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J. Math. Pures Appl. 94 (2010) 131–169

JOURNAL DE MATHÉMATIQUES PURES ET APPLIQUÉES

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Mathematical study of the β-plane model for rotating fluids in a thin layer

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Received 10 July 2009

Available online 19 February 2010

Abstract

This article is concerned with an oceanographic model describing the asymptotic behaviour of a rapidly rotating and incompressible fluid with an inhomogeneous rotation vector; the motion takes place in a thin layer. We first exhibit a stationary solution of the system which consists of an interior part and a boundary layer part. The spatial variations of the rotation vector generate strong singularities within the boundary layer, which have repercussions on the interior part of the solution. The second part of the article is devoted to the analysis of two-dimensional and three-dimensional waves. It is shown that the thin layer effect modifies the propagation of three-dimensional Poincaré waves by creating small scales. Using tools of semiclassical analysis, we prove that the energy propagates at speeds of order one, i.e. much slower than in traditional rotating fluid models.

Résumé

On étudie ici le comportement asymptotique d'un fluide incompressible tournant à grande vitesse dans une couche mince, avec un vecteur rotation inhomogène ; ce type de modèle apparaît en océanographie. On commence par exhiber une solution stationnaire du système, obtenue comme la somme d'un terme intérieur et d'un terme de couche limite. Les variations spatiales du vecteur rotation génèrent de fortes singularités dans la couche limite, qui se répercutent dans la partie intérieure de la solution. Dans un second temps, on caractérise le comportement des ondes bi- et tri-dimensionnelles. L'effet de couche mince modifie la propagation des ondes de Poincaré (3D) en favorisant l'apparition de petites échelles. Grâce à une analyse de type semi-classique, on montre que la vitesse de propagation de l'énergie est d'ordre un, soit beaucoup plus faible que dans les modèles classiques de fluides tournants.

Keywords: Coriolis factor; Navier–Stokes system; Boundary layer

1. Introduction

The goal of this article is to study the behaviour of a rotating, incompressible and homogeneous fluid, whose **rotation vector depends on the (horizontal) space variable**. We also assume that the motion of the fluid takes place in a **thin layer**. These two features are inspired from models of oceanic circulation, which are the main physical

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0021-7824/$ – see front matter
doi:10.1016/j.matpur.2010.02.007

motivation for our study. We will explain more thoroughly the physical assumptions and scalings leading to our model in Section 1.1.

The mathematical framework of our analysis is the following: consider the equation:

$$\partial_t u + \frac{1}{\epsilon} b(x_h) e_3 \wedge u + \begin{pmatrix} \nabla_h p \\ \frac{1}{\eta^2}\partial_z p \end{pmatrix} - \nu_h \Delta_h u - \nu_z \partial_{zz} u = 0, \quad (x_h, z) \in \omega_h \times (0,1), \tag{1.1}$$

where the horizontal domain ω_h is either $\mathbf{T}^2$ or $\mathbf{T} \times \mathbf{R}$. Eq. (1.1) is endowed with Navier conditions at the bottom of the domain,

$$\partial_z u_{h|z=0} = 0, \qquad u_{3|z=0} = 0, \tag{1.2}$$

and we assume that there is a shear stress at the surface of the fluid, described by the boundary condition,

$$\partial_z u_{h|z=1}(t, x_h) = \gamma \sigma(x_h), \qquad u_{3|z=1} = 0. \tag{1.3}$$

Above, ϵ, η, ν_h, ν_z, γ are positive parameters, whose relative size will be precised later on. Let us merely announce that ϵ, η, ν_h, ν_z are meant to be small, whereas γ will be taken large. We emphasize that Eq. (1.1), supplemented with (1.2)–(1.3), is already in rescaled form. Hence all quantities are dimensionless. We refer to the next subsection for a derivation of this equation, and for a definition of the various parameters in terms of the physical quantities involved in the model.

Notice that the rotation is of order ϵ^{-1}, with $\epsilon \ll 1$; hence we focus on the limit of **high rotation**. As we will see in Section 1.1, the parameter η is the aspect ratio of the domain: assuming that $\eta \ll 1$ means that the characteristic horizontal length scale is much larger than the vertical one. In other words, the motion is set in a **thin layer**.

In this article, we are primarily interested in two topics:

- the computation of stationary solutions of our model;
- the analysis of the local stability of these stationary solutions in the case $\omega_h = \mathbf{T} \times \mathbf{R}$.

In particular, we will not address the full Cauchy problem here. Indeed, it can be proved that in the scaling which is the most relevant for our study, the energy estimates for the system (1.1)–(1.2)–(1.3) blow up in finite time. In a similar way, the stationary solution that we build has a size which becomes arbitrarily large as ϵ, η vanish. Hence the problem (1.1)–(1.2)–(1.3) is highly singular.

To our knowledge, the asymptotic analysis of the system (1.1) has not been addressed before: in the papers [8] by A. Dutrifoy and A. Majda, [11] by I. Gallagher and the second author, and [9] by A. Dutrifoy, A. Majda and S. Schochet, the authors study the asymptotic behaviour of a shallow water system within a β-plane model (i.e. in the case $b(x_h) = \beta x_2$). This shallow water system can be obtained by considering the limit $\eta \to 0$ in (1.1) (see [15]). Thus the studies of [11,9] are concerned with the successive limits $\eta \to 0$, $\epsilon \to 0$. In [7], B. Desjardins and E. Grenier take into account the thin layer effect within the original Navier–Stokes system, but they assume that $b(x_h) = 1 + \epsilon x_2$; hence the penalization is constant at first order. Our goal is to study a crossed limit $(\epsilon, \eta) \to (0, 0)$, with a rotation vector which has variations at the main order.

Let us now make precise the main novelties of our work: first, the construction of stationary solutions involves the definition of **boundary layer terms with a varying Coriolis factor** b. Since the size of the boundary layer is directly related to the amplitude of b, singularities appear at the vanishing points of b. These singularities in the boundary layer have repercussions on the interior part of the stationary solution, and make the construction much more involved than in the constant case. On the other hand, studying the stability of stationary solution when $\omega_h = \mathbf{T} \times \mathbf{R}$ amounts to describing the **waves in the β-plane model with a thin layer effect**. We exhibit new types of behaviour for the Poincaré waves, for which we prove that dispersion takes place on a time scale much larger than usual: for instance, in Chapter 2 of [12], the group velocity associated with Poincaré waves (i.e. the speed at which energy propagates) is of order ϵ^{-1}, while the group velocity in the present setting is of order one. The proof of this fact uses tools of semiclassical analysis, in the spirit of the recent papers by C. Cheverry, I. Gallagher, T. Paul and the second author (see [5,6]). Notice also that the presence of dispersion in our model reflects the fact that the domain is unbounded in the y-direction; however, in the case of the shallow water system within the β-plane model (see [8,11,9]), in which the domain is also $\mathbf{T} \times \mathbf{R}$, no dispersion occurs because equatorial waves are trapped into a waveguide. Hence the structure of the waves is physically quite different in the shallow water system and in our model.

In the next sections, we explain which physical assumptions led to the system (1.1). We then present our main results. Eventually, let us point out that the structure of the stationary solution which will be built in this article enforces particular shapes for the isothermal surfaces inside the fluid (the so-called "thermocline"). We present a few results in this regard in Section 1.4.

1.1. Physical derivation of Eq. (1.1)

Let us now explain in which regime oceanic currents can be modeled by Eq. (1.1). In this subsection, we denote by u' the velocity of oceanic currents in dimensional variables.

• As a starting point, we recall that the ocean can be considered as an incompressible fluid with variable density ρ'. In order to simplify the analysis, we **neglect the variations of density**, which are of order 10^{-3} in the ocean. Consequently, the velocity u' satisfies the Navier–Stokes equations, with a Coriolis term accounting for the rotation of the Earth

$$\rho'_0\big[\partial_t u' + \big(u'\cdot\nabla\big)u'\big] + \nabla p' = \mathcal{F} + \rho'_0 u'\wedge\Omega, \quad \nabla\cdot u' = 0, \tag{1.4}$$

where $\mathcal{F}$ denotes the frictional force acting on the fluid, Ω is the Earth rotation vector, p' is the pressure defined as the Lagrange multiplier associated with the incompressibility constraint, and ρ'_0 is the (constant) value of the density.

Since we have chosen to work on large horizontal scales (see below), Eq. (1.4) should be written in spherical coordinates. However, computations involving spherical coordinates are much lengthier, and do not change substantially the physical phenomena we wish to highlight, at least at a formal level (see [25]). Thus in the rest of the article, we neglect the **curvature of the Earth** (but we keep a varying Coriolis factor nonetheless). Note also that we neglect the influence of the horizontal component of the Earth rotation vector, which is classical in an oceanographic framework (see [12]). In fact, it is proved in [7,19] that when the aspect ratio is small, the effect of the vertical component (the so-called "cosine effect") can be neglected at first order, which justifies the present assumption. However, the cosine terms may modify the behaviour of the waves; this issue is left aside in the present paper.

The observed persistence over several days of large-scale waves in the oceans shows that **frictional forces** $\mathcal{F}$ are weak, almost everywhere, when compared with the Coriolis acceleration and the pressure gradient, but large when compared with the kinematic viscous dissipation of water. One common but not very precise notion is that small-scale motions, which appear sporadic or on longer time scales, act to smooth and mix properties on the larger scales by processes analogous to molecular, diffusive transports. For the present purposes it is only necessary to note that one way to estimate the dissipative influence of smaller-scale motions is to retain the same representation of the frictional force

$$\mathcal{F} = A_h\Delta_h u' + A_z\partial_{zz}u'$$

where A_z and A_h are respectively the vertical and horizontal turbulent viscosities, of much larger magnitude than the molecular value, supposedly because of the greater efficiency of momentum transport by macroscopic chunks of fluid. Notice that $A_z \neq A_h$ is therefore natural in a geophysical framework (see [25]). Moreover, models of oceanic circulation usually assume that the vertical viscosity A_z is not constant (see [2,24]); we choose to retain only the mean boundary value of the vertical viscosity A_z, since one of the motivations for our work was to compute the boundary layer terms in a context where Ω is not constant.

• Let us now describe the boundary conditions associated with (1.4): typically, Dirichlet boundary conditions are enforced at the bottom of the ocean and on the lateral boundaries of the horizontal domain ω'_h (the coasts), i.e.

$$\begin{aligned} u'_{|z'=h_B(x'_h)} &= 0 \quad \text{(bottom)},\\ u'_{|x'\in\partial\omega'_h} &= 0 \quad \text{(coasts)}. \end{aligned} \tag{1.5}$$

In Eq. (1.1), we have **neglected the effects of the lateral boundary conditions** by considering the case when ω_h is either $\mathbf{T}\times\mathbf{R}$ or $\mathbf{T}^2$. By doing so, we have deliberately prohibited the apparition of strong western boundary currents, which play a crucial role in the oceanic circulation (e.g. the Gulf Stream, the Kuroshio current). These horizontal boundary layers are believed to be responsible for the vertical structure of the ocean, and for the creation of large eddies. In the linear case, the mathematical treatment of these layers, called *Munk layers*, is performed by B. Desjardins and E. Grenier in [7]. Their study could probably be mimicked in the present paper without strong modifications;

however, we have chosen to leave this issue aside in order to focus on the other features of the model. Note that in the nonlinear case, the analysis of lateral boundary layers is completely open from a mathematical point of view.

In a similar fashion, for the sake of simplicity, we did not take into account the **topography of the bottom** in (1.2) (i.e. we have taken $h_B \equiv 0$), and we took Navier instead of Dirichlet boundary conditions, meaning that oceanic currents achieve perfect slip on the bottom. This choice simplifies the mathematical analysis, since it avoids the apparition of Ekman boundary layers on the lower boundary. The treatment of Ekman boundary layers in the case of a Dirichlet boundary condition with $h_B \equiv 0$ is in fact completely similar to the one of Ekman boundary layers due to the wind at the surface of the fluid, which is performed in Section 2. Hence changing Dirichlet into Navier boundary conditions is not a strong mathematical restriction. The case of Ekman boundary layers with a non-zero h_B has been addressed by B. Desjardins and E. Grenier [7], N. Masmoudi [22], and D. Gérard-Varet [14] in the case of a constant b, when h_B is of the order of the Ekman boundary layer (see below). In the present case, if the same assumption is satisfied, it can be checked that the case of a non-constant h_B can be treated with the same arguments as the ones in Section 2. To our knowledge, the case when $h_B = O(D)$ (where D is the average depth of the ocean) has never been addressed mathematically, although vertical sections of the Atlantic ocean show that this scaling is in fact relevant. In fact, formal calculations show that variations of h_B of order one deeply modify the asymptotic analysis; therefore the treatment of the case $h_B = O(D)$ most likely requires new mathematical techniques.

We assume that the upper surface, which we denote by Γ_s, has an equation of the type $z' = h_S(t, x'_h)$. As boundary conditions on Γ_s, we enforce (see [15])

$$\begin{gathered}\Sigma \cdot n_{|\Gamma_s} = \sigma_w,\\ \frac{\partial}{\partial t}\mathbf{1}_{0\leqslant z'\leqslant h_S(t,x'_h)} + \operatorname{div}_{x'}(\mathbf{1}_{0\leqslant z'\leqslant h_S(t,x'_h)}u) = 0,\end{gathered} \tag{1.6}$$

where Σ is the total stress tensor of the fluid, and σ_w is a given stress tensor describing the **wind on the surface of the ocean**. In general, Γ_s is a free surface, and a moving interface between air and water, which has its own self-consistent motion. In (1.3), we have assumed that

$$h_S(t, x'_h) \equiv D,$$

where D is the typical depth of the ocean. Hence (1.3) is a **rigid lid approximation**, which is a drastic, but standard simplification. The justification of (1.3) starting from a free surface is mainly open from a mathematical point of view; we refer to [1] for the derivation of Navier-type wall laws for the Laplace equation, under general assumptions on the interface, and to [17] for some elements of justification in the case of the great lake equations. Nevertheless, from a physical point of view, the simplification does not appear so dramatic, since in any case the free surface is so turbulent with waves and foam, that only modelization is tractable and meaningful. Condition (1.3) is a simple modelization which already catches most of the physical phenomena (see [25]).

• Let us now evaluate the order of magnitude of the different parameters occurring in (1.4), and write the equations in a dimensionless form. We set:

$$\begin{aligned} u'_h &= U u_h, \qquad & u'_3 &= W u_3,\\ x'_h &= H x_h, \qquad & z' &= D z,\end{aligned}$$

where U (resp. W) is the typical value of the horizontal (resp. vertical) velocity, H is the horizontal length scale, and D is the depth of the ocean. In order that $u'(x')$ remains divergence-free, we choose,

$$W = \frac{UD}{H}.$$

A typical value of the horizontal velocity for the mesoscale eddies that have been observed in the western Atlantic (see for instance [25]) is $U \sim 1\ \mathrm{cm\,s^{-1}}$. Moreover, the typical horizontal and vertical scales which we are interested in are:

$$H \sim 10^4\ \mathrm{km}, \quad \text{and} \quad D \sim 4\ \mathrm{km}.$$

Notice that we work on an almost planetary scale, which justifies the use of a varying rotation vector. Concerning the rotation, we write $\Omega = \Omega_0 \sin(\theta)$, where θ is the latitude, and $\Omega_0 = 2\pi/\text{day} \sim 7 \cdot 10^{-5}\ \text{s}^{-1}$. Eventually, we consider the motion on a typical time scale T, with T of the order of a few months ($T \sim 10^7$ s). With these values, we get:

$$\epsilon := \frac{1}{T\Omega_0} \sim 10^{-3},$$

and hence $\epsilon \ll 1$ (notice that the parameter ϵ is dimensionless). Thus the asymptotic of fast rotation (small Rossby number) is valid.

The dimensionless system (see for instance [25,16]) becomes:

$$\partial_t u + \frac{TU}{H} u \cdot \nabla u + \frac{1}{\epsilon} b(x_h) e_3 \wedge u + \begin{pmatrix} \nabla_h p \\ \frac{1}{\eta^2}\partial_z p \end{pmatrix} - \nu_h \Delta_h u - \nu_z \partial_{zz} u = 0,$$
$$\nabla \cdot u = 0, \tag{1.7}$$

where $\eta := D/H \sim 4 \cdot 10^{-4}$ is the aspect ratio, and the vertical and horizontal viscosities are defined by,

$$\nu_z := \frac{TA_z}{\rho_0 D^2}, \qquad \nu_h = \frac{A_h T}{\rho_0 H^2}.$$

Typical values for the turbulent viscosities are (see [16]) $A_z/\rho_0 \sim 10^{-4}$–$10^{-3}\ \text{m}^2\,\text{s}^{-1}$, and $A_h/\rho_0 \sim 10^4$–$10^5\ \text{m}^2\,\text{s}^{-1}$, which yields in the present case $\nu_z \sim 10^{-3}$ and $\nu_h \sim 10^{-10}$–10^{-9}.

The boundary conditions are (1.3), (1.2), with

$$\gamma := \frac{|\sigma_w| D}{A_z U}.$$

Notice that with the time scale chosen above, the **convective term** is of order $10^{-2} \ll 1$; hence we neglect it in the rest of the study. Note however that the effect of this term is expected to be large if the waves associated with (1.7) are resonant, and small if they are dispersive. Thus the rigorous treatment of the convective term requires a mathematical analysis which goes beyond the scope of this article, and which we deliberately leave aside from now on.

In the rest of the article, the **relative size of the parameters** will be chosen as follows: the most important feature of our analysis is that η and ϵ are chosen of the same order. In order to keep the number of different small parameters to a minimum, we also choose to take $\nu_z = \epsilon$, and $\gamma = \epsilon^{-2}$; with this last choice, the interior part of the stationary solution built in the next sections will be of order one. Concerning the size of ν_h, our analysis allows us to consider horizontal viscosities $\nu_h = o(\epsilon)$, which is compatible with the orders of magnitude given above.

The next subsections are devoted to the presentation of our main results: the existence of approximate stationary solutions, their stability, and the computation of asymptotic temperature profiles.

1.2. Stationary solutions of the system

Our first result is concerned with the construction of an approximate "stationary" solution of the system (1.1), endowed with the boundary conditions (1.2)–(1.3). The problem under consideration is rather different from the Cauchy problem, since no initial data is prescribed. The goal is merely to compute a solution of (1.1), and to investigate its asymptotic behaviour as ϵ vanishes. Furthermore, the stationary solution we compute is the sum of a boundary layer term and an interior term. As we will see, the solution obtained by this construction is uniquely determined, up to terms of the type $(u(y), 0, 0)$ (or lower order terms, since we only compute an approximate solution). We do not claim, however, that the stationary solution we build is unique among the whole class of approximate solutions. In some cases, it is possible to prove that all approximate stationary solutions are close to one another; we refer to Remark 1.3 below Theorem 1.2 for more details.

Remark 1.1 *(Influence of the convective term).* Of course, the analysis we perform is valid only in the linear case. If the convective term is kept, then the relevant equation for the waves should include a linearization of the quadratic term around the stationary solution, which substantially modifies the analysis. Moreover, if the convective term is of order one (i.e. $TU/H = O(1)$ with the notation of the previous section), the boundary layer terms should satisfy a non-linear degenerate elliptic equation, whose treatment is completely open from a mathematical point of view.

