_+ \mapsto \mathbb{C}$ is *W-continuous at a point* $x \in \mathbb{R}_+$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(x \oplus y) - f(x)| < \varepsilon$ for $0 < y < \delta$. For example, each Walsh polynomial is W-continuous (see [22, Section 9.2], [9, Section 2.3]).

Denote by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ the inner product and the norm in $L^2(\mathbb{R}_+)$, respectively.

Proposition 1 (*See [9, Chap. 6]*). *The following properties hold:*

(a) *if* $f \in L^1(\mathbb{R}_+)$, *then* $\widehat{f}$ *is a* W*-continuous function and* $\widehat{f}(\omega) \to 0$ *as* $\omega \to \infty$;

(b) *if both f and $\widehat{f}$ belong to $L^1(\mathbb{R}_+)$ and f is W-continuous, then*

$$f(x) = \int_{\mathbb{R}_+} \widehat{f}(\omega)\chi(x,\omega)\,\mathrm{d}\omega \quad \textit{for all } x \in \mathbb{R}_+;$$

(c) *if $f, g \in L^2(\mathbb{R}_+)$, then $\langle f, g\rangle = \langle \widehat{f}, \widehat{g}\rangle$ (Parseval's relation).*

Let $\mathcal{E}_n(\mathbb{R}_+)$ be the space of p-adic entire functions of order n on $\mathbb{R}_+$, that is, the set of functions which are constant on all p-adic intervals of range n. Then for every $f \in \mathcal{E}_n(\mathbb{R}_+)$ we have

$$f(x) = \sum_{\alpha=0}^{\infty} f(\alpha p^{-n})\mathbf{1}_{[\alpha p^{-n},(\alpha+1)p^{-n})}(x), \qquad x \in \mathbb{R}_+.$$

For example, the mask m of Eq. (1.1) belongs to $\mathcal{E}_n(\mathbb{R}_+)$.

Proposition 2 (*[9, Section 6.2]*). *The following properties hold:*
(a) *if $f \in L^1(\mathbb{R}_+) \cap \mathcal{E}_n(\mathbb{R}_+)$, then* $\operatorname{supp}\widehat{f} \subset [0, p^n]$;
(b) *if $f \in L^1(\mathbb{R}_+)$ and* $\operatorname{supp} f \subset [0, p^n]$, *then $\widehat{f} \in \mathcal{E}_n(\mathbb{R}_+)$.*

Now we prove the following analogue of Theorem 1 in [8]:

Proposition 3. *Let $\varphi \in L^2(\mathbb{R}_+)$ be a compactly supported solution of equation (1.1) such that $\widehat{\varphi}(0) = 1$. Then*

$$\sum_{\alpha=0}^{p^n-1} a_\alpha = 1 \quad \textit{and} \quad \operatorname{supp}\varphi \subset [0, p^{n-1}].$$

This solution is unique, is given by the formula

$$\widehat{\varphi}(\omega) = \prod_{j=1}^{\infty} m(p^{-j}\omega)$$

and possesses the following properties:
(1) *$\widehat{\varphi}(k) = 0$ for all $k \in \mathbb{N}$ (the modified Strang–Fix condition);*
(2) *$\sum_{k\in\mathbb{Z}_+} \varphi(x \oplus k) = 1$ for almost all $x \in \mathbb{R}_+$ (the partition of unity property).*

Proof. Using the Walsh–Fourier transform, we have

$$\widehat{\varphi}(\omega) = m(\omega/p)\widehat{\varphi}(\omega/p). \tag{2.4}$$

Observe that $w_\alpha(0) = \widehat{\varphi}(0) = 1$. Hence, letting $\omega = 0$ in (1.2) and (2.4), we obtain

$$\sum_{\alpha=0}^{p^n-1} a_\alpha = 1.$$

Further, let s be the greatest integer such that

$$\mu\{x \in [s-1, s) | \varphi(x) \neq 0\} > 0,$$

where μ is the Lebesgue measure on $\mathbb{R}_+$. Suppose that $s \geq p^{n-1} + 1$. Choose an arbitrary p-adic irrational $x \in [s-1, s)$. Applying (2.1), we have

$$x = [x] + \{x\} = \sum_{j=1}^{k} x_{-j}p^{j-1} + \sum_{j=1}^{\infty} x_j p^{-j}, \tag{2.5}$$

where $\{x\} > 0$, $x_{-k} \neq 0$, $k = k(x) \geq n$. For any $\alpha \in \{0, 1, \dots, p^n - 1\}$ we set $y^{(\alpha)} = p\,x \ominus \alpha$. Then

$$y^{(\alpha)} = \sum_{j=1}^{k+1} y_{-j}^{(\alpha)} p^{j-1} + \sum_{j=1}^{\infty} y_j^{(\alpha)} p^{-j},$$

where $y_{-k-1}^{(\alpha)} = x_{-k}$ and among the digits $y_1^{(\alpha)}, y_2^{(\alpha)}, \dots$, there is a nonzero one. Therefore,

$$px \ominus \alpha > p^n \quad \text{for a.e. } x \in [s-1, s). \tag{2.6}$$

Now assume that $s \leq p^n$. Then it is easy to see from (2.6) that $\varphi(p\,x \ominus \alpha) = 0$ for a.e. $x \in [s-1, s)$. Therefore by (1.1) we get $\varphi(x) = 0$ for a.e. $x \in [s-1, s)$, contrary to our choice of s. Thus $s \geq p^n + 1$. Hence, if x given by (2.5), then for any $\alpha \in \{0, 1, \dots, p^n - 1\}$ we have

$$px \ominus \alpha > p(s-1) - (p^n - 1) \geq 2(s-1) - (s-2) = s,$$

where the first inequality is strong because $\{x\} > 0$. As above, we conclude that $\varphi(x) = 0$ for a.e. $x \in [s-1, s)$. Consequently, $s \leq p^{n-1}$ and $\operatorname{supp} \varphi \subset [0, p^{n-1}]$.

