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ANOSOV DIFFEOMORPHISMS ARE TOPOLOGICALLY STABLE

PETER WALTERS

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§0

ANOSOV [1] and Moser [3] have shown that an Anosov diffeomorphism f of a compact manifold M is structurally stable. This means that in the space of all C^1 diffeomorphisms of M, with the C^1 topology, there is a neighbourhood of f such that every member of this neighbourhood is topologically equivalent to f. We show in this paper that f is also topologically stable. This means that in the space of all homeomorphisms of M, with the C^0 topology, there is a neighbourhood U of f in which f is the "simplest" map from a topological point of view, in the sense that if $g \in U$ then f is a continuous image of g. (See definition 2). The idea of the proof follows that of Moser [3].

§1

M will always denote a compact C^∞ manifold without boundary.

Definition 1. A C^1 diffeomorphism $f: M \to M$ is an Anosov diffeomorphism if there exists a Riemannian metric $\|\,.\,\|$ on M and constants $c>0$, $0<\lambda<1$ such that $TM = E^s \oplus E^u$ (Whitney bundle sum), $dfE^s = E^s$, $dfE^u = E^u$,

$$\|df^n w\| \le c\lambda^n \|w\| \text{ if } w \in E^s, \quad n > 0$$

and

$$\|df^m w\| \le c\lambda^{-m} \|w\| \text{ if } w \in E^u, \quad m < 0.$$

If a different Riemannian metric is chosen the same conditions hold with different constants c, λ. It is easily shown that the splitting $TM = E^s \oplus E^u$ is continuous.

$\mathfrak{X}^0(M)$, or $\mathfrak{X}^0$, will denote the real Banach space of continuous vector fields on M with

$$\|v\| = \sup_{x \in M} \|v(x)\| \quad v \in \mathfrak{X}^0.$$

(A continuous vector field on M is a continuous section of $\pi: TM \to M$ where π is the natural projection). If $f: M \to M$ is a diffeomorphism $F: \mathfrak{X}^0 \to \mathfrak{X}^0$ will denote the linear transformation defined by $Fv = dfvf^{-1}$. An equivalent way of defining an Anosov diffeomorphism is as follows: f is an Anosov diffeomorphism if there exists a Riemannian metric $\|\,.\,\|$ on M and constants $c > 0$, $0 < \lambda < 1$, such that $\mathfrak{X}^0 = \mathfrak{X}^0_s \oplus \mathfrak{X}^0_u$ (vector space direct sum), $F\mathfrak{X}^0_s = \mathfrak{X}^0_s$, $F\mathfrak{X}^0_u = \mathfrak{X}^0_u$,

$$\|F^n v\| \le c\lambda^n \|v\| \text{ if } v \in \mathfrak{X}^0_s, \quad n > 0$$

and

$$\|F^m v\| \leq c\lambda^{-m}\|v\| \text{ if } v \in \mathfrak{X}_u^0, \quad m < 0.$$

We assume that we have some fixed Riemannian metric $\|.\|$ on M. We denote by $d(x, y)$ the distance between $x, y \in M$ given by this Riemannian metric. $\rho > 0$ will denote a fixed number with the property that for each $x \in M$ the exponential map at x, $\exp_x$, is a diffeomorphism of the open ρ-ball about the origin in TM_x onto the open ρ-ball about x in M. Such a number exists by the compactness of M.

If f, g are continuous maps of M then $d(f, g) = \sup_{x \in M} d(f(x), g(x))$. *id* will denote the identity mapping of M. The following two definitions are meaningful for any homeomorphism f of a compact metric space (M, d).

Definition 2. $f: M \to M$ is topologically stable if $\exists \delta > 0$ with the property that if g is a homeomorphism of M with $d(f, g) < \delta$ there exists a continuous map φ of M onto M with $\varphi g = f\varphi$.

We shall in fact prove that Anosov diffeomorphisms are topologically stable in a stronger sense:

Definition 3. $f: M \to M$ is topologically stable in the strong sense if $\exists \varepsilon_0 > 0$ such that if $0 < \varepsilon < \varepsilon_0 \exists \delta > 0$ with the property that if g is a homeomorphism of M with $d(f, g) < \delta$ there exists a unique continuous map φ of M onto M with $\varphi g = f\varphi$ and $d(\varphi, id) < \varepsilon$.