Let us now state our result about stationary solutions: since the vertical viscosity is small (we take $\nu_z = \epsilon \ll 1$), it disappears from the asymptotic system. As a consequence, solutions of the limit system cannot satisfy the boundary conditions. Thus boundary layer terms are introduced, which restore the correct boundary conditions. Hence the stationary solution built here is composed of an interior part and a boundary layer part.

We state our result in the case $\omega_h = \mathbf{T} \times \mathbf{R}$, and explain below the theorem the main differences when $\omega_h = \mathbf{T}^2$. Throughout the paper, we set,

$$\omega := \omega_h \times (0,1).$$

Theorem 1.2 *(Stationary solutions of (1.1)). Let $\omega_h = \mathbf{T} \times \mathbf{R}$.*

Assume that $\nu_h = o(\epsilon)$ and that $\eta = \nu_z = \epsilon$, $\gamma = \epsilon^{-2}$.

Let $\sigma \in H^2(\omega_h) \cap W^{2,\infty}(\omega_h)$ such that

$$\begin{aligned} &\left|\sigma(x,y)\right|, \qquad \left|\partial_x \sigma(x,y)\right| \leqslant C y^2 \quad \forall (x,y) \in \omega_h, \\ &\left|\partial_y \sigma(x,y)\right| \leqslant C|y| \quad \forall (x,y) \in \omega_h, \end{aligned} \tag{1.8}$$

and such that the following compatibility condition is satisfied,

$$\int_{\mathbf{T}} \sigma_1(x,y)\,dx = 0 \quad \forall y. \tag{1.9}$$

Assume that the Coriolis factor b satisfies the following assumptions:

$$\begin{aligned} &b(x,y) = b(y) \quad \forall (x,y) \in \omega_h, \quad \text{with } b \in W^{3,\infty}_{\mathrm{loc}}(\mathbf{R}), \\ &\exists c > 0, \quad c^{-1} \leqslant b'(y) \leqslant c \quad \forall y, \qquad b(y) \sim \beta y \quad \text{for } y \to 0. \end{aligned} \tag{1.10}$$

Then there exists stationary functions $(u^{\mathrm{stat}}, p^{\mathrm{stat}}) \in L^2(\omega) \cap H^1(\omega)$, such that u^{stat} satisfies (1.3), (1.2), *and*

$$\begin{aligned} &\frac{1}{\epsilon} b(y)\big(u_h^{\mathrm{stat}}\big)^{\perp} + \nabla_h p^{\mathrm{stat}} - \epsilon \partial_{zz} u_h^{\mathrm{stat}} - \nu_h \Delta_h u_h^{\mathrm{stat}} = r_h^1 + r_h^2, \\ &\frac{1}{\epsilon^2} \partial_z p^{\mathrm{stat}} - \epsilon \partial_{zz} u_3^{\mathrm{stat}} - \nu_h \Delta_h u_3^{\mathrm{stat}} = r_3^2, \end{aligned}$$

with

$$\begin{aligned} &r_h^1 = O\big(\epsilon^{5/4}\big) \quad \text{in } L^2(\omega), \qquad r_h^2 = O\left(\frac{\nu_h}{\sqrt{\epsilon}}\right) \quad \text{in } L^2\big([0,1], H^{-1}(\omega_h)\big), \\ &r_3^2 = O\left(\frac{\nu_h}{\epsilon^{1/4}}\right) \quad \text{in } L^2\big([0,1], H^{-1}(\omega_h)\big). \end{aligned}$$

Moreover, u^{stat} can be decomposed as

$$u^{\mathrm{stat}} = u^{\mathrm{BL}} + u^{\mathrm{int}},$$

where u^{BL} is a term located in a boundary layer of size ϵ, in the vicinity of the surface, and u^{int} is an interior term. The functions u^{BL} and u^{int} satisfy the following estimates:

$$\begin{aligned} &\big\|u^{\mathrm{int}}\big\|_{L^2(\omega)} \leqslant C\|\sigma\|_{H^2(\omega_h)}, \\ &\big\|u_h^{\mathrm{BL}}\big\|_{L^2(\omega)} \leqslant \frac{C}{\sqrt{\epsilon}}\|\sigma\|_{H^1(\omega_h)}, \\ &\big\|u_3^{\mathrm{BL}}\big\|_{L^2(\omega)} \leqslant C\|\sigma\|_{H^1(\omega_h)}. \end{aligned} \tag{1.11}$$

The above theorem ensures that the function u^{stat} is stable by the semi-group associated with Eq. (1.1) in the following sense: if $u_{ini} \in L^2(\omega)^3$ is such that $\|u_{ini} - u^{\mathrm{stat}}\|_{L^2_\epsilon} = o(1)$, then for all $T > 0$,

$$\sup_{t \in [0,T]} \big\|u(t) - u^{\mathrm{stat}}\big\|_{L^2_\epsilon} = o(1),$$

where $u(t)$ is the solution of (1.1)–(1.2)–(1.3) with initial data $u_{|t=0} = u_{ini}$. Above, the norm L^2_ϵ is the relevant norm for the energy conservation, namely

$$\|u\|^2_{L^2_\epsilon} := \|u_h\|^2_{L^2(\omega)} + \epsilon^2 \|u_3\|^2_{L^2(\omega)}.$$

Corollary 1.7 below gives a more refined version of this stability result.

Remark 1.3. As we have already pointed out, the stationary solutions of (1.1)–(1.2)–(1.3) are not unique in general. However, if u^{stat}, v^{stat} are two approximate stationary solutions of (1.1) with error terms which satisfy the estimates of Theorem 1.2, then a simple energy inequality shows that $w = u^{\text{stat}} - v^{\text{stat}}$ satisfies,

$$\nu_h \int_\omega \left(|\nabla_h w_h|^2 + \epsilon^2 |\nabla_h w_3|^2\right) + \epsilon \int_\omega \left(|\partial_z w_h|^2 + \epsilon^2 |\partial_z w_3|^2\right) \leqslant C\left(\frac{\nu_h}{\epsilon} + \epsilon^{5/4} \|w_h\|_{L^2(\omega)}\right).$$

If we know furthermore that $\|w_h\|_{L^2} = O(\epsilon^{-1/2})$, then we infer that

$$\nu_h \int_\omega \left(|\nabla_h w_h|^2 + \epsilon^2 |\nabla_h w_3|^2\right) + \epsilon \int_\omega \left(|\partial_z w_h|^2 + \epsilon^2 |\partial_z w_3|^2\right) \leqslant C\left(\frac{\nu_h}{\epsilon} + \epsilon^{3/4}\right) = o(1),$$

so that

$$u^{\text{stat}} - v^{\text{stat}} = o\left(\left\|u^{\text{stat}}\right\|\right). \tag{1.12}$$

If no estimates on w in L^2 are available, we need to derive one from the equation directly. For instance, one may use the Poincaré–Wirtinger inequality: setting

$$w_h = \bar{w}_h + \tilde{w}_h, \quad \text{where } \bar{w}_h := \int_0^1 w_h(\cdot, z)\, dz,$$

there holds,

$$\|\tilde{w}_h\|_{L^2} \leqslant C \|\partial_z w_h\|_{L^2}.$$

Thus there only remains to find a bound on $\bar{w}_h$. In the case of the β-plane approximation, for instance (i.e. $b(y) = \beta y$), it is convenient to work in Fourier space. We refer to Section 4 for the details of the calculation. We retrieve eventually,

$$\|\bar{w}_h\|_{L^2} \leqslant C\left(\frac{\epsilon^{5/4}}{\nu_h} + \epsilon^{-1/2}\right).$$

Thus as long as ν_h is not too small (i.e. if $\epsilon^{5/2} \ll \nu_h \ll \epsilon$), the (1.12) holds true.

If $\omega_h = \mathbf{T}^2$, the result of Theorem 1.2 remains true under slightly different conditions on σ and b. More precisely, we assume that $\sigma \in H^2(\mathbf{T}^2)$ satisfies (1.8), (1.9), and that

$$d\left(\operatorname{Supp} \sigma, \mathbf{T} \times \left\{\frac{1}{2}\right\}\right) > 0.$$

In other words, σ vanishes in a neighbourhood of $(x, 1/2)$ for all $x \in \mathbf{T}$ (and by periodicity, in a neighbourhood of $(x, -1/2)$ also).

We assume furthermore that $b(x, y) = b(y)$ with

$$\begin{gathered}
b \in L^\infty(\mathbf{T}) \quad \text{and} \quad b \in W^{3,\infty}(K) \quad \forall K \subset \mathbf{T} \text{ compact s.t. } d(K, 1/2) > 0, \\
b(y) \neq 0 \quad \text{for } y \neq 0, \quad \text{and} \quad \exists C > 0, \quad \left|b(y)\right| \geqslant C \quad \text{for } |y| \geqslant 1/4, \\
b(y) \sim \beta y \quad \text{for } y \to 0, \\
\forall K \subset \mathbf{T} \text{ compact s.t. } d(K, 1/2) > 0, \quad \exists c_K > 0, \quad c_K^{-1} \leqslant b'(y) \leqslant c_K \quad \forall y \in c_K.
\end{gathered} \tag{1.13}$$

In other words, we do not assume that $b \in W^{2,\infty}(\mathbf{T})$: b may have a discontinuity at $y = 1/2$; an example of a function b satisfying the assumptions above is given in Fig. 1.

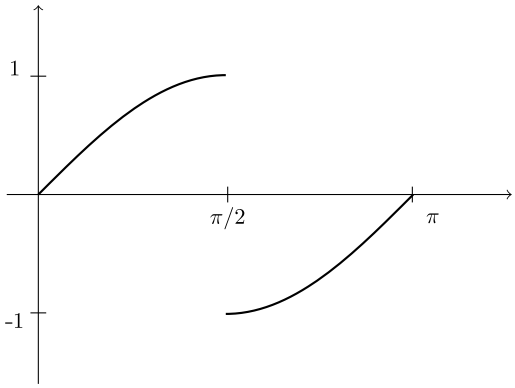

Fig. 1. Example of a Coriolis factor satisfying assumptions (1.13).

But we require that σ vanishes in a neighbourhood of that singularity, so that all terms of the type σb, σ/b, σ/b' are well-defined and $\mathbf{T}^2$-periodic.

Remark 1.4.

(i) The assumptions (1.10) and (1.13) on the **Coriolis factor** b are satisfied in two particular cases:
- $b(y) = \beta y$, with $\omega_h = \mathbf{T} \times \mathbf{R}$: this approximation is especially relevant for the motion of equatorial currents, and is used in [11,9].
- $b(y) = \sin(\pi y)$ for $y \in (-1/2, 1/2)$, with $\omega_h = \mathbf{T}^2$ (see Fig. 1): this is the case of a "real" ocean, whose study takes place on a planetary scale. Of course, in this case, the effect of the curvature of the Earth should be taken into account, which we have chosen not to do here (see the discussion in the previous section). The choice of periodic boundary conditions is also clearly a strong mathematical simplification, which is not realistic from a physical point of view.

(ii) Notice that in the above theorem, it is assumed that the **surface stress** vanishes near $y = 0$. Although this assumption stems from mathematical considerations, it is in fact quite reasonable in an oceanographic context. Indeed, it is a well-known phenomenon that there are no steady surface winds near the equator: as trade winds coming from the North and South meet, they are heated and produce upward winds. The area of calm in the vicinity of the equator is called the *Doldrums*.

(iii) The **compatibility condition (1.9)** means that there is no zonal average wind. This condition is of course not realistic from a physical point of view, but it is the price to pay for working with a domain with no boundary in x. If the horizontal domain ω_h is replaced by $[0, 1] \times \mathbf{T}$ or $[0, 1] \times \mathbf{R}$, this condition disappears; the (mathematical) counterpart lies in the construction of the horizontal boundary layer terms, the so-called Munk layers discussed in the previous section.

(iv) In general, the size of the boundary layer term u^{BL} is much larger than that of the interior term. This means that the greatest part of the **energy is concentrated in a boundary layer** located in the vicinity of the surface. In the original variables, it can be checked that the boundary layer carries an energy of order $\rho_0 U^2 H^3$, while the energy contained in the interior of the domain is of order $\rho_0 U^2 H^2 D$.
This is in fact a consequence on the requirements on u^{BL}, u^{int}, and not an artefact of our model. Indeed, assume that the functions u^{BL}, u^{int} are such that

$$\left\| u^{\mathrm{int}}_{3|z=1} \right\|_{L^2(\omega_h)} \sim \left\| u^{\mathrm{int}}_{h|z=1} \right\|_{L^2(\omega_h)} \sim \left\| u^{\mathrm{int}}_{h} \right\|_{L^2(\omega)},$$
$$u^{\mathrm{int}}_{3|z=1} = -u^{\mathrm{BL}}_{3|z=1},$$

and assume that u^{BL}, u^{int} are divergence free and that u^{BL} is located in a boundary layer of size δ_E (where E stands for 'Ekman') near the surface. Denote by A_h^{BL}, A_3^{BL} the size of u_h^{BL}, u_3^{BL} in L^∞, and by A^{int} the size of u^{int} in $L^2(\omega)$.

The assumptions above entail that $A_3^{\mathrm{BL}} = A^{\mathrm{int}}$; on the other hand, since u^{BL} is divergence free, we have:

$$A_h^{\mathrm{BL}} = \frac{1}{\delta_E} A_3^{\mathrm{BL}} = \frac{1}{\delta_E} A^{\mathrm{int}}.$$

Consequently, since

$$u_h^{\mathrm{BL}} \sim A_h^{\mathrm{BL}} \exp\left(-\frac{1-z}{\delta_E}\right),$$

we infer that

$$\left\|u_h^{\mathrm{BL}}\right\|_{L^2(\omega)} \sim \sqrt{\delta_E} A_h^{\mathrm{BL}} = \frac{1}{\sqrt{\delta_E}} A^{\mathrm{int}}.$$

Thus the energy in the boundary layer is always larger than the energy in the interior of the fluid with this type of model. The assumption that $\|u_{3|z=1}^{\mathrm{int}}\| \sim \|u_{h|z=1}^{\mathrm{int}}\|$ stems from observations of the isothermal surfaces in the ocean, as we will explain in the next section. From a physical point of view, having $\|u^{\mathrm{BL}}\|$ much larger than $\|u^{\mathrm{int}}\|$ is in fact quite reasonable: indeed, it is observed that subsurface currents generally travel at a much slower speed when compared to surface flows.

Let us also emphasize that in the case of the f-plane model (i.e. when the rotation vector b is constant), the result of Theorem 1.2 is false in general. Indeed, the interior part of the solution must satisfy the geostrophic system, namely:

$$\begin{gathered} u_h^{\perp} + \nabla_h p = 0, \\ \operatorname{div}_h u_h + \partial_z u_3 = 0, \\ \partial_z p = 0, \end{gathered}$$

and thus u is a two-dimensional divergence-free vector field. In other words, $u_3 \equiv 0$ and thus the Ekman pumping velocities must be zero at first order. Consequently, the interior part of the solution cannot be wind-driven at first order. Notice that in the present case, the stationary solution is given by the so-called **Sverdrup relation** (see Eq. (3.3)).

1.3. Stability issues and propagation of waves

Once the behaviour of the stationary solution is understood, we address the question of its local stability; since Eq. (1.1) is linear, this is equivalent to studying the Cauchy problem for Eq. (1.1), with homogeneous Navier conditions at $z = 0$ and $z = 1$. We then exhibit Rossby waves, which are essentially two-dimensional, and Poincaré waves, which are fluctuations around the three-dimensional part of the initial data, and which take place on a much larger time scale.

Theorem 1.5 *(Waves associated with Eq. (1.1)). Assume that $\omega_h = \mathbf{T} \times \mathbf{R}$, and that $b(x_h) = \beta y$ for all $x_h = (x, y) \in \omega_h$.*

For any $\epsilon > 0$, let v^ϵ be a solution to the propagation equation,

$$\begin{gathered} \partial_t u + \frac{1}{\epsilon} b(x_h) \wedge u + \begin{pmatrix} \nabla_h p \\ \frac{1}{\epsilon^2}\partial_z p \end{pmatrix} - \nu_h \Delta_h u - \epsilon \partial_{zz} u = 0, \\ (x_h, z) \in \omega_h \times (0, 1), \end{gathered}$$

with $\nu_h = O(\epsilon^2)$, supplemented with homogeneous boundary conditions:

$$\partial_z u_{h|z=0} = \partial_z u_{h|z=1} = 0, \qquad u_{3|z=0} = u_{3|z=1} = 0.$$

Then v^ϵ can be decomposed as the sum of

- *a stationary part $\bar{v}^\epsilon(t, y) = \int_{\mathbf{T}} \int_0^1 v^\epsilon(t, x, y, z)\, dx\, dz$, which satisfies:*

$$\partial_t \bar{v}^\epsilon - \nu_h \partial_y^2 \bar{v}^\epsilon = 0,$$

- **Rossby waves** $v_R^\epsilon = \int v^\epsilon dx_3 - \bar{v}^\epsilon$ *corresponding to the 2D vorticity propagation,*
$$\partial_t \zeta_R^\epsilon + \frac{\beta}{\epsilon}\partial_x \Delta_h^{-1}\zeta_R^\epsilon - \nu_h \Delta_h \zeta_R^\epsilon = 0,$$
where $\zeta_R^\epsilon = \mathrm{rot}_h\, v_R^\epsilon$,
- *and* **gravity waves** $v_G^\epsilon = v^\epsilon - \int v^\epsilon dx_3$.

Rossby and Gravity waves have a dispersive behaviour as ϵ *vanishes*:

- *Rossby waves disperse on a small time scale,*
$$\forall t > 0,\ \forall K \Subset \omega, \quad \left\| v_R^\epsilon(t)\right\|_{L^2(K)} \to 0 \quad \text{as } \epsilon \to 0,$$
since we have assumed that $y \in \mathbf{R}$;
- *Gravity waves generate fast oscillations with respect to* y, *which slows down the propagation*:
$$\forall K \Subset \omega, \quad \left\| v_G^\epsilon(t)\right\|_{L^2(K)} \to 0 \quad \text{as } (\epsilon, t) \to (0, \infty).$$

Remark 1.6.

- The energy associated with gravity (or Poincaré) waves propagates on a time scale much larger than the one of Rossby waves; we refer to Sections 4 and 5 for details. This is due to the **thin layer effect**, which causes the apparition of small scales in the variable y.
- The field $\bar{v}^\epsilon$ is said to be **"stationary"** because the horizontal viscosity ν_h is small: hence
$$\bar{v}^\epsilon(t) \approx \bar{v}^\epsilon_{|t=0} \quad \text{in } L^2$$
on time scales of order one.
- The assumption $\nu_h = O(\epsilon^2)$ is discussed in Section 5.6. In particular, we prove that if $\nu_h \gg \epsilon^2$, the energy contained in the Poincaré modes is in fact dissipated in a very short time.