Let us prove that

$$\widehat{\varphi}(\omega) = \prod_{j=1}^{\infty} m(p^{-j}\omega). \tag{2.7}$$

We note that φ belongs to $L^1(\mathbb{R}_+)$ because it lies in $L^2(\mathbb{R}_+)$ and has a compact support. Since $\operatorname{supp} \varphi \subset [0, p^{n-1}]$, by Proposition 2 we get $\widehat{\varphi} \in \mathcal{E}_{n-1}(\mathbb{R}_+)$. Also, by virtue of $\widehat{\varphi}(0) = 1$, we obtain $\widehat{\varphi}(\omega) = 1$ for all $\omega \in [0, p^{1-n})$. On the other hand, $m(\omega) = 1$ for all $\omega \in [0, p^{1-n})$. Hence, for every positive integer l,

$$\widehat{\varphi}(\omega) = \widehat{\varphi}(p^{-l-n}\omega) \prod_{j=1}^{l+n} m(p^{-j}\omega) = \prod_{j=1}^{\infty} m(p^{-j}\omega), \quad \omega \in [0, p^l).$$

Therefore, (2.7) is valid and a solution φ is unique.

By Proposition 1, for any $k \in \mathbb{N}$ we have

$$\widehat{\varphi}(k) = \widehat{\varphi}(k) \prod_{s=0}^{j-1} m(p^s k) = \widehat{\varphi}(p^j k) \to 0$$

as $j \to \infty$ (since $\varphi \in L^1(\mathbb{R}_+)$ and $m(p^s k) = 1$ because $m(0) = 1$ and m is 1-periodic). It follows that

$$\widehat{\varphi}(k) = 0 \quad \text{for all } k \in \mathbb{N}. \tag{2.8}$$

By the Poisson summation formula we get

$$\sum_{k \in \mathbb{Z}_+} \varphi(x \oplus k) = \sum_{k \in \mathbb{Z}_+} \widehat{\varphi}(k) \chi(x, k).$$

Hence, since $\widehat{\varphi}(0) = 1$, from (2.8) we obtain

$$\sum_{k \in \mathbb{Z}_+} \varphi(x \oplus k) = 1 \quad \text{for a.e. } x \in \mathbb{R}_+. \quad \square$$

The proposition is proved.

A function $f \in L^2(\mathbb{R}_+)$ is said to be *stable* if there exist positive constants A and B such that

$$A \left(\sum_{\alpha=0}^{\infty} |a_\alpha|^2 \right)^{1/2} \le \left\| \sum_{\alpha=0}^{\infty} a_\alpha f(\cdot \ominus \alpha) \right\| \le B \left(\sum_{\alpha=0}^{\infty} |a_\alpha|^2 \right)^{1/2}$$

for each sequence $\{a_\alpha\} \in \ell^2$. In other words, f is stable if functions $f(\cdot \ominus k)$, $k \in \mathbb{Z}_+$, form a Riesz system in $L^2(\mathbb{R}_+)$. We note also, that a function f is stable in $L^2(\mathbb{R}_+)$ with constants A and B if and only if

$$A \le \sum_{k \in \mathbb{Z}_+} |\widehat{f}(\omega \ominus k)|^2 \le B \quad \text{for a.e. } \omega \in \mathbb{R}_+ \tag{2.9}$$

(the proof of this fact is quite similar to that of Theorem 1.1.7 in [21]).

We say that a function $g : \mathbb{R}_+ \to \mathbf{C}$ has a *periodic zero* at a point $\omega \in \mathbb{R}_+$ if $g(\omega \oplus k) = 0$ for all $k \in \mathbb{Z}_+$.

Proposition 4 (*cf.* *[8, Theorem 2]*). *For a compactly supported function $f \in L^2(\mathbb{R}_+)$ the following statements are equivalent:*

(a) *f is stable in $L^2(\mathbb{R}_+)$;*
(b) *$\{f(\cdot \ominus k) | k \in \mathbb{Z}_+\}$ is a linearly independent system in $L^2(\mathbb{R}_+)$;*
(c) *$\widehat{f}$ does not have periodic zeros.*

Proof. The implication (a) $\Rightarrow$ (b) follows from the well-known property of the Riesz systems (see, e.g., [21, Theorem 1.1.2]). Our next claim is that $f \in L^1(\mathbb{R}_+)$, since f has compact support and $f \in L^2(\mathbb{R}_+)$. Let us choose a positive integer n such that $\operatorname{supp} f \subset [0, p^{n-1}]$. Then by Proposition 2 we have $\widehat{f} \in \mathcal{E}_{n-1}(\mathbb{R}_+)$. Besides, if $k > p^{n-1}$, then

$$\mu\{\operatorname{supp} f(\cdot \ominus k) \cap [0, p^{n-1}]\} = 0$$

(as above, μ denotes the Lebesgue measure on $\mathbb{R}_+$). Therefore, the linearly independence of the system $\{f(\cdot \ominus k) | k \in \mathbb{Z}_+\}$ in $L^2(\mathbb{R}_+)$ is equivalent to that for the finite system $\{f(\cdot \ominus k) | k = 0, 1, \dots, p^{n-1} - 1\}$. Further, if some vector $(a_0, \dots, a_{p^{n-1}-1})$ satisfies conditions

$$\sum_{\alpha=0}^{p^{n-1}-1} a_\alpha f(\cdot \ominus \alpha) = 0 \quad \text{and} \quad |a_0| + \cdots + |a_{2^{n-1}-1}| > 0, \tag{2.10}$$

then using the Walsh–Fourier transform we obtain

$$\widehat{f}(\omega) \sum_{\alpha=0}^{p^{n-1}-1} a_\alpha \overline{w_\alpha(\omega)} = 0 \quad \text{for a.e. } \omega \in \mathbb{R}_+.$$