§2

In this section we prove some lemmas which are used in the proof of the theorem.

If F^1 and F^2 are vector bundles over M with fibres F_x^1 and F_x^2 over $x \in M$, $L(F_x^1, F_x^2)$ denotes the collection of all linear transformations of F_x^1 to F_x^2. $L(F^1, F^2) = \bigcup_{x \in M} L(F_x^1, F_x^2)$ is a vector bundle over M with charts induced in a natural way from those of F^1 and F^2.

LEMMA 1. *Suppose TM is a continuous Whitney sum of two subbundles E^s and E^u. Let $\mathfrak{X}_s^0$ denote the space of continuous sections of E^s and $\mathfrak{X}_u^0$ the space of continuous sections of E^u. There exist real members $\tau_1, \tau_2 > 0$ with the following properties:*

(i) *If h is a homeomorphism of M with $d(h, id) < \tau_1$ there exists an invertible bounded linear transformation $J_h: \mathfrak{X}^0 \to \mathfrak{X}^0$ such that $J_h \mathfrak{X}_s^0 = \mathfrak{X}_s^0$ and $J_h \mathfrak{X}_u^0 = \mathfrak{X}_u^0$*

(ii) *If $v \in \mathfrak{X}^0$ with $\|v\| < \tau_2$ and h is as above there exists $t(v, h) \in \mathfrak{X}^0$ such that*

$$\exp_{h(x)} v(h(x)) = \exp_x[(J_h v)(x) + t(v, h)(x)], \qquad x \in M$$
$$\exp_x t(o, h)(x) = h(x)$$
$$\|J_h v + t(v, h)\| < \rho$$
$$\|t(v, h) - t(v', h)\| \leq K(h)\|v - v'\| \quad \textit{if} \quad \|v\|, \|v'\| < \tau_2,$$
$$\textit{where } K(h) \to 0 \textit{ as } d(h, id) \to 0.$$

(iii) *If $\|.\|_1$ is any continuous Riemannian metric on M then $\|J_h\|_1 \to 1$ and $\|J_h^{-1}\|_1 \to 1$ as $d(h, id) \to 0$.*

Proof. Let $p, q : M \times M \to M$ be defined by $p(x, y) = x$, $q(x, y) = y$. Let U be a neighbourhood of the diagonal in $M \times M$ so that $(x, y) \in U$ implies $d(x, y) < \rho/2$. $p^*(TM) = \{(x,y,v) \mid x = \pi(v)\}$ ($\pi : TM \to M$ is the natural projection)will denote the pull-back of TM by p, $p^*(TM)|_U$ the restriction *of* $p^*(TM)$ to U and $p^*(TM)|_{U, \rho/2}$ will denote those elements of $p^*(TM)|_U$ with length less than or equal to $\rho/2$ in the pull back metric (which we also denote by $\| . \|$). We define a map α: $p^*(TM)|_{U, \rho/2} \to q^*(TM)|_U$ by $\alpha(x, y, w) = (x, y, \exp_y^{-1} \exp_x w)$. α is well defined by the choice of U and is a "fibre map." The fibre derivative of α at the origin varies continuously in the following sense.

The map $U \to L(p^*(TM), q^*(TM))$ given by $(x, y) \to [d(\alpha|_{p^*(TM)(x, y)})]_0$ is continuous. Let $G_{(x, y)} = [d(\alpha|_{p^*(TM)(x, y)})]_0 \in L(p^*(TM)_{(x, y)}, q^*(TM)_{(x, y)})$. By the definition of derivative

$$\alpha(x, y, v) = G_{(x, y)}v + \alpha(x, y, 0) + \|v\|\beta(x, y, v)$$

if $(x, y, v) \in p^*(TM)_{(x, y)}$ and $\|v\| < \rho/2$, where $\beta(x, y, v) \to 0$ as $v \to 0$. Also $\beta(x, x, v) = 0$ since α is the identity over the diagonal of $M \times M$. Let $\pi_1 : q^*(TM) \to q^*(E^s)$ and $\pi_2 : q^*(TM) \to q^*(E^u)$ denote the natural projections. Define

$$\begin{aligned} A &: p^*(E^s) \to q^*(E^s) \quad \text{by} \quad A(x, y, v) = \pi_1 G_{(x, y)}v, \\ B &: p^*(E^u) \to q^*(E^s) \quad \text{by} \quad B(x, y, w) = \pi_1 G_{(x, y)}w, \\ C &: p^*(E^s) \to q^*(E^u) \quad \text{by} \quad C(x, y, v) = \pi_2 G_{(x, y)}v, \text{ and} \\ D &: p^*(E^u) \to q^*(E^u) \quad \text{by} \quad D(x, y, w) = \pi_2 G_{(x, y)}w. \end{aligned}$$

