On wavelets related to the Walsh series

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Abstract

For any integers \( p, n \geq 2 \) necessary and sufficient conditions are given for scaling filters with \( p^n \) many terms to generate a \( p \)-multiresolution analysis in \( L^2(\mathbb{R}_+) \). A method for constructing orthogonal compactly supported \( p \)-wavelets on \( \mathbb{R}_+ \) is described. Also, an adaptive \( p \)-wavelet approximation in \( L^2(\mathbb{R}_+) \) is considered.

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1. Introduction

In the wavelet literature, there is some interest in the study of compactly supported orthonormal scaling functions and wavelets with an arbitrary dilation factor \( p \in \mathbb{N}, p \geq 2 \) (see, e.g., [3, Section 10.2], [21, Section 2.4], [4, and references therein]). Such wavelets can have very small support and multifractal structure, features which may be important in signal processing and numerical applications. In this paper we study compactly supported orthogonal \( p \)-wavelets related to the generalized Walsh functions \( \{w_l\} \). There are two ways of considering these functions; either they may be defined on the positive half-line \( \mathbb{R}_+ = [0, \infty) \), or, following Vilenkin [24], they may be identified with the characters of the locally compact Abelian group \( G_p \) which is a weak direct product of a countable set of the cyclic groups of order \( p \). The classical Walsh functions correspond to the case \( p = 2 \), while the group \( G_2 \) is isomorphic to the Cantor

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dyadic group $C$ (see [22,9]). Orthogonal compactly supported wavelets on the group $C$ (and relevant wavelets on $\mathbb{R}_+$) are studied in [15–17,8]. Decimation by an integer different from 2 is discussed in [5,6], but construction for a general $p$ is not completely treated. Here we review some of the elements of that construction on $\mathbb{R}_+$ and give an approach to the $p > 2$ case in a concrete fashion. An essential new element is the matrix extension in Section 4. Finally, in Section 5, we describe an adaptive $p$-wavelet approximation in $L^2(\mathbb{R}_+)$. Let us consider the half-line $\mathbb{R}_+$ with the $p$-adic operations $\oplus$ and $\ominus$ (see Section 2 for the definitions). We say that a compactly supported function $\varphi \in L^2(\mathbb{R}_+)$ is a $p$-refinable function if it satisfies an equation of the type

$$\varphi(x) = p \sum_{\alpha=0}^{p^n-1} a_{\alpha} \varphi(px \ominus \alpha)$$

with complex coefficients $a_{\alpha}$. Further, the generalized Walsh polynomial

$$m(\omega) = \sum_{\alpha=0}^{p^n-1} a_{\alpha} w_{\alpha}(\omega)$$

is called the mask of Eq. (1.1) (or its solution $\varphi$).

An interval $I \subset \mathbb{R}_+$ is a $p$-adic interval of range $n$ if $I = I_s^{(n)} = [s p^{-n}, (s + 1) p^{-n})$ for some $s \in \mathbb{Z}_+$. Since $w_{\alpha}$ is constant on $I_s^{(n)}$ whenever $0 \leq \alpha, s < p^n$, it is clear that the mask $m$ is a $p$-adic step function. If $b_s = m(s p^{-n})$ are the values of $m$ on $p$-adic intervals, i.e.,

$$b_s = \sum_{\alpha=0}^{p^n-1} a_{\alpha} w_{\alpha}(s p^{-n}), \quad 0 \leq s \leq p^n - 1,$$

then

$$a_{\alpha} = \frac{1}{p^n} \sum_{s=0}^{p^n-1} b_s w_{\alpha}(s/p^n), \quad 0 \leq \alpha \leq p^n - 1,$$

and, conversely, equalities (1.3) follow from (1.4). These discrete transforms can be realized by the fast Vilenkin–Chrestenson algorithm (see, for instance, [22, p.463], [19]). Thus, an arbitrary choice of the values of the mask on $p$-adic intervals defines also the coefficients of Eq. (1.1).

It was claimed in [6] that if a $p$-refinable function $\varphi$ satisfies the condition $\hat{\varphi}(0) = 1$ and the system $\{\varphi(\cdot \ominus k) | k \in \mathbb{Z}_+\}$ is orthonormal in $L^2(\mathbb{R}_+)$, then

$$m(0) = 1 \quad \text{and} \quad \sum_{l=0}^{p-1} |m(\omega + l/p)|^2 = 1 \quad \text{for all} \ \omega \in [0, 1/p).$$

From this it follows that the equalities

$$b_0 = 1, \quad |b_j|^2 + |b_{j + p^{-1}j}|^2 + \cdots + |b_{j + (p-1)p^{n-1}}|^2 = 1, \quad 0 \leq j \leq p^{n-1} - 1,$$

are necessary (but not sufficient, see Example 4) for the system $\{\varphi(\cdot \ominus k) | k \in \mathbb{Z}_+\}$ to be orthonormal in $L^2(\mathbb{R}_+)$. Denote by $1_E$ the characteristic function of a subset $E$ of $\mathbb{R}_+$. 
Example 1. If \( a_0 = \cdots = a_{p-1} = 1/p \) and \( a_\alpha = 0 \) for all \( \alpha \geq p \), then a solution of Eq. (1.1) is \( \varphi = 1_{[0, p^{n-1}]} \). Therefore the Haar function \( \varphi = 1_{[0, 1]} \) satisfies this equation for \( n = 1 \) (compare with [5, Remark 1.3] and [1, Section 5.1]).

Example 2. If we take \( p = n = 2 \) and put
\[
b_0 = 1, b_1 = a, b_2 = 0, b_3 = b,
\]
where \(|a|^2 + |b|^2 = 1\), then by (1.4) we have
\[
a_0 = (1 + a + b)/4, \quad a_1 = (1 + a - b)/4, \\
a_2 = (1 - a - b)/4, \quad a_3 = (1 - a + b)/4.
\]
In particular, for \( a = 1 \) and \( a = -1 \) the Haar function: \( \varphi(x) = 1_{[0, 1]}(x) \) and the displaced Haar function: \( \varphi(x) = 1_{[0, 1]}(x \ominus 1) \) are obtained. If \( 0 < |a| < 1, \) then
\[
\varphi(x) = (1/2)1_{[0, 1]}(x/2) \left( 1 + a \sum_{j=0}^{\infty} b^j w_{2j+1-1}(x/2) \right)
\]
and
\[
\varphi(x) = \begin{cases} 
(1 + a - b)/2 + b\varphi(2x), & 0 \leq x < 1, \\
(1 - a + b)/2 - b\varphi(2x - 2), & 1 \leq x < 2
\end{cases}
\]
(see [15,17]). Moreover, it was proved in [16] that, if \(|b| < 1/2\), then the corresponding wavelet system \( \{\psi_{jk}\} \) is an unconditional basis in all spaces \( L^q(\mathbb{R}_+), 1 < q < \infty \). When \( a = 0 \) the system \( \{\varphi(\cdot \ominus k)\mid k \in \mathbb{Z}_+\} \) is linear dependence (since \( \varphi(x) = (1/2)1_{[0, 1]}(x/2) \) and \( \varphi(x \ominus 1) = \varphi(x) \)).

We recall that a collection of closed subspaces \( V_j \subset L^2(\mathbb{R}_+) \), \( j \in \mathbb{Z} \), is called a \textit{p-multiresolution analysis} (p-MRA) in \( L^2(\mathbb{R}_+) \) if the following hold:

(i) \( V_j \subset V_{j+1} \) for all \( j \in \mathbb{Z} \);
(ii) \( \bigcup_{j} V_j = L^2(\mathbb{R}_+) \) and \( \bigcap_{j} V_j = \{0\} \);
(iii) \( f(\cdot) \in V_j \iff f(p\cdot) \in V_{j+1} \) for all \( j \in \mathbb{Z} \);
(iv) \( f(\cdot) \in V_0 \implies f(\cdot \ominus k) \in V_0 \) for all \( k \in \mathbb{Z}_+ \);
(v) there is a function \( \varphi \in L^2(\mathbb{R}_+) \) such that the system \( \{\varphi(\cdot \ominus k)\mid k \in \mathbb{Z}_+\} \) is an orthonormal basis of \( V_0 \).

The function \( \varphi \) in condition (v) is called a \textit{scaling function} in \( L^2(\mathbb{R}_+) \).