Corollary 1.7. *Assume that* $\omega_h = \mathbf{T} \times \mathbf{R}$, *and that* $b(x_h) = \beta y$ *for all* $x_h = (x, y) \in \omega_h$. *Assume that* $\nu_h = O(\epsilon^2)$.
For any $\epsilon > 0$, *let* u^ϵ *be a solution of* (1.1) *supplemented with* (1.2)–(1.3), *and assume that*
$$\sup_{\epsilon>0}\left(\left\| u^\epsilon_{h|t=0} - u_h^{\mathrm{stat}}\right\|_{L^2(\omega)} + \epsilon\left\| u^\epsilon_{3|t=0} - u_3^{\mathrm{stat}}\right\|_{L^2(\omega)}\right) < +\infty.$$
Then for any finite time $t > 0$,
$$u^\epsilon(t) - u^{\mathrm{stat}} \sim \bar{v}^\epsilon(t) + v_G^\epsilon(t) \quad \text{in } L^2_{\mathrm{loc}}(\omega),$$
where v_G^ϵ *is the* (*slow propagating and fast oscillating*) *gravity part of the velocity field* v^ϵ *defined in Theorem* 1.5, *and* $\bar{v}^\epsilon$ *is the stationary part of* v^ϵ, *with* $v^\epsilon_{|t=0} = u^\epsilon_{h|t=0} - u_h^{\mathrm{stat}}$ *before the full stop.*

The above corollary is an immediate consequence of Theorems 1.2 and 1.5, together with the energy inequality.

An important consequence of our analysis is that **the vertical component of the velocity** u^ϵ **is not expected to be bounded** – as is usually claimed for shallow water approximation. We refer to Section 5 for further details regarding that point. This is due to the smallness of the horizontal velocity ν_h: indeed, if ν_h is of order one, then the energy inequality entails that $\nabla_h u_h^\epsilon$ is bounded in $L^2([0, T] \times \omega)$ for all $T > 0$. Since u^ϵ is divergence free, $\partial_z u_3^\epsilon$ is also bounded in L^2, and thus u_3^ϵ is bounded.

1.4. Towards a mathematical derivation of the thermocline

In this section, we try to justify the shape of the surfaces of equal temperature in the ocean, in view of the results of Theorem 1.2.

The **isothermal surface** which is located just below the Ekman boundary layer is of special interest to oceanographers, due to its importance on the global oceanic circulation (see [25,26,20]). Fig. 2 shows the longitudinal variations of the temperature in the Pacific ocean in a layer of 1000 m depth below the surface.

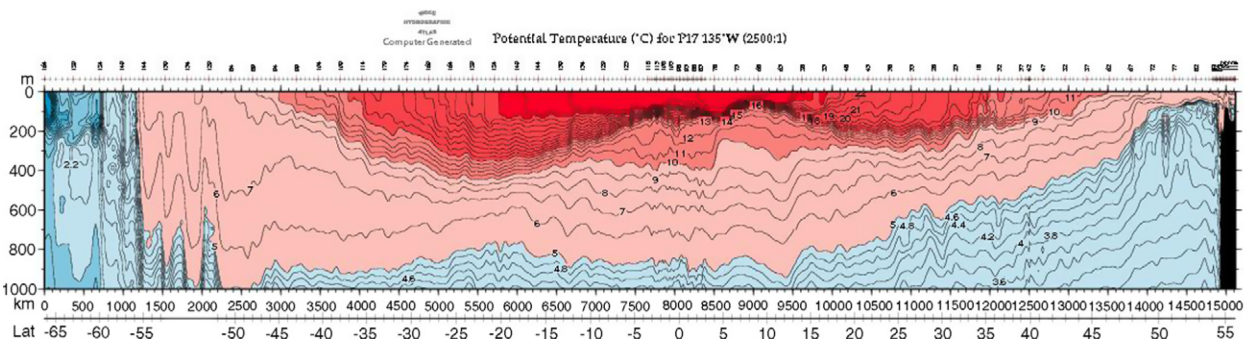

Fig. 2. Longitudinal section of the surfaces of equal temperature in the Pacific ocean (*from the WOCE Pacific Ocean Atlas*).

In particular, there are zones in which the temperature surfaces are tilted up (that is, there is a flux of cold water towards the surface); this phenomenon cannot always be accounted for by the heating differences at the surface, as shows the upward flux of cold water in the equatorial zone. The physical justification of these particular shapes is the following: inside the ocean, the temperature T solves an equation of the kind,

$$u' \cdot \nabla T - \kappa \Delta T = 0,$$

where u' is the velocity of oceanic currents (in dimensional variables) and κ is the heat conductivity coefficient. If the temperature diffusion can be neglected, this equation takes the form,

$$u' \cdot \nabla T = 0,$$

which means that u' is tangent to the isothermal surfaces. Consequently, the temperature surfaces are tilted up (or down) if and only if $u'_{3|\text{surface}} \neq 0$, or more precisely, if $|u_{3|z=1}|/|u_{h|z=1}| = O(1)$ in rescaled variables. This justifies the assumption,

$$\left\| u^{\text{int}}_{3|z=1} \right\| \sim \left\| u^{\text{int}}_{h|z=1} \right\|,$$

in the previous section (see Remark 1.4(iv)).

In that regard, the special solution constructed in Theorem 1.2 is of particular interest. Indeed, in rescaled variables, we have (see Section 3),

$$u^{\text{int}}_{3|z=1} = -\frac{1}{b} \operatorname{rot}_h \sigma - \frac{b'}{b^2} \sigma_1.$$

Hence $u_{3|\text{surface}} \neq 0$, and our model predicts that the temperature surfaces are indeed **modified by the Ekman pumping velocity.**

We now give a rigorous result about the asymptotic shape of the temperature in our model. We denote with a prime the original (dimensional) variables. We write

$$T\left(x'_h, z'\right) = T_0 + T_1 \theta\left(\frac{x'_h}{H}, \frac{z'}{D}\right),$$

with the same notations as in Section 1.1. The temperature T_0 is a reference temperature (for instance, $T_0 = 10\,^\circ\text{C}$), whereas T_1 is the order of magnitude of the variations of the temperature. Performing the same change of variables as in Section 1.1, we obtain,

$$u \cdot \nabla \theta - \lambda \eta^2 \Delta_h \theta - \lambda \partial_{zz} \theta = 0,$$

where the diffusion coefficient λ is given by,

$$\lambda = \frac{\kappa L}{D^2 U}.$$

We recall that η is the aspect ratio of the domain; as in Theorems 1.2 and 1.5, we take $\eta = \epsilon$. With the notation of Theorem 1.2, our result is the following:

Proposition 1.8. *Let $\lambda > 0$. Assume that the wind stress $\sigma \in H^s(\omega_h)$ is such that*

$$\begin{aligned} &\left|\sigma(x,y)\right| \leqslant C y^k \quad \forall (x,y) \in \omega_h, \\ &\left|\nabla\sigma(x,y)\right| \leqslant C|y|^{k-1} \quad \forall (x,y) \in \omega_h, \end{aligned} \tag{1.14}$$

for some $k, s \geqslant 2$ chosen sufficiently large, and assume that

$$\left\| \nabla_h u_h^{\mathrm{int}} \right\|_{L^\infty(\omega)} \leqslant \frac{\lambda}{4}. \tag{1.15}$$

Let θ be the solution of the equation,

$$u^{\mathrm{stat}} \cdot \nabla\theta - \eta^2 \lambda \Delta_h \theta - \lambda \partial_{zz}\theta = 0, \tag{1.16}$$

supplemented with the boundary conditions

$$\theta_{|z=1} = \theta_1, \qquad \partial_z \theta_{|z=0} = 0, \tag{1.17}$$

for some function $\theta_1 \in H^2(\omega_h)$.

Define the function θ^{app} by

$$\theta^{\mathrm{app}}(x_h, z) = \bar\theta(x_h, z) + \epsilon\theta^{\mathrm{BL}}\left(x_h, \frac{1-z}{\epsilon}\right),$$

where $\bar\theta$, θ^{BL} are solutions of

$$\begin{aligned} &-\lambda\partial_{zz}\bar\theta + u^{\mathrm{int}} \cdot \nabla\bar\theta = 0 \quad \textit{in } \omega, \\ &\bar\theta_{|z=1} = \theta_1, \qquad \partial_z\bar\theta_{|z=0} = 0, \end{aligned} \tag{1.18}$$

and

$$\begin{aligned} &-\lambda\partial_{\zeta\zeta}\theta^{\mathrm{BL}}(x_h, \zeta) + \epsilon u_h^{\mathrm{BL}}(x_h, 1 - \epsilon\zeta) \cdot \nabla_h \theta_1 = 0, \\ &\theta^{\mathrm{BL}}(x_h, \zeta) \underset{\zeta\to\infty}{\longrightarrow} 0. \end{aligned}$$

Then as $\epsilon \to 0$,

$$\left\|\theta - \theta^{\mathrm{app}}\right\|_{L^2(\omega)} + \left\|\partial_z\left(\theta - \theta^{\mathrm{app}}\right)\right\|_{L^2(\omega)} \to 0.$$

Remark 1.9.

(i) If the Sobolev exponent s is chosen sufficiently large, then $\sigma \in W^{1,\infty}(\omega_h)$. Hence assumption (1.14) merely specifies the behaviour of σ near $y = 0$. In general, the assumptions of Proposition 1.8 are more stringent than (1.8).

(ii) The assumption (1.15) on the size of $\nabla_h u_h^{\mathrm{int}}$ is purely technical, and does not have any physical interpretation. It rises from the fact that Eq. (1.18) on $\bar\theta$ is degenerate in the horizontal variable; we refer to Section 6 for more details. We emphasize in particular that if (1.15) is not satisfied, Eq. (1.18) is still well-posed in $L^2(\omega_h, H^1(0,1))$; however, in this case, we are no longer able to prove the convergence.

(iii) The boundary conditions (1.17) mean that the atmosphere acts like a thermostat for the ocean, and that there is no heat flux at the bottom of the ocean. Both assumptions seem reasonable from a physical point of view, although other boundary conditions might also make sense: for instance, it could also be assumed that the heat flux at the surface (caused by heating by the sun) is a given function of the latitude.

(iv) Let us mention a last direction towards which the physical accuracy of our model could be improved. When considering the spatial variations of the temperature, it would be more reasonable to consider a model which couples the velocity of ocean currents and the temperature, in the spirit of [3]. However, the relevant scalings within such models are not completely clear. Furthermore, the analysis in Chapter 6 of [25] shows that for such problems, the curvature of the Earth should be taken into account. Hence we leave this issue aside in the present paper.

The construction of the article is as follows: in the next two sections, we construct the stationary solution of Eq. (1.1), starting with the boundary layer part, and then building the interior part by solving the geostrophic equations with a Dirichlet boundary condition on the vertical component. Then, we prove Theorem 1.5 in Sections 4 and 5, by treating separately the two-dimensional and three-dimensional parts of the initial data. Eventually, Section 6 is dedicated to the proof of Proposition 1.8.

2. The boundary layer part of the stationary solution

In this section, we construct functions u^{BL}, p^{BL} which are approximate stationary solutions of Eq. (1.1) (in the sense of Theorem 1.2), and which satisfy the horizontal part of the boundary condition (1.3). These functions are located in a boundary layer in the vicinity of the surface $z=1$. Our methodology is the following: we first assume that $\nu_h=0$, and we use the classical construction of Ekman layers in this case. We then derive several estimates on the functions thus obtained. Eventually, we estimate the error terms in Eq. (1.1) which are due to the fact that ν_h is non-zero.

2.1. Construction in the case $\nu_h=0$

When the horizontal viscosity vanishes, the construction of the boundary layer is exactly the same as in the f-plane model, i.e. when the function b does not depend on x_h. Indeed, in this case the variable x_h is merely a parameter of the equation, and building the boundary layer term amounts to solving an equation on the rate of exponential decay. For more results regarding classical boundary layers, we refer to [4,22,23,27]. Nonetheless, let us stress that even though the construction itself is the same, the estimates become much more involved than in the case of the f-plane model. Indeed, the vanishing points of b create singularities, and prevent the boundary layer terms to belong to L^2 in general. Hence, assumptions on the stress σ have to be introduced in order to handle these singularities.

The construction of the boundary layer term is as follows: we wish to construct an approximate solution (u^{BL}, p^{BL}) of (1.1), such that (1.3) is satisfied. Furthermore, we assume that this approximate solution is small outside a boundary layer located in the vicinity of the surface $z=1$. Hence, we look for u^{BL}, p^{BL} in the form:

$$u^{\mathrm{BL}}(t,x_h,z)=U^{\mathrm{BL}}\left(x_h,\frac{1-z}{\epsilon}\right),$$
$$p^{\mathrm{BL}}(t,x_h,z)=P^{\mathrm{BL}}\left(x_h,\frac{1-z}{\epsilon}\right).$$

We assume that U^{BL}, P^{BL} together with all their derivatives vanish as $\zeta\to\infty$, where ζ stands for the rescaled variable $(1-z)/\epsilon$. Inserting the above ansatz into Eq. (1.1) yields,

$$\begin{cases} b(x_h)\big(U_h^{\mathrm{BL}}\big)^{\perp}-\partial_\zeta^2 U_h^{\mathrm{BL}}+\epsilon\nabla_h P^{\mathrm{BL}}=0,\\ -\partial_\zeta^2 U_3^{\mathrm{BL}}-\dfrac{1}{\epsilon^2}\partial_\zeta P^{\mathrm{BL}}=0,\\ \operatorname{div}_h U_h^{\mathrm{BL}}-\dfrac{1}{\epsilon}\partial_\zeta U_3=0. \end{cases}\tag{2.1}$$

The last two equations entail that

$$P^{\mathrm{BL}}=-\epsilon^2\partial_\zeta U_3^{\mathrm{BL}}=-\epsilon^3\operatorname{div}_h U_h^{\mathrm{BL}}.$$

We henceforth neglect the pressure term in the equation on U_h^{BL}. Then, we set, as usual (see for instance [22]),

$$U_h^{\pm}:=U_h\pm iU_h^{\perp}.$$

Above and in the rest of the article, for all $u=(u_1,u_2)\in\mathbf{R}^2$, $u^{\perp}:=(-u_2,u_1)$.

An easy calculation leads to,

$$-\partial_\zeta^2 U_h^{\pm}\mp ibU_h^{\pm}=0,$$
$$\partial_\zeta U^{\pm}_{h|\zeta=0}=-\frac{1}{\epsilon}\big(\sigma\pm i\sigma^{\perp}\big).$$

Consequently, $U_h^\pm$ is an exponentially decaying function of the form,

$$U_h^\pm(x_h,\zeta)=\frac{1}{\epsilon\lambda^\pm(x_h)}\big(\sigma\pm i\sigma^\perp\big)(x_h)\exp\big(-\lambda^\pm(x_h)\zeta\big),$$

where the decay rate $\lambda^\pm$ is defined by:

$$\big(\lambda^\pm\big)^2=\mp ib\quad\text{and}\quad\Re\big(\lambda^\pm\big)>0,$$

i.e.

$$\lambda^\pm(x_h)=\lambda^\pm(y)=\frac{1\mp i\,\mathrm{sign}(b)}{\sqrt{2}}\big|b(y)\big|^{1/2}. \tag{2.2}$$

Notice in particular that the decay rates $\lambda^\pm$ vanish at $y=0$ and depend only on y.

Going back to the definition of $U_h^\pm$, we infer that

$$U_h^{\mathrm{BL}}(x_h,\zeta)=\frac{U_h^+ + U_h^-}{2}=\frac{1}{2\epsilon}\sum_\pm\frac{(\sigma\pm i\sigma^\perp)(x_h)}{\lambda^\pm(x_h)}\exp\big(-\lambda^\pm(x_h)\zeta\big). \tag{2.3}$$

Hence, in order that u^{BL} is divergence free, we set:

$$\begin{aligned}U_3^{\mathrm{BL}}(x_h,\zeta)&=-\epsilon\int_\zeta^\infty \mathrm{div}_h\,U_h^{\mathrm{BL}}\big(x_h,\zeta'\big)\,d\zeta'\\&=-\frac{1}{2}\sum_\pm(\mathrm{div}_h\,\sigma\mp i\,\mathrm{rot}_h\,\sigma)(x_h)\big(\lambda^\pm(x_h)\big)^{-2}e^{-\lambda^\pm(x_h)\zeta}\\&\quad+\frac{1}{2}\sum_\pm\big(\sigma\pm i\sigma^\perp\big)(x_h)\cdot\frac{\nabla_h\lambda^\pm(x_h)}{(\lambda^\pm(x_h))^3}\big(2+\zeta\lambda^\pm(x_h)\big)e^{-\lambda^\pm(x_h)\zeta}.\end{aligned} \tag{2.4}$$

We have used the convention,

$$\mathrm{rot}_h\,u_h=-\,\mathrm{div}_h\,u_h^\perp,$$

for two-dimensional vector fields.

The remaining flux term is then given by:

$$\begin{aligned}u^{\mathrm{BL}}_{3|z=1}=U^{\mathrm{BL}}_{3|\zeta=0}(x_h)&=-\frac{1}{2}\sum_\pm(\mathrm{div}_h\,\sigma\mp i\,\mathrm{rot}_h\,\sigma)(x_h)\big(\lambda^\pm(x_h)\big)^{-2}\\&\quad+\sum_\pm\big(\sigma\pm i\sigma^\perp\big)(x_h)\cdot\frac{\nabla_h\lambda^\pm(x_h)}{(\lambda^\pm(x_h))^3}.\end{aligned} \tag{2.5}$$

We now wish to point out a particular difficulty steaming from the above construction. If the Coriolis factor b has vanishing points, which occurs in particular in the case of the β-plane approximation ($b(x_h)=\beta y$), then the functions U_h^{BL}, U_3^{BL} may not be square integrable if the function σ is arbitrary. Hence, the function σ should vanish at a sufficiently high order near $y=0$ so that the singularity disappears. We will check that (1.8) entails that the functions U_h^{BL}, U_3^{BL} defined by (2.3), (2.4) are square integrable for a Coriolis factor satisfying (1.10). For further purposes, we also require that the function $\nabla_h U^{\mathrm{BL}}$ belongs to $L^2(\omega_h\times[0,\infty)_\zeta)$. Unfortunately, assumption (1.8) is not sufficient to ensure such a result. Thus we introduce an approximate boundary layer term, in which the low values of b have been truncated.

2.2. Estimates on the boundary layer terms

We begin with a short justification of the need for a truncation. Using the definition of $\lambda^\pm$ together with assumption (1.10), we infer that if y is close to zero, then

$$\left\|\nabla_{x_h} U_3^{\mathrm{BL}}(x_h)\right\|_{L^2([0,\infty)_\zeta)} \leqslant C\left(\frac{|D^2\sigma(x_h)|}{y^{5/4}} + \frac{|\nabla\sigma(x_h)|}{y^{9/4}} + \frac{|\sigma(x_h)|}{y^{13/4}}\right) \leqslant C y^{-5/4}.$$

Hence $\nabla_{x_h} U_3^{\mathrm{BL}}$ does not belong to $L^2(\omega_h \times [0,\infty)_\zeta)$ in general. We thus define, for any $\delta > 0$, the function

$$b_\delta(y) = b(y)\psi\left(\frac{|y|}{\delta}\right), \quad y \neq 0, \tag{2.6}$$

where $\psi \in \mathcal{C}^\infty((0,\infty))$ is such that

$$\begin{aligned} &\psi(y) \geqslant \frac{1}{2} && \text{for } y \in (0,\infty),\\ &\psi(y) = 1 && \text{if } y \geqslant 2,\\ &\psi(y) = y^{-\alpha} && \text{if } y \in (0,1), \end{aligned}$$

for some exponent $\alpha \in (0,1)$ to be chosen later on. Notice that with this choice of ψ, the function b_δ behaves like $\delta^\alpha y^{1-\alpha}$ for $y > 0$ near zero. Consequently, b_δ vanishes with a weaker rate than b, and thus σ/b_δ vanishes more strongly than σ/b.

