Non-Abelian Aharonov–Bohm effect with the time-dependent gauge fields

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A B S T R A C T

We investigate the non-Abelian Aharonov–Bohm (AB) effect for time-dependent gauge fields. We prove that the non-Abelian AB phase shift related to time-dependent gauge fields, in which the electric and magnetic fields are written in the adjoint representation of $SU(N)$ generators, vanishes up to the first order expansion of the phase factor. Therefore, the flux quantization in a superconductor ring does not appear in the time-dependent Abelian or non-Abelian AB effect.

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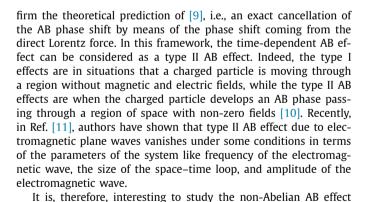
1. Introduction

In 1959, Y. Aharonov and D. Bohm proposed an experiment to test the effect of the electromagnetic gauge potential on the quantum wave function [1]. Later, Chambers performed the proposed experiment and proved that the effect did exist [2]. The AB effect is indeed a quantum-mechanical phenomenon in which the wave function of a charged particle traveling around an extremely long solenoid undergoes a phase shift depending on the magnetic field between the paths albeit $B = 0$ along the paths themselves [1,2].

Over the past few years, considerable interest has been shown in the AB effect in Abelian gauge fields with a time-independent magnetic field. Recently, the AB effect with a time-dependent magnetic field has been investigated [3–7] to show that a cancellation of phases occurs in the AB effect with a time-dependent magnetic field. Strictly speaking, an extra phase coming from the electric field, $E = -\partial_t A$, outside the solenoid cancels out the phase shift of the time-dependent magnetic field. The experimental results of Marton et al. [8], where the effect of the time variation of the magnetic field was not seen in the interference pattern, also con-

firm the theoretical prediction of [9], i.e., an exact cancellation of the AB phase shift by means of the phase shift coming from the direct Lorentz force. In this framework, the time-dependent AB effect can be considered as a type II AB effect. Indeed, the type I effects are in situations that a charged particle is moving through a region without magnetic and electric fields, while the type II AB effects are when the charged particle develops an AB phase passing through a region of space with non-zero fields [10]. Recently, in Ref. [11], authors have shown that type II AB effect due to electromagnetic plane waves vanishes under some conditions in terms of the parameters of the system like frequency of the electromagnetic wave, the size of the space–time loop, and amplitude of the electromagnetic wave.

It is, therefore, interesting to study the non-Abelian AB effect [12] with a time-dependent magnetic field. Recently, the AB effect has been studied for time-dependent non-Abelian fields by using two specific, known time-dependent solutions [13] such as the Coleman plane wave solutions [14] and the time-dependent Wu–Yang monopole [15]. Here, we prove that when the non-Abelian gauge field A^a is a function of space–time, the AB phase shift coming from the electric and magnetic non-Abelian fields will be canceled out up to the first order. Our results also show that the "single-valuedness of the wave function" does not constrain the flux of a time-dependent magnetic field to be quantized in a superconducting ring. It will be interesting to verify this result experimentally.

The outline of this paper is as follows. Section 2 presents a description of the AB phase shift. In Section 3, we study the quantization of the magnetic flux in a superconducting ring. We will show that there is no "single-valuedness" condition for the wave function because the phase shift will be zero in a time-dependent AB effect. In Section 4, we generalize the Abelian AB effect to a time-dependent non-Abelian field configuration. We prove that the AB phase factor remains equal to zero up to the first order when considering the time-varying vector fields. Conclusions will be presented in the last section.