For any \( \varphi \in L^2(\mathbb{R}_+) \), we set
\[
\varphi_{j,k}(x) = p^{j/2} \varphi(p^j x \ominus k), \quad j \in \mathbb{Z}, k \in \mathbb{Z}_+.
\]

We say that \( \varphi \) \textit{generates a p-MRA in} \( L^2(\mathbb{R}_+) \) if the system \( \{\varphi(\cdot \ominus k)\mid k \in \mathbb{Z}_+\} \) is orthonormal in \( L^2(\mathbb{R}_+) \) and, in addition, the family of subspaces
\[
V_j = \text{clos}_{L^2(\mathbb{R}_+)} \text{span} \{\varphi_{j,k} \mid k \in \mathbb{Z}_+\}, \quad j \in \mathbb{Z},
\]
is a p-MRA in \( L^2(\mathbb{R}_+) \). Any \( p \)-refinable function \( \varphi \) which generates a p-MRA in \( L^2(\mathbb{R}_+) \) can be written as a sum of lacunary series by the generalized Walsh functions (see [5,6]).
The results of this paper are concerned mainly with the following two problems:

1. Find necessary and sufficient conditions in order that a \( p \)-refinable function \( \varphi \) generates a \( p \)-MRA in \( L^2(\mathbb{R}_+) \).

2. Describe a method for constructing orthogonal compactly supported \( p \)-wavelets on \( \mathbb{R}_+ \).

Note that similar problems can be considered in framework of the biorthogonal \( p \)-wavelet theory (see [7] for the \( p = 2 \) case).

If a function \( \varphi \) generates a \( p \)-MRA, then it is a scaling function in \( L^2(\mathbb{R}_+) \). In this case, the system \( \{ \varphi_{j,k} | k \in \mathbb{Z}_+ \} \) is an orthonormal basis of \( V_j \) for each \( j \in \mathbb{Z} \), and moreover, one can define orthogonal \( p \)-wavelets \( \psi_1, \ldots, \psi_{p-1} \) in such a way that the functions

\[
\psi_{l,j,k}(x) = p^{l/2} \varphi_l(p^j x \ominus k), \quad 1 \leq l \leq p-1, j \in \mathbb{Z}, k \in \mathbb{Z}_+,
\]

form an orthonormal basis of \( L^2(\mathbb{R}_+) \). If \( p = 2 \), only one wavelet \( \psi \) is obtained and the system \( \{ 2^{l/2} \varphi(2^j \cdot \ominus k) | j \in \mathbb{Z}, k \in \mathbb{Z}_+ \} \) is an orthonormal basis of \( L^2(\mathbb{R}_+) \). In Section 4 we give a practical method to design orthogonal \( p \)-wavelets \( \psi_1, \ldots, \psi_{p-1} \), which is based on an algorithm for matrix extension and on the following

**Theorem.** Suppose that equation (1.1) possesses a compactly supported \( L^2 \)-solution \( \varphi \) such that its mask \( m \) satisfies conditions (1.5) and \( \widehat{\varphi}(0) = 1 \). Then the following are equivalent:

(a) \( \varphi \) generates a \( p \)-MRA in \( L^2(\mathbb{R}_+) \);

(b) \( m \) satisfies modified Cohen’s condition;

(c) \( m \) has no blocked sets.

We review some notation and terminology. Let \( M \subset [0, 1) \) and let

\[
T_p M = \bigcup_{l=0}^{p-1} \{ l/p + \omega/p \omega \in M \}.
\]

The set \( M \) is said to be **blocked** (for the mask \( m \)) if it is a union of \( p \)-adic intervals of range \( n-1 \), does not contain the interval \([0, p^{-n+1})\), and satisfies the condition

\[
T_p M \setminus M \subset \text{Null } m,
\]

where \( \text{Null } m := \{ \omega \in [0, 1) | m(\omega) = 0 \} \). It is clear that each mask can have only a finite number of blocked sets. In Section 3 we shall prove that if \( \varphi \) is a \( p \)-refinable function in \( L^2(\mathbb{R}_+) \) such that \( \widehat{\varphi}(0) = 1 \), then the system \( \{ \varphi(\cdot \ominus k) | k \in \mathbb{Z}_+ \} \) is linearly dependent if and only if its mask possesses a blocked set. The notion of blocked set (in the case \( p = 2 \)) was introduced in the recent paper [8].

The family \( \{ [0, p^{-j}) | j \in \mathbb{Z} \} \) forms a fundamental system of the \( p \)-adic topology on \( \mathbb{R}_+ \). A subset \( E \) of \( \mathbb{R}_+ \) that is compact in the \( p \)-adic topology is said to be \( W \)-compact. It is easy to see that the union of a finite family of \( p \)-adic intervals is \( W \)-compact.

A \( W \)-compact set \( E \) is said to be **congruent to** \([0, 1)\) \( \text{modulo } \mathbb{R}_+ \) if its Lebesgue measure is 1 and, for each \( x \in [0, 1) \), there is an element \( k \in \mathbb{Z}_+ \) such that \( x \ominus k \in E \). As before, let \( m \) be the mask of refinable equation (1.1). We say that \( m \) satisfies the **modified Cohen condition** if there exists a \( W \)-compact subset \( E \) of \( \mathbb{R}_+ \) congruent to \([0, 1)\) \( \text{modulo } \mathbb{Z}_+ \) and containing a neighbourhood of zero such that

\[
\inf_{j \in \mathbb{N}} \inf_{\omega \in E} |m(p^{-j} \omega)| > 0 \quad (1.7)
\]
(cf. [3, Section 6.3], [16, Sect. 2]). Since \( E \) is \( W \)-compact, it is evident that if \( m(0) = 1 \) then there exists a number \( j_0 \) such that \( m(p^{-j_0} \omega) = 1 \) for all \( j > j_0, \omega \in E \). Therefore (1.7) holds if \( m \) does not vanish on the sets \( E/p, \ldots, E/p^{-j_0} \). Moreover, one can choose \( j_0 \leq p^n \) because \( m \) is \( 1 \)-periodic and completely defined by the values (1.3).

Now we illustrate the theorem with the following two examples.

**Example 3.** Let \( p = 3, n = 2 \) and

\[
\begin{align*}
    b_0 &= 1, b_1 = a, b_2 = \alpha, b_3 = 0, b_4 = b, b_5 = \beta, b_6 = 0, b_7 = c, b_8 = \gamma,
\end{align*}
\]

where

\[
|a|^2 + |b|^2 + |c|^2 = |\alpha|^2 + |\beta|^2 + |\gamma|^2 = 1.
\]

Then (1.4) implies precisely that

\[
\begin{align*}
    a_0 &= \frac{1}{9}(1 + a + b + c + \alpha + \beta + \gamma), \\
    a_1 &= \frac{1}{9}(1 + a + \alpha + (b + \beta)\varepsilon_3^2 + (c + \gamma)\varepsilon_3), \\
    a_2 &= \frac{1}{9}(1 + a + \alpha + (b + \beta)\varepsilon_3 + (c + \gamma)\varepsilon_3^2), \\
    a_3 &= \frac{1}{9}(1 + (a + b + c)\varepsilon_3^2 + (\alpha + \beta + \gamma)\varepsilon_3), \\
    a_4 &= \frac{1}{9}(1 + c + \beta + (a + \gamma)\varepsilon_3^2 + (b + \alpha)\varepsilon_3), \\
    a_5 &= \frac{1}{9}(1 + b + \gamma + (a + \beta)\varepsilon_3^2 + (c + \alpha)\varepsilon_3), \\
    a_6 &= \frac{1}{9}(1 + (a + b + c)\varepsilon_3 + (\alpha + \beta + \gamma)\varepsilon_3^2), \\
    a_7 &= \frac{1}{9}(1 + b + \gamma + (a + \beta)\varepsilon_3 + (c + \alpha)\varepsilon_3^2), \\
    a_8 &= \frac{1}{9}(1 + c + \beta + (a + \gamma)\varepsilon_3 + (b + \alpha)\varepsilon_3^2),
\end{align*}
\]

where \( \varepsilon_3 = \exp(2\pi i/3) \). Further, if

\[
\gamma(1, 0) = a, \gamma(2, 0) = \alpha, \gamma(1, 1) = b, \gamma(2, 1) = \beta, \gamma(1, 2) = c, \gamma(2, 2) = \gamma
\]

and \( v_j \in \{1, 2\} \), then we let

\[
\begin{align*}
    c_l &= \gamma(v_0, 0) & \text{for } l = v_0; \\
    c_l &= \gamma(v_1, 0)\gamma(v_0, v_1) & \text{for } l = v_0 + 3v_1; \\
    \cdots \\
    c_l &= \gamma(v_k, 0)\gamma(v_{k-1}, v_k) \cdots \gamma(v_0, v_1) & \text{for } l = \sum_{j=0}^{k} v_j 3^j, k \geq 2.
\end{align*}
\]

The solution of Eq. (1.1) can be decomposed (see [6]) as follows:

\[
\varphi(x) = (1/3)1_{[0,1]}(x/3) \left( 1 + \sum_l c_l w_l(x/3) \right).
\]
The blocked sets are: (1) \([1/3, 2/3]\) for \(a = c = 0\), (2) \([2/3, 1]\) for \(\alpha = \beta = 0\), (3) \([1/3, 1]\) for \(a = \alpha = 0\). Hence, \(\varphi\) generates a MRA in \(L^2(\mathbb{R}_+)\) in the following cases: (1) \(a \neq 0, \alpha \neq 0\), (2) \(a = 0, \alpha \neq 0, c \neq 0\), (3) \(\alpha = 0, a \neq 0, \beta \neq 0\).