We now define approximated decay rates $\lambda_\delta^\pm$ by replacing b by b_δ in the expression (2.2); eventually, we define approximated boundary layer terms by the formulas (2.3)–(2.4), in which the decay rates $\lambda^\pm$ have been replaced by $\lambda_\delta^\pm$.

Remark 2.1. We will eventually choose $\delta = \epsilon$ and $\alpha > 3/5$, which allows us to obtain the result stated in Theorem 1.2.

We then have the following result:

Lemma 2.2. *Assume that hypotheses* (1.8), (1.10) *are satisfied. Then there exists a constant C, depending only on σ and b, such that for all $\alpha > 0$, $\delta > 0$,*

$$\begin{aligned} \left\|U_{\delta,h}^{\mathrm{BL}}\right\|_{L^2(\omega_h\times[0,\infty)_\zeta)} &\leqslant \frac{C}{\epsilon},\\ \left\|U_{\delta,3}^{\mathrm{BL}}\right\|_{L^2(\omega_h\times[0,\infty)_\zeta)} &\leqslant C. \end{aligned}$$

Additionally, if $\alpha > 3/5$, there exists a constant C_α, depending only on α, σ and b, such that for all $\delta > 0$,

$$\begin{aligned} \left\|\nabla_h U_{\delta,h}^{\mathrm{BL}}\right\|_{L^2(\omega_h\times[0,\infty)_\zeta)} &\leqslant \frac{C_\alpha}{\epsilon},\\ \left\|\nabla_h U_{\delta,3}^{\mathrm{BL}}\right\|_{L^2(\omega_h\times[0,\infty)_\zeta)} &\leqslant \frac{C_\alpha}{\delta^{3/4}}. \end{aligned}$$

Moreover, for all $\delta > 0$,

$$\begin{aligned} \left\|(b-b_\delta)U_{\delta,h}^{\mathrm{BL}}\right\|_{L^2(\omega_h\times[0,\infty)_\zeta)} &\leqslant C\frac{\delta^{11/4}}{\epsilon},\\ \left\|\nabla_h P_\delta^{\mathrm{BL}}\right\|_{L^2(\omega_h\times[0,\infty)_\zeta)} &\leqslant C_\alpha\frac{\epsilon^2}{\delta^{1/4}}. \end{aligned}$$

Remark 2.3. The above estimates are given for the rescaled boundary layer profiles U^{BL}, P^{BL}, which are defined on $\omega_h \times [0,\infty)_\zeta$. Remember that the boundary layer part of the stationary solution is defined on $\omega_h \times [0,1]$ by

$$u^{\mathrm{BL}}(x_h, z) = U_h^{\mathrm{BL}}\left(x_h, \frac{1-z}{\epsilon}\right).$$

Hence

$$\left\|u_h^{\mathrm{BL}}\right\|_{L^2(\omega_h\times(0,1))} \leqslant \epsilon^{1/2}\left\|U_h^{\mathrm{BL}}\right\|_{L^2(\omega_h\times[0,\infty))}.$$

Similar estimates hold for p^{BL}, u_3^{BL}.

Proof. • L^2 **estimates:** According to (1.10), and to the definition of b_δ, we have:

$$\left|\lambda^\pm(x_h)\right| \neq 0 \quad \text{if } y \neq 0, \qquad \left|\lambda^\pm(x_h)\right| \sim \sqrt{\beta|y|} \quad \text{as } y \to 0,$$

and thus there exists a constant C such that

$$\left|\Re\left(\lambda^\pm(x_h)\right)\right|^{-1}, \left|\lambda^\pm(x_h)\right|^{-1} \leqslant C|y|^{-1/2} \quad \forall x, \forall y \in [-1,1].$$

We recall that

$$\lambda^\pm = \left|\lambda^\pm\right| \exp\left(\mp i \operatorname{sign}(b)\frac{\pi}{4}\right).$$

Similarly, for all $\delta > 0$, we have, for $|y| \leqslant \delta$

$$\left|\Re\left(\lambda_\delta^\pm(x_h)\right)\right|^{-1}, \left|\lambda_\delta^\pm(x_h)\right|^{-1} \leqslant C|y|^{-(1-\alpha)/2}\delta^{-\alpha/2}.$$

If $|y| \geqslant \delta$, then $\lambda_\delta^\pm(x_h)$ satisfies the same estimates as $\lambda^\pm(x_h)$. A careful computation leads to

$$\int_0^\infty \left|U_{\delta,h}^{\mathrm{BL}}(x_h,\zeta)\right|^2 d\zeta = \frac{1}{2\epsilon^2}\left|\sigma(x_h)\right|^2 \sum_\pm \frac{1}{|\lambda_\delta^\pm(y)|^2 \Re(\lambda_\delta^\pm(y))}. \tag{2.7}$$

Hence we obtain, using (1.8),

$$\left(\int_0^\infty \left|U_{\delta,h}^{\mathrm{BL}}(x_h,\zeta)\right|^2 d\zeta\right)^{1/2} \leqslant \frac{C}{\epsilon}\begin{cases} |y|^{\frac{5+3\alpha}{4}}\delta^{-\frac{3\alpha}{4}} & \text{if } |y| \leqslant \delta, \\ |y|^{5/4} & \text{if } \delta \leqslant |y| \leqslant 1, \\ |\sigma| & \text{else.}\end{cases}$$

Eventually, we infer that

$$\left\|U_{\delta,h}^{\mathrm{BL}}\right\|_{L^2(\omega_h\times[0,\infty)_\zeta)} \leqslant \frac{C_0}{\epsilon},$$

where the constant C_0 depends only on b and σ. Notice that the truncation does not play any role at this stage: the same arguments show that $U_h^{\mathrm{BL}} \in L^2(\omega_h \times [0,\infty)_\zeta)$.

Similarly, we have,

$$\int_0^\infty \left|U_{\delta,3}^{\mathrm{BL}}(x_h,\zeta)\right|^2 d\zeta \leqslant C\sum_\pm\left(\frac{|\nabla\sigma(x_h)|^2}{|\lambda_\delta^\pm(y)|^5} + \frac{|\sigma(x_h)|^2|\nabla\lambda_\delta^\pm(y)|^2}{|\lambda_\delta^\pm(y)|^7}\right). \tag{2.8}$$

Using the definition of the decay rates $\lambda_\delta^\pm$ together with the definition of the function ψ, we obtain:

$$\left|\nabla\lambda_\delta^\pm\right| = \frac{|b_\delta'|}{2|b_\delta|^{1/2}} \leqslant C\begin{cases} |y|^{-\frac{\alpha+1}{2}}\delta^{\alpha/2} & \text{if } |y| \leqslant \delta, \\ |y|^{-1/2} & \text{if } \delta \leqslant |y| \leqslant 1, \\ 1 & \text{else.}\end{cases}$$

Thus

$$\left\|U_{\delta,3}^{\mathrm{BL}}\right\|_{L^2(\omega_h\times[0,\infty)_\zeta)} \leqslant C_0.$$

• H_h^1 **estimates:**

We begin with the bound on $\nabla_h U_{\delta,h}^{\mathrm{BL}}$; the calculations are very similar to the ones which led to the L^2 bound on $U_{\delta,3}$, and are therefore left to the reader. In fact, the situation is even a little less singular than in the case of $U_{\delta,3}$ (we "gain" one integration with respect to the variable ζ, and thus one factor $(\lambda_\delta^\pm)^{-1}$). The bounds on $\lambda_\delta^\pm$ and σ entail that

$$\left\|\nabla_h U_{\delta,h}^{\mathrm{BL}}\right\|_{L^2(\omega_h\times[0,\infty)_\zeta)} \leqslant \frac{C_0}{\epsilon}.$$

We now tackle the bound on $\nabla_h U_{\delta,3}^{\mathrm{BL}}$. We differentiate Eq. (2.4) with respect to x_h and use the following rules,

$$\exp\left(-\zeta\lambda_\delta^\pm\right), \zeta\lambda_\delta^\pm\exp\left(-\zeta\lambda_\delta^\pm\right) = O\left(|\lambda_\delta|^{-1/2}\right) \quad \text{in } L^2\left([0,\infty)_\zeta\right),$$

where $|\lambda_\delta|$ denotes the common size of $|\lambda_\delta^+|$ and $|\lambda_\delta^-|$. We obtain:

$$\begin{aligned}\big\|\nabla_h U^{\mathrm{BL}}_{\delta,3}(x_h)\big\|_{L^2([0,\infty)_\zeta)} &\leqslant C\left(\frac{|D^2\sigma(x_h)|}{|\lambda_\delta|^{5/2}(y)} + \big|\nabla\sigma(x_h)\big|\frac{|\partial_y\lambda_\delta^{\pm}(y)|}{|\lambda_\delta(y)|^{7/2}}\right)\\ &\quad + C\left(\big|\sigma(x_h)\big|\frac{|\partial_{yy}\lambda_\delta(y)|}{|\lambda_\delta(y)|^{7/2}} + \big|\sigma(x_h)\big|\frac{|\partial_y\lambda_\delta(y)|^2}{|\lambda_\delta(y)|^{9/2}}\right).\end{aligned}$$

Notice that due to the sign change in b at $y=0$, there is in general a Dirac mass at $y=0$ in the term $\partial_{yy}\lambda_\delta$; more precisely, the part of $\nabla_h U^{\mathrm{BL}}_{\delta,3}$ which is not absolutely continuous with respect to the Lebesgue measure is of the type:

$$\delta_{y=0}|\sigma|\frac{|b_\delta'|}{|b_\delta|^{1/2}|\lambda_\delta|^{7/2}} = \delta_{y=0}|\sigma||b_\delta|^{-9/4}.$$

At this stage, the need for a truncation is clear: if b_δ is replaced by b, then $|\sigma||b|^{-9/4}\sim|y|^{-1/4}$ near $y=0$, and thus the singular part of $\nabla_h U^{\mathrm{BL}}_3$ is not well-defined in the sense of distributions. Conversely, if $\alpha>1/9$, then

$$|\sigma||b_\delta|^{-9/4}\sim|y|^{\frac{9\alpha-1}{4}}\delta^{-\frac{9\alpha}{4}}\quad\text{as } y\to 0,$$

and thus the singular part of $\nabla_h U^{\mathrm{BL}}_{\delta,3}$ is zero.

Gathering all the terms, we deduce that

$$\big\|\nabla_h U^{\mathrm{BL}}_{\delta,3}(x_h)\big\|_{L^2_\zeta}\leqslant C\begin{cases}|y|^{-\frac{5(1-\alpha)}{4}}\delta^{-\frac{5\alpha}{4}} & \text{if } |y|\leqslant\delta,\\ |y|^{-5/4} & \text{if } \delta\leqslant|y|\leqslant 1,\\ |\sigma(x_h)|+|\nabla\sigma(x_h)|+|D^2\sigma(x_h)| & \text{else.}\end{cases}$$

Thus $\nabla_h U^{\mathrm{BL}}_{\delta,3}\in L^2(\omega_h\times[0,\infty))$ if and only if $\alpha>3/5$, and in this case there exists a constant C_α, depending on σ, b and α, such that for all $\delta>0$,

$$\big\|\nabla_h U^{\mathrm{BL}}_{\delta,3}(x_h)\big\|_{L^2(\omega_h\times[0,\infty))}\leqslant\frac{C_\alpha}{\delta^{3/4}}.$$

• **Error estimates:** First, by definition of b_δ, we have

$$\big\|(b-b_\delta)U^{\mathrm{BL}}_{\delta,h}\big\|^2_{L^2} = \int\limits_{\omega_h\cap\{|y|\leqslant 2\delta\}}\int\limits_0^\infty\big|b(y)-b_\delta(y)\big|^2\big|U^{\mathrm{BL}}_{\delta,h}(x,y,\zeta)\big|^2 d\zeta\,dy\,dx.$$

Notice that for all $y\in\mathbf{R}\setminus\{0\}$,

$$\begin{aligned}\big|b(y)-b_\delta(y)\big| &= \big|b(y)\big|\left|1-\psi\left(\frac{y}{\delta}\right)\right|\\ &= \mathbf{1}_{|y|\leqslant\delta}\big|b(y)\big|\left(\frac{\delta^\alpha}{|y|^\alpha}-1\right) + \mathbf{1}_{\delta\leqslant|y|\leqslant 2\delta}\big|b(y)\big|\left|1-\psi\left(\frac{y}{\delta}\right)\right|\\ &\leqslant C\big(\mathbf{1}_{|y|\leqslant\delta}|y|^{1-\alpha}\big(\delta^\alpha-|y|^\alpha\big)+\mathbf{1}_{\delta\leqslant|y|\leqslant 2\delta}|y|\big)\\ &\leqslant C\mathbf{1}_{|y|\leqslant 2\delta}|y|^{1-\alpha}\delta^\alpha.\end{aligned}$$

Using (2.7), we infer

$$\begin{aligned}\big\|(b-b_\delta)U^{\mathrm{BL}}_{\delta,h}\big\|^2_{L^2(\omega_h\times[0,\infty)_\zeta)} &\leqslant\frac{C}{\epsilon^2}\int\limits_{x\in\mathbf{T}}\int\limits_{|y|\leqslant 2\delta}|y|^{2(1-\alpha)}\delta^{2\alpha}|y|^4|y|^{-3/2}\left|\frac{y}{\delta}\right|^{3\alpha/2}dy\\ &\leqslant\frac{C}{\epsilon^2}\int\limits_{|y|\leqslant 2\delta}|y|^{(9-\alpha)/2}\delta^{\alpha/2}\,dy\\ &\leqslant\frac{C\delta^{11/2}}{\epsilon^2}.\end{aligned}$$

There remains to evaluate P_δ^{BL}. By definition:

$$P_\delta^{\mathrm{BL}} = -\epsilon^3 \operatorname{div}_h U_{h,\delta}^{\mathrm{BL}}.$$

Using the same kinds of calculations as the ones which led to the bound on $\nabla_h U_{\delta,3}^{\mathrm{BL}}$, we deduce that

$$\left\| \nabla_h P_\delta^{\mathrm{BL}} \right\|_{L^2(\omega_h \times [0,\infty)_\zeta)} \leqslant C \frac{\epsilon^2}{\delta^{1/4}}. \qquad \square$$

2.3. Error estimates in the case $\nu_h \neq 0$ and conditions on the parameter δ

If $\nu_h \neq 0$, we keep the construction of the previous section, and we merely treat the viscous terms as error terms. The function $u_{\delta,h}^{\mathrm{BL}}$ is an approximate solution of the horizontal part of Eq. (1.1), with the error term,

$$\frac{1}{\epsilon}(b - b_\delta)\big(u_{\delta,h}^{\mathrm{BL}}\big)^\perp - \nu_h \Delta_h u_{\delta,h}^{\mathrm{BL}} + \nabla_h p_\delta^{\mathrm{BL}}.$$

According to the estimates of the previous section (see Lemma 2.2), we have,

$$\left\| \frac{1}{\epsilon}(b - b_\delta)\big(u_{\delta,h}^{\mathrm{BL}}\big)^\perp \right\|_{L^2(\omega)} \leqslant C \frac{\delta^{11/4}}{\epsilon^{3/2}}$$

and

$$\left\| \nu_h \Delta_h u_{\delta,h}^{\mathrm{BL}} \right\|_{L^2([0,1], H^{-1}(\omega_h))} \leqslant C \frac{\nu_h}{\sqrt{\epsilon}},$$

$$\left\| \nabla_h p^{\mathrm{BL}} \right\|_{L^2(\omega)} \leqslant C \frac{\epsilon^{5/2}}{\delta^{1/4}}.$$

Recall that because of the boundary layer scaling, there is a factor $\epsilon^{1/2}$ between the L^2 norms of u_δ^{BL} and U_δ^{BL}. With the choice,

$$\delta = \epsilon, \qquad \alpha > \frac{3}{5},$$

we infer that the error terms,

$$r_h^1 := \frac{1}{\epsilon}(b - b_\delta)\big(u_{\delta,h}^{\mathrm{BL}}\big)^\perp + \nabla_h p^{\mathrm{BL}},$$
$$r_h^2 := -\nu_h \Delta_h u_{\delta,h}^{\mathrm{BL}},$$
$$r_3^2 := -\nu_h \Delta_h u_{\delta,3}^{\mathrm{BL}}$$

satisfy the estimates of Theorem 1.2, namely

$$\left\| r_h^1 \right\|_{L^2(\omega)} = O\big(\epsilon^{5/4}\big),$$
$$\left\| r_h^2 \right\|_{L^2((0,1), H_h^{-1})} = O\big(\nu_h \epsilon^{-1/2}\big), \qquad \left\| r_3^2 \right\|_{L^2((0,1), H_h^{-1})} = O\big(\nu_h \epsilon^{-1/4}\big).$$

Furthermore, u_δ^{BL} satisfies the horizontal part of the boundary condition (1.3) at $z = 1$; on the other hand, u_3^{BL} does not satisfy the non-penetration condition at $z = 1$. Hence, we construct in the next section an interior term, which is also an approximate solution of (1.1), and which lifts the trace of u_3^{BL} at $z = 1$.

Notice that u^{BL} also has a non-vanishing trace at $z = 0$; however, this trace is exponentially small on the set where b is bounded away from zero, and can thus be lifted thanks to an exponentially small corrector. This will be taken care of after the construction of the interior term u^{int}, in the next section.

3. The interior part of the stationary solution

In this section, we construct a stationary solution u^{int} of Eq. (1.1), which is such that $u^{\text{int}} + u^{\text{BL}}$ satisfies the boundary conditions (1.2), (1.3). Going back to Eq. (1.1), it can be readily checked that the function u^{int} should satisfy the system:

$$\begin{aligned} b(y)\big(u_h^{\text{int}}\big)^\perp + \nabla_h p = 0,\\ \partial_z p = 0,\\ \operatorname{div} u^{\text{int}} = 0, \end{aligned} \tag{3.1}$$

together with the boundary conditions,

$$\begin{aligned} \partial_z u^{\text{int}}_{h|z=1} = 0, \qquad u^{\text{int}}_{3|z=1} = -u^{\text{BL}}_{3|z=1},\\ \partial_z u^{\text{int}}_{h|z=0} = 0, \qquad u^{\text{int}}_{3|z=0} = 0. \end{aligned} \tag{3.2}$$

We recall that since the function u_3^{BL} depends on the small parameter δ, the function u^{int} also depends on δ in general, and thus will be denoted by u_δ^{int} in the sequel. Hence we also investigate the asymptotic behaviour of u_δ^{int} as $\delta \to 0$.