Then the map $U \to L(p^*(E^s), q^*(E^s))$ given by $(x, y) \to A|_{p^*(E^s)(x, y)}$ is continuous, and similarly for B, C and D.

If $v \in TM_x$ let $v = v_1 + v_2$ $v_1 \in E^s_x$ $v_2 \in E^u_x$. Then

$$\alpha(x, y, v) = A(x, y, v_1) + D(x,y,v_2) + B(x,y,v_2) + C(x,y,v_1) + \alpha(x,y,0) + \|v\|\beta(x,y,v)$$

if $\|v\| < \rho/2$, i.e.

$$\alpha(x, y, v) = A(x, y, v_1) + D(x, y, v_2) + \gamma(x, y, v),$$

and if v, $v^1 \in TM_x$ and $\|v\|$, $\|v'\| \le \rho/3$ then

$$\|\gamma(x, y, v) - \gamma(x, y, v')\| \le \|B|_{p^*(E^u)(x, y)}(v_2 - v_2')\| + \|C|_{p^*(E^s)(x, y)}(v_1 - v_1')\| + \|v - v'\| \sup_{\substack{x \in TM_x \\ \|w\| \le \rho/3}} \|\beta(x, y, w)\| \le K(x, y)\, \|v - v'\|$$

where $K(x, y) \to 0$ as $d(x, y) \to 0$ since $B|_{p^*(E^u)(x, x)} = 0$, $C|p_{*(E^s)(x, x)} = 0$ and $\beta(x, x, v) = 0$.

Let U_0 be a neighbourhood of the diagonal in $M \times M$ so that $A|_{p^*(E^s)U_0}$ and $D|_{p^*(E^u)U_0}$ are invertible. This is possible since $A|_{p^*(E^s)(x, x)} = I$ and $D|_{p^*(E^u)(x, x)} = I$. Choose $\tau_1 > 0$ so that $d(x,y) < \tau_1$ implies $(x, y) \in U_0$. Put $\tau_2 = \rho/3$. Suppose h is a homeomorphism of M with $d(h, id) < \tau_1$. Then $(h(y), y) \in U_0$, $y \in M$. Let $v \in \mathfrak{X}^0$ and $\|v\| < \tau_2$. Then

$$\begin{aligned} \alpha(h(y), y, v(h(y))) &= A(h(y), y, v_1(h(y))) + D(h(y), y, v_2(h(y))) + \gamma(h(y), y, v(h(y))) \\ &= (h(y), y, \exp_y^{-1} \exp_{h(y)}v(h(y))). \end{aligned}$$

Therefore $\exp_y^{-1} \exp_{h(y)}v(h(y)) = L_{h, y}\, v(h(y)) + t(v, h)(y)$ where $L_{h, y} : TM_{h(y)} \to TM_y$ is linear and sends $E^s_{h(y)}$ to E^s_y, $E^u_{h(y)}$ to E^u_y, and $(h(y), y, t(v, h)(y)) = \gamma(h(y), y, v(h(y)))$.

By the continuity of $L_{h,y}$ we can define $J_h : \mathfrak{X}^0 \to \mathfrak{X}^0$ by $(J_h v)(y) = L_{h,y} v(h(y))$, and if $v \in \mathfrak{X}^0$ then $t(v, h) \in \mathfrak{X}^0$. Hence $\exp_{h(y)} v(h(y)) = \exp_y[(J_h v)(y) + t(v, h)(y)]$. By construction $J_h \mathfrak{X}_s^0 = \mathfrak{X}_s^0$ and $J_h \mathfrak{X}_u^0 = \mathfrak{X}_u^0$. Also $\exp_y t(0, h)(y) = h(y)$ and $\|J_h v + t(v, h)\| < \rho$ if $\|v\| < \tau_2$, $v \in \mathfrak{X}^0$. By the above estimate on γ, $\|t(v, h) - t(v', h)\| \leq K(h) \|v - v'\|$ if $v, v' \in \mathfrak{X}^0$ and $\|v\|$, $\|v'\| < \tau_2$ where $K(h) = \sup_{y \in M} K(h(y), y)$. Hence $K(h) \to 0$ as $d(h, id) \to 0$.