**Example 4.** Suppose that for some numbers \(b_s, 0 \leq s \leq p^n - 1\), equalities (1.5) are true. Using (1.4), we find the mask

\[
m(\omega) = \sum_{a=0}^{2^n-1} a \cdot w_\alpha(\omega),
\]

which takes the values \(b_s\) on the intervals \(I_s^{(n)}, 0 \leq s \leq p^n - 1\). When \(b_j \neq 0\) for \(1 \leq j \leq p^{n-1} - 1\) Eq. (1.1) has a solution, which generates a \(p\)-MRA in \(L^2(\mathbb{R}_+)\) (the modified Cohen condition is fulfilled for \(E = [0, 1]\)). The expansion of this solution in a lacunary series by generalized Walsh functions is contained in [6].

2. Preliminaries

For the integer and the fractional parts of a number \(x\) we are using the standard notations, \([x]\) and \(\{x\}\), respectively. For any \(s \in \mathbb{Z}\) let us denote by \(\langle s \rangle_p\) the remainder upon dividing \(s\) by \(p\). Then for \(x \in \mathbb{R}_+\) we set

\[
x_j = \langle [p^j x] \rangle_p, \quad x_j = \langle [p^{1-j} x] \rangle_p, \quad j \in \mathbb{N}.
\]

For each \(x \in \mathbb{R}_+\), these numbers are the digits of the \(p\)-ary expansion

\[
x = \sum_{j<0} x_j p^{-j-1} + \sum_{j>0} x_j p^{-j}
\]

(for a \(p\)-adic rational \(x\) we obtain an expansion with finitely many nonzero terms). It is clear that

\[
[x] = \sum_{j=1}^\infty x_j p^{j-1}, \quad \{x\} = \sum_{j=1}^\infty x_j p^{-j},
\]

and there exists \(k = k(x)\) in \(\mathbb{N}\) such that \(x_j = 0\) for all \(j > k\).

Consider the \(p\)-adic addition defined on \(\mathbb{R}_+\) as follows: if \(z = x \oplus y\), then

\[
z = \sum_{j<0} (x_j + y_j) p^{j-1} + \sum_{j>0} (x_j + y_j) p^{-j}.
\]

As usual, the equality \(z = x \oplus y\) means that \(z \oplus y = x\). According to our notation

\[
[x \oplus y] = [x] \oplus [y] \quad \text{and} \quad \{x \oplus y\} = \{x\} \oplus \{y\}.
\]

Note that for \(p = 2\) we have

\[
x \oplus y = \sum_{j<0} |x_j - y_j| 2^{-j-1} + \sum_{j>0} |x_j - y_j| 2^{-j}.
\]

Letting \(\varepsilon_p = \exp(2\pi i/p)\), we define a function \(w_1\) on \([0, 1)\) by

\[
w_1(x) = \begin{cases} 1, & x \in [0, 1/p), \\ \varepsilon_p^l, & x \in (lp^{-1}, (l+1)p^{-1}), l \in \{1, \ldots, p-1\} \end{cases}
\]
and extend it to $\mathbb{R}_+$ by periodicity: $w_1(x + 1) = w_1(x)$ for all $x \in \mathbb{R}_+$. Then the generalized Walsh system $\{w_l|l \in \mathbb{Z}_+\}$ is defined by

$$w_0(x) \equiv 1, \quad w_l(x) = \prod_{j=1}^{k} (w_1(p^{j-1}x))^{l_j}, \quad l \in \mathbb{N}, \, x \in \mathbb{R}_+,$$

where the $l_j$ are the digits of the $p$-ary expansion of $l$:

$$l = \sum_{j=1}^{k} l_j p^{j-1}, \quad l_j \in \{0, 1, \ldots, p-1\}, l_k \neq 0, k = k(l).$$

For any $x, y \in \mathbb{R}_+$, let

$$\chi(x, y) = \varepsilon_p^{t(x, y)}, \quad t(x, y) = \sum_{j=1}^{\infty} (x_j y_{-j} + x_{-j} y_j), \quad (2.2)$$

where $x_j, y_j$ are given by (2.1). Note that

$$\chi(x, p^{-s}l) = \chi(p^{-s}x, l) = w_l(p^{-s}x), \quad l, s \in \mathbb{Z}_+, \, x \in [0, p^s),$$

and

$$\chi(x, z)\chi(y, z) = \chi(x \oplus y, z), \quad \chi(x, z)\overline{\chi(y, z)} = \chi(x \ominus y, z), \quad (2.3)$$

if $x, y, z \in \mathbb{R}_+$ and $x \oplus y$ is $p$-adic irrational. Thus, for fixed $x$ and $z$, equalities (2.3) hold for all $y \in \mathbb{R}_+$ except countably many of them (see [9, Section 1.5]).

It is known also that Lebesgue measure is translation invariant on $\mathbb{R}_+$ with respect to $p$-adic addition, and so we can write

$$\int_{\mathbb{R}_+} f(x \oplus y) \, dx = \int_{\mathbb{R}_+} f(x) \, dx, \quad f \in L^1(\mathbb{R}_+),$$

for all $y \in \mathbb{R}_+$ (see [22, Section 1.3], [9, Section 6.1]).

The *Walsh–Fourier transform* of a function $f \in L^1(\mathbb{R}_+)$ is defined by

$$\widehat{f}(\omega) = \int_{\mathbb{R}_+} f(x) \overline{\chi(x, \omega)} \, dx,$$

where $\chi(x, \omega)$ is given by (2.2). If $f \in L^2(\mathbb{R}_+)$ and

$$J_a f(\omega) = \int_{0}^{a} f(x) \overline{\chi(x, \omega)} \, dx, \quad a > 0,$$

then $\widehat{f}$ is the limit of $J_a f$ in $L^2(\mathbb{R}_+)$ as $a \to \infty$. We say that a function $f : \mathbb{R}_+ \mapsto \mathbb{C}$ is $W$-continuous at a point $x \in \mathbb{R}_+$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(x \oplus y) - f(x)| < \varepsilon$ for $0 < y < \delta$. For example, each Walsh polynomial is $W$-continuous (see [22, Section 9.2], [9, Section 2.3]).

Denote by $(\cdot, \cdot)$ and $\| \cdot \|$ the inner product and the norm in $L^2(\mathbb{R}_+)$, respectively.

**Proposition 1** (See [9, Chap. 6]). The following properties hold:

(a) if $f \in L^1(\mathbb{R}_+)$, then $\widehat{f}$ is a $W$-continuous function and $\widehat{f}(\omega) \to 0$ as $\omega \to \infty$;
(b) if both $f$ and $\hat{f}$ belong to $L^1(\mathbb{R}_+)$ and $f$ is $W$-continuous, then
\[
f(x) = \int_{\mathbb{R}_+} \hat{f}(\omega) \chi(x, \omega) \, d\omega \quad \text{for all } x \in \mathbb{R}_+;
\]

(c) if $f, g \in L^2(\mathbb{R}_+)$, then $\langle f, g \rangle = \langle \hat{f}, \hat{g} \rangle$ (Parseval’s relation).

Let $E_n(\mathbb{R}_+)$ be the space of $p$-adic entire functions of order $n$ on $\mathbb{R}_+$, that is, the set of functions which are constant on all $p$-adic intervals of range $n$. Then for every $f \in E_n(\mathbb{R}_+)$ we have
\[
f(x) = \sum_{\alpha=0}^{\infty} f(\alpha p^{-n}) \mathbf{1}_{[\alpha p^{-n}, (\alpha+1)p^{-n})}(x), \quad x \in \mathbb{R}_+.
\]

For example, the mask $m$ of Eq. (1.1) belongs to $E_n(\mathbb{R}_+)$.  

Proposition 2 ([9, Section 6.2]). The following properties hold:
(a) if $f \in L^1(\mathbb{R}_+) \cap E_n(\mathbb{R}_+)$, then $\text{supp} \, \hat{f} \subset [0, p^n]$;
(b) if $f \in L^1(\mathbb{R}_+)$ and $\text{supp} \, f \subset [0, p^n]$, then $\hat{f} \in E_n(\mathbb{R}_+)$.  

Now we prove the following analogue of Theorem 1 in [8]:

Proposition 3. Let $\varphi \in L^2(\mathbb{R}_+)$ be a compactly supported solution of equation (1.1) such that $\hat{\varphi}(0) = 1$. Then
\[
\sum_{\alpha=0}^{p^n-1} a_\alpha = 1 \quad \text{and} \quad \text{supp} \, \varphi \subset [0, p^{n-1}].
\]

This solution is unique, is given by the formula
\[
\hat{\varphi}(\omega) = \prod_{j=1}^{\infty} m(p^{-j} \omega)
\]
and possesses the following properties:
(1) $\hat{\varphi}(k) = 0$ for all $k \in \mathbb{N}$ (the modified Strang–Fix condition);
(2) $\sum_{k \in \mathbb{Z}_+} \varphi(x \oplus k) = 1$ for almost all $x \in \mathbb{R}_+$ (the partition of unity property).