The Walsh polynomial

$$w(\omega) = \sum_{\alpha=0}^{p^{n-1}-1} a_\alpha \overline{w_\alpha(\omega)}$$

is not identically equal to zero; hence among $I_s^{(n-1)}$, $0 \le s \le p^{n-1} - 1$, there exists an interval (denote it by I) for which $w(I \oplus k) \ne 0$, $k \in \mathbb{Z}_+$. Since $\widehat{f} \in \mathcal{E}_{n-1}(\mathbb{R}_+)$, it follows that (2.10) holds if and only if there exists a p-adic interval I of range $n - 1$, such that $\widehat{f}(I \oplus k) = 0$ for all $k \in \mathbb{Z}_+$. Thus, (b)$\Leftrightarrow$ (c). It remains to prove that (c) $\Rightarrow$ (a). Suppose that $\widehat{f}$ does not have periodic zeros. Then

$$F(\omega) := \sum_{k\in\mathbb{Z}_+} |\widehat{f}(\omega \ominus k)|^2, \quad \omega \in \mathbb{R}_+,$$

is positive and 1-periodic function. Moreover, since $\widehat{f} \in \mathcal{E}_{n-1}(\mathbb{R}_+)$, we see that F is constant on each $I_s^{(n-1)}$, $0 \le s \le p^{n-1} - 1$. Hence (2.9) is satisfied and so Proposition 4 is established. □

The following two propositions are proved in [6]:

Proposition 5. *Let $\varphi \in L^2(\mathbb{R}_+)$. Then the system $\{\varphi(\cdot \ominus k) \mid k \in \mathbb{Z}_+\}$ is orthonormal in $L^2(\mathbb{R}_+)$ if and only if*

$$\sum_{k\in\mathbb{Z}_+} |\widehat{\varphi}(\omega \ominus k)|^2 = 1 \quad \textit{for a.e. } \omega \in \mathbb{R}_+.$$

Proposition 6. *Let $\{V_j\}$ be the family of subspaces defined by (1.6) with given $\varphi \in L^2(\mathbb{R}_+)$. If $\{\varphi(\cdot \ominus k) \mid k \in \mathbb{Z}_+\}$ is an orthonormal basis in V_0, then $\bigcap V_j = \{0\}$.*

We shall use also the following

Proposition 7. *Let*

$$m(\omega) = \sum_{\alpha=0}^{p^n-1} a_\alpha \overline{w_\alpha(\omega)}$$

be a polynomial such that

$$m(0) = 1 \quad \textit{and} \quad \sum_{l=0}^{p-1} |m(\omega \oplus l/p)|^2 = 1 \quad \textit{for all } \omega \in \mathbb{R}_+.$$

Suppose φ is a function defined by the Walsh–Fourier transform

$$\widehat{\varphi}(\omega) = \prod_{j=1}^{\infty} m(p^{-j}\omega).$$

Then the system $\{\varphi(\cdot \ominus k) \mid k \in \mathbb{Z}_+\}$ is orthonormal in $L^2(\mathbb{R}_+)$ if and only if m satisfies the modified Cohen condition.

The proof of this proposition is similar to that of Theorem 6.3.1 in [3] (cf. [15, Theorem 2.1], [5, Proposition 3.3]).

3. Proof of the theorem

The next lemma gives a relation between stability and blocked sets.

Lemma 1. *Let φ be a p-refinable function in $L^2(\mathbb{R}_+)$ such that $\widehat{\varphi}(0) = 1$. Then φ is not stable if and only if its mask m has a blocked set.*

Proof. Using Propositions 2 and 3, we have $\operatorname{supp}\varphi \subset [0, p^{n-1})$ and $\widehat{\varphi} \in \mathcal{E}_{n-1}(\mathbb{R}_+)$. Suppose that the function φ is not stable. As noted in the proof of Proposition 4, then there exists an interval $I = I_s^{(n-1)}$ consisting entirely of periodic zeros of the Walsh–Fourier transform $\widehat{\varphi}$ (and each periodic zero $\omega \in [0, 1)$ of $\widehat{\varphi}$ lies in some such I). Thus, the set

$$M_0 = \{\omega \in [0, 1) | \widehat{\varphi}(\omega + k) = 0 \quad \text{for all } k \in \mathbb{Z}_+\}$$

is a union of some intervals $I_s^{(n-1)}$, $0 \le s \le p^{n-1} - 1$. Since $\widehat{\varphi}(0) = 1$, it follows that M_0 does not contain $I_0^{(n-1)}$. Furthermore, if $\omega \in M_0$, then by (2.4)

$$m(\omega/p + k/p)\widehat{\varphi}(\omega/p + k/p) = 0 \quad \text{for all } k \in \mathbb{Z}_+$$

and hence $\omega/p + l/p \in M_0 \cup \operatorname{Null} m$ for $l = 0, 1, \dots, p - 1$. Thus, if φ is not stable, then M_0 is a blocked set for m.

Conversely, let m possess a blocked set M. Then we will show that each element of M is a periodic zero for $\widehat{\varphi}$ (and by Proposition 4 φ is not stable). Assume that there exist $\omega \in M$ and $k \in \mathbb{Z}_+$ such that $\widehat{\varphi}(\omega + k) \ne 0$. Choose a positive integer j for which $p^{-j}(\omega + k) \in [0, p^{1-n})$ and, for every $r \in \{0, 1, \dots, j\}$, set

$$u_r = [p^{-r}(\omega + k)], \qquad v_r = \{p^{-r}(\omega + k)\}.$$

Further, let $u_r/p = l_r/p + s_r$, where $l_r \in \{0, 1, \dots, p - 1\}$ and $s_r \in \mathbb{Z}_+$. It is clear that for all $r \in \{0, 1, \dots, j - 1\}$