By the continuity of A and D and the fact that $A|_{p^*(E^s)(x,x)} = I$ and $D|_{p^*(E^u)(x,x)} = I$ it follows that $\|J_h\|_1 \to 1$ and $\|J_h^{-1}\|_1 \to 1$ as $d(h, id) \to 0$ for any continuous Riemannian metric on M.

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COROLLARY. *Let f be a Anosov diffeomorphism of M with splitting $TM = E^s + E^u$, Anosov constants c, λ and let F: $\mathfrak{X}^0 \to \mathfrak{X}^0$ be defined by $(Fv)(x) = dfv(f^{-1}x)$. Using this splitting in Lemma 1 $\exists \tau_1 > 0$ such that J_h exists for each homeomorphism of M with $d(h, id) < \tau_1$. There exists $\tau_3 > 0$ such that if h is a homeomorphism of M with $d(h, id) < \tau_3$ then $I - J_h^{-1}F$ is invertible, where I is the identity mapping of $\mathfrak{X}^0$. Also*

$$\|(I - J_h^{-1}F)^{-1}\| \leq \frac{c\sqrt{N}}{1 - \lambda_1 \mu(h)}$$

where $\mu(h) \to 1$ as $d(h, id) \to 0$, $0 < \lambda_1 < 1$, N is an integer, and λ_1 and N are constants depending only on the Anosov constants c and λ.

Proof. We define a new norm $\| . \|_1$ on $\mathfrak{X}^0$ using a technique of Mather [2]. Choose an integer N so that $\lambda^N < \frac{1}{c}$. If $v_1 \in \mathfrak{X}_s^0$ let $\|v_1\|_1^2 = \sum_{k=0}^{N-1} \|F^k v_1\|^2$ and if $v_2 \in \mathfrak{X}_u^0$ let $\|v_2\|_1^2 = \sum_{k=0}^{N-1} \|F^{-k} v_2\|^2$. For $v \in \mathfrak{X}^0$, $v = v_1 + v_2$, $v_1 \in \mathfrak{X}_s^0$, $v_2 \in \mathfrak{X}_u^0$ put $\|v\|_1^2 = \|v_1\|_1^2 + \|v_2\|_1^2$. If $v \in \mathfrak{X}_s^0$ then $\|v\|_1 \leq \sqrt{Nc}\|v\|$ and

$$\begin{aligned}\|Fv\|_1^2 = \sum_{k=1}^{N} \|F^k v\|^2 &= \|v\|_1^2 - \|v\|^2 + \|F^N v\|^2 \\ &\leq \|v\|_1^2 - (1 - c^2\lambda^{2N})\|v\|^2 \\ &\leq \left(1 - \frac{(1 - c^2\lambda^{2N})}{Nc^2}\right)\|v\|_1^2.\end{aligned}$$

Hence $\|Fv\|_1 \leq \lambda_1 \|v\|_1$ if $\lambda_1^2 = 1 - \dfrac{(1 - c^2\lambda^{2N})}{Nc^2} < 1$. Similarly $\|F^{-1}v\|_1 \leq \lambda_1 \|v\|_1$ if $v \in \mathfrak{X}_u^0$. N and λ_1 depend on c and λ only.