Proof. Using the Walsh–Fourier transform, we have
\[
\hat{\varphi}(\omega) = m(\omega/p) \hat{\varphi}(\omega/p).
\]  (2.4)

Observe that $w_\omega(0) = \hat{\varphi}(0) = 1$. Hence, letting $\omega = 0$ in (1.2) and (2.4), we obtain
\[
\sum_{\alpha=0}^{p^n-1} a_\alpha = 1.
\]

Further, let $s$ be the greatest integer such that
\[
\mu\{x \in [s - 1, s) | \varphi(x) \neq 0\} > 0,
\]
where $\mu$ is the Lebesgue measure on $\mathbb{R}_+$. Suppose that $s \geq p^{n-1} + 1$. Choose an arbitrary $p$-adic irrational $x \in [s - 1, s)$. Applying (2.1), we have
\[
x = [x] + \{x\} = \sum_{j=1}^{k} x_{-j} p^{j-1} + \sum_{j=1}^{\infty} x_j p^{-j},
\]  (2.5)
where \( \{x\} > 0, x_{-k} \neq 0, k = k(x) \geq n \). For any \( \alpha \in \{0, 1, \ldots, p^n - 1\} \) we set \( y^{(\alpha)} = px \ominus \alpha \). Then

\[
y^{(\alpha)} = \sum_{j=1}^{k+1} y^{(\alpha)}_{j-1} p^{j-1} + \sum_{j=1}^{\infty} y^{(\alpha)}_{j} p^{-j},
\]

where \( y^{(\alpha)}_{-k-1} = x_{-k} \) and among the digits \( y^{(\alpha)}_{1}, y^{(\alpha)}_{2}, \ldots \), there is a nonzero one. Therefore,

\[
px \ominus \alpha > p^n \quad \text{for a.e.} \; x \in [s - 1, s).
\] (2.6)

Now assume that \( s \leq p^n \). Then it is easy to see from (2.6) that \( \varphi(px \ominus \alpha) = 0 \) for a.e. \( x \in [s - 1, s) \). Therefore by (1.1) we get \( \varphi(x) = 0 \) for a.e. \( x \in [s - 1, s) \), contrary to our choice of \( s \). Thus \( s \geq p^n + 1 \). Hence, if \( x \) given by (2.5), then for any \( \alpha \in \{0, 1, \ldots, p^n - 1\} \) we have

\[
px \ominus \alpha > p(s - 1) - (p^n - 1) \geq 2(s - 1) - (s - 2) = s,
\]

where the first inequality is strong because \( \{x\} > 0 \). As above, we conclude that \( \varphi(x) = 0 \) for a.e. \( x \in [s - 1, s) \). Consequently, \( s \leq p^n - 1 \) and supp \( \varphi \subset [0, p^{n-1}] \).

Let us prove that

\[
\hat{\varphi}(\omega) = \prod_{j=1}^{\infty} m(p^{-j} \omega).
\] (2.7)

We note that \( \varphi \) belongs to \( L^1(\mathbb{R}_+) \) because it lies in \( L^2(\mathbb{R}_+) \) and has a compact support. Since supp \( \varphi \subset [0, p^{n-1}] \), by Proposition 2 we get \( \hat{\varphi} \in \mathcal{E}_{n-1}(\mathbb{R}_+) \). Also, by virtue of \( \hat{\varphi}(0) = 1 \), we obtain \( \hat{\varphi}(\omega) = 1 \) for all \( \omega \in [0, p^{1-n}] \). On the other hand, \( m(\omega) = 1 \) for all \( \omega \in [0, p^{1-n}] \). Hence, for every positive integer \( l \),

\[
\hat{\varphi}(\omega) = \hat{\varphi}(p^{-l-n} \omega) \prod_{j=1}^{l+n} m(p^{-j} \omega) = \prod_{j=1}^{\infty} m(p^{-j} \omega), \quad \omega \in [0, p^l).
\]

Therefore, (2.7) is valid and a solution \( \varphi \) is unique.

By Proposition 1, for any \( k \in \mathbb{N} \) we have

\[
\hat{\varphi}(k) = \hat{\varphi}(j) \prod_{s=0}^{j-1} m(p^s k) = \hat{\varphi}(p^j k) \to 0
\]

as \( j \to \infty \) (since \( \varphi \in L^1(\mathbb{R}_+) \) and \( m(p^s k) = 1 \) because \( m(0) = 1 \) and \( m \) is 1-periodic). It follows that

\[
\hat{\varphi}(k) = 0 \quad \text{for all} \; k \in \mathbb{N}.
\] (2.8)

By the Poisson summation formula we get

\[
\sum_{k \in \mathbb{Z}_+} \varphi(x \ominus k) = \sum_{k \in \mathbb{Z}_+} \hat{\varphi}(k) \chi(x, k).
\]

Hence, since \( \hat{\varphi}(0) = 1 \), from (2.8) we obtain

\[
\sum_{k \in \mathbb{Z}_+} \varphi(x \ominus k) = 1 \quad \text{for a.e.} \; x \in \mathbb{R}_+. \quad \Box
\]

The proposition is proved.
A function \( f \in L^2(\mathbb{R}_+) \) is said to be stable if there exist positive constants \( A \) and \( B \) such that
\[
A \left( \sum_{\alpha=0}^{\infty} |a_{\alpha}|^2 \right)^{1/2} \leq \left\| \sum_{\alpha=0}^{\infty} a_{\alpha} f(\cdot \ominus \alpha) \right\| \leq B \left( \sum_{\alpha=0}^{\infty} |a_{\alpha}|^2 \right)^{1/2}
\]
for each sequence \( \{a_{\alpha}\} \in \ell^2 \). In other words, \( f \) is stable if functions \( f(\cdot \ominus k), k \in \mathbb{Z}_+ \), form a Riesz system in \( L^2(\mathbb{R}_+) \). We note also, that a function \( f \) is stable in \( L^2(\mathbb{R}_+) \) with constants \( A \) and \( B \) if and only if
\[
A \leq \sum_{k \in \mathbb{Z}_+} |\hat{f}(\omega \ominus k)|^2 \leq B \quad \text{for a.e. } \omega \in \mathbb{R}_+
\]
(2.9)
(the proof of this fact is quite similar to that of Theorem 1.1.7 in [21]).

We say that a function \( g : \mathbb{R}_+ \to \mathbb{C} \) has a periodic zero at a point \( \omega \in \mathbb{R}_+ \) if \( g(\omega \oplus k) = 0 \) for all \( k \in \mathbb{Z}_+ \).

**Proposition 4** (cf. [8, Theorem 2]). For a compactly supported function \( f \in L^2(\mathbb{R}_+) \) the following statements are equivalent:

(a) \( f \) is stable in \( L^2(\mathbb{R}_+) \);
(b) \( \{f(\cdot \ominus k) \mid k \in \mathbb{Z}_+\} \) is a linearly independent system in \( L^2(\mathbb{R}_+) \);
(c) \( \hat{f} \) does not have periodic zeros.

**Proof.** The implication (a) \( \Rightarrow \) (b) follows from the well-known property of the Riesz systems (see, e.g., [21, Theorem 1.1.2]). Our next claim is that \( f \in L^1(\mathbb{R}_+) \), since \( f \) has compact support and \( f \in L^2(\mathbb{R}_+) \). Let us choose a positive integer \( n \) such that \( \text{supp } f \subset [0, p^n-1] \). Then by **Proposition 2** we have \( \hat{f} \in \mathcal{E}_{n-1}(\mathbb{R}_+) \). Besides, if \( k > p^n-1 \), then
\[
\mu\{\text{supp } f(\cdot \ominus k) \cap [0, p^n-1]\} = 0
\]
(as above, \( \mu \) denotes the Lebesgue measure on \( \mathbb{R}_+ \)). Therefore, the linearly independence of the system \( \{f(\cdot \ominus k) \mid k \in \mathbb{Z}_+\} \) in \( L^2(\mathbb{R}_+) \) is equivalent to that for the finite system \( \{f(\cdot \ominus k) \mid k = 0, 1, \ldots, p^n-1 - 1\} \). Further, if some vector \( (a_0, \ldots, a_{p^n-1-1}) \) satisfies conditions
\[
\sum_{\alpha=0}^{p^n-1-1} a_{\alpha} f(\cdot \ominus \alpha) = 0 \quad \text{and} \quad |a_0| + \cdots + |a_{2^{n-1}-1}| > 0,
\]
then using the Walsh–Fourier transform we obtain
\[
\hat{f}(\omega) \sum_{\alpha=0}^{p^n-1-1} a_{\alpha} w_{\alpha}(\omega) = 0 \quad \text{for a.e. } \omega \in \mathbb{R}_+.
\]
The Walsh polynomial
\[
w(\omega) = \sum_{\alpha=0}^{p^n-1-1} a_{\alpha} w_{\alpha}(\omega)
\]
is not identically equal to zero; hence among \( I_s^{(n-1)}, 0 \leq s \leq p^n-1 - 1 \), there exists an interval (denote it by \( I \)) for which \( w(I \oplus k) \neq 0, k \in \mathbb{Z}_+ \). Since \( f \in \mathcal{E}_{n-1}(\mathbb{R}_+) \), it follows that (2.10) holds if and only if there exists a \( p \)-adic interval \( I \) of range \( n - 1 \), such that \( \hat{f}(I \oplus k) = 0 \) for
all \( k \in \mathbb{Z}_+ \). Thus, (b) \( \iff \) (c). It remains to prove that (c) \( \implies \) (a). Suppose that \( \hat{f} \) does not have periodic zeros. Then