$$u_{r+1} + v_{r+1} = (p^{-1}v_r + p^{-1}l_r) + s_r$$

and hence $v_{r+1} = p^{-1}(v_r + l_r)$. From this it follows that if $v_r \in M$, then $v_{r+1} \in T_p M$. Besides, from the equalities

$$\widehat{\varphi}(\omega + k) = \widehat{\varphi}(p^{-j}(\omega + k)) \prod_{r=1}^{j} m(p^{-r}(\omega + k)) = \widehat{\varphi}(v_j) \prod_{r=1}^{j} m(v_r)$$

we see that all $v_r \notin \operatorname{Null} m$. Thus, if $v_r \in M$, then $v_{r+1} \in M$. Since $v_0 = \omega \in M$, we conclude that $v_j \in M$. But this is impossible because $v_j = p^{-j}(\omega + k) \in [0, p^{1-n})$ and $M \cap [0, p^{1-n}) = \emptyset$. This contradiction completes the proof of Lemma 1. □

Corollary. *If φ is a p-refinable function in $L^2(\mathbb{R}_+)$ such that $\widehat{\varphi}(0) = 1$, then the system $\{\varphi(\cdot \ominus k) | k \in \mathbb{Z}_+\}$ is linearly dependent if and only if the mask of φ possesses a blocked set.*

Lemma 2. *Suppose that the mask of refinable equation (1.1) satisfies*

$$m(0) = 1 \quad \text{and} \quad \sum_{l=0}^{p-1} |m(\omega \oplus l/p)|^2 = 1 \quad \text{for all } \omega \in \mathbb{R}_+. \tag{3.1}$$

Then the function φ given by

$$\widehat{\varphi}(\omega) = \prod_{j=1}^{\infty} m(p^{-j}\omega) \tag{3.2}$$

is a solution of Eq. (1) *and* $\|\varphi\| \le 1$.

Proof. The pointwise convergence of product in (3.2) follows from the fact that m is equal to 1 on $[0, p^{1-n})$ (and for any $\omega \in \mathbb{R}_+$ only finitely many of the factors in (3.2) cannot be equal to 1). Denote by $g(\omega)$ the right part of (3.2). From (3.1) we see that $|m(\omega)| \le 1$ for all $\omega \in \mathbb{R}_+$.

Therefore, for any $s \in \mathbb{N}$ we have

$$|g(\omega)|^2 \le \prod_{j=1}^{s} |m(p^{-j}\omega)|^2$$

and hence

$$\int_0^{p^l} |g(\omega)|^2 \, \mathrm{d}\omega \le \int_0^{p^l} \prod_{j=1}^{s} |m(B^{-j}\omega)|^2 \, \mathrm{d}\omega = 2^s \int_0^1 \prod_{j=0}^{s-1} |m(B^j\omega)|^2 \, \mathrm{d}\omega. \tag{3.3}$$

Further, from the equalities

$$m(\omega) = \sum_{\alpha=0}^{p^n-1} a_\alpha \overline{w_\alpha(\omega)}, \quad w_\alpha(\omega)\overline{w_\beta(\omega)} = w_{\alpha\ominus\beta}(\omega),$$

it follows that

$$|m(\omega)|^2 = \sum_{\alpha=0}^{p^n-1} c_\alpha w_\alpha(\omega), \tag{3.4}$$

where the coefficients c_α may be expressed via a_α. Now, we substitute (3.4) into the second equality of (3.1) and observe that if α is multiply to p, then

$$\sum_{l=0}^{p-1} w_\alpha(l/p) = p,$$

and this sum is equal to 0 for the rest α. As a result, we obtain $c_0 = 1/p$ and $c_\alpha = 0$ for nonzero α, which are multiply to p. Hence,

$$|m(\omega)|^2 = \frac{1}{p} + \sum_{\alpha=0}^{p^{n-1}-1} \sum_{l=1}^{p-1} c_{p\alpha+l} w_{p\alpha+l}(\omega).$$

This gives

$$\prod_{j=0}^{s-1} |m(p^j\omega)|^2 = p^{-s} + \sum_{\gamma=1}^{\sigma(s)} b_\gamma w_\gamma(\omega), \quad \sigma(s) \le sp^{n-1}(p-1),$$

where each coefficient b_γ equals to the product of some coefficients $c_{p\alpha+l}$, $l = 1, \ldots, p-1$. Taking into account that

$$\int_0^1 w_\gamma(\omega) \, \mathrm{d}\omega = 0, \quad \gamma \in \mathbb{N},$$

we have

$$\int_0^1 \prod_{j=0}^{s-1} |m(p^j\omega)|^2 \, \mathrm{d}\omega = p^{-s}.$$

Substituting this into (3.3), we deduce

$$\int_0^{p^l} |g(\omega)|^2 \, \mathrm{d}\omega \le 1, \quad l \in \mathbb{N},$$

which is due to the inequality

$$\int_{\mathbb{R}_+} |g(\omega)|^2 \, \mathrm{d}\omega \le 1. \tag{3.5}$$

Now, let $\varphi \in L^2(\mathbb{R}_+)$ and $\widehat{\varphi} = g$. Then from (3.2) it follows that

$$\widehat{\varphi}(\omega) = m(p^{-1}\omega)\widehat{\varphi}(p^{-1}\omega),$$

and hence φ satisfies (1.1). Moreover, from (3.5), by Proposition 1, we get $\|\varphi\| \le 1$. □

Lemma 3. *Let φ be a p-refinable function with a mask m and let $\widehat{\varphi}(0) = 1$. Then the system $\{\varphi(\cdot \ominus k) \mid k \in \mathbb{Z}_+\}$ is orthonormal in $L^2(\mathbb{R}_+)$ if and only if the mask m has no blocked sets and satisfies*

$$\sum_{l=0}^{p-1} |m(\omega \oplus l/p)|^2 = 1 \quad \textit{for all } \omega \in \mathbb{R}_+. \tag{3.6}$$