If $d(h, id) < \tau_1$ let $\mu(h) = \max, \{\|J_h\|_1, \|J_h^{-1}\|_1\}$. by lemma 1 $\mu(h) \to 1$ as $d(h, id) \to 0$. Choose $\tau_3 > 0$ so that $\tau_3 < \tau_1$ and so that $d(h, id) < \tau_3$ implies $\mu(h)\lambda_1 < 1$. Let $F_s = F|\mathfrak{X}_s^0$, $F_u = F|\mathfrak{X}_u^0$, $J_{hs} = J_h|\mathfrak{X}_s^0$ and $J_{hu} = J_h|\mathfrak{X}_u^0$. Then

$$\|(J_{hs}^{-1}F_s)^k\|_1 \leq (\mu(h)\lambda_1)^k \quad k \geq 0$$

and

$$(I - J_{hs}^{-1}F_s)^{-1} = \sum_{k=0}^{\infty} (J_{hs}^{-1}F_s)^k$$

exists. Also

$$\|(F_u^{-1}J_{hu})^k\|_1 \leq (\mu(h)\lambda_1)^k \quad k \geq 0$$

and

$$(F_u^{-1}J_{hu} - I)^{-1} = -\sum_{k=0}^{\infty}(F_u^{-1}J_{hu})^k$$

exists. Hence

$$(J_{hu}^{-1}F_u - I)^{-1} = -(F_u^{-1}J_{hu} - I)^{-1}F_u^{-1}J_{hu}$$

exists and therefore $I - J_h^{-1}F$ is invertible.

We have

$$\|(I - J_h^{-1}F)^{-1}\|_1 \leq \sum_{k=0}^{\infty}(\mu(h)\lambda_1)^k = \frac{1}{1-\mu(h)\lambda_1},$$

and since $\frac{1}{\sqrt{Nc}}\|v\|_1 \leq \|v\| \leq \|v\|_1$ we have $\|(I - J_h^{-1}F)^{-1}\| \leq \frac{c\sqrt{N}}{1-\lambda_1\mu(h)}$.

The following two lemmas are well-known.

LEMMA 2. *Let $f: M \to M$ be a C^1 diffeomorphism. There exists τ_4 such that if $v \in \mathfrak{X}^0$ and $\|v\| < \tau_4$ there exists $s(v) \in \mathfrak{X}^0$ with*

$$f\exp_{f^{-1}(x)} v(f_{(x)}^{-1}) = \exp_x[dfv(f^{-1}(x)) + s(v)(x)], \qquad s(0) = 0,$$

$\|dfvf^{-1} + s(v)\| < \rho$ and $\|s(v) - s(v')\| < C(\tau_4)\|v - v'\|$ if $\|v\|, \|v'\| < \tau_4$, where $C(\tau_4) \to 0$ as $\tau_4 \to 0$.

Proof. Choose $\tau_4 < \rho$ so that $d(x, y) \leq \tau_4$ implies $d(f(x), f(y)) < \rho$. Let $x \in M$. If $T_{\tau_4}M_{f^{-1}(x)}$ denotes those elements $u \in TM_{f^{-1}(x)}$ with $\|u\| \leq \tau_4$ then the map $\beta_x : T_{\tau_4}M_{f^{-1}(x)} \to TM_x$ defined by $\beta_x(u) = \exp_x^{-1} f\exp_{f^{-1}(x)} u$ is well defined and differentiable at $u = 0$. The linear approximation of β_x at $u = 0$ is $df|_{TMf^{-1}(x)}$ and hence $\exp_x^{-1} f\exp_{f^{-1}(x)} u = dfu + \|u\|\sigma_x(u)$ where $\sigma_x(u) \to o$ *as* $u \to 0$. Let $v = \mathfrak{X}^0$, $\|v\| < \tau_4$. By the above

$$\exp_x^{-1} f\exp_{f^{-1}(x)} v(f^{-1}(x)) = dfv(f^{-1}(x)) + \|v(f^{-1}(x))\|\sigma_x(v(f^{-1}(x))$$

and if we put $s(v)(x) = \|v(f^{-1}x)\|\sigma_x(v(f^{-1}(x))$ then $s(v) \in \mathfrak{X}^0$ and

$$f\exp_{f^{-1}(x)} v(f^{-1}(x)) = \exp_x[dfv(f^{-1}(x)) + s(v)(x)].$$

Moreover $s(0) = 0$, $\|dfvf^{-1} + s(v)\| < \rho$ and if $v, v' \in \mathfrak{X}^0$, $\|v\|, \|v'\| < \tau_4$ then

$$\|s(v) - s(v')\| \leq \sup_{\substack{x \in M \\ w \in \mathfrak{X}^0 \\ \|X\| \leq \tau_4}} \|\sigma_x(w(f^{-1(x)}))\| \, \|v - v'\| = C(\tau_4)\|v - v'\|.$$

and $C(\tau_4) \to 0$ as $\tau_4 \to 0$.