\[
F(\omega) := \sum_{k \in \mathbb{Z}_+} |\hat{f}(\omega \ominus k)|^2, \quad \omega \in \mathbb{R}_+.
\]

is positive and 1-periodic function. Moreover, since \( \hat{f} \in \mathcal{E}_{n-1}(\mathbb{R}_+) \), we see that \( F \) is constant on each \( I^{(n-1)}_s, 0 \leq s \leq p^n - 1 \). Hence (2.9) is satisfied and so Proposition 4 is established. \( \square \)

The following two propositions are proved in [6]:

**Proposition 5.** Let \( \varphi \in L^2(\mathbb{R}_+) \). Then the system \( \{ \varphi(\cdot \ominus k) \mid k \in \mathbb{Z}_+ \} \) is orthonormal in \( L^2(\mathbb{R}_+) \) if and only if

\[
\sum_{k \in \mathbb{Z}_+} |\hat{\varphi}(\omega \ominus k)|^2 = 1 \quad \text{for a.e.} \ \omega \in \mathbb{R}_+.
\]

**Proposition 6.** Let \( \{ V_j \} \) be the family of subspaces defined by (1.6) with given \( \varphi \in L^2(\mathbb{R}_+) \). If \( \{ \varphi(\cdot \ominus k) \mid k \in \mathbb{Z}_+ \} \) is an orthonormal basis in \( V_0 \), then \( \bigcap V_j = \{0\} \).

We shall use also the following

**Proposition 7.** Let

\[
m(\omega) = \sum_{\alpha=0}^{p^n - 1} a_\alpha w_\alpha(\omega)
\]

be a polynomial such that

\[
m(0) = 1 \quad \text{and} \quad \sum_{l=0}^{p-1} |m(\omega \oplus l/p)|^2 = 1 \quad \text{for all} \ \omega \in \mathbb{R}_+.
\]

Suppose \( \varphi \) is a function defined by the Walsh–Fourier transform

\[
\hat{\varphi}(\omega) = \prod_{j=1}^{\infty} m(p^{-j} \omega).
\]

Then the system \( \{ \varphi(\cdot \ominus k) \mid k \in \mathbb{Z}_+ \} \) is orthonormal in \( L^2(\mathbb{R}_+) \) if and only if \( m \) satisfies the modified Cohen condition.

The proof of this proposition is similar to that of Theorem 6.3.1 in [3] (cf. [15, Theorem 2.1], [5, Proposition 3.3]).

**3. Proof of the theorem**

The next lemma gives a relation between stability and blocked sets.

**Lemma 1.** Let \( \varphi \) be a \( p \)-refinable function in \( L^2(\mathbb{R}_+) \) such that \( \hat{\varphi}(0) = 1 \). Then \( \varphi \) is not stable if and only if its mask \( m \) has a blocked set.

**Proof.** Using Propositions 2 and 3, we have \( \text{supp} \ \varphi \subset [0, p^{n-1}] \) and \( \hat{\varphi} \in \mathcal{E}_{n-1}(\mathbb{R}_+) \). Suppose that the function \( \varphi \) is not stable. As noted in the proof of Proposition 4, then there exists an interval \( I = I^{(n-1)}_s \) consisting entirely of periodic zeros of the Walsh–Fourier transform \( \hat{\varphi} \) (and
each periodic zero \( \omega \in [0, 1) \) of \( \hat{\varphi} \) lies in some such \( I \). Thus, the set
\[
M_0 = \{ \omega \in [0, 1) | \hat{\varphi}(\omega + k) = 0 \text{ for all } k \in \mathbb{Z}_+ \}
\]
is a union of some intervals \( I_s^{(n-1)} \), \( 0 \leq s \leq p^{n-1} - 1 \). Since \( \hat{\varphi}(0) = 1 \), it follows that \( M_0 \) does not contain \( I_0^{(n-1)} \). Furthermore, if \( \omega \in M_0 \), then by (2.4)
\[
m(\omega/p + k/p)\hat{\varphi}(\omega/p + k/p) = 0 \text{ for all } k \in \mathbb{Z}_+
\]
and hence \( \omega/p + l/p \in M_0 \cup \text{Null } m \) for \( l = 0, 1, \ldots, p - 1 \). Thus, if \( \varphi \) is not stable, then \( M_0 \) is a blocked set for \( m \).

Conversely, let \( m \) possess a blocked set \( M \). Then we will show that each element of \( M \) is a periodic zero for \( \hat{\varphi} \) (and by Proposition 4 \( \varphi \) is not stable). Assume that there exist \( \omega \in M \) and \( k \in \mathbb{Z}_+ \) such that \( \hat{\varphi}(\omega + k) \neq 0 \). Choose a positive integer \( j \) for which \( p^{-j}(\omega + k) \in [0, p^{1-n}) \) and, for every \( r \in \{0, 1, \ldots, j \} \), set
\[
u_r = [p^{-r}(\omega + k)], \quad v_r = [p^{-r}(\omega + k)].
\]
Further, let \( u_r/p = l_r/p + s_r \), where \( l_r \in \{0, 1, \ldots, p - 1 \} \) and \( s_r \in \mathbb{Z}_+ \). It is clear that for all \( r \in \{0, 1, \ldots, j - 1 \} \)
\[
u_{r+1} + v_{r+1} = (p^{-1}v_r + p^{-1}l_r) + s_r
\]
and hence \( v_{r+1} = p^{-1}(v_r + l_r) \). From this it follows that if \( v_r \in M \), then \( v_{r+1} \in T_p M \). Besides, from the equalities
\[
\hat{\varphi}(\omega + k) = \hat{\varphi}(\omega) \prod_{r=1}^{j} m(p^{-r}(\omega + k)) = \hat{\varphi}(v_j) \prod_{r=1}^{j} m(v_r)
\]
we see that all \( v_r \notin \text{Null } m \). Thus, if \( v_r \in M \), then \( v_{r+1} \in M \). Since \( v_0 = \omega \in M \), we conclude that \( v_j \in M \). But this is impossible because \( v_j = p^{-j}(\omega + k) \in [0, p^{1-n}) \) and \( M \cap [0, p^{1-n}) = \emptyset \). This contradiction completes the proof of Lemma 1. \( \square \)

**Corollary.** If \( \varphi \) is a \( p \)-refinable function in \( L^2(\mathbb{R}_+) \) such that \( \hat{\varphi}(0) = 1 \), then the system \( \{ \varphi(\cdot \oplus k) | k \in \mathbb{Z}_+ \} \) is linearly dependent if and only if the mask of \( \varphi \) possesses a blocked set.

**Lemma 2.** Suppose that the mask of refinable equation (1.1) satisfies
\[
m(0) = 1 \quad \text{and} \quad \sum_{l=0}^{p-1} |m(\omega \oplus l/p)|^2 = 1 \quad \text{for all } \omega \in \mathbb{R}_+.
\]
Then the function \( \varphi \) given by
\[
\hat{\varphi}(\omega) = \prod_{j=1}^{\infty} m(p^{-j}\omega)
\]
is a solution of Eq. (1) and \( \|\varphi\| \leq 1 \).