2. Time-dependent AB effect for Abelian gauge fields

The relativistic form of the AB phase factor can be written as follows:

$$\beta = \exp\left[\frac{e}{\hbar}\oint A_\mu dx^\mu\right] = \exp\left[\frac{e}{\hbar}\oint \varphi dt - A.dx\right] \quad (1)$$

where A^μ is the Abelian gauge field that might be transformed under the $U(1)$ group as follows:

$$A^\mu \rightarrow A^{\mu'} = A^\mu + \partial^\mu \xi \quad (2)$$

where ξ is a transformation function of space–time coordinates [16,17]. We may rewrite the phase factor in a 2-form structure by making use of Stokes' theorem, stating that the integral of a differential form ω over the boundary of some orientable manifold Ω is equal to the integral of its exterior derivative $d\omega$ over the whole of Ω, which may be expressed as follows:

$$\int_{\partial\Omega} \omega = \int_\Omega d\omega \quad (3)$$

where ω and $d\omega$ are p-form and $(p+1)$-form, respectively. One could also define the 1-form as $\omega = A = A_\mu dx^\mu$ and 2-form $d\omega = dA$ as the Faraday 2-form F by:

$$dA = F = \frac{1}{2}F_{\mu\nu}dx^\mu \wedge dx^\nu$$
$$= (E_x dx + E_y dy + E_z dz) \wedge dt + B_x dy \wedge dz$$
$$+ B_y dz \wedge dx + B_z dx \wedge dy \quad (4)$$

where E and B are the electric and magnetic fields, respectively. Therefore, Eq. (1) can be rewritten as in (5) below:

$$\beta = \exp\left[-\frac{e}{2\hbar}\int F_{\mu\nu}dx^\mu \wedge dx^\nu\right] \quad (5)$$

This expression plays a key role in the study of the AB phase factor when considering the time-dependent Abelian gauge fields. Time-dependent AB effect is based on constructing a subspace in a space–time in which the four-vector potential depends on time [18,19]. Both the electric and the magnetic effects depend on the particle's particular path in this subspace [4]. We assume that the magnetic field inside the solenoid is time-dependent so that the vector potential A will be time-dependent outside the solenoid. However, based on Maxwell's equation, i.e., $E = -\partial_t A$, an electric field is also created outside the solenoid (we have assumed the scalar potential field φ to be zero). Thus, from Eqs. (4) and (5), the magnetic phase factor is obtained by:

$$\frac{e}{\hbar}\int \left[B_x dy \wedge dz + B_y dz \wedge dx + B_z dx \wedge dy\right] = \frac{e}{\hbar}\int B(x,t).dS \quad (6)$$

and the electric part of the phase is given by:

$$\frac{e}{\hbar}\int \left[E_x dx \wedge dt + E_y dy \wedge dt + E_z dz \wedge dt\right] = -\frac{e}{\hbar}\oint A.dx$$
$$= -\frac{e}{\hbar}\int B(x,t).dS \quad (7)$$

where we have replaced the electric field by $-\partial_t A$. It is clear that the AB phase shift for a time-varying magnetic field vanishes. This means that the magnetic AB phase shift is canceled out by a phase shift coming from the Lorentz force associated with the electric field, $E = -\partial_t A$, outside the solenoid [3].

3. Non-flux quantization in superconducting rings for time-dependent magnetic fields

Let us now consider a superconducting ring with rigid walls which is exposed to an external uniform magnetic field. Assuming a particle of charge e completely confined in the interior shell of the superconducting ring, one can obtain the relevant energy eigenvalues and wave functions [20–22]. However, care must be taken to ensure that the value of the wave function at any given point in the ring has the same value as the wave function obtained by traveling around the ring to return back to the original point. In other words, the wave function must have a single value at a given point in the ring.

In this case, the variation of the wave function phase is

$$\delta\alpha_B = \frac{2e}{\hbar}\oint A.dl = \frac{2e}{\hbar}\int \nabla \times A.dS = \frac{2e}{\hbar}\int B.dS = \frac{2e}{\hbar}\Phi \quad (8)$$

where Φ is the magnetic flux and the factor $2e$ shows that the Cooper pairs [23] in the superconductor have charges twice that of an electron. In order to maintain the single-valuedness of the wave function, this phase factor must be equal to $2\pi n$ ($n = 1, 2, 3, ...$), so that we can obtain the following quantum flux:

$$\Phi_n = \frac{\hbar\pi n}{e} = \frac{hn}{2e} \qquad n = 1, 2, 3, ... \quad (9)$$

Now, we consider a time-dependent magnetic field. According to Maxwell's equation, there is an electric field ($E = -\nabla\varphi - \partial_t A$) which creates an additional phase factor. Moreover, for this case, the scalar potential is still zero and the vector potential is a function of time and space. Based on Eq. (4), the relativistic phase shift will be zero due to the cancellation of the magnetic phase shift due to a phase shift coming from the electric field, $E = -\partial_t A$. As a result, there is no constraint on the magnetic flux Φ. It will be interesting to design an experimental plan to examine this effect.