Proof. If the system $\{\varphi(\cdot \ominus k) \mid k \in \mathbb{Z}_+\}$ is orthonormal in $L^2(\mathbb{R}_+)$, then (3.6) holds (see [6]) and a lack of blocked sets follows from Lemma 1 and Proposition 4. Conversely, suppose that m has no blocked sets and (3.6) is fulfilled. Then we set

$$\Phi(\omega) := \sum_{k \in \mathbb{Z}_+} |\widehat{\varphi}(\omega \ominus k)|^2. \tag{3.7}$$

Obviously, Φ is nonnegative and 1-periodic function. According to Proposition 5, it suffices to show that $\Phi(\omega) \equiv 1$. Let

$$a = \inf\{\Phi(\omega) | \omega \in [0, 1)\}.$$

From Propositions 2 and 3 it follows that Φ is constant on each $I_s^{(n-1)}$, $0 \le s \le p^{n-1} - 1$. Moreover, if Φ vanishes on one of these intervals, then $\widehat{\varphi}$ has a periodic zero, and hence φ is unstable. On account of Proposition 4 and Lemma 1, this assertion contradicts a lack of blocked sets for m. Hence, a is positive. Also, by the modified Strang–Fix condition (see Proposition 3), we have $\Phi(0) = 1$. Thus, $0 < a \le 1$.

Further, by (2.4) and (3.7) we obtain

$$\Phi(\omega) = \sum_{l=0}^{p-1} |m(p^{-1}\omega \ominus p^{-1}l)|^2 \Phi(p^{-1}\omega \ominus p^{-1}l). \tag{3.8}$$

Now, let $M_a = \{\Phi(\omega) = a | \omega \in [0, 1)\}$. In the case $0 < a < 1$ from (3.6) and (3.8) we see that for any $\omega \in M_a$ the elements $p^{-1}\omega \ominus p^{-1}l$, $l = 0, 1, \ldots, p - 1$, belong either M_a or $\mathrm{Null}\, m$. Therefore, M_a is a blocked set, which contradicts the assumption. Thus, $\Phi(\omega) \ge 1$ for all $\omega \in [0, 1)$. Hence from the equalities

$$\int_0^1 \Phi(\omega) \, \mathrm{d}\omega = \sum_{k \in \mathbb{Z}_+} \int_k^{k+1} |\widehat{\varphi}(\omega)|^2 \, \mathrm{d}\omega = \int_{\mathbb{R}_+} |\widehat{\varphi}(\omega)|^2 \, \mathrm{d}\omega = \|\varphi\|^2$$

by Lemma 2 we have

$$\int_0^1 \Phi(\omega)\,\mathrm{d}\omega = 1.$$

Once again applying the inequality $\Phi(\omega) \geq 1$ and using the fact that Φ is constant on each $I_s^{(n-1)}$, $0 \leq s \leq p^{n-1} - 1$, we conclude that $\Phi(\omega) \equiv 1$. □

Proof of the theorem. Suppose that m satisfies condition (b) or (c). Then, by Proposition 7 and Lemma 3, the system $\{\varphi(\cdot \ominus k) \mid k \in \mathbb{Z}_+\}$ is orthonormal in $L^2(\mathbb{R}_+)$. Let us define the subspaces V_j, $j \in \mathbb{Z}_+$ by the formula (1.6). By Proposition 6 we have $\bigcap V_j = \{0\}$. The embeddings $V_j \subset V_{j+1}$ follow from the fact that φ satisfies the Eq. (1.1). The equality

$$\overline{\bigcup V_j} = L^2(\mathbb{R}_+)$$

is proved in just the same way as (2.14) in [5] (cf. [3, Section 5.3]). Thus, the implications (b) $\Rightarrow$ (a) and (c) $\Rightarrow$ (a) are true. The inverse implications follow directly from Proposition 7 and Lemma 3. □

4. On matrix extension and p-wavelet construction

Following the standard approach (e.g., [11,18]), we reduce the problem of p-wavelet decomposition to the problem of matrix extension. More precisely, we shall discuss the following *procedure to construct orthogonal p-wavelets in $L^2(\mathbb{R}_+)$*:

1. Choose numbers b_s such that equalities (1.5) are true.
2. Compute a_α by (1.4) and verify that the mask

$$m_0(\omega) = \sum_{\alpha=0}^{p^n-1} a_\alpha \overline{w_\alpha(\omega)}$$

has no blocked sets.

3. Find

$$m_l(\omega) = \sum_{\alpha=0}^{p^n-1} a_\alpha^{(l)} \overline{w_\alpha(\omega)}, \quad 1 \leq l \leq p-1,$$

such that $(m_l(\omega + k/p))_{l,k=0}^{p-1}$ is an unitary matrix.

4. Define $\psi_1, \ldots, \psi_{p-1}$ by the formula

$$\psi_l(x) = p \sum_{\alpha=0}^{p^n-1} a_\alpha^{(l)} \varphi(p\,x \ominus \alpha), \quad 1 \leq l \leq p-1. \tag{4.1}$$

In the $p = 2$ case, one can choose $a_\alpha^{(1)} = (-1)^\alpha a_{\alpha\oplus 1}$ for $0 \leq \alpha \leq 2^n - 1$ (and $a_\alpha^{(1)} = 0$ for the rest α). Then $m_1(\omega) = -w_1(\omega)\overline{m_0(\omega \oplus 1/2)}$, the matrix

$$\begin{pmatrix} m_0(\omega) & m_0(\omega \oplus 1/2) \\ m_1(\omega) & m_1(\omega \oplus 1/2) \end{pmatrix}$$

is unitary and, as in [8], we obtain

$$\psi(x) = 2\sum_{\alpha=0}^{2^n-1}(-1)^\alpha \bar{a}_{\alpha\oplus 1}\varphi(2x\ominus\alpha).$$

In particular, if $n = 1$ and $a_0 = a_1 = 1/2$, then ψ is the classical Haar wavelet.