**Proof.** The pointwise convergence of product in (3.2) follows from the fact that \( m \) is equal to 1 on \( [0, p^{1-n}) \) (and for any \( \omega \in \mathbb{R}_+ \) only finitely many of the factors in (3.2) cannot be equal to 1). Denote by \( g(\omega) \) the right part of (3.2). From (3.1) we see that \( |m(\omega)| \leq 1 \) for all \( \omega \in \mathbb{R}_+ \).
Therefore, for any $s \in \mathbb{N}$ we have
\[ |g(\omega)|^2 \leq \prod_{j=1}^{s} |m(p^{-j}\omega)|^2 \]
and hence
\[ \int_{0}^{p^l} |g(\omega)|^2 \, d\omega \leq \int_{0}^{p^l} \prod_{j=1}^{s} |m(B^{-j}\omega)|^2 \, d\omega = 2^s \int_{0}^{p^{s-1}} \prod_{j=0}^{s-1} |m(B^j\omega)|^2 \, d\omega. \tag{3.3} \]
Further, from the equalities
\[ m(\omega) = \sum_{\alpha=0}^{p^n-1} a_\alpha w_\alpha(\omega), \quad w_\alpha(\omega)w_\beta(\omega) = w_{\alpha \oplus \beta}(\omega), \]
it follows that
\[ |m(\omega)|^2 = \sum_{\alpha=0}^{p^n-1} c_\alpha w_\alpha(\omega), \tag{3.4} \]
where the coefficients $c_\alpha$ may be expressed via $a_\alpha$. Now, we substitute (3.4) into the second equality of (3.1) and observe that if $\alpha$ is multiply to $p$, then
\[ \sum_{l=0}^{p-1} w_\alpha(l/p) = p, \]
and this sum is equal to 0 for the rest $\alpha$. As a result, we obtain $c_0 = 1/p$ and $c_\alpha = 0$ for nonzero $\alpha$, which are multiply to $p$. Hence,
\[ |m(\omega)|^2 = \frac{1}{p} + \sum_{\alpha=0}^{p^n-1} \sum_{l=1}^{p-1} c_{p\alpha+l} w_{p\alpha+l}(\omega). \]
This gives
\[ \prod_{j=0}^{s-1} |m(p^j\omega)|^2 = p^{-s} + \sum_{\gamma=1}^{\sigma(s)} b_\gamma w_\gamma(\omega), \quad \sigma(s) \leq sp^{n-1}(p-1), \]
where each coefficient $b_\gamma$ equals to the product of some coefficients $c_{p\alpha+l}, l = 1, \ldots, p-1$.
Taking into account that
\[ \int_{0}^{1} w_\gamma(\omega) \, d\omega = 0, \quad \gamma \in \mathbb{N}, \]
we have
\[ \int_{0}^{1} \prod_{j=0}^{s-1} |m(p^j\omega)|^2 \, d\omega = p^{-s}. \]
Substituting this into (3.3), we deduce
\[ \int_{0}^{p^l} |g(\omega)|^2 \, d\omega \leq 1, \quad l \in \mathbb{N}, \]
which is due to the inequality
\[ \int_{\mathbb{R}^+} |g(\omega)|^2 \, d\omega \leq 1. \tag{3.5} \]

Now, let \( \varphi \in L^2(\mathbb{R}_+) \) and \( \hat{\varphi} = g \). Then from (3.2) it follows that
\[ \hat{\varphi}(\omega) = m(p^{-1}\omega)\hat{\varphi}(p^{-1}\omega), \]
and hence \( \varphi \) satisfies (1.1). Moreover, from (3.5), by Proposition 1, we get \( \|\varphi\| \leq 1. \)

**Lemma 3.** Let \( \varphi \) be a p-refinable function with a mask \( m \) and let \( \hat{\varphi}(0) = 1 \). Then the system \( \{ \varphi(\cdot \ominus k) | k \in \mathbb{Z}_+ \} \) is orthonormal in \( L^2(\mathbb{R}_+) \) if and only if the mask \( m \) has no blocked sets and satisfies
\[ \sum_{l=0}^{p-1} |m(\omega \oplus l/p)|^2 = 1 \quad \text{for all } \omega \in \mathbb{R}_+. \tag{3.6} \]

**Proof.** If the system \( \{ \varphi(\cdot \ominus k) | k \in \mathbb{Z}_+ \} \) is orthonormal in \( L^2(\mathbb{R}_+) \), then (3.6) holds (see [6]) and a lack of blocked sets follows from Lemma 1 and Proposition 4. Conversely, suppose that \( m \) has no blocked sets and (3.6) is fulfilled. Then we set
\[ \Phi(\omega) := \sum_{k \in \mathbb{Z}_+} |\hat{\varphi}(\omega \ominus k)|^2. \tag{3.7} \]
Obviously, \( \Phi \) is nonnegative and 1-periodic function. According to Proposition 5, it suffices to show that \( \Phi(\omega) \equiv 1 \). Let
\[ a = \inf \{ \Phi(\omega) | \omega \in [0, 1) \}. \]
From Propositions 2 and 3 it follows that \( \Phi \) is constant on each \( I_s^{(n-1)}, 0 \leq s \leq p^{n-1} - 1 \). Moreover, if \( \Phi \) vanishes on one of these intervals, then \( \hat{\varphi} \) has a periodic zero, and hence \( \varphi \) is unstable. On account of Proposition 4 and Lemma 1, this assertion contradicts a lack of blocked sets for \( m \). Hence, \( a \) is positive. Also, by the modified Strang–Fix condition (see Proposition 3), we have \( \Phi(0) = 1 \). Thus, \( 0 < a \leq 1 \).

Further, by (2.4) and (3.7) we obtain
\[ \Phi(\omega) = \sum_{l=0}^{p-1} |m(p^{-1}\omega \ominus p^{-1}l)|^2 \Phi(p^{-1}\omega \ominus p^{-1}l). \tag{3.8} \]
Now, let \( M_a = \{ \Phi(\omega) = a | \omega \in [0, 1) \} \). In the case \( 0 < a < 1 \) from (3.6) and (3.8) we see that for any \( \omega \in M_a \) the elements \( p^{-1}\omega \ominus p^{-1}l, l = 0, 1, \ldots, p - 1 \), belong either \( M_a \) or \( \text{Null } m \). Therefore, \( M_a \) is a blocked set, which contradicts the assumption. Thus, \( \Phi(\omega) \geq 1 \) for all \( \omega \in [0, 1) \). Hence from the equalities
\[ \int_0^1 \Phi(\omega) \, d\omega = \sum_{k \in \mathbb{Z}_+} \int_k^{k+1} |\hat{\varphi}(\omega)|^2 \, d\omega = \int_{\mathbb{R}_+} |\hat{\varphi}(\omega)|^2 \, d\omega = \|\varphi\|^2 \]
by Lemma 2 we have

$$\int_0^1 \Phi(\omega) \, d\omega = 1.$$ 

Once again applying the inequality $\Phi(\omega) \geq 1$ and using the fact that $\Phi$ is constant on each $I_s^{(n-1)}$, $0 \leq s \leq p^{n-1} - 1$, we conclude that $\Phi(\omega) \equiv 1$. \qed

**Proof of the theorem.** Suppose that $m$ satisfies condition (b) or (c). Then, by Proposition 7 and Lemma 3, the system $\{\phi(\cdot \ominus k) \mid k \in \mathbb{Z}_+\}$ is orthonormal in $L^2(\mathbb{R}_+)$. Let us define the subspaces $V_j$, $j \in \mathbb{Z}_+$ by the formula (1.6). By Proposition 6 we have $\bigcap V_j = \{0\}$. The embeddings $V_j \subset V_{j+1}$ follow from the fact that $\phi$ satisfies the Eq. (1.1). The equality

$$\bigcup V_j = L^2(\mathbb{R}_+)$$

is proved in just the same way as (2.14) in [5] (cf. [3, Section 5.3]). Thus, the implications (b) $\Rightarrow$ (a) and (c) $\Rightarrow$ (a) are true. The inverse implications follow directly from Proposition 7 and Lemma 3. \qed

4. **On matrix extension and $p$-wavelet construction**

Following the standard approach (e.g., [11,18]), we reduce the problem of $p$-wavelet decomposition to the problem of matrix extension. More precisely, we shall discuss the following procedure to construct orthogonal $p$-wavelets in $L^2(\mathbb{R}_+)$:

1. Choose numbers $b_s$ such that equalities (1.5) are true.
2. Compute $a_\alpha$ by (1.4) and verify that the mask

$$m_0(\omega) = \sum_{\alpha=0}^{p^{n-1}} a_\alpha w_\alpha(\omega)$$

has no blocked sets.
3. Find

$$m_l(\omega) = \sum_{\alpha=0}^{p^{n-1}} a_\alpha^{(l)} w_\alpha(\omega), \quad 1 \leq l \leq p - 1,$$

such that $(m_l(\omega + k/p))_{l,k=0}^{p-1}$ is an unitary matrix.
4. Define $\psi_1, \ldots, \psi_{p-1}$ by the formula

$$\psi_l(x) = p \sum_{\alpha=0}^{p^{n-1}} a_\alpha^{(l)} \varphi(p \cdot \ominus \alpha), \quad 1 \leq l \leq p - 1. \quad (4.1)$$

In the $p = 2$ case, one can choose $a_\alpha^{(1)} = (-1)^\alpha a_{\alpha \ominus 1}$ for $0 \leq \alpha \leq 2^n - 1$ (and $a_\alpha^{(1)} = 0$ for the rest $\alpha$). Then $m_1(\omega) = -w_1(\omega)m_0(\omega \oplus 1/2)$, the matrix

$$\begin{pmatrix}
m_0(\omega) & m_0(\omega \oplus 1/2) \\
m_1(\omega) & m_1(\omega \oplus 1/2)
\end{pmatrix}$$
is unitary and, as in [8], we obtain

$$\psi(x) = 2 \sum_{\alpha=0}^{2^n-1} (-1)^{\alpha} \tilde{a}_\alpha \phi(2x \oplus \alpha).$$

In particular, if $n = 1$ and $a_0 = a_1 = 1/2$, then $\psi$ is the classical Haar wavelet.