It turns out that the solution of the system (3.1)–(3.2) is unique, up to a function of the type $(v(y), 0, 0)$. Hence we give in this section a straightforward way of building the solution, and then we derive L^2 estimates on the function u_δ^{int}. The main result of this section is the following:

Lemma 1. *Assume that assumptions* (1.8)–(1.10) *are fulfilled. Then there exists a solution $u_\delta^{\text{int}} \in L^2(\omega)$ of the system* (3.1). *Moreover, there exists a positive constant C, depending only on σ and b, such that*

$$\big\| u_\delta^{\text{int}} \big\|_{L^2(\omega)} \leqslant C \quad \forall \delta > 0.$$

3.1. Construction of u_δ^{int}

To begin with, we differentiate the first equation of (3.1) with respect to z, and we obtain

$$b(y)\partial_z\big(u^{\text{int}}_{\delta,h}\big)^\perp = 0.$$

Since u_δ^{int} is divergence-free, we infer that $\partial_{zz} u^{\text{int}}_{\delta,3} = 0$. Hence the third component $u^{\text{int}}_{\delta,3}$ is uniquely determined; in order to lighten the notation, set

$$w_\delta(x_h) = -u^{\text{BL}}_{\delta,3|z=1}(x_h).$$

We have,

$$u^{\text{int}}_{\delta,3}(x_h, z) = z w_\delta(x_h).$$

Then, taking the two-dimensional vorticity of the first equation in (3.1), we derive,

$$\operatorname{rot}_h\big(b\big(u^{\text{int}}_{\delta,h}\big)^\perp\big) = \operatorname{div}_h\big(b u^{\text{int}}_{\delta,h}\big) = 0.$$

Since the Coriolis factor only depends on the latitude y, we are led to,

$$b'(y) u^{\text{int}}_{\delta,2} = -b(y) \operatorname{div}_h u^{\text{int}}_{\delta,h} = +b(y)\partial_z u^{\text{int}}_{\delta,3} = b(y) w_\delta(x_h).$$

Consequently, the second component is also uniquely determined. In the case when $b(y) = \beta y$, one has in particular,

$$u^{\text{int}}_{\delta,2}(x_h) = y w_\delta(x_h). \tag{3.3}$$

This equation is known as the **Sverdrup relation** (see [25,26]).

There remains to compute the first component of u^{int}; the divergence-free condition entails that

$$\partial_x u^{\text{int}}_{\delta,1} = -\partial_y u^{\text{int}}_{\delta,2} - \partial_z u^{\text{int}}_{\delta,3} = -\partial_y\left(\frac{b}{b'} w_\delta\right) - w_\delta = -\left(2 - \frac{bb''}{b'^2}\right) w_\delta - \frac{b}{b'}\partial_y w_\delta.$$

Notice that this equation has a solution in ω_h if and only if the right-hand side has zero average in x, for all y. This is satisfied in particular, if

$$\int_{\mathbf{T}} w_\delta(x,y)\,dx = 0 \quad \forall y. \tag{3.4}$$

We assume that this assumption is satisfied for the time being, and we will prove that it is in fact equivalent to (1.9). Integrating the equality giving $\partial_x u^{\mathrm{int}}_{\delta,1}$ with respect to x, we deduce that $u^{\mathrm{int}}_{\delta,1}$ is defined up to a function of y only, provided (3.4) is satisfied.

Now, let us compute w_δ in terms of σ and b. Using Eq. (2.5), we infer that

$$w_\delta(x_h) = \frac{1}{2}\sum_\pm (\mathrm{div}_h\,\sigma \mp i\,\mathrm{rot}_h\,\sigma)\frac{1}{(\lambda^\pm_\delta)^2} - \sum_\pm \left(\sigma \pm i\sigma^\perp\right)\cdot \frac{\nabla_h \lambda^\pm_\delta}{(\lambda^\pm_\delta)^3}.$$

By definition of $\lambda^\pm$ (see (2.2)), we have,

$$\nabla\lambda^\pm_\delta = \left(0, \frac{1 \mp i\,\mathrm{sign}(b)}{2\sqrt{2}}\frac{\mathrm{sign}(b)b'_\delta}{|b_\delta|^{1/2}}\right).$$

Hence

$$w_\delta(x_h) = \frac{1}{b_\delta}\,\mathrm{rot}_h\,\sigma + \frac{1}{b_\delta^2}\sigma^\perp\cdot\nabla b_\delta = \frac{\partial_x\sigma_2}{b_\delta} - \partial_y\left(\frac{\sigma_1}{b_\delta}\right). \tag{3.5}$$

(Recall that b_δ only depends on the latitude y.)

We now prove the equivalence of (1.9) and (3.4). It is clear that (1.9) $\Rightarrow$ (3.4). Conversely, if (3.4) is satisfied, then (3.5) leads to the existence of a constant $\alpha_\delta \in \mathbf{R}$ such that

$$\int_{\mathbf{T}} \frac{\sigma_1(x,y)}{b_\delta(y)}\,dx = \alpha_\delta \quad \forall y.$$

Since σ_1 vanishes quadratically near $y = 0$, we deduce that the left-hand side of the above equality vanishes at least linearly near $y = 0$. Consequently, $\alpha_\delta = 0$ for all δ, and thus (1.9) is satisfied.

3.2. *Bounds on* u^{int}

We begin with a bound on the function w_δ given by (3.5). We recall that

$$b(y) \sim \beta y \quad \text{near } y = 0,$$

and

$$\sigma_1(x,y) = O\left(|y|^2\right) \quad \text{as } y \to 0,$$
$$\partial_x\sigma_2(x,y) = O\left(|y|^2\right) \quad \text{as } y \to 0.$$

Thus

$$\partial_y\left(\frac{\sigma_1}{b_\delta}\right) = \frac{b_\delta\partial_y\sigma_1 - \sigma_1\partial_y b_\delta}{b_\delta^2} = O\left(|y|^\alpha\delta^{-\alpha}\right) \quad \text{for } y \to 0,\ |y| \leqslant \delta.$$

The exponent α was introduced in the previous section, see (2.6).

Consequently, there exists a constant C (independent of δ) such that

$$\|w_\delta\|_{L^2(\omega_h)} \leqslant C.$$

This entails immediately that $u^{\mathrm{int}}_{\delta,3}$ and $u^{\mathrm{int}}_{\delta,2}$ are bounded in $L^2(\omega)$, uniformly in δ.

As for $u^{\mathrm{int}}_{\delta,1}$, we have, by definition

$$\begin{aligned}\partial_x u_{\delta,1}^{\text{int}} &= -\partial_y\left(\frac{b\partial_x\sigma_2}{b'b_\delta}\right) - \frac{\partial_x\sigma_2}{b_\delta} + \partial_y\left(\frac{b}{b'}\partial_y\frac{\sigma_1}{b_\delta}\right) + \partial_y\frac{\sigma_1}{b_\delta}\\ &= -\partial_y\left(\frac{\partial_x\sigma_2}{\psi(\frac{\cdot}{\delta})b'}\right) - \frac{\partial_x\sigma_2}{b_\delta} + \partial_y\frac{\partial_y\sigma_1}{b'\psi(\frac{\cdot}{\delta})} - \partial_y\left(\frac{\sigma_1}{b_\delta}\left(\frac{bb'_\delta}{b'b_\delta}-1\right)\right).\end{aligned}$$

Integrating with respect to x, we deduce that

$$u_{\delta,1}^{\text{int}} = -\partial_y\left(\frac{\sigma_2}{\psi b'}\right) - \frac{\sigma_2}{b_\delta} + \partial_y\frac{\partial_y S_1}{b'\psi} - \partial_y\left(\frac{S_1}{b_\delta}\left(\frac{bb'_\delta}{b'b_\delta}-1\right)\right),$$

where $S_1(x,y) = \int_0^x \sigma_1(x',y)\,dx'$. Using the definition of b_δ, we obtain

$$\frac{bb'_\delta}{b'b_\delta} - 1 = \frac{1}{\delta}\frac{\psi'}{\psi}\left(\frac{\cdot}{\delta}\right)\frac{b}{b'}.$$

It can be checked that the function in the right-hand side is bounded in $W^{2,\infty}(\mathbf{R})$. Moreover, its support is included in $[-2\delta, 2\delta]$. As a consequence, the term

$$\partial_y\left(\frac{S_1}{b_\delta}\left(\frac{bb'_\delta}{b'b_\delta}-1\right)\right)$$

is $o(1)$ in $H^1(\omega)$ as $\delta\to 0$. The other terms can be evaluated in a similar fashion. Using the assumptions on σ and b together with the definition of ψ, we deduce that there exists a constant $C[\sigma]$ such that

$$\left\|u_\delta^{\text{int}}\right\|_{L^2(\omega)} \leqslant C[\sigma]. \tag{3.6}$$

• We now derive estimates in $L^2([0,1], H^1(\omega_h))$, which are needed to bound the error term $\nu_h\Delta_h u_\delta^{\text{int}}$. First, using the definition of b_δ together with assumptions (1.8), (1.10), it can be proved that

$$\partial_y w_\delta = O\left(y^{\alpha-1}\delta^{-\alpha}\right) \quad \text{as } y\to 0, |y|\leqslant\delta.$$

Hence $\partial_y w_\delta \in L^2(\omega)$ (recall that $\alpha > 3/5 > 1/2$) and

$$\|\partial_y w_\delta\|_{L^2(\omega)} = O\left(\delta^{-1/2}\right).$$

The term $\partial_x w_\delta$, on the other hand, is bounded in $L^2(\omega)$, uniformly in δ. Consequently, there exists a constant C, depending only on σ, b and α, such that

$$\left\|\nabla_h u_{\delta,3}^{\text{int}}\right\|_{L^2(\omega)} \leqslant \frac{C}{\delta^{1/2}}.$$

Similarly, we prove that $\partial_y u_{\delta,2} = O(|y|^\alpha\delta^{-\alpha})$ for y in a neighbourhood of zero, and thus there exists a constant C such that

$$\left\|\nabla_h u_{\delta,2}^{\text{int}}\right\|_{L^2(\omega)} \leqslant C.$$

We now tackle the term $u_{\delta,1}$; using either the expression of $\partial_x u_{\delta,1}$ in terms of w_δ or the final definition in terms of σ_2 and S_1, it can be checked that

$$\partial_y u_{\delta,1} = O\left(y^{\alpha-1}\delta^{-\alpha}\right) \quad \text{as } y\to 0, |y|\leqslant\delta.$$

The largest terms are those coming from S_1 (or from $b\partial_y w_\delta/b'$); for instance, the above calculations show that

$$\frac{b}{b'}\partial_y w_\delta = O\left(|y|^\alpha\delta^{-\alpha}\right);$$

since one power of y is lost with each differentiation with respect to y, we obtain the desired bound on $u_{\delta,1}$. Eventually, we are led to

$$\left\|\nabla_h u_{\delta,1}^{\text{int}}\right\|_{L^2(\omega)} \leqslant \frac{C}{\delta^{1/2}}.$$

• Notice that

$$u_\delta^{\text{int}} \to u^{\text{int}} \quad \text{in } L^2(\omega),$$

as $\delta \to 0$, where $u^{\rm int}$ is the function defined by the same expressions as $u_\delta^{\rm int}$, but replacing every occurrence of w_δ by

$$w = \frac{\operatorname{rot}_h \sigma}{b} + \frac{\sigma^\perp \cdot \nabla b}{b^2}.$$

By definition of b_δ, w and w_δ coincide on the set $\{|y| \geqslant 2\delta\}$. Moreover, w is bounded in L^2 and w, $y\partial_y w$ have finite limits as $y \to 0$, while

$$\int_{\mathbf{T}} \int_{|y| \leqslant \delta} |w_\delta|^2 + |y|^2 |\partial_y w_\delta|^2 = o(1).$$

Consequently, w_δ (resp. $b\partial_y w_\delta$) converges towards w (resp $b\partial_y w$) in $L^2(\omega)$ as $\delta \to 0$. The convergence of $u_\delta^{\rm int}$ follows. However, in general, $u^{\rm int}$ does not belong to $H^1(\omega)$, except if the surface stress σ vanishes at sufficiently high order.

3.3. Proof of Theorem 1.2

Let us first evaluate the error terms in Eq. (1.1). To begin with, notice that $\partial_{zz} u_\delta^{\rm int} = 0$, so that there is no error term associated with the vertical Laplacian. Consequently, the only error terms in Eq. (1.1) are those coming from the term $\nu_h \Delta_h u_\delta^{\rm int}$.

According to the H^1 estimates of the previous section, we have

$$\left\| \nu_h \Delta_h u_{\delta,h}^{\rm int} \right\|_{L^2([0,1], H^{-1}(\omega_h))} \leqslant C \frac{\nu_h}{\sqrt{\delta}},$$
$$\left\| \nu_h \Delta_h u_{\delta,3}^{\rm int} \right\|_{L^2([0,1], H^{-1}(\omega_h))} \leqslant C \frac{\nu_h}{\sqrt{\delta}}.$$

With the choice $\delta = \epsilon$, this yields another error term $r^2 \in L^2([0,1], H^{-1}(\omega_h))$ such that

$$\left\| r^2 \right\|_{L^2([0,1], H^{-1}(\omega_h))} = O\left(\frac{\nu_h}{\sqrt{\epsilon}} \right).$$

• The proof of Theorem 1.2 is now almost complete. There only remains to take care of the boundary conditions: indeed, as we have explained at the end of the previous section, the trace of $\partial_z u_{\delta,h}^{\rm BL}$ and $u_{\delta,3}^{\rm BL}$ is non-zero at $z = 0$. Hence, we define a corrector $v_\delta^{\rm int}$, which is small in H^1, and which lifts the remaining boundary conditions. The result is the following:

Lemma 3.1. *Assume that $\nu_h = o(\epsilon)$. Then there exists a divergence-free function $v_\delta^{\rm int}$, such that $v_\delta^{\rm int} = o(1)$ in $L^2(\omega)$, which satisfies the conditions,*

$$\partial_z v_{\delta,h|z=1}^{\rm int} = 0, \qquad v_{\delta,3|z=1}^{\rm int} = 0,$$
$$\partial_z v_{\delta,h|z=0}^{\rm int} = -\partial_z u_{h|z=0}^{\rm BL}, \qquad v_{\delta,3|z=0}^{\rm int} = -u_{3|z=0}^{\rm BL}.$$

Furthermore, with the choice $\delta = \epsilon$ and $\alpha > 3/5$, we have

$$\frac{1}{\epsilon} b e_3 \wedge v_\delta^{\rm int}, -\epsilon \partial_{zz} v_{\delta,h}^{\rm int} = O(\epsilon^3) \quad \text{in } L^2(\omega),$$
$$-\epsilon \partial_{zz} v_{\delta,3}^{\rm int} = O(\epsilon^2) \quad \text{in } L^2([0,1], H^{-1}(\omega_h)),$$
$$\nu_h \Delta_h v_{\delta,h}^{\rm int} = O(\epsilon \nu_h) \quad \text{in } L^2([0,1], H^{-1}(\omega_h)),$$
$$\nu_h \Delta_h v_{\delta,3}^{\rm int} = O(\nu_h \epsilon^{-1/2}) \quad \text{in } L^2([0,1], H^{-1}(\omega_h)).$$

Before proving the lemma, let us complete the proof of Theorem 1.2: we choose $\delta = \epsilon$, $\alpha > 3/5$. We set

$$u^{\rm stat} = u_\delta^{\rm BL} + u_\delta^{\rm int} + v_\delta^{\rm int};$$

by construction, u^{stat} satisfies the boundary conditions (1.2), (1.3), and it is an approximate solution of Eq. (1.1) in the sense of Theorem 1.2. The bounds on u_δ^{int} and u_δ^{BL} were proved in the previous sections. Notice that the error terms steaming from the corrector v_δ^{int} are all of lower order than the ones coming from u_δ^{int}, u_δ^{BL}. Hence Theorem 1.2 is proved.

Proof of Lemma 3.1. Throughout the proof, we drop all indices δ in order not to burden the notation.

The construction of the corrector v^{int} follows the one given in Lemma 1 in Appendix B of [10]: setting

$$\phi_h := -\partial_z u^{\mathrm{BL}}_{h|z=0}, \qquad \phi_3 := -u^{\mathrm{BL}}_{3|z=0},$$

we define:

$$v_h^{\mathrm{int}} = \frac{(1-z)^2}{2}\phi_h + \nabla_h \chi,$$

where the potential $\chi \in H^2(\omega_h)$ is defined by,

$$\Delta_h \chi = \int_0^1 \mathrm{div}_h\, v_h^{\mathrm{int}} - \frac{1}{6}\mathrm{div}_h\, \phi_h = -\big[v_3^{\mathrm{int}}\big]_{z=0}^{z=1} - \frac{1}{6}\mathrm{div}_h\, \phi_h = \phi_3 - \frac{1}{6}\mathrm{div}_h\, \phi_h.$$

We will check later on that the function ϕ_3 has zero mean value on ω_h, so that χ is well-defined. The third component of v^{int} is then determined by

$$v_3^{\mathrm{int}}(x_h, z) = -\int_z^1 \partial_z v_3^{\mathrm{int}}\big(x_h, z'\big)\, dz' = \int_z^1 \mathrm{div}_h\, v_h^{\mathrm{int}}\big(x_h, z'\big)\, dz'.$$

By construction, v^{int} is divergence free and satisfies the correct boundary conditions. There remains to evaluate v^{int} in $L^2(\omega)$ and $L^2([0,1], H^1(\omega_h))$.

The boundary conditions ϕ_h, ϕ_3 are given by,

$$\phi_h = -\frac{1}{2\epsilon}\sum_\pm \big(\sigma \pm i\sigma^\perp\big)\exp\biggl(-\frac{\lambda^\pm}{\epsilon}\biggr),$$

$$\phi_3 = -\frac{1}{2}\sum_\pm (\mathrm{div}_h\, \sigma \mp i\, \mathrm{rot}_h\, \sigma)\frac{1}{(\lambda^\pm)^2}\exp\biggl(-\frac{\lambda^\pm}{\epsilon}\biggr) + \frac{1}{2}\sum_\pm \big(\sigma \pm i\sigma^\perp\big)\cdot \frac{\nabla_h \lambda^\pm}{(\lambda^\pm)^3}\biggl(2 + \frac{\lambda^\pm}{\epsilon}\biggr)\exp\biggl(-\frac{\lambda^\pm}{\epsilon}\biggr).$$

Recall that in the expressions above, the functions $\lambda^\pm$ are in fact $\lambda_\delta^\pm$. Notice that

$$\phi_3 = \mathrm{div}_h\, \varphi,$$

where

$$\varphi = -\frac{1}{2}\sum_\pm \big(\sigma \pm i\sigma^\perp\big)\frac{1}{(\lambda^\pm)^2}\exp\biggl(-\frac{\lambda^\pm}{\epsilon}\biggr);$$

this proves that ϕ_3 has zero mean value on ω_h, and will be used several times in the proof.

We now derive three type of estimates: first, estimates of $\nabla_h \phi_h$ and ϕ_3 in $L^2(\omega_h)$ will yield $H^2(\omega_h)$-bounds on χ, and thus bounds in $L^2([0,1], H^1(\omega_h))$ for the function v_h^{int}, and in $L^2(\omega)$ for the function v_3^{int}. Then, estimates of ϕ_h and φ will provide $L^2(\omega)$-bounds on v_h^{int}. Eventually, L^2 estimates of $\nabla_h \phi_3$, $D_h^2 \phi_h$ will allow us to derive bounds on v_3^{int} in $L^2([0,1], H^1(\omega_h))$.

• **Estimates of $\nabla_h \phi_h$ and ϕ_3 in $L^2(\omega_h)$:**

The main difficulty lies in the fact that $\lambda_\delta^\pm$ does not have the same behaviour for $|y| \leqslant \delta$ and $|y| \geqslant \delta$. We merely explain how the term $\mathrm{div}_h\, \phi_h$ is evaluated; the treatment of the term ϕ_3 is left to the reader.