4. Time-dependent AB effect for non-Abelian gauge fields

In section 2, we investigated the time-dependent AB effect [3, 5,6], and showed that there is no phase shift in this case. In this section, we will verify the claim that the phase shift of the non-Abelian AB effect is zero for time-dependent gauge fields. The concept of the non-Abelian gauge field was first introduced in 1954 by Yang and Mills [24]. The 4-vector gauge fields A_a were introduced with N internal components labeled by $a = 1, 2, 3, ..., N$, corresponding to the N-generators of the gauge group closed under the following commutation:

$$[L_a, L_b] = iC_{ab}^c L_c \quad (10)$$

where the constants C_{ab}^c are real numbers called structure constants. For simplicity, we shall use the shorthand $A^\mu = A^\mu_a L_a$, which is a matrix. Moreover, A^μ_a under an infinitesimally local gauge transformation can be written in the following form:

$$A^\mu_a(x) \rightarrow A^\mu_a(x) + \frac{1}{g}\partial^\mu \omega_a(x) + C_{abc}\omega_b(x)A^\mu_c(x) \quad (11)$$

where g and ω are the gauge coupling constant and arbitrary real functions, respectively. In Maxwell's $U(1)$ gauge theory, a gauge-invariant field tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is defined, whose components are the electric and magnetic fields; in the non-Abelian

case, however, such a field tensor is not gauge invariant or, indeed, there is no gauge-invariant field tensor. In order to define a gauge-invariant field tensor for the non-Abelian gauge fields, the representation must be the adjoint representation [25]. The following equation satisfies our requirements:

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g C^{abc} A_\mu^b A_\nu^c \qquad (12)$$

where the antisymmetric constants $C_{abc} = -i(L_b)_{ac}$ are defined in the adjoint representation. Using the above equation, the electric and the magnetic fields can be written as in the following equations [16]:

$$E_a = -\nabla A_a^0 - \partial_t A_a - g C_{abc} A_b A_c^0 \qquad (13)$$

$$B_a = \nabla \times A_a + \frac{1}{2} g C_{abc} A_b \times A_c \qquad (14)$$

It is surprising that the AB experiment can also be used to examine the existence of non-Abelian gauge fields [26,27]. One can generalize the phase factor of the Abelian AB effect to the non-Abelian AB one [12] using the following relation:

$$\beta = P\left[\exp\left[\frac{g}{\hbar} \oint A \right] \right] \qquad (15)$$

where $A = A_\mu^a L_a dx^\mu$ and P is the path-ordering operator. This phase factor is quite similar to Wilson loop [28,29]. Expanding this phase shift up to the second order, we will have:

$$\beta \simeq 1 + \frac{g}{\hbar} L_a \oint A_\mu^a dx^\mu +$$
$$P(\frac{g}{\hbar})^2 \oint \oint dx^\mu A^a{}_\mu(x) dx^\nu . A_\nu{}^b(x) L_a L_b + \dots \qquad (16)$$

We will now go on to show that the time-dependent non-Abelian AB phase shift vanishes up to the first order, while the other orders indicate a non-zero non-Abelian AB phase factor. Let us consider a 4-vector potential in the non-Abelian AB effect as $A_a^\mu \equiv (A_a^0, A_a^i) = (0, A_a^i(x,t))$. Therefore, from Eq. (13), the electric field will be a non-zero term ($E_a = -\partial_t A_a$). Applying the Stokes theorem (Eq. (3)), we will have:

$$d\omega = dA = dA^a L_a = \left(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a \right) L_a dx^\mu \wedge dx^\nu \qquad (17)$$