In the $p > 2$ case, we take the coefficients $a_\alpha$ as in Step 2 (so that $b_\alpha$ satisfy (1.5) and $m_0$ has no blocked sets). Then

$$\sum_{\alpha=0}^{p^n-1} |a_\alpha|^2 = \frac{1}{p^n}.$$  \hspace{1cm} (4.2)

In fact, Parseval’s relation for the discrete transforms (1.3) and (1.4) can be written as

$$\sum_{\alpha=0}^{p^n-1} |a_\alpha|^2 = \frac{1}{p^n} \sum_{\alpha=0}^{p^n-1} |b_\alpha|^2.$$  \hspace{1cm} (4.3)

Therefore (4.2) follows from (1.5). Now we define

$$A_{0k}(z) = \sum_{l=0}^{p^n-1-1} a_{k+p/l} z^l, \quad 0 \leq k \leq p - 1,$$

and introduce the polynomials $A_{lk}(z)$, $\deg A_{lk} \leq p^n - 1$, such that

$$m_l(\omega) = \sum_{k=0}^{p-1} w_k(\omega) A_{lk}(\overline{w_p(\omega)}), \quad 1 \leq l \leq p - 1.$$  \hspace{1cm} (4.4)

It follows immediately that

$$M(\omega) = A(w_p(\omega)) W(\omega),$$

where $M(\omega) := (m_l(\omega + k/p))_{l,k=0}^{p-1}$, $A(z) := (A_{lk}(z))_{l,k=0}^{p-1}$, and $W(\omega) := (w_l(\omega + k/p))_{l,k=0}^{p-1}$. The matrix $p^{-1/2} W(\omega)$ is unitary. Thus, by (4.4), unitarity of $M(\omega)$ is equivalent to that of the matrix $p^{-1/2} A(z)$ with $z = w_p(\omega)$. From this we claim that Step 3 of the procedure can be realized by some modification of the algorithm for matrix extension suggested by Lawton, Lee and Shen in [18] (see also [2, Theorem 2.1]).

We illustrate the described procedure by the following examples.

**Example 5.** Let

$$m_0(\omega) = \frac{1}{p} \sum_{\alpha=0}^{p-1} w_\alpha(\omega)$$

so that $a_0 = \cdots = a_{p-1} = 1/p$. Then, as in Example 1, we have $\varphi = 1_{[0, p^n-1)}$. Setting

$$m_l(\omega) = \frac{1}{p} \sum_{\alpha=0}^{p-1} \varepsilon_{l\alpha} w_\alpha(\omega), \quad 1 \leq l \leq p - 1,$$
we observe that \((m_1(\omega + k/p))_{|k|=0}^{p-1}\) is unitary for all \(\omega \in [0, 1)\). Indeed, the constant matrix
\[
(p^{-1}(e^{ik}_p))_{|k|=0}^{p-1}
\]
may be taken as \(A(z)\) in (4.4). Therefore we obtain from (4.1)
\[
\psi_1(x) = \sum_{\alpha=0}^{p-1} e^{ik}_p \varphi(p x \ominus \alpha), \quad 1 \leq l \leq p - 1.
\]

Note that the similar wavelets in the space \(L^2(\mathbb{Q}_p)\) were introduced by Kozyrev in [13]; in connection with these wavelets see also [1, p.450] and Example 4.1 in [12].

**Example 6.** Let \(p = 3, n = 2\). As in Example 3, we take \(a, b, c, \alpha, \beta, \gamma\) such that
\[
|a|^2 + |b|^2 + |c|^2 = |\alpha|^2 + |\beta|^2 + |\gamma|^2 = 1
\]
and then define \(a_0, a_1, \ldots, a_8\) using (1.4). In this case we have
\[
A_{00}(z) = a_0 + a_3z + a_6z^2, \quad A_{01}(z) = a_1 + a_4z + a_7z^2, \quad A_{02}(z) = a_2 + a_5z + a_8z^2.
\]
Now, we require
\[
a \neq 0, \quad \alpha = \overline{\alpha}, \quad a\overline{\alpha} + b\overline{\beta} + c\overline{\gamma} = \overline{a}.
\]
In particular, for \(0 < a < 1\) we can choose numbers \(\theta, t\) such that
\[
\cos(\theta - t) = \frac{a - a^2}{1 - a^2}
\]
and then set \(\alpha = a, r = \sqrt{1 - a^2}, \beta = r \cos \theta, \gamma = r \sin \theta, b = r \cos t, c = r \sin t\).

Under our assumptions the mask \(m_0\) has no blocked sets (see Example 3). Moreover, it follows from (4.2) and (4.5) that
\[
|A_{00}(z)|^2 + |A_{01}(z)|^2 + |A_{02}(z)|^2 = \frac{1}{3}
\]
for all \(z\) on the unit circle \(\mathbb{T}\). To see this, note that by a direct calculation
\[
|A_{00}(z)|^2 + |A_{01}(z)|^2 + |A_{02}(z)|^2 = \sum_{\alpha=0}^{8} |a_\alpha|^2 + 2\mathcal{R} \left[ (a_0\alpha_3 + a_1\alpha_4 + a_2\alpha_5)z \right] \\
+ 2\mathcal{R} \left[ (a_0\alpha_6 + a_1\alpha_7 + a_2\alpha_8)z^2 \right] + 2\mathcal{R} \left[ (a_3\alpha_6 + a_4\alpha_7 + a_5\alpha_8)z^2 \right],
\]
where
\[
27(a_0\alpha_3 + a_1\alpha_4 + a_2\alpha_5) = a + \alpha + (\alpha + a\alpha + b\beta + c\gamma)\epsilon_3 + (\alpha + a\alpha + b\beta + c\gamma)\epsilon_3^2, \\
27(a_0\alpha_6 + a_1\alpha_7 + a_2\alpha_8) = a + \alpha + (\alpha + a\alpha + b\beta + c\gamma)\epsilon_3 + (\alpha + a\alpha + b\beta + c\gamma)\epsilon_3^2, \\
27(a_0\alpha_6 + a_1\alpha_7 + a_2\alpha_8) = 2\epsilon_3 \mathcal{R} a + 2\epsilon_3^2 \mathcal{R} \alpha + 2\mathcal{R} (a\alpha + b\beta + c\gamma).
\]

Further, if
\[
\alpha_0 = \sqrt{3} (a_0, a_1, a_2), \quad \alpha_1 = \sqrt{3} (a_3, a_4, a_5), \quad \alpha_2 = \sqrt{3} (a_6, a_7, a_8),
\]
then
\[
|\alpha_0|^2 + |\alpha_1|^2 + |\alpha_2|^2 = 1, \quad \langle \alpha_0, \alpha_1 \rangle = \langle \alpha_0, \alpha_2 \rangle = \langle \alpha_1, \alpha_2 \rangle = 0,
\]
where $\langle \cdot, \cdot \rangle$ is the inner product in $\mathbb{C}^3$. It is clear that
\[
\alpha_0 + \alpha_1 z + \alpha_2 z^2 = \sqrt{3} \left( A_{00}(z), A_{01}(z), A_{02}(z) \right).
\]

Let $P_2$ be the orthogonal projection onto $\alpha_2$, i.e.,
\[
P_2 w = \frac{\langle w, \alpha_2 \rangle}{\langle \alpha_2, \alpha_2 \rangle} \alpha_2, \quad w \in \mathbb{C}^3.
\]

Then we have
\[
\left( I - P_2 + z^{-1} P_2 \right) \left( \alpha_0 + \alpha_1 z + \alpha_2 z^2 \right)
= \left( I - P_2 \right) \alpha_0 + P_2 \alpha_1 + z \left( P_2 \alpha_2 + (I - P_2) \alpha_1 \right) =: \beta_0 + \beta_1 z.
\]

One now verifies that
\[
|\beta_0|^2 + |\beta_1|^2 = 1, \quad (\beta_0, \beta_1) = 0.
\]

Furthermore, if $P_1$ is the orthogonal projection onto $\beta_1$, then
\[
\left( I - P_1 + z^{-1} P_1 \right) (\beta_0 + \beta_1 z) = (I - P_1) \beta_0 + P_1 \beta_1 =: \gamma_0.
\]

By the Gram–Schmidt orthogonalization, we can find an unitary matrix $I_0$ once the first row of this matrix is the unit vector $\gamma_0$. Then we set
\[
I'_1(z) = (I - P_1 + z P_1) I_0 \quad \text{and} \quad I'_2(z) = (I - P_2 + z P_2) I'_1(z).
\]

The first row of $I'_2(z)$ coincides with $\alpha_0 + \alpha_1 z + \alpha_2 z^2$. Putting
\[
(A_{l k}(z))_{l, k = 0}^2 = \frac{1}{\sqrt{3}} I'_2(z),
\]
we see that $m_1$ and $m_2$ can be defined as follows:
\[
m_1(\omega) = \sum_{k=0}^{2} w_k(\omega) A_{l k}(w_3(\omega)) = \sum_{a=0}^{8} a^{(l)}_a w_a(\omega), \quad l = 1, 2.
\]

Finally, we find
\[
\psi_l(x) = 3 \sum_{a=0}^{8} a^{(l)}_a \varphi(3 x \ominus \alpha), \quad l = 1, 2.
\]

Note that for the space $L^2(\mathbb{Q}_p)$ the corresponding wavelets were introduced recently in [12].