If $|y| \geqslant 1$, we have

$$\big(\lambda^\pm\big)^2 = \mp i b(y), \quad \text{with } \big|b(y)\big| \geqslant C \quad (\text{see (1.10)}).$$

Thus

$$\left|\exp\left(-\frac{\lambda^\pm}{\epsilon}\right)\right| \leqslant \exp\left(-\frac{C}{\epsilon}\right),$$

and

$$\int_{|y|\geqslant 1}\int_{x\in\mathbf{T}} \left|\nabla_h\phi_h(x,y)\right|^2 dx\,dy \leqslant \frac{C}{\epsilon^4}\|\sigma\|^2_{H^1(\omega_h)}\exp\left(-\frac{2C}{\epsilon}\right).$$

On the set where $\delta \leqslant |y| \leqslant 1$, the assumptions on the truncation function ψ entail that there exists a constant c such that

$$c^{-1}|y|^{1/2} \leqslant \Re\left(\lambda_\delta^\pm(y)\right), \qquad \left|\lambda_\delta^\pm(y)\right| \leqslant c|y|^{1/2},$$
$$\left|\partial_y\lambda_\delta^\pm(y)\right| \leqslant c|y|^{-1/2}.$$

As a consequence,

$$\int_{\delta\leqslant|y|\leqslant 1}\int_{x\in\mathbf{T}} \left|\nabla_h\phi_h(x,y)\right|^2 dx\,dy \leqslant \frac{C}{\epsilon^2}\int_\delta^1 |y|^2\exp\left(-2c\frac{\sqrt{y}}{\epsilon}\right)dy + \frac{C}{\epsilon^4}\int_\delta^1 |y|^4\frac{1}{|y|}\exp\left(-2c\frac{\sqrt{y}}{\epsilon}\right)dy \leqslant C\epsilon^4.$$

There remains to treat the set where $|y| \leqslant \delta$; because of the truncation function ψ, this part is the most complicated. The definition of the function ψ and the fact that $b(y) \sim \beta y$ for y close to zero entail that

$$c^{-1}|y|^{\frac{1-\alpha}{2}}\delta^{\frac{\alpha}{2}} \leqslant \left|\lambda_\delta^\pm(y)\right|, \qquad \Re\left(\lambda_\delta^\pm(y)\right) \leqslant c|y|^{\frac{1-\alpha}{2}}\delta^{\frac{\alpha}{2}},$$
$$\left|\partial_y\lambda_\delta^\pm(y)\right| \leqslant c|y|^{-\frac{1+\alpha}{2}}\delta^{\frac{\alpha}{2}}.$$

Thus, for instance

$$\int_{|y|\leqslant\delta}\int_{x\in\mathbf{T}} \left|\operatorname{div}_h\sigma\exp\left(-\frac{\lambda^\pm}{\epsilon}\right)\right|^2 \leqslant C\int_0^\delta |y|^2\exp\left(-c\frac{|y|^{\frac{1-\alpha}{2}}\delta^{\frac{\alpha}{2}}}{\epsilon}\right)dy \leqslant C\left(\frac{\epsilon^{\frac{2}{1-\alpha}}}{\delta^{\frac{\alpha}{1-\alpha}}}\right)^3.$$

The other terms in $\operatorname{div}_h\phi_h$ are evaluated in the same way. Gathering all the terms, we infer that

$$\|\nabla_h\phi_h\|_{L^2(\omega_h)} \leqslant C\left(\epsilon^{-2}\exp(-C/\epsilon) + \epsilon^2 + \epsilon^{-1}\left(\frac{\epsilon^{\frac{1}{1-\alpha}}}{\delta^{\frac{\alpha}{2(1-\alpha)}}}\right)^3\right).$$

With the choice $\delta = \epsilon$, we obtain

$$\|\nabla_h\phi_h\|_{L^2(\omega_h)} \leqslant C\left(\epsilon^{-2}\exp(-C/\epsilon) + \epsilon^2 + \epsilon^{\frac{4-\alpha}{2(1-\alpha)}}\right).$$

Since $\frac{4-\alpha}{2(1-\alpha)} \geqslant 2$, we deduce eventually that $\|\nabla_h\phi_h\|_{L^2(\omega_h)} = O(\epsilon^2)$.

Using the same arguments, it can be checked that $\|\phi_3\|_{L^2(\omega)} = O(\epsilon)$. As a consequence,

$$\left\|\nabla_h v_h^{\text{int}}\right\|_{L^2((0,1),H^{-1}(\omega_h))} = O(\epsilon), \qquad \left\|\partial_{zz}v_3^{\text{int}}\right\|_{L^2((0,1),H^{-1}(\omega_h))} = O(\epsilon).$$

Similarly, we show that with $\delta = \epsilon$,

$$\|\phi_h\|_{L^2(\omega)} = O\left(\epsilon^4\right), \qquad \|\varphi\|_{L^2(\omega)} = O\left(\epsilon^3\right).$$

Consequently, $\|v_h^{\text{int}}\|_{L^2} = O(\epsilon^3)$. Notice that this is not entirely sufficient to prove the assertion of the lemma since the Coriolis factor b is unbounded when $\omega_h = \mathbf{T}\times\mathbf{R}$. However, using the fact that ϕ_h and φ decay like $\exp(-|b|^{1/2}/\epsilon)$ for $|y| \geqslant 1$, it can be easily proved that

$$\Delta_h(b\chi) = b\Delta_h\chi + 2b'\partial_2\chi + b''\chi = O\left(\epsilon^3\right) \quad \text{in } H^{-1}(\omega_h).$$

Hence $b\chi = O(\epsilon^3)$ in $H^1(\omega_h)$, and $b\nabla\chi = \nabla(b\chi) - b'\chi = O(\epsilon^3)$ in L^2. Eventually, we infer that $bv_h^{\text{int}} = o(\epsilon)$ in $L^2(\omega)$.

• **Estimates of $D^2\phi_h$ and $\nabla_h\phi_3$ in $L^2(\omega_h)$:**
Calculations similar to the ones led above show that if $\delta=\epsilon$,

$$\left\|D^2\phi_h\right\|_{L^2(\omega_h)}\leqslant C\|\sigma\|_{H^2(\omega_h)}\left(\frac{\exp(-C/\epsilon)}{\epsilon^3}+1+\epsilon^{\frac{\alpha}{2(1-\alpha)}}\right).$$

Hence $\|D^2\phi_h\|_{L^2(\omega)}=O(1)$.

The term $\nabla_h\phi_3$ is the most singular of all. Indeed, it can be proved that with $\delta=\epsilon$,

$$\|\nabla_h\phi_3\|_{L^2(\omega_h)}\leqslant C\|\sigma\|_{H^2(\omega_h)}\left(\frac{\exp(-C/\epsilon)}{\epsilon^2}+1+\epsilon^{-1/2}+\epsilon^{\frac{3\alpha-2}{1-\alpha}}\right).$$

Since $\alpha>3/5$, $(3\alpha-2)/(1-\alpha)>-1/2$, and thus

$$\|\nabla_h\phi_3\|_{L^2(\omega_h)}=O\big(\epsilon^{-1/2}\big).$$

Eventually, we infer that

$$\left\|\Delta_h v_3^{\mathrm{int}}\right\|_{L^2((0,1),H^{-1}(\omega_h))}=O\big(\epsilon^{-1/2}\big). \qquad \square$$

4. Two-dimensional propagation

We recall that throughout this section and the following, we assume that $b(x_h)=\beta y$, and that $\omega_h=\mathbf{T}\times\mathbf{R}$. The object of this section is to prove the “two-dimensional part” of Theorem 1.5. In particular, we prove that a two-dimensional perturbation of the solution u^{stat} creates waves, propagating at a speed of order ϵ^{-1}, with frequencies given by

$$\beta\frac{k}{|k|^2+|\xi_y|^2},$$

where (k,ξ_y) is the wavelength.

Remark 4.1 *(Strong stability in 2D).* A consequence of our result is that if u^{stat} is initially perturbed by a two-dimensional function u^0 such that $u^0=O(1)$ in L^2 and such that the x-average of u^0 is zero (i.e. u^0 has no Fourier mode corresponding to $k=0$), then the solution of (1.1) with initial data $u^{\mathrm{stat}}+u^0$ becomes close to u^{stat} for finite times, with an error term which is $o(1)$ in $L^2([T_0,T]\times\omega)$ for all $T>T_0>0$.

Definition 4.2. Denote by $\mathbb{P}_{2D}:L^2(\omega_h)^2\to L^2(\omega_h)^2$ the projection on two-dimensional divergence-free vector fields. The Rossby propagation operator, denoted by L_R, is defined by

$$L_RV=\mathbb{P}_{2D}\big(bV^\perp\big).$$

Lemma 4.3. *Let $\bar v_h^0\in L^2(\omega_h)$ be a two-dimensional divergence-free vector field, and let $v\in\mathcal{C}(\mathbf{R}_+,L^2(\omega))$ be the solution of Eq.* (1.1) *with initial data*

$$v_{|t=0}=\begin{pmatrix}\bar v_h^0\\0\end{pmatrix},$$

supplemented with the boundary conditions,

$$\partial_z v_{h|z=1}=0,\qquad v_{3|z=1}=0,$$
$$\partial_z v_{h|z=0}=0,\qquad v_{3|z=0}=0.$$

Then $v=(v_h,0)$, where v_h is a two-dimensional divergence-free vector field given by

$$v_h(t)=\frac{1}{2\pi}\sum_{k\in\mathbf{Z}}\int_{\mathbf{R}}\exp\left(\frac{i\beta t}{\epsilon}\frac{k}{|k_h|^2}-\nu_h|k_h|^2t+ix_h\cdot k_h\right)\hat v_h^0(k,\xi_y)\,d\xi_y,$$

where $k_h=(k,\xi_y)$ *and*

$$\hat{v}_h^0(k,\xi_y)=\frac{1}{2\pi}\int_{\omega_h}\exp\bigl(-i(xk+y\xi_y)\bigr)\bar{v}_h^0(x,y)\,dx\,dy,\quad \forall(k,\xi_y)\in\mathbf{Z}\times\mathbf{R}.$$

Proof. Let us first prove that the property $\partial_z u=0$ is propagated by Eq. (1.1). Using the same arguments as Chemin, Desjardins, Gallagher and Grenier in [4] for classical rotating fluids, one can introduce some kind of Fourier variable with respect to z, denoted by k_3. Since Eq. (1.1) is linear, it can be easily checked that there is no resonance between Fourier modes in k_3; in other words, since the only Fourier mode at time $t=0$ is $k_3=0$, there is no Fourier mode corresponding to $k_3\neq 0$ for $t>0$, which means exactly that $\partial_z v=0$.

We infer that for all $t\geqslant 0$, $v(t)$ is a two-dimensional vector field which satisfies

$$\begin{gathered}\partial_t v_h+\frac{1}{\epsilon}L_R v_h-\nu_h\Delta_h v_h=0,\qquad \operatorname{div}_h v_h=0,\\ \partial_z v_h=0,\qquad v_3=0.\end{gathered}\tag{4.1}$$

This leads to

$$v_h(t)=\exp\biggl(t\biggl(-\frac{L_R}{\epsilon}+\nu_h\Delta_h\biggr)\biggr)v_{h|t=0}.$$

Let us now investigate the precise expression of the operator L_R. First, since v_h is divergence free, we have, for all $y\in\mathbf{R}$,

$$\partial_y\int_{\mathbf{T}}v_2(\cdot,y)=-\int_{\mathbf{T}}\partial_x v_1(\cdot,y)=0.$$

Consequently, since $v_h\in L^2(\mathbf{T}\times\mathbf{R})$,

$$\int_{\mathbf{T}}v_2(t,\cdot,y)=0\quad\forall t\geqslant 0,\ y\in\mathbf{R}.$$

Taking the x-average of the first component of (4.1), we obtain

$$\partial_t\int_{\mathbf{T}}v_1-\nu_h\partial_y^2\int_{\mathbf{T}}v_1=0.$$

This corresponds to the "stationary part" of v^ϵ in Theorem 1.5.

Hence Lemma 4.3 is proved for the Fourier modes such that $k=0$, where k is the Fourier variable associated with x. Thus we now focus on the modes such that $k\neq 0$, or, in other words, on initial data such that $\int_{\mathbf{T}}\bar{v}_h^0=0$. For such vector fields, we have, since $v_h\in L^2(\mathbf{T}\times\mathbf{R})$ is divergence free,

$$v_h=\nabla_h^\perp\Delta_h^{-1}\zeta,$$

where $\zeta:=\operatorname{rot}_h v_h$. On the other hand,

$$\operatorname{rot}_h\bigl(bv_h^\perp\bigr)=\operatorname{div}_h(bv_h)=v_h\cdot\nabla b=\beta v_2=\beta\partial_x\Delta_h^{-1}\zeta.$$

In Fourier space, this leads to

$$\partial_t\hat{\zeta}(k,\xi_y)-i\frac{\beta k}{\epsilon(|k|^2+|\xi_y|^2)}\hat{\zeta}(k,\xi_y)+\nu_h\bigl(|k|^2+|\xi_y|^2\bigr)\hat{\zeta}=0,$$

and thus, setting $k_h=(k,\xi_y)$,

$$\hat{v}_h(t,k,\xi_y)=-\frac{ik_h^\perp}{|k_h|^2}\exp\biggl(i\frac{\beta k}{\epsilon|k_h|^2}t-\nu_h|k_h|^2t\biggr)\hat{\zeta}_{|t=0}(k,\xi_y)=\exp\biggl(i\frac{\beta k}{\epsilon|k_h|^2}t-\nu_h|k_h|^2t\biggr)\frac{k_h^\perp\cdot\hat{v}_h^0(k,\xi_y)}{|k_h|^2}k_h^\perp.$$

Since v is a two-dimensional divergence-free vector field, for all $k_h \in \mathbf{Z} \times \mathbf{R}$, we have $k_h \cdot \hat{v}_h^0(k_h) = 0$, and thus

$$\hat{v}_h^0(k_h) = \frac{k_h^\perp \cdot \hat{v}_h^0(k_h)}{|k_h|^2} k_h^\perp.$$

Eventually, we retrieve

$$\hat{v}_h(t, k, \xi_y) = \exp\left(i \frac{\beta k}{\epsilon(|k|^2 + |\xi_y|^2)} t - \nu_h |k_h|^2 t \right) v_h^0(k, \xi_y) \quad \forall t, k, \xi_y.$$

Using the Fourier inversion formula, the proof of the lemma is complete. $\square$

• We now prove the dispersion result for Rossby waves. The argument is quite classical: we want to prove that the function

$$v_R(x, t) := \frac{1}{2\pi} \sum_{k \in \mathbf{Z} \setminus \{0\}} \int_{\mathbf{R}} \exp\left(\frac{i\beta t}{\epsilon} \frac{k}{|k_h|^2} - \nu_h |k_h|^2 t + i x_h \cdot k_h \right) \hat{v}_h^0(k, \xi_y)\, d\xi_y \tag{4.2}$$

satisfies, for all $t > 0$ and for all compact sets $K \subset \mathbf{R}^2$,

$$\left\| v_R(t) \right\|_{L^2(K)} \to 0 \quad \text{as } \epsilon \to 0.$$

We first localize the problem in Fourier space. Let $n \in \mathbf{N}$ arbitrary. There exists a function $\hat{v}_h^n \in L^2(\mathbf{Z} \times \mathbf{R})$ with compact support in $\mathbf{Z} \times \mathbf{R}$ such that

$$\operatorname{Supp} \hat{v}_h^n \subset \mathbf{Z} \times \mathbf{R}^*,$$
$$\left\| \hat{v}_h^n - \hat{v}_h^0 \right\|_{L^2(\mathbf{Z} \times \mathbf{R})} \leqslant \frac{1}{n}.$$

Notice that the assumptions on the support of $\hat{v}_h^n$ mean that we have truncated both the large and the small frequencies. Without loss of generality, we can also assume that $\hat{v}_h^n(k, \cdot) \in \mathcal{C}_0^\infty(\mathbf{R})$ for all $k \in \mathbf{Z}$. We denote by v_R^n the function defined as in (4.2) by replacing $\hat{v}_h^0$ by $\hat{v}_h^n$. Then the Plancherel formula yields, for all $t \geqslant 0$,

$$\left\| v_R(t) - v_R^n(t) \right\|_{L^2(\mathbf{T} \times \mathbf{R})} = \left\| \hat{v}_h^0 - \hat{v}_h^n \right\|_{L^2(\mathbf{Z} \times \mathbf{R})} \leqslant \frac{1}{n}.$$

Hence we work with v_R^n from now on.

Notice that for $\xi_y \in \mathbf{R}^*$, $k \in \mathbf{Z}^*$

$$\exp\left(\frac{i\beta t k}{\epsilon |k_h|^2} \right) = -\frac{|k_h|^4 \epsilon}{2i\beta t k \xi_y} \frac{d}{d\xi_y} \exp\left(\frac{i\beta t k}{\epsilon |k_h|^2} \right).$$

Thus, integrating by parts, we infer

$$\begin{aligned} v_R^n(t, x) = \frac{1}{2\pi} \sum_{k \in \mathbf{Z}^*} \int_{\mathbf{R}} d\xi_y \frac{|k_h|^4 \epsilon}{2i\beta t k} \exp\left(\frac{i\beta t}{\epsilon} \frac{k}{|k_h|^2} - \nu_h |k_h|^2 t + i x_h \cdot k_h \right) \\ \times \left\{ \left[-\frac{1}{\xi_y^2} + \frac{1}{\xi_y} (-2\nu_h \xi_y t + i y) \right] \hat{v}_h^n(k, \xi_y) + \frac{1}{\xi_y} \partial_{\xi_y} \hat{v}_h^n(k, \xi_y) \right\}. \end{aligned}$$

According to the assumptions on the support of $\hat{v}_h^n$, there exists a constant C_n such that for all $k_h = (k, \xi_y) \in \operatorname{Supp} \hat{v}_h^n$,

$$\left| \xi_y^{-1} \right| \leqslant C_n, \quad |k_h| \leqslant C_n.$$

Thus we deduce that for all $t > 0$, $x \in K$,

$$\left| v_R^n(t, x) \right| \leqslant C_{n,K} \frac{\epsilon}{t} (1 + t),$$

so that eventually

$$\left\| v_R(t) \right\|_{L^2(K)} \leqslant \frac{1}{n} + C_{n,K} \epsilon \left(1 + \frac{1}{t} \right).$$

The result announced in Theorem 1.5 follows.

5. Three-dimensional propagation

5.1. Remarks about the qualitative behaviour of three-dimensional waves

We are now interested in waves having vertical oscillations, that is in the solutions to

$$\begin{gathered}\partial_t u+\frac{1}{\epsilon}\beta y u^{\perp}+\begin{pmatrix}\nabla_h p\\ \frac{1}{\epsilon^2}\partial_z p\end{pmatrix}-\nu_h\Delta_h u-\epsilon\partial_{zz}u=0,\\ \nabla\cdot u=0,\\ \partial_z u_{h|z=0}=\partial_z u_{h|z=1}=0,\qquad u_{3|z=0}=u_{3|z=1}=0,\end{gathered}\tag{5.1}$$

having zero average with respect to z.

Once again, we introduce a kind of Fourier variable with respect to z (see [4]), denoted by k_3, which here is different from zero. The Fourier variable associated with the first coordinate x is still denoted by k.