In the $p > 2$ case, we take the coefficients a_α as in Step 2 (so that b_s satisfy (1.5) and m_0 has no blocked sets). Then

$$\sum_{\alpha=0}^{p^n-1}|a_\alpha|^2 = \frac{1}{p}. \tag{4.2}$$

In fact, Parseval's relation for the discrete transforms (1.3) and (1.4) can be written as

$$\sum_{\alpha=0}^{p^n-1}|a_\alpha|^2 = \frac{1}{p^n}\sum_{\alpha=0}^{p^n-1}|b_\alpha|^2.$$

Therefore (4.2) follows from (1.5). Now we define

$$A_{0k}(z) = \sum_{l=0}^{p^{n-1}-1} a_{k+pl}z^l, \quad 0 \le k \le p-1,$$

and introduce the polynomials $A_{lk}(z)$, $\deg A_{lk} \le p^{n-1}-1$, such that

$$m_l(\omega) = \sum_{k=0}^{p-1}\overline{w_k(\omega)}A_{lk}(\overline{w_p(\omega)}), \quad 1 \le l \le p-1. \tag{4.3}$$

It follows immediately that

$$M(\omega) = A(\overline{w_p(\omega)})W(\omega), \tag{4.4}$$

where $M(\omega) := (m_l(\omega + k/p))_{l,k=0}^{p-1}$, $A(z) := (A_{lk}(z))_{l,k=0}^{p-1}$, and $W(\omega) := (\overline{w_l(\omega + k/p)})_{l,k=0}^{p-1}$. The matrix $p^{-1/2}W(\omega)$ is unitary. Thus, by (4.4), unitarity of $M(\omega)$ is equivalent to that of the matrix $p^{-1/2}A(z)$ with $z = \overline{w_p(\omega)}$. From this we claim that Step 3 of the procedure can be realized by some modification of the algorithm for matrix extension suggested by Lawton, Lee and Shen in [18] (see also [2, Theorem 2.1]).

We illustrate the described procedure by the following examples.

Example 5. Let

$$m_0(\omega) = \frac{1}{p}\sum_{\alpha=0}^{p-1}\overline{w_\alpha(\omega)}$$

so that $a_0 = \cdots = a_{p-1} = 1/p$. Then, as in Example 1, we have $\varphi = \mathbf{1}_{[0,\,p^{n-1})}$. Setting

$$m_l(\omega) = \frac{1}{p}\sum_{\alpha=0}^{p-1}\varepsilon_p^{l\alpha}\overline{w_\alpha(\omega)}, \quad 1 \le l \le p-1,$$

we observe that $(m_l(\omega + k/p))_{l,k=0}^{p-1}$ is unitary for all $\omega \in [0, 1)$. Indeed, the constant matrix $p^{-1}(\varepsilon_p^{lk})_{l,k=0}^{p-1}$ may be taken as $A(z)$ in (4.4). Therefore we obtain from (4.1)

$$\psi_l(x) = \sum_{\alpha=0}^{p-1} \varepsilon_p^{l\alpha} \varphi(p\,x \ominus \alpha), \quad 1 \le l \le p-1.$$

Note that the similar wavelets in the space $L^2(\mathbb{Q}_p)$ were introduced by Kozyrev in [13]; in connection with these wavelets see also [1, p.450] and Example 4.1 in [12].

Example 6. Let $p = 3, n = 2$. As in Example 3, we take $a, b, c, \alpha, \beta, \gamma$ such that

$$|a|^2 + |b|^2 + |c|^2 = |\alpha|^2 + |\beta|^2 + |\gamma|^2 = 1$$

and then define $a_0, a_1, \ldots, a_8$ using (1.4). In this case we have

$$A_{00}(z) = a_0 + a_3 z + a_6 z^2, \quad A_{01}(z) = a_1 + a_4 z + a_7 z^2, \quad A_{02}(z) = a_2 + a_5 z + a_8 z^2.$$

Now, we require

$$a \neq 0, \quad \alpha = \overline{a}, \quad a\overline{\alpha} + b\overline{\beta} + c\overline{\gamma} = \overline{a}. \tag{4.5}$$

In particular, for $0 < a < 1$ we can choose numbers θ, t such that

$$\cos(\theta - t) = \frac{a - a^2}{1 - a^2}$$

and then set $\alpha = a, r = \sqrt{1 - a^2}, \beta = r\cos\theta, \gamma = r\sin\theta, b = r\cos t, c = r\sin t$.

Under our assumptions the mask m_0 has no blocked sets (see Example 3). Moreover, it follows from (4.2) and (4.5) that

$$|A_{00}(z)|^2 + |A_{01}(z)|^2 + |A_{02}(z)|^2 = \frac{1}{3}$$

for all z on the unit circle $\mathbb{T}$. To see this, note that by a direct calculation

$$|A_{00}(z)|^2 + |A_{01}(z)|^2 + |A_{02}(z)|^2 = \sum_{\alpha=0}^{8} |a_\alpha|^2 + 2\mathrm{Re}\,[(a_0\overline{a}_3 + a_1\overline{a}_4 + a_2\overline{a}_5)z]$$
$$+2\mathrm{Re}\,[(a_0\overline{a}_6 + a_1\overline{a}_7 + a_2\overline{a}_8)z^2] + 2\mathrm{Re}\,[(a_3\overline{a}_6 + a_4\overline{a}_7 + a_5\overline{a}_8)z\overline{z}^2,$$

where

$$27(a_0\overline{a}_3 + a_1\overline{a}_4 + a_2\overline{a}_5) = a + \alpha + (\overline{\alpha} + a\overline{\alpha} + b\overline{\beta} + c\overline{\gamma})\varepsilon_3 + (\overline{a} + \overline{a}\alpha + \overline{b}\beta + \overline{c}\gamma)\varepsilon_3^2,$$
$$27(a_0\overline{a}_6 + a_1\overline{a}_7 + a_2\overline{a}_8) = a + \alpha + (\overline{a} + \overline{a}\alpha + \overline{b}\beta + \overline{c}\gamma)\varepsilon_3 + (\overline{\alpha} + a\overline{\alpha} + b\overline{\beta} + c\overline{\gamma})\varepsilon_3^2,$$
$$27(a_3\overline{a}_6 + a_4\overline{a}_7 + a_5\overline{a}_8) = 2\varepsilon_3\mathrm{Re}\,a + 2\varepsilon_3^2\mathrm{Re}\,\alpha + 2\mathrm{Re}\,(a\overline{\alpha} + b\overline{\beta} + c\overline{\gamma}).$$