5. Adapted $p$-wavelet approximation

Suppose that a $p$-refinable function $\varphi$ generates a $p$-MRA in $L^2(\mathbb{R}_+)$ and subspaces $V_j$ are given by (1.6). For each $j \in \mathbb{Z}$ denote by $P_j$ the orthogonal projection of $L^2(\mathbb{R}_+)$ onto $V_j$. Given $f$ in $L^2(\mathbb{R}_+)$ it is naturally to choose parameters $b_s$ in (1.5) such that the approximation method $f \approx P_j f$ will be optimal. If $f$ belongs to some class $\mathcal{M}$ in $L^2(\mathbb{R}_+)$ then it is possible to seek the parameters $b_s$, which minimize for some fixed $j$ the quantity
\[
\sup \{ \| f - P_j f \| \mid f \in \mathcal{M} \}
\]
and to study the behavior of this quantity as \( j \to +\infty \). Also, it is very interesting investigate \( p \)-wavelet approximation in the \( p \)-adic Hardy spaces (cf. \([10,14]\)).

By analogy with \([23]\) we discuss here another approach to the problem on optimization of the approximation method \( f \approx P_j f \). For every \( j \in \mathbb{Z} \) denote by \( W_j \) the orthogonal complement of \( V_j \) in \( V_{j+1} \) and let \( Q_j \) be the orthogonal projection of \( L^2(\mathbb{R}_+) \) to \( W_j \). Since \( \{V_j\} \) is a \( p \)-MRA, for any \( f \in L^2(\mathbb{R}_+) \) we have

\[
 f = \sum_j Q_j f = P_0 f + \sum_{j \geq 0} Q_j f
\]

and

\[
 \lim_{j \to +\infty} \| f - P_j f \| = 0, \quad \lim_{j \to -\infty} \| P_j f \| = 0.
\]

It is easily seen, that

\[
 P_j f = Q_{j-1} f + Q_{j-2} f + \cdots + Q_{j-s} f + P_{j-s} f, \quad j \in \mathbb{Z}, \ s \in \mathbb{N}.
\]

The equality \( V_j = V_{j-1} \oplus W_{j-1} \) means that \( W_{j-1} \) contains the “details” which are necessary to get over the \((j-1)\)th level of approximation to the more exact \( j \)th level. Since

\[
 \| P_j f \|^2 = \| P_{j-1} f \|^2 + \| Q_{j-1} f \|^2,
\]

it is natural to choose the parameters \( b_s \) to maximize \( \| P_{j-1} f \| \) (or, equivalently, to minimize \( \| Q_{j-1} f \| \)). To this end let us write Eq. (1.1) in the form

\[
 \varphi(x) = \sqrt{p} \sum_{\alpha=0}^{p^n-1} \tilde{a}_\alpha \varphi(p x \ominus h_\alpha),
\]

where \( \tilde{a}_\alpha = \sqrt{p} \ a_\alpha \). Putting \( \varphi_j(x) = p^{j/2} \varphi(p^j x) \), we have

\[
 \varphi_{j-1}(x) = \sum_{\alpha=0}^{p^{n-1}} \tilde{a}_\alpha \varphi_j(x \ominus p^{-j} \alpha), \quad \text{(5.1)}
\]

where \( \varphi_j(x \ominus p^{-j} \alpha) = \varphi_{j,k}(x) \). Further, given \( f \in L^2(\mathbb{R}_+) \) we set

\[
 f(j,k) := \langle f, \varphi_{j,k} \rangle = \int_{\mathbb{R}_+} f(x) \varphi_j(x \ominus p^{-j} k) \, dx.
\]

Applying (5.1), we obtain

\[
 f(j-1,k) = \int_{\mathbb{R}_+} \varphi_{j-1}(x \ominus p^{-j+1} k) \, dx
 = \sum_{\alpha=0}^{p^{n-1}} \tilde{a}_\alpha \int_{\mathbb{R}_+} f(x) \varphi_j(x \ominus p^{-j}(p k \oplus \alpha)) \, dx
\]

and hence

\[
 f(j-1,k) = \sum_{\alpha=0}^{p^{n-1}} \tilde{a}_\alpha f(j, p k \oplus \alpha). \quad \text{(5.2)}
\]
Since
\[ P_j f = \sum_{k \in \mathbb{Z}_+} f(j, k)\varphi_{j,k} , \]
we see from (5.2) that
\[ \| P_{j-1} f \|^2 = \sum_{k \in \mathbb{Z}_+} |f(j - 1, k)|^2 = \sum_{k \in \mathbb{Z}_+} \left| \sum_{\alpha = 0}^{p^n - 1} \bar{a}_\alpha f(j, p k \oplus \alpha) \right|^2 = \sum_{k \in \mathbb{Z}_+} \left( \sum_{\alpha, \beta = 0}^{p^n - 1} \bar{a}_\alpha \bar{a}_\beta f(j, p k \oplus \alpha) f(j, p k \oplus \beta) \right) . \]  
(5.3)

For \( 0 \leq \alpha, \beta \leq p^n - 1 \) we let
\[ F_{\alpha, \beta}(j) := \sum_{k \in \mathbb{Z}_+} f(j, p k \oplus \alpha) f(j, p k \oplus \beta) . \]
Then \( F_{\beta, \alpha}(j) = \overline{F_{\alpha, \beta}(j)} \) and (5.3) implies
\[ \| P_{j-1} f \|^2 = \sum_{\alpha, \beta = 0}^{p^n - 1} F_{\alpha, \beta}(j) \bar{a}_\alpha \bar{a}_\beta . \]  
(5.4)

Denote by \( \mathcal{U}(p, n) \) the set of vectors \( u = (u_0, u_1, \ldots, u_{p^n - 1}) \) such that
\[ u_0 = 1, \quad u_j = 0 \quad \text{for} \quad j \in \{p^n - 1, 2p^n - 1, \ldots, (p - 1)p^n - 1\} , \]
and
\[ \sum_{l=0}^{p^n - 1} |u_{(p^n - 1) + l}|^2 = 1 \quad \text{for} \quad j \in \{1, 2, \ldots, p^n - 1\} . \]
For every \( u = (u_0, u_1, \ldots, u_{p^n - 1}) \) in \( \mathcal{U}(p, n) \) we define \( a_\alpha(u) \) by the formulas
\[ a_\alpha(u) = \frac{1}{p^n} \sum_{s=0}^{p^n - 1} u_s w_\alpha(s/p^n) , \quad 0 \leq \alpha \leq p^n - 1 . \]

Fix a positive integer \( j_0 \). If a vector \( u^* \) is a solution of the extremal problem
\[ \sum_{\alpha, \beta = 0}^{p^n - 1} F_{\alpha, \beta}(j_0) \bar{a}_\alpha(u^*) \bar{a}_\beta(u^*) \rightarrow \max, \quad u \in \mathcal{U}(p, n) , \]  
(5.5)
then \( \varphi_{j_0 - 1}^* \) is defined by
\[ \varphi_{j_0 - 1}^*(x) = \sum_{\alpha = 0}^{p^n - 1} a_\alpha(u^*) \varphi_{j_0}(x \ominus p^{-j_0} \alpha) . \]

It is seen from (5.4) and (5.5) that \( \| P_{j_0}^* f \| \geq \| P_j f \| \) for \( j = j_0 - 1 \). Now, if the mask of \( \varphi_{j_0 - 1}^* \) has no blocked sets, then \( \varphi_{j_0 - 2}^* \) is constructed by \( \varphi_{j_0 - 1}^* \) and so on. Finally, we fix \( s \) and for each
j ∈ {j_0 − 1, . . . , j_0 − s} replace P_j f by the orthogonal projection P^*_j f of f to the subspace
\[ V^*_j = \text{clos}_{L^2(\mathbb{R}^+)} \text{span} \{ \varphi^*_j k | k ∈ \mathbb{Z}^+ \}. \]

The effectiveness of this method of adaptation can be illustrated by numerical examples in terms (cf. [20]) of the entropy estimates.

References