If ν_h is sufficiently small, we then expect the main dynamics to be given by the Poincaré propagation operator

$$L_P u=\beta y e_3\wedge u+\begin{pmatrix}\epsilon\nabla_h p\\ \frac{1}{\epsilon}\partial_z p\end{pmatrix}\tag{5.2}$$

where p is such that both the incompressibility constraint and the boundary condition are satisfied.

• A very rough analysis shows that **fast oscillations with respect to** y **should appear** for times greater than ϵ. Indeed, as long as the solution (u,p) to

$$\epsilon\partial_t u+L_P u=0$$

depends slowly on y, the pressure which satisfies

$$-\left(\partial_{xx}+\partial_{yy}p+\frac{1}{\epsilon^2}\partial_{zz}\right)p=-\frac{1}{\epsilon}\beta y\partial_x u_2+\frac{1}{\epsilon}\partial_y(\beta y u_1)$$

can be approximated in the following way

$$\hat p=\frac{\epsilon}{k_3^2}\big(-ik\beta y\hat u_2+\partial_y(\beta y\hat u_1)\big)=O(\epsilon).$$

In particular, at leading order, the singular penalization behaves as in the compressible case

$$(L_P u)_h\sim\beta y u_h^{\perp}.$$

Plugging this ansatz in the evolution equation leads to

$$u_h\sim\sum_{\pm}u_h^{0,\pm}\exp\left(\pm i\frac{\beta y t}{\epsilon}\right),$$

which is relevant only for very small times, but indicates that a fast dependence with respect to y can be expected.

• On the other hand, we do not expect (u,p) to behave as a function of y/ϵ only. Such a property, together with usual integrability conditions, would indeed imply that the solution (u,p) concentrates on small times in the vicinity of $y=0$. Notice that this is not a consequence of the equation, but follows from the two properties,

$$u(t,y)\sim\varphi\left(t,\frac{y}{\epsilon}\right),$$

for some function φ and for small times, and $\|u(t)\|_{L^2}\leqslant C$.

As previously, a rough analysis based on the change of variable $Y=y/\epsilon$ and on some asymptotic expansion of $L_P u$,

$$\widehat{L_P u}\sim\big(0,ik_3\big(k_3^2-\partial_{YY}\big)^{-1}(\beta Y\partial_Y\hat u_1)\big),$$

shows that "concentrated functions" are not stable under the penalization L_P.

The mechanism we want to study **involves therefore both scales** y **and** y/ϵ, and results from a balance between rotation and vertical oscillations, which is the main novelty here. Note indeed that previous works on rotating fluids consider either the case when the effect of rotation is dominating (macroscopic layer of fluid) [4] or the case when vertical oscillations hold on very small scales and can be averaged (shallow water approximation) [11].

Semiclassical analysis seems therefore to be the relevant tool to study this problem, insofar as it allows to separate both scales in a systematic way.

• Note finally that, if the horizontal viscosity is such that $\nu_h \gg \epsilon^2$, then because of the small scale in y, we expect **all the energy to be dissipated on a small time interval**, leading to some boundary layer effect (see the discussion in Section 5.6).

In order to exhibit a non-trivial propagation, we will assume in all the sequel that

$$\nu_h = o\big(\epsilon^2\big).$$

We therefore start with the study of the 3D propagation without dissipation. We will then check *a posteriori* that the viscous dissipation introduces only small error terms for any finite time.

5.2. Semiclassical analysis of the three-dimensional propagation

In order to study the propagation of energy by 3D waves, a natural idea is then to get a polarization of Poincaré waves, i.e. to obtain a diagonalization of the system,

$$\epsilon \partial_t u + L_P u = 0,$$

in the limit $\epsilon \to 0$. We first use the incompressibility constraint to rewrite the propagator in the form of a 2×2 matrix of pseudo-differential operators. We indeed have:

$$-\Delta_\epsilon p := -\epsilon^2\left(\Delta_h + \frac{1}{\epsilon^2}\partial_{zz}\right)p = -\epsilon\beta y \partial_x u_2 + \epsilon \partial_y(\beta y u_1),$$

from which we deduce that

$$\epsilon \partial_t u_h + \begin{pmatrix} -\epsilon^2 \partial_x \partial_y \Delta_\epsilon^{-1}(\beta y \cdot) & -\beta y \cdot + \epsilon^2 \partial_{xx}^2 \Delta_\epsilon^{-1}(\beta y \cdot) \\ \beta y \cdot - \epsilon^2 \partial_{yy}^2 \Delta_\epsilon^{-1}(\beta y \cdot) & \epsilon^2 \partial_x \partial_y \Delta_\epsilon^{-1}(\beta y \cdot) \end{pmatrix} u_h = 0.$$

Our first goal is then to perform a suitable change of variables leading to,

$$\epsilon \partial_t v + \begin{pmatrix} H_\epsilon^+(\partial_x, \partial_z, y, \epsilon\partial_y) & 0 \\ 0 & H_\epsilon^-(\partial_x, \partial_z, y, \epsilon\partial_y) \end{pmatrix} v = O\big(\epsilon^\infty\big).$$

In all the sequel, for the sake of simplicity, we will consider a single Fourier mode in (x, z), and denote by $(k, k_3) \in \mathbf{Z} \times \mathbf{Z}^*$ the associated wavenumber. Any solution is indeed a superposition of such waves. We will denote abusively $H_\epsilon^\pm(k, k_3, y, \epsilon\partial_y)$ the Fourier transform of $H_\epsilon^\pm(\partial_x, \partial_z, y, \epsilon\partial_y)$.

We are then brought back to study the propagation of waves by the scalar pseudo-differential operator $H_\epsilon^\pm(k, k_3, y, \epsilon\partial_y)$, which can be done for instance using classical results on the Wigner transform. For such scalar skew-symmetric pseudo-differential operators, we indeed know [13] that **energy is propagated according to the Hamiltonian transport equations**

$$\partial_t f + \big\{h^\pm, f\big\} = 0,$$

where $ih^\pm(k, k_3, y, \xi)$ is the semiclassical principal symbol of $H_\epsilon^\pm(k, k_3, y, \epsilon\partial_y)$.

Note that the time scale over which one has a macroscopic propagation of the energy is inversely proportional to the size of the oscillations. Such a property can be seen very simply on equations with constant coefficients,

$$\epsilon \partial_t v + h(\epsilon\partial_y) v = 0.$$

Indeed, denote by λ the time scale of the energy propagation (i.e. the inverse of the group velocity). Then by definition,

$$\frac{\epsilon}{\lambda} = \frac{dh(i\epsilon k_2)}{dk_2} = i\epsilon h'(i\epsilon k_2).$$

Thus λ has a finite limit as $\epsilon k_2 \to \xi$.

What we are finally able to establish is the following proposition:

Proposition 5.1. *Let $u^0 \in L^2(\omega)$ be a compactly supported divergence-free vector field such that $\int u^0\,dz = 0$, and let $u \in \mathcal{C}(\mathbf{R}_+, L^2(\omega))$ be the solution of Eq.* (5.1) *with initial data u^0.*
Then the L^2 norm of $u_h(t)$ on any fixed compact converges to 0 *as $t \to \infty$.*

In other words, 3D waves are dispersive, but only on times of order 1. Note that, in the case of a macroscopic layer of fluid, the group velocity of Poincaré waves is much larger (typically of order $1/\epsilon$); see for instance [4,12].

Furthermore the vertical component u_3 of the velocity will not remain bounded, as is usually claimed in formal derivations leading to shallow water models.

5.3. Reduction to a scalar situation

The first step of the proof follows a method initiated in [5].

• We first compute **a kind of characteristic polynomial** for the matrix of pseudo-differential operators:

$$\begin{pmatrix} \epsilon(-\Delta_\epsilon)^{-1} ik\epsilon\partial_y(\beta y\cdot) & -\beta y - \epsilon(-\Delta_\epsilon)^{-1}k^2\beta y \\ \beta y + \epsilon\partial_y(-\Delta_\epsilon)^{-1}\epsilon\partial_y(\beta y\cdot) & -\epsilon\partial_y(-\Delta_\epsilon)^{-1}(i\epsilon k\beta y\cdot) \end{pmatrix}.$$

A simple way to obtain a scalar equation is to proceed by linear combination and substitution.

Because the solution is expected to depend both on y and y/ϵ (whatever the initial data), $\epsilon\partial_y$ is a $O(1)$ operator like multiplication by any function of y. We then apply usual rules of semiclassical analysis (see [21]):

$$\epsilon\partial_y = O(1), \qquad y = O(1),$$

and any commutator has smaller order,

$$[\epsilon\partial_y, y] = O(\epsilon).$$

Keeping only leading order terms, we get

$$\begin{gathered} i\tau\hat{u}_1 - \beta y\hat{u}_2 = O(\epsilon), \\ \beta y\hat{u}_1 + \epsilon\partial_y\big(k_3^2 - (\epsilon\partial_y)^2\big)^{-1}\epsilon\partial_y(\beta y\hat{u}_1) + i\tau\hat{u}_2 = O(\epsilon), \end{gathered}$$

so that

$$\beta^2 y^2\hat{u}_2 + \epsilon\partial_y\big(k_3^2 - (\epsilon\partial_y)^2\big)^{-1}\epsilon\partial_y\big(\beta^2 y^2\hat{u}_2\big) - \tau^2\hat{u}_2 = O(\epsilon)$$

or equivalently

$$k_3^2(\beta y)^2\hat{u}_2 - \tau^2\big(k_3^2 - (\epsilon\partial_y)^2\big)\hat{u}_2 = O(\epsilon), \tag{5.3}$$

since commutators provide higher order terms with respect to ϵ. Note that one can also compute an exact pseudodifferential relation (which is actually a polynomial of degree 6 with respect to τ) by keeping all the terms,

$$P(\epsilon, y, \epsilon\partial_y, \tau)\hat{u}_2 = 0. \tag{5.4}$$

Note that, contrarily to [5], as we will only consider times of order 1, we do not need to compute subsymbols, so that we could also proceed directly using symbolic calculation and diagonalize the matrix:

$$\begin{pmatrix} 0 & -\beta y \\ \beta y - \frac{\xi^2\beta y}{k_3^2+\xi^2} & 0 \end{pmatrix}.$$

Anyway, we expect the roots to the following polynomial to play a special role in the propagation:

$$P(0, y, \xi, \tau) = k_3^2(\beta y)^2 - \big(k_3^2 + \xi^2\big)\tau^2. \tag{5.5}$$

• We can actually prove that there exist **pseudo-differential operators** $H_\epsilon^\pm$ with principal symbols,

$$ih^\pm = \pm i\sqrt{\frac{(k_3\beta y)^2}{k_3^2+\xi^2}}$$

such that $\epsilon\partial_t\mu^\pm + H_\epsilon^\pm\mu^\pm = 0$ implies that

$$v_h^\pm := \begin{pmatrix} (\beta y - \epsilon \partial_y \hat{\Delta}_\epsilon^{-1} \epsilon \partial_y (\beta y \cdot))^{-1} (H_\epsilon^\pm - \epsilon \partial_y \hat{\Delta}_\epsilon^{-1} (i \epsilon k \beta y \cdot)) \\ 1 \end{pmatrix} \mu^\pm$$

$$v_3^\pm := \frac{i}{\epsilon k_3} \epsilon \partial_y \mu^\pm - \frac{k}{k_3} v_1^\pm,$$

satisfies (5.1) up to $O(\epsilon^\infty)$, where μ^+, μ^- are scalar functions.

This result is actually a variant of the main lemma in [5]. (Indeed the exact dispersion relation depends here explicitly on ϵ.)

Lemma 5.2. *(See [5].) Let $P_\epsilon = P(\epsilon, y, \xi, \tau)$ be a polynomial function such that $\partial_\tau P_{0|P=0} \neq 0$, and let $h = h(y, \xi)$ be any continuous root of*

$$P\big(0, y, \xi, h(y, \xi)\big) = 0.$$

Then there exists a pseudo-differential operator $H_\epsilon = H_\epsilon(y, -i\epsilon\partial_y)$ with principal symbol $ih(y, \xi)$ such that

$$H_\epsilon \psi = i\tau\psi \quad \Longrightarrow \quad \mathbf{P}_{\epsilon,\tau}\psi = O\big(\epsilon^\infty\big), \tag{5.6}$$

where $\mathbf{P}_{\epsilon,\tau}$ is a pseudo-differential operator of full symbol $P(\epsilon, y, \xi, \tau)$.

The proof of this lemma relies on pseudo-differential functional calculus, and uses various quantifications to make the computations as simple as possible. For the sake of completeness, we recall here the main arguments, but refer to [5] for details.

At first order, we have:

$$\begin{aligned} \mathbf{P}_{\epsilon,\tau}\psi &\equiv \int e^{i\frac{\xi(y-y')}{\epsilon}} P\big(\epsilon, y, \xi, H_\epsilon\big(y', -i\epsilon\partial_y\big)\big)\psi\big(y'\big) \frac{d\xi\, dy'}{\epsilon} \\ &= \int e^{i\frac{\xi(y-y')}{\epsilon}} e^{i\frac{\xi'(y'-y'')}{\epsilon}} P\big(\epsilon, y, \xi, h\big(y'', \xi'\big)\big)\psi\big(y''\big) \frac{d\xi\, d\xi'\, dy'\, dy''}{\epsilon^2} \\ &= \int e^{i\frac{\xi(y-y')}{\epsilon}} P\big(\epsilon, y, \xi, h\big(y', \xi\big)\big)\psi\big(y'\big) \frac{d\xi\, dy'}{\epsilon}. \end{aligned}$$

So the principal symbol of $\mathbf{P}_{\epsilon,\tau}$ is $P(0, y, \xi, h(y, \xi))$ which, by assumption, is 0.

For the ϵ^∞ result, it is enough to repeat the same argument with $h_\epsilon \sim h + \sum \epsilon^k h_k$. We obtain:

$$P(\epsilon, y, \xi, h_\epsilon) + \sum_{k\geqslant 1} \epsilon^k Q_k\big(h, \ldots, \partial_y^l \partial_\xi^m h_\epsilon\big) = 0,$$

that can be solved recursively under the condition $\partial_\tau P_{0|P=0} \neq 0$.

• We further obtain a **decomposition of any initial data** on the eigenstates of the scalar propagators $H_\epsilon^\pm$.

For all u_h^0, there exist $\mu_\epsilon^{0,\pm}$ such that

$$\begin{aligned} u_{0,h} &= \sum_\pm \begin{pmatrix} -(\beta y - \epsilon \partial_y \hat{\Delta}_\epsilon^{-1} \epsilon \partial_y (\beta y \cdot))^{-1} (H_\epsilon^\pm - \epsilon \partial_y \hat{\Delta}_\epsilon^{-1} (i \epsilon k \beta y \cdot)) \\ 1 \end{pmatrix} \mu_\epsilon^{0,\pm} + O\big(\epsilon^\infty\big) \\ &=: \sum_\pm \mathbb{Q}_\epsilon^\pm \mu_\epsilon^{0,\pm} + O\big(\epsilon^\infty\big), \end{aligned}$$

where $\hat{\Delta}_\epsilon := \epsilon^2 \partial_{yy}^2 - \epsilon^2 k^2 - k_3^2$. The vertical component is then entirely determined by the divergence-free condition.

To prove this result, one first remarks that the leading order symbol of the matrix $(\mathbb{Q}_\epsilon^+ \mathbb{Q}_\epsilon^-)$, namely

$$\begin{pmatrix} -\frac{i\sqrt{k_3^2+\xi^2}}{|k_3|\operatorname{sgn}(y)} & \frac{i\sqrt{k_3^2+\xi^2}}{|k_3|\operatorname{sgn}(y)} \\ 1 & 1 \end{pmatrix}$$

is invertible.

The inversion of the matrix $(\mathbb{Q}_\epsilon^+ \mathbb{Q}_\epsilon^-)$ can then be done symbolically at any order.

Remark 5.3. Notice that the operators $\mathbb{Q}_\epsilon^\pm$ have a singularity at $y = 0$. Hence it is necessary to consider initial data whose support is bounded away from zero. In fact, we can always restrict ourselves to this case: indeed, the analysis we perform in the next section shows that such a property is preserved by the evolution. Moreover, functions whose support is bounded away from zero are dense in L^2, and the operator L_P preserves the L^2 norm.

5.4. Dispersion of energy

Standard arguments of semiclassical analysis allow then to control the propagation of energy for the scalar equations,

$$\epsilon \partial_t \mu_\epsilon^\pm + H_\epsilon^\pm \mu_\epsilon^\pm = 0.$$

• Because $H_\epsilon^\pm$ is skew-symmetric, we have a uniform control on the L^2 norm of $\mu_\epsilon^\pm$,

$$\left\|\mu_\epsilon^\pm(t)\right\|_{L^2(\omega)} = \left\|\mu_\epsilon^{0,\pm}\right\|_{L^2(\omega)}.$$

These uniform *a priori* estimates allow to establish the **convergence of the remainders in the equations for the Wigner transforms**:

$$f_\epsilon^\pm(t, y, \xi) := \frac{1}{\pi} \int e^{2i\xi y'} \mu_\epsilon^\pm\big(y - \epsilon y'\big) \bar{\mu}_\epsilon^\pm\big(y + \epsilon y'\big)\, dy'.$$

We therefore have:

$$\partial_t f_\epsilon^\pm + \big\{h^\pm, f_\epsilon^\pm\big\} = O(\epsilon).$$

For detailed computations leading to that estimate, we refer for instance to [18] or [13]:

Lemma 5.4. *(See [13].) Let $\mu^{0,\pm}$ be any fixed function of L^2 (non-oscillatory).*
Assume that

- *$H_\epsilon^\pm$ is skew-symmetric on L^2;*
- *there exists $\sigma \in \mathbf{R}$ such that $H_\epsilon^\pm$ is of order σ uniformly as $\epsilon \to 0$;*
- *the Weyl symbol of $H_\epsilon^\pm$ satisfies*

$$i h_\epsilon^\pm = i h^\pm + \epsilon i h_1^\pm + o(\epsilon) \quad \textit{uniformly in } C^\infty_{\mathrm{loc}}.$$

Then the Wigner transform $f_\epsilon^\pm(t, y, \xi)$ of $\mu_\epsilon^\pm(t)$ converges locally uniformly in t to the continuously t-dependent positive measure $f^\pm$, solution to

$$\partial_t f^\pm + \big\{h^\pm, f^\pm\big\} = 0.$$

In other words, the energy associated to the $\pm$ mode is transported along the characteristics of the Hamiltonian $h^\pm$:

$$\begin{aligned} \frac{dY^\pm}{dt} &= \frac{\partial h^\pm}{\partial \xi}\big(Y^\pm, \varXi^\pm\big), \\ \frac{d\varXi^\pm}{dt} &= -\frac{\partial h^\pm}{\partial y}\big(Y^\pm, \varXi^\pm\big). \end{aligned} \tag{5.7}$$

• The previous **1D Hamiltonian systems** are of course integrable. The bicharacteristics are indeed included in the level lines of $h^\pm$, which are hyperbola as shown in Fig. 3. In particular, the sign of $Y^\pm$ is constant along a trajectory ($Y^\pm$ does not vanish as long as $Y^\pm_{|t=0} \neq 0$).

A rapid inspection of the large time asymptotics shows that trajectories cannot be trapped in some compact. This would indeed imply that there exists either some stationary point or some turning point. But $\varXi^\pm(t)$ is a monotonic function,

$$\frac{d\varXi^\pm}{dt} = \mp \frac{\beta k_3 \operatorname{sgn}(Y(t))}{\sqrt{k_3^2 + \varXi^2(t)}},$$

which converges necessarily to infinity.