LEMMA 3. *There exists $\tau_5 > 0$, depending only on the manifold M, with the property that if $\varphi : M \to M$ is continuous and $d(\varphi, id) < \tau_5$ then φ maps M onto M.*

Proof. Choose $\tau_5 > 0$ so that $d(\varphi, id) < \tau_5$ implies φ is homotopic to *id* and the result then follows by easy homology theory.

§3

THEOREM 1. *An Anosov diffeomorphism f of a compact manifold M is topologically stable in the strong sense i.e.* $\exists \varepsilon_0 > 0$ *such that* $\forall\, 0 < \varepsilon < \varepsilon_0\ \exists \delta > 0$ *with the property* $d(g,f) < \delta$, *g a homeomorphism of M,* $\Rightarrow \exists$ *a unique continuous map φ of M onto M with* $\varphi g = f\varphi$ *and* $d(\varphi, id) < \varepsilon$.

Proof. c, λ will denote the Anosov constants of f (using our fixed Riemannian metric), N, λ_1, τ_3, $\mu(h)$ denote the numbers obtained from the corollary of Lemma 1, τ_1 and τ_2 those obtained from Lemma 1 using the given splitting of TM, τ_4 and $C(\tau_4)$ those given by Lemma 2 applied to f and τ_5 the number determined by Lemma 3.

Choose $\varepsilon_0 > 0$ so that $\varepsilon_0 < \min(\rho, \tau_2, \tau_4, \tau_5)$ and so that $0 < \varepsilon < \varepsilon_0$ implies $c\sqrt{N}\,\frac{3}{2}\left(\frac{1}{1-\lambda_1}\right)C(\varepsilon) < \frac{1}{4}$. Let $0 < \varepsilon < \varepsilon_0$. Choose $\delta > 0$ so that $\delta < \min(\tau_1, \tau_3, \rho)$, $c\sqrt{N}\,\frac{3}{2}\left(\frac{1}{1-\lambda_1}\right)\delta < \frac{\varepsilon}{4}$ and so that $d(h, id) < \delta$, where h is a homeomorphism of M, implies $c\sqrt{N}\,\frac{3}{2}\left(\frac{1}{1-\lambda_1}\right)K(h) < \frac{1}{4}$ and $c\sqrt{N}\left(\frac{1}{1-\lambda_1\mu(h)}\right)\mu(h) < c\sqrt{N}\,\frac{3}{2}\left(\frac{1}{1-\lambda_1}\right)$.

Let g be a homeomorphism of M with $d(g,f) < \delta$. Put $h = gf^{-1}$, then $d(h, id) < \delta$ and h is a homeomorphism of M. Since $\varepsilon < \rho$ we wish to show that the equation $\varphi g = f\varphi$ has a unique solution of the form $\varphi(x) = \exp_x v(x)$ with $v \in \mathfrak{X}^0$ and $\|v\| < \varepsilon$. Therefore we have to solve the following equation uniquely for $v \in \mathfrak{X}^0$ with $\|v\| < \varepsilon$:

$$\exp_{h(x)} v(h(x)) = f \exp_{f^{-1}(x)} v(f^{-1}(x))$$

i.e. $\exp_x[(J_h v)(x) + t(v, h)(x)] = \exp_x[(Fv)(x) + s(v)(x)]$ by Lemmas 1 and 2. i.e. $J_h v + t(v, h) = Fv + s(v)$, since each side of this equation has length less than ρ. We now have an equation in the Banach space $\mathfrak{X}^0$ and rearrangement gives $(I - J_h^{-1}F)v = J_h^{-1}(s(v) - t(v, h))$. Since $\delta < \tau_3$ the corollary to Lemma 1 implies that $P = (I - J_h^{-1}F)^{-1}$ exists and