Further, if

$$\alpha_0 = \sqrt{3}\,(a_0, a_1, a_2), \quad \alpha_1 = \sqrt{3}\,(a_3, a_4, a_5), \quad \alpha_2 = \sqrt{3}\,(a_6, a_7, a_8),$$

then

$$|\alpha_0|^2 + |\alpha_1|^2 + |\alpha_2|^2 = 1, \quad \langle\alpha_0, \alpha_1\rangle = \langle\alpha_0, \alpha_2\rangle = \langle\alpha_1, \alpha_2\rangle = 0,$$

where $\langle\cdot,\cdot\rangle$ is the inner product in $\mathbb{C}^3$. It is clear that

$$\alpha_0+\alpha_1 z+\alpha_2 z^2=\sqrt{3}\,(A_{00}(z),A_{01}(z),A_{02}(z)).$$

Let P_2 be the orthogonal projection onto α_2, i.e.,

$$P_2 w=\frac{\langle w,\alpha_2\rangle}{\langle\alpha_2,\alpha_2\rangle}\alpha_2,\quad w\in\mathbb{C}^3.$$

Then we have

$$\begin{aligned}&(I-P_2+z^{-1}P_2)(\alpha_0+\alpha_1 z+\alpha_2 z^2)\\&\quad=(I-P_2)\alpha_0+P_2\alpha_1+z(P_2\alpha_2+(I-P_2)\alpha_1)=:\beta_0+\beta_1 z.\end{aligned}$$

One now verifies that

$$|\beta_0|^2+|\beta_1|^2=1,\quad\langle\beta_0,\beta_1\rangle=0.$$

Furthermore, if P_1 is the orthogonal projection onto β_1, then

$$(I-P_1+z^{-1}P_1)(\beta_0+\beta_1 z)=(I-P_1)\beta_0+P_1\beta_1=:\gamma_0.$$

By the Gram–Schmidt orthogonalization, we can find an unitary matrix Γ_0 once the first row of this matrix is the unit vector γ_0. Then we set

$$\Gamma_1(z)=(I-P_1+zP_1)\Gamma_0\quad\text{and}\quad\Gamma_2(z)=(I-P_2+zP_2)\Gamma_1(z).$$

The first row of $\Gamma_2(z)$ coincides with $\alpha_0+\alpha_1 z+\alpha_2 z^2$. Putting

$$(A_{lk}(z))_{l,k=0}^2=\frac{1}{\sqrt{3}}\,\Gamma_2(z),$$

we see that m_1 and m_2 can be defined as follows:

$$m_l(\omega)=\sum_{k=0}^{2}\overline{w_k(\omega)}A_{lk}(\overline{w_3(\omega)})=\sum_{\alpha=0}^{8}a_\alpha^{(l)}\overline{w_\alpha(\omega)},\quad l=1,2.$$

Finally, we find

$$\psi_l(x)=3\sum_{\alpha=0}^{8}a_\alpha^{(l)}\varphi(3x\ominus\alpha),\quad l=1,2.$$

Note that for the space $L^2(\mathbb{Q}_p)$ the corresponding wavelets were introduced recently in [12].

5. Adapted p-wavelet approximation

Suppose that a p-refinable function φ generates a p-MRA in $L^2(\mathbb{R}_+)$ and subspaces V_j are given by (1.6). For each $j\in\mathbb{Z}$ denote by P_j the orthogonal projection of $L^2(\mathbb{R}_+)$ onto V_j. Given f in $L^2(\mathbb{R}_+)$ it is naturally to choose parameters b_s in (1.5) such that the approximation method $f\approx P_j f$ will be optimal. If f belongs to some class $\mathcal{M}$ in $L^2(\mathbb{R}_+)$ then it is possible to seek the parameters b_s, which minimize for some fixed j the quantity

$$\sup\{\|f-P_j f\|\mid f\in\mathcal{M}\}$$

and to study the behavior of this quantity as $j \to +\infty$. Also, it is very interesting investigate p-wavelet approximation in the p-adic Hardy spaces (cf. [10,14]).

By analogy with [23] we discuss here another approach to the problem on optimization of the approximation method $f \approx P_j f$. For every $j \in \mathbb{Z}$ denote by W_j the orthogonal complement of V_j in V_{j+1} and let Q_j be the orthogonal projection of $L^2(\mathbb{R}_+)$ to W_j. Since $\{V_j\}$ is a p-MRA, for any $f \in L^2(\mathbb{R}_+)$ we have

$$f = \sum_j Q_j f = P_0 f + \sum_{j \geq 0} Q_j f$$

and

$$\lim_{j \to +\infty} \| f - P_j f \| = 0, \qquad \lim_{j \to -\infty} \| P_j f \| = 0.$$

It is easily seen, that

$$P_j f = Q_{j-1} f + Q_{j-2} f + \cdots + Q_{j-s} f + P_{j-s} f, \quad j \in \mathbb{Z}, s \in \mathbb{N}.$$