Based on Eq. (12), the above equation can be rewritten as:

$$dA^a = \frac{1}{2} F_{\mu\nu}^a dx^\mu \wedge dx^\nu - g C^{abc} A_\mu^b A_\nu^c dx^\mu \wedge dx^\nu \qquad (18)$$

where the factor $1/2$ comes from the anti-symmetry property of $F_{\mu\nu}$ and $dx^\mu \wedge dx^\nu$ [25]. Therefore, the second term of the expansion in Eq. (16) may be replaced with the following equation:

$$\frac{g}{\hbar} L_a \oint A_\mu^a dx^\mu$$
$$= \frac{g}{2\hbar} L_a \int F_{\mu\nu}^a dx^\mu \wedge dx^\nu - \frac{g^2}{\hbar} C^{abc} L_a \int A_\mu^b A_\nu^c dx^\mu \wedge dx^\nu \qquad (19)$$

where the 2-form tensor can be defined as

$$\frac{1}{2} F_{\mu\nu}{}^a dx^\mu \wedge dx^\nu = (E^a{}_x dx + E^a{}_y dy + E^a{}_z dz) \wedge dt$$
$$+ B^a{}_x dy \wedge dz + B^a{}_y dz \wedge dx + B_{az} dx \wedge dy \qquad (20)$$

One can then divide up the above equation into the two magnetic and electric parts. In this way, the phase difference associated with the magnetic field terms is given by:

$$\frac{g}{\hbar} \int \left[B_x^a dy \wedge dz + B_y^a dz \wedge dx + B_z^a dx \wedge dy \right] = \frac{g}{\hbar} \int B^a(x,t).dS \qquad (21)$$

Substituting B^a from Eq. (14) and using the Stokes theorem, we have:

$$\frac{g}{\hbar} \int B^a(x,t).dS = \frac{g}{\hbar} \oint A^a.dl + \frac{g^2}{2\hbar} C^{abc} \int (A^b \times A^c).dS \qquad (22)$$

On the other hand, the electric field part is given by:

$$\frac{g}{\hbar} \int \left[E_x^a dx \wedge dt + E_y^a dy \wedge dt + E_z^a dz \wedge dt \right] = -\frac{g}{\hbar} \oint A^a.dl \qquad (23)$$

where the electric field is $E_a = -\partial_t A_a$. Finally, one can obtain the phase shift from the 2-form tensor as follows:

$$\frac{g}{2\hbar} L_a \int F_{\mu\nu}^a dx^\mu \wedge dx^\nu = +\frac{g^2}{2\hbar} C^{abc} L_a \int (A^b \times A^c).dS \qquad (24)$$

Moreover, when considering the 4-vector potential $A_a^\mu \equiv (A_a^0, A_a^i) = (0, A_a^i(x,t))$, we can rewrite the second part of relation (19) as follows:

$$-\frac{g^2}{\hbar} C^{abc} L_a \int A_\mu^b A_\nu^c dx^\mu \wedge dx^\nu$$
$$= -\frac{g^2}{\hbar} C^{abc} L_a \int (A_j^b)(A_k^c) dx^j \wedge dx^k$$
$$= -\frac{g^2}{2\hbar} C^{abc} L_a \int (A^b \times A^c).dS \qquad (25)$$

in which the following wedge product is used:

$$dx^j \wedge dx^k = \frac{\varepsilon^{ijk}}{2} dx^j \otimes dx^k \equiv \frac{\varepsilon^{ijk}}{2} dS_i. \qquad (26)$$

Finally, using Eqs. (19), (24), and (25), we arrive at the following interesting result:

$$\frac{g}{\hbar} L_a \oint A_\mu^a dx^\mu = 0 \qquad (27)$$

Therefore, the phase shift related to the time-dependent non-Abelian AB effect vanishes up to the first order expansion of the phase factor. This is a generally valid result. For future research, it will be interesting to investigate the higher order terms of gauge fields. It may be anticipated that all higher order terms of gauge fields will also vanish. This conjecture cannot, however, be proved presently.

5. Conclusion

In this paper, we studied time-dependent Abelian and non-Abelian AB effects. We showed that for a superconductor exposed to a time-varying magnetic field, there is no constraint on the magnetic flux due to the presence of zero phase factor in both Abelian and non-Abelian AB effects. We also investigated the non-Abelian AB effect and proved that, for time-dependent non-Abelian magnetic fields, the AB phase disappears up to the first order in the expansion of the phase factor.

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