$$\|P\| \le c\sqrt{N}\left(\frac{1}{1-\lambda_1\mu(h)}\right).$$

We now have the equation

$$v = PJ_h^{-1}(s(v) - t(v, h)).$$

$\Phi(v) = PJ_h^{-1}(s(v) - t(v, h))$ is defined on the open ε_0-ball around the origin in $\mathfrak{X}^0$ and we wish to show that it has a unique fixed point in the open ε-ball. Let denote the closed $\frac{\varepsilon+\varepsilon_0}{2}$-ball about the origin in $\mathfrak{X}^0$. B is a complete metric space. We shall show that Φ is a contraction of B and that the unique fixed point determined by the contraction mapping theorem lies in the open ε-ball. We first show that Φ maps B to itself. If $v \in B$ then

$$\begin{aligned}\|\Phi(v)\| &= \|PJ_h^{-1}(s(v) - t(v, h))\| \\ &\le \|P\|\,\|J_h^{-1}\|[\|s(v)\| + \|t(v, h) - t(0, h)\| + \|t(0, h)\|] \\ &\le c\sqrt{N}\left(\frac{1}{1-\lambda_1\mu(h)}\right)\mu(h)[C(\varepsilon_0)\|v\| + K(h)\|v\| + \delta]\end{aligned}$$

by lemmas 1 and 2

$$\leq c\sqrt{N}\frac{3}{2}\left(\frac{1}{1-\lambda_1}\right)[C(\varepsilon_0)\|v\| + K(h)\|v\| + \delta]$$

$$\leq \frac{1}{4}\|v\| + \frac{1}{4}\|v\| + \frac{\varepsilon}{4} \leq \frac{\varepsilon+\varepsilon_0}{2}.$$

Φ is a contraction of B, since if v, $v' \in B$ then

$$\begin{aligned}\|\Phi(v) - \Phi(v')\| &\leq \|P\|\ \|J_h^{-1}\|[\|s(v) - s(v')\| + \|t(v,h) - t(v',h)\|]\\ &\leq c\sqrt{N}\left(\frac{1}{1-\lambda_1\mu(h)}\right)\mu(h)[C(\varepsilon_0)\|v-v'\| + K(h)\|v-v'\|]\\ &\leq \frac{3}{2}c\sqrt{N}\left(\frac{1}{1-\lambda_1}\right)[C(\varepsilon_0)\|v-v'\| + K(h)\|v-v'\|]\\ &\leq \tfrac{1}{2}\|v-v'\|.\end{aligned}$$

By the contraction mapping theorem Φ has a unique fixed point $v_0 \in B$. We have to show that $\|v_0\| < \varepsilon$.

$$\|v_0 - \Phi(0)\| = \|\Phi(v_0) - \Phi(0)\| \leq \tfrac{1}{2}\|v_0\|$$

by the above, and therefore

$$\|v_0\| \leq \|v_0 - \Phi(0)\| + \|\Phi(0)\| < \tfrac{1}{2}\|v_0\| + \|\Phi(0)\|.$$

Hence

$$\|v_0\| < 2\|\Phi(0)\| = 2\|PJ_h^{-1}(s(0) - t(0,h)\| \leq 2\|P\|\ \|J_h^{-1}\|\ \|t(0,h)\|$$

since $s(0) = 0$

$$< 2c\sqrt{N}\frac{3}{2}\left(\frac{1}{1-\lambda_1}\right)\delta < \frac{\varepsilon}{2} < \varepsilon.$$

We have obtained a unique continuous map φ such that $\varphi g = f\varphi$ and $d(\varphi, id) < \varepsilon$. φ maps M onto M by Lemma 3.

One cannot always expect the continuous map φ constructed above to be a homeomorphism. A hyperbolic toral automorphism, which has only finitely many fixed points, can be C^0-perturbed into a homeomorphism with an increased number of fixed points. However, we can put a condition on the perturbation g of f to ensure φ is a homeomorphism.

THEOREM 2. *If in Theorem* 1 *the perturbation g of f has the property that $x \neq y$ implies $d(g^n(x), g^n(y)) > 2\varepsilon$ for some integer n, then φ is a homeomorphism.*

(*This condition says g is an expansive homeomorphism of the metric space* (M, d) *with expansive constant* 2ε.)

Proof. For any integer n, $\varphi g^n = f^n\varphi$. Suppose $\varphi(x) = \varphi(y)$ and $x \neq y$. Then $\varphi g^n(x) = \varphi g^n(y)$ for every integer n. But for some integer n_0 $d(g^{n_0}(x), g^{n_0}(y)) > 2\varepsilon$ and this contradicts the fact that $\varphi(a) = \varphi(b)$ implies $d(a, b) < 20$.