The equality $V_j = V_{j-1} \oplus W_{j-1}$ means that W_{j-1} contains the "details" which are necessary to get over the $(j-1)$th level of approximation to the more exact jth level. Since

$$\| P_j f \|^2 = \| P_{j-1} f \|^2 + \| Q_{j-1} f \|^2,$$

it is natural to choose the parameters b_s to maximize $\| P_{j-1} f \|$ (or, equivalently, to minimize $\| Q_{j-1} f \|$). To this end let us write Eq. (1.1) in the form

$$\varphi(x) = \sqrt{p} \sum_{\alpha=0}^{p^n - 1} \tilde{a}_\alpha \varphi(p x \ominus h_\alpha),$$

where $\tilde{a}_\alpha = \sqrt{p}\, a_\alpha$. Putting $\varphi_j(x) = p^{j/2} \varphi(p^j x)$, we have

$$\varphi_{j-1}(x) = \sum_{\alpha=0}^{p^n - 1} \tilde{a}_\alpha \varphi_j(x \ominus p^{-j} \alpha), \tag{5.1}$$

where $\varphi_j(x \ominus p^{-j} k) = \varphi_{j,k}(x)$. Further, given $f \in L^2(\mathbb{R}_+)$ we set

$$f(j, k) := \langle f, \varphi_{j,k} \rangle = \int_{\mathbb{R}} f(x) \overline{\varphi_j(x \ominus p^{-j} k)}\, \mathrm{d}x.$$

Applying (5.1), we obtain

$$\begin{aligned} f(j-1, k) &= \int_{\mathbb{R}_+} f(x) \overline{\varphi_{j-1}(x \ominus p^{-j+1} k)}\, \mathrm{d}x \\ &= \sum_{\alpha=0}^{p^n - 1} \overline{\tilde{a}}_\alpha \int_{\mathbb{R}_+} f(x) \overline{\varphi_j(x \ominus p^{-j}(p k \oplus \alpha))}\, \mathrm{d}x \end{aligned}$$

and hence

$$f(j-1, k) = \sum_{\alpha=0}^{p^n - 1} \overline{\tilde{a}}_\alpha f(j, p k \oplus \alpha). \tag{5.2}$$

Since

$$P_j f = \sum_{k\in\mathbb{Z}_+} f(j,k)\varphi_{j,k},$$

we see from (5.2) that

$$\|P_{j-1}f\|^2 = \sum_{k\in\mathbb{Z}_+} |f(j-1,k)|^2 = \sum_{k\in\mathbb{Z}_+}\left|\sum_{\alpha=0}^{p^n-1}\bar{\tilde{a}}_\alpha f(j,p\,k\oplus\alpha)\right|^2$$
$$= \sum_{k\in\mathbb{Z}_+}\left(\sum_{\alpha,\beta=0}^{p^n-1}\bar{\tilde{a}}_\alpha\tilde{a}_\beta f(j,p\,k\oplus\alpha)\overline{f(j,p\,k\oplus\beta)}\right). \tag{5.3}$$

For $0\le\alpha,\ \beta\le p^n-1$ we let

$$F_{\alpha,\beta}(j) := \sum_{k\in\mathbb{Z}_+} f(j,p\,k\oplus\alpha)\overline{f(j,p\,k\oplus\beta)}.$$

Then $F_{\beta,\alpha}(j)=\overline{F}_{\alpha,\beta}(j)$ and (5.3) implies

$$\|P_{j-1}f\|^2 = \sum_{\alpha,\beta=0}^{p^n-1} F_{\alpha,\beta}(j)\bar{\tilde{a}}_\alpha\tilde{a}_\beta. \tag{5.4}$$

Denote by $\mathcal{U}(p,n)$ the set of vectors $u=(u_0,u_1,\ldots,u_{p^n-1})$ such that

$$u_0=1,\qquad u_j=0\quad\text{for } j\in\{p^{n-1},2p^{n-1},\ldots,(p-1)p^{n-1}\},$$

and

$$\sum_{l=0}^{p-1}|u_{lp^{n-1}+j}|^2=1\quad\text{for } j\in\{1,2,\ldots,p^{n-1}-1\}.$$

For every $u=(u_0,u_1,\ldots,u_{p^n-1})$ in $\mathcal{U}(p,n)$ we define $a_\alpha(u)$ by the formulas

$$a_\alpha(u)=\frac{1}{p^n}\sum_{s=0}^{p^n-1}u_s w_\alpha(s/p^n),\quad 0\le\alpha\le p^n-1.$$

Fix a positive integer j_0. If a vector u^* is a solution of the extremal problem

$$\sum_{\alpha,\beta=0}^{p^n-1}F_{\alpha,\beta}(j_0)\overline{a_\alpha(u)}a_\beta(u)\to\max,\quad u\in\mathcal{U}(p,n), \tag{5.5}$$

then $\varphi^*_{j_0-1}$ is defined by

$$\varphi^*_{j_0-1}(x)=\sum_{\alpha=0}^{p^n-1}a_\alpha(u^*)\varphi_{j_0}(x\ominus\ p^{-j_0}\alpha).$$

It is seen from (5.4) and (5.5) that $\|P^*_j f\|\ge\|P_j f\|$ for $j=j_0-1$. Now, if the mask of $\varphi^*_{j_0-1}$ has no blocked sets, then $\varphi^*_{j_0-2}$ is constructed by $\varphi^*_{j_0-1}$ and so on. Finally, we fix s and for each

$j \in \{j_0 - 1, \ldots, j_0 - s\}$ replace $P_j f$ by the orthogonal projection $P_j^* f$ of f to the subspace

$$V_j^* = \text{clos}_{L^2(\mathbb{R}_+)} \text{span}\,\{\varphi_{j,k}^* |\; k \in \mathbb{Z}_+\}.$$

The effectiveness of this method of adaptation can be illustrated by numerical examples in terms (cf. [20]) of the entropy estimates.

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