The following result is a generalisation of Theorem 1.

THEOREM 3. *Let $f : M \to M$ be a Anosov diffeomorphism and suppose the homeomorphism $g_1 : M \to M$ is topologically conjugate to f by a homeomorphism ψ i.e. $\psi g_1 = f\psi$. $\exists \varepsilon_0 > 0$ such that $\forall 0 < \varepsilon < \varepsilon_0$, $\exists \delta > 0$ with the property that if g_2 is a homeomorphism of M with $d(g_1, g_2) < \delta$ then there exists a unique continuous map φ of M onto M so that $\varphi g_2 = f\varphi$ and $d(\varphi, \psi) < \varepsilon$.*

Proof. We wish to solve $\varphi g_2 = f\varphi$, or equivalently $\varphi g_2 g_1^{-1} = f\varphi \psi^{-1} f \psi$. Rearrangement gives $\varphi \psi^{-1}(\psi g_2 g_1^{-1} \psi^{-1}) = f\varphi \psi f^{-1}$. Putting $\pi = \varphi \psi^{-1}$ and $h = \psi g_2 g_1^{-1} \psi^{-1}$ we have to solve $\pi h f = f\pi$, for π near id, since $d(\pi, id) = d(\varphi, \psi)$. Also $d(hf, f) = d(\psi g_2, \psi g_1)$. By Theorem 1 $\exists \varepsilon_0 > 0$ such that $\forall\, 0 < \varepsilon < \varepsilon_0$, $\exists \delta_1 > 0$ so that if l is a homeomorphism of M with $d(l, f) < \delta_1$ there exists a unique continuous map π of M onto M with $\pi l = f\pi$ and $d(\pi, id) < \varepsilon$. Choose $\delta > 0$ so that $d(g_2, g_1) < \delta$ implies $d(\psi g_2, \psi g_1) < \delta_1$. Then $d(hf, f) < \delta_1$ and so there is a unique solution of $\pi h f = f\pi$ with $d(\pi, id) < \varepsilon$.

L. Zsido pointed out to me that Theorem 1 could be used to give a simple proof of the following known result. The previous proofs used either stable manifold theory or the (difficult to prove) C^1-closing lemma and structural stability.

THEOREM 4. *Let $f : M \to M$ be an Anosov diffeomorphism of a compact manifold. Then the periodic points are dense in the non-wandering set, $\Omega(f)$, of f.*

Proof. Let $\varepsilon > 0$ be given. Let $x_0 \in \Omega(f)$. We shall produce a periodic point of f within 2ε of x_0.

By Theorem 1, assuming $\varepsilon < \varepsilon_0$, $\exists \delta > 0$ with the property that $d(g, f) < \delta$ for g a homeomorphism of M implies the existence of a continuous map φ of M with $\varphi g = f\varphi$ and $d(\varphi, id) < \varepsilon$. We suppose $\delta < \varepsilon$ and that δ is so small that the $\delta/2$-ball, U, about x_0 is a coordinate chart. Since $x_0 \in \Omega(f)$ $\exists n > 0$ with $f^{-n}(U) \cap U \neq \varphi$ and we let n denote the least positive integer with this property. Let $y \in f^{-n}(U) \cap U$. Then $y \in U$, $f^i(y) \notin U$ for $0 < i < n$ and $f^n(y) \in U$. Choose a homeomorphism h of M which is the identity outside U and maps $f^{-n}(y)$ to y. Then $h \circ f$ is a homeomorphism of M and $d(h \circ f, f) = d(h, id) < \delta$. Hence $\exists$ a continuous map φ of M with $\varphi(h \circ f) = f\varphi$ and $d(\varphi, id) < \varepsilon$. But $(h \circ f)^n(y) = y$, by choice of n and h, and therefore $f^n \varphi(y) = \varphi(y)$ i.e. $\varphi(y)$ is a periodic point of f. Also $d(\varphi(y), x_0) \leq d(\varphi(y), y) + d(y, x_0) < \varepsilon + \delta/2 < 2\varepsilon$.

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Mathematics Institute,
University of Warwick