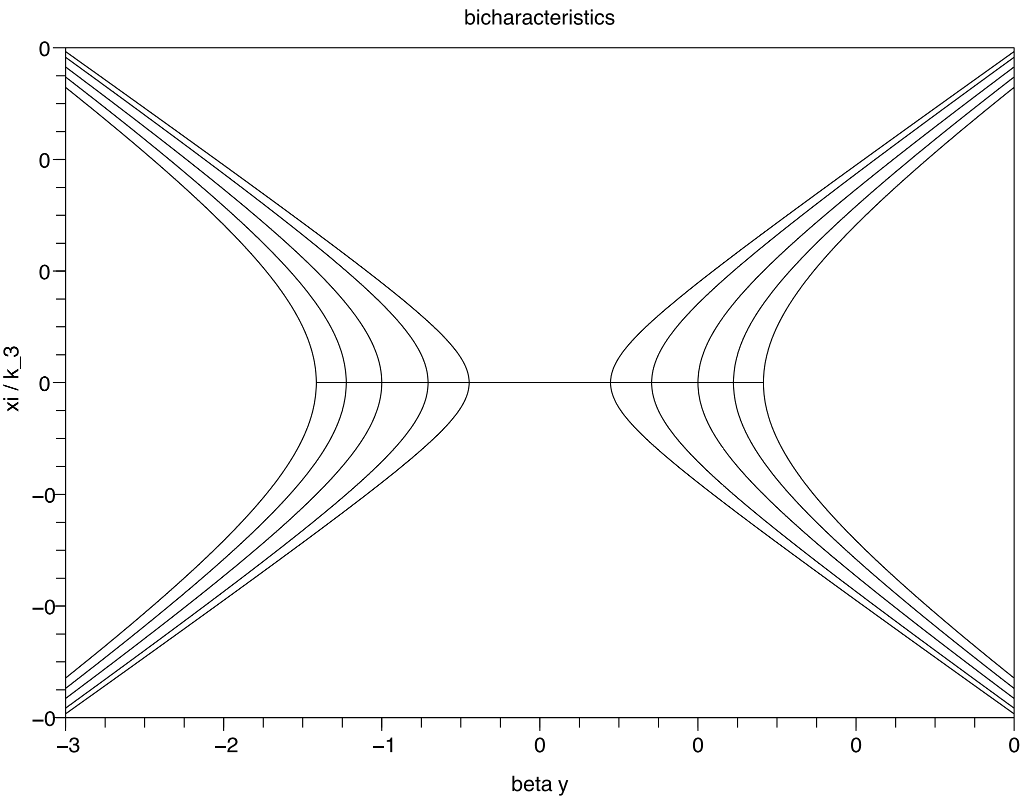

Fig. 3. Bicharacterictics associated to Poincaré waves.

More precisely, for large t, we have,

$$\left|\Xi^{\pm}(t)\right| \sim \sqrt{\beta|k_3|t},$$

from which we deduce that

$$Y^{\pm}(t) \sim \frac{h^0|\Xi^{\pm}(t)|}{|k_3|\beta} \sim h^0\sqrt{\frac{t}{|k_3|\beta}}.$$

5.5. Proof of Proposition 5.1

Combining the previous results, we are now able to establish Proposition 5.1. Without loss of generality, we can assume that u^0 has only a finite number of Fourier modes k and k_3 (truncation of high frequencies). By linearity, we can then restrict our attention to the case when there is only one mode (k, k_3).

We start by decomposing the initial data u^0 on the eigenstates of the scalar propagators $H_\epsilon^{\pm}$:

$$u_h^0 = \sum \mathbb{Q}_\epsilon^{\pm}\mu_\epsilon^{0,\pm}.$$

We then propagate $\mu_\epsilon^{0,\pm}$ according to $H_\epsilon^{\pm}$:

$$\epsilon\partial_t\mu_\epsilon^{\pm} = H_\epsilon^{\pm}\mu_\epsilon^{\pm}.$$

By Lemma 5.2, the velocity field v_ϵ defined by,

$$v_{\epsilon,h} = \sum \mathbb{Q}_\epsilon^{\pm}\mu_\epsilon^{\pm}, \quad \text{and} \quad v_{\epsilon,3} = \frac{i}{k_3}\nabla_h \cdot v_h,$$

satisfies the original system up to $O(\epsilon^\infty)$ error terms,

$$\epsilon\partial_t v_\epsilon + L_P v_\epsilon = r_\epsilon = O\left(\epsilon^\infty\right).$$

Remark 5.5. Proving that $v_{\epsilon,1}$ and $\epsilon v_{\epsilon,3}$ are in $L^2(\omega)$ is not immediate. Indeed $(\mathbb{Q}_\epsilon^+, \mathbb{Q}_\epsilon^-)$ is a pseudodifferential operator of order 1, while $(\mathbb{Q}_\epsilon^+, \mathbb{Q}_\epsilon^-)^{-1}$ is of order 0. The L^2 bounds rely actually on symmetry considerations which can be seen by simple computations on the principal symbols.

By a standard energy inequality, we then have,

$$\begin{aligned}
&\left\|(u_{\epsilon,h} - v_{\epsilon,h})(t)\right\|_{L^2(\omega)} + \epsilon^2 \left\|(u_{\epsilon,3} - v_{\epsilon,3})(t)\right\|_{L^2(\omega)} \\
&\quad \leqslant \left\|\left(u^0_{\epsilon,h} - v^0_{\epsilon,h}\right)(t)\right\|_{L^2(\omega)} + \epsilon^2 \left\|\left(u^0_{\epsilon,3} - v^0_{\epsilon,3}\right)(t)\right\|_{L^2(\omega)} + \int_0^t \left\|r_\epsilon(s)\right\|^2_{L^2(\omega)} = O\left(\epsilon^\infty\right).
\end{aligned} \tag{5.8}$$

Now, for v_ϵ, we can use the orthogonal L^2-decomposition $v_\epsilon = v_\epsilon^+ + v_\epsilon^-$, together with the semiclassical approximation of the Wigner transforms of $\mu_\epsilon^\pm$. By Lemma 5.4, we indeed have for all compact subset K of ω,

$$\left| \|v_\epsilon\|^2_{L^2(K)} - \sum_\pm \left\|f^\pm\right\|_{L^1(K\times\mathbf{R}^2)} \right| \to 0 \quad \text{as } \epsilon \to 0. \tag{5.9}$$

The solutions $f^\pm$ to the transport equations,

$$\partial_t f^\pm + \left\{h^\pm, f^\pm\right\} = 0,$$

are explicitly given by the method of characteristics,

$$f^\pm\left(t, X^\pm(t,y,\xi), \Xi^\pm(t,y,\xi)\right) = f^{0,\pm}(y,\xi).$$

As u^0 is compactly supported in y, $f^{0,\pm}$ is microlocalized in the vicinity of $\operatorname{supp} u^0 \times \{0\}$. For large times t, the spatial support of $f^\pm$ is transported at a distance of the order of $\sqrt{t}$ from the initial support. We then have the dispersion estimate:

$$\left\|f^\pm\right\|_{L^1(K\times\mathbf{R}^2)} \to 0 \quad \text{as } t \to \infty. \tag{5.10}$$

Gathering together (5.8), (5.9) and (5.10), we get the expected dispersion result,

$$\left\|u_{\epsilon,h}(t)\right\|_{L^2(K)} \to 0 \quad \text{as } t \to \infty,\ \epsilon \to 0,$$

which concludes the proof.

Remark 5.6.

(i) The qualitative behaviour of Rossby and Poincaré waves obtained here, i.e. in the case of a thin layer of fluid with rigid lid, is very different from the one exhibited in shallow water approximations (see [5]). Note that, in both cases, Rossby waves are easily identified because they are directly linked to the inhomogeneity of the Coriolis force, in particular they always propagate eastwards.
Here the energy associated to Poincaré waves propagates much slower than the energy associated to Rossby waves. The point is that fast oscillations with respect to latitude y, which are generated spontaneously for vertical modes but not for purely 2D Rossby waves, slow down the propagation. Maybe it would be physically relevant to consider initial data that depend already on the fast variable y/ϵ.
The other points which should be discussed are the influence of the free surface and the effect of the bottom topography. But, at the present time, we have no convenient mathematical tool to study the propagation of waves in such a complex geometry.

(ii) The formula,

$$v_3^\pm = -\frac{k}{k_3}\left(H_\epsilon^\pm\right)^{-1}(\beta y\cdot) + \frac{i}{\epsilon k_3}(\epsilon \partial_y \cdot)\mu^\pm,$$

shows that the vertical component of the velocity does not remain bounded, as is usually claimed. Notice that this is due to the apparition of small scales in y.

5.6. Influence of the viscosity

In the case when $\nu_h = o(\epsilon^2)$, an easy computation based on the energy estimate shows that the viscous dissipation does not modify the propagation for finite times.

More generally, we could extend the previous study considering the whole viscous Poincaré propagation operator,

$$L_P u = \beta y u^\perp + \begin{pmatrix} \epsilon \nabla_h p \\ \frac{1}{\epsilon}\partial_z p \end{pmatrix} - \nu_h \Delta_h u - \epsilon \partial_{zz} u, \tag{5.11}$$

where p is such that both the incompressibility constraint and the boundary condition are satisfied.

The diagonalization process is of course unchanged since the dissipation operator is scalar. The only difference is therefore that one has now to control the propagation of energy for the scalar equations,

$$\epsilon \partial_t \mu_\epsilon^\pm + H_\epsilon^\pm \mu_\epsilon^\pm - \nu_h \Delta_h \mu_\epsilon^\pm - \epsilon \partial_{zz} \mu_\epsilon^\pm = 0.$$

A standard computation (reported for instance in Proposition 1.8 of [13]) shows that the Wigner transform then satisfies the following damped transport equation:

$$\partial_t f_\epsilon^\pm + 4\frac{\nu_h}{\epsilon^2}|\xi|^2 f_\epsilon^\pm + \{h^\pm, f_\epsilon^\pm\} = o(1).$$

(Note that the symmetric part of the propagator occurs at leading order in ϵ, which can be seen by easy symmetry considerations.)

We then deduce that

- if $\nu_h \ll \epsilon^2$, the energy is propagated according to the bicharacteristics associated to $h^\pm$, as stated in Proposition 5.1;
- if $\nu_h \gg \epsilon^2$, the energy contained initially in the Poincaré modes is dissipated on a very short time, leading to some initial layer phenomenon;
- if $\nu_h \sim \epsilon^2$, the dynamics is a combination of both phenomena, as shown by Duhamel's formula

$$f^\pm\big(t, Y^\pm(t,y,\xi), \Xi^\pm(t,y,\xi)\big)\exp\left(4\frac{\nu_h}{\epsilon^2}\big|\Xi(t,y,\xi)\big|^2 t\right) = f^0(t,y,\xi).$$

  Note in particular that the energy associated to Poincaré modes has a super exponential decay, since $|\Xi(t,y,\xi)| \to \infty$ along any trajectory.

6. Derivation of the thermocline

This section is devoted to the proof of Proposition 1.8, which relies on classical elliptic arguments. The main difficulty lies in the fact that the equation on θ is degenerate in the horizontal variables. We first prove the existence of $\bar\theta$, along with some H^1 estimates, and then we prove the convergence.

Throughout the proof, we assume that the wind stress σ vanishes at sufficiently high order near $y = 0$, so that there is no need for a truncation (see Section 2) and the function u^{stat} does not have any singularity. If such an assumption is not satisfied, the function $\bar\theta$ (solution of (1.18)) will be well-defined nonetheless, but our proof of convergence fails: indeed, our arguments require estimates of $\nabla_h\bar\theta$, and these are available only if $\nabla_h u_h^{\mathrm{int}}$ is small in $L^\infty(\omega_h)$. On the other hand, a close look at the calculations in Section 3 shows that when σ vanishes quadratically at the origin, $\nabla_h u_h^{\mathrm{int}}$ does not belong to $L^\infty(\omega_h)$, and thus the proof below is no longer valid.

• ***A priori* estimates on the function $\bar\theta$:**

Let $\bar\theta \in L^2(\omega_h, H^1([0,1]))$ be any solution of

$$-\lambda \partial_{zz}\bar\theta + u^{\mathrm{int}} \cdot \nabla\bar\theta = 0 \quad \text{in } \omega,$$
$$\bar\theta_{|z=1} = \theta_1, \qquad \partial_z\bar\theta_{|z=0} = 0.$$

Multiplying the above equation by $\bar\theta$ and integrating on ω, we obtain:

$$\lambda\int|\partial_z\bar\theta|^2=-\frac{1}{2}\int\limits_{\partial\omega}u^{\text{int}}\cdot n_\omega\bar\theta^2+\lambda\int\limits_{\omega_h}\partial_z\bar\theta_{|z=1}\bar\theta_{|z=1}-\lambda\int\limits_{\omega_h}\partial_z\bar\theta_{|z=0}\bar\theta_{|z=0}$$
$$=-\frac{1}{2}\int\limits_{\omega_h}u^{\text{int}}_{3|z=1}\theta_1^2+\lambda\int\limits_{\omega_h}\theta_1\partial_z\bar\theta_{|z=1}. \tag{6.1}$$

According to Section 3, we have,

$$u^{\text{int}}_{3|z=1}=\frac{\partial_x\sigma_2}{b}-\partial_y\frac{\sigma_1}{b}.$$

We assume that σ is such that $u^{\text{int}}_{3|z=1}$ belongs to $L^\infty(\omega_h)$. We now evaluate $\partial_z\bar\theta_{|z=1}$: we have,

$$\lambda\partial_z\bar\theta_{|z=1}=\lambda\int_0^1\partial_{zz}\bar\theta=\int_0^1u^{\text{int}}\cdot\nabla\bar\theta=\operatorname{div}_h\left(u_h^{\text{int}}\int_0^1\bar\theta\right)+u^{\text{int}}_{3|z=1}\theta_1. \tag{6.2}$$

Recall that u_h^{int}, defined in Section 3, is independent of z, while u_3^{int} is linear with respect to z. Consequently, the function $\bar\theta$ depends on x_h and z, and

$$\lambda\int\limits_{\omega_h}\theta_1\partial_z\bar\theta_{|z=1}=\int\limits_{\omega_h}\theta_1\left(\operatorname{div}_h\left(u_h^{\text{int}}\int_0^1\bar\theta\right)+u^{\text{int}}_{3|z=1}\theta_1\right)=-\int\limits_\omega\bar\theta u_h^{\text{int}}\cdot\nabla_h\theta_1+\int\limits_{\omega_h}u^{\text{int}}_{3|z=1}\theta_1^2. \tag{6.3}$$

Using the identity

$$\bar\theta(\cdot,z)=\theta_1-\int_z^1\partial_z\bar\theta(\cdot,z')\,dz',$$

we deduce that

$$\|\bar\theta\|_{L^2(\omega)}\leqslant\|\theta_1\|_{L^2(\omega_h)}+\|\partial_z\bar\theta\|_{L^2(\omega)}. \tag{6.4}$$

Gathering (6.1), (6.3) and (6.4), we infer that

$$\lambda\int\limits_\omega|\partial_z\bar\theta|^2=\frac{1}{2}\int\limits_{\omega_h}u^{\text{int}}_{3|z=1}\theta_1^2-\int\limits_\omega\bar\theta u_h^{\text{int}}\cdot\nabla_h\theta_1$$
$$\leqslant\frac{1}{2}\left\|u^{\text{int}}_{3|z=1}\right\|_{L^\infty(\omega_h)}\|\theta_1\|_{L^2}^2+\left\|u_h^{\text{int}}\right\|_{L^\infty}\|\nabla_h\theta_1\|_{L^2}\big(\|\theta_1\|_{L^2}+\|\partial_z\bar\theta\|_{L^2}\big).$$

Using the Cauchy–Schwarz inequality, we obtain eventually:

$$\lambda\int\limits_\omega|\partial_z\bar\theta|^2\leqslant\left\|u^{\text{int}}_{3|z=1}\right\|_{L^\infty}\|\theta_1\|_{L^2}^2+\left\|u_h^{\text{int}}\right\|_{L^\infty}\|\nabla_h\theta_1\|_{L^2}\|\theta_1\|_{L^2}+\frac{1}{\lambda}\left\|u_h^{\text{int}}\right\|_{L^\infty}^2\|\nabla_h\theta_1\|_{L^2}^2. \tag{6.5}$$

Inequalities (6.5) and (6.4) entail that any solution $\bar\theta$ of (1.18) is bounded in $L^2(\omega_h,H^1([0,1]))$ by a constant depending only on λ, θ_1 and u^{int}.

We now derive estimates on the horizontal derivatives in a similar fashion: we have,

$$-\lambda\partial_{zz}\nabla_h\bar\theta+\big(u^{\text{int}}\cdot\nabla\big)\nabla_h\bar\theta=-\big(\nabla_hu_h^{\text{int}}\big)\cdot\nabla_h\bar\theta-\nabla_hu_3^{\text{int}}\partial_z\bar\theta. \tag{6.6}$$

Multiplying the above equation by $\nabla_h\bar\theta$ and integrating by parts, we have, using the boundary conditions,

$$-\int\limits_\omega\partial_{zz}\nabla_h\bar\theta\cdot\nabla_h\bar\theta=\int\limits_\omega|\partial_z\nabla_h\bar\theta|^2-\int\limits_{\omega_h}\partial_z\nabla_h\bar\theta_{|z=1}\cdot\nabla_h\theta_1=\int\limits_\omega|\partial_z\nabla_h\bar\theta|^2+\int\limits_{\omega_h}\partial_z\bar\theta_{|z=1}\Delta_h\theta_1.$$

Using Eq. (6.2), we express $\partial_z\bar\theta_{|z=1}$ in terms of $\bar\theta$ and θ_1. Integrating by parts once again leads to

$$\left|\int_{\omega_h}\partial_z\bar\theta_{|z=1}\Delta_h\theta_1\right|\leqslant\frac{1}{\lambda}\left(\left\|u_h^{\mathrm{int}}\right\|_{L^\infty}\|\bar\theta\|_{L^2}\|\theta_1\|_{H^3}+\left\|u_3^{\mathrm{int}}\right\|_{L^\infty}\|\theta_1\|_{H^2}\|\theta_1\|_{L^2}\right).$$

On the other hand, since $u^{\mathrm{int}}_{3|z=0}=0$, we have

$$2\int_\omega\left[\left(u^{\mathrm{int}}\cdot\nabla\right)\nabla_h\bar\theta\right]\cdot\nabla_h\bar\theta=\int_{\omega_h}u^{\mathrm{int}}_{3|z=1}|\nabla_h\bar\theta_{|z=1}|^2-\int_{\omega_h}u^{\mathrm{int}}_{3|z=0}|\nabla_h\bar\theta_{|z=0}|^2=\int_{\omega_h}u^{\mathrm{int}}_{3|z=1}|\nabla_h\theta_1|^2.$$

The two terms in the right-hand side of (6.6) can easily be evaluated in L^2 using the estimate on $\partial_z\bar\theta$; there remains,

$$\lambda\int_\omega|\partial_z\nabla_h\bar\theta|^2\leqslant C+\left\|\nabla_h u_h^{\mathrm{int}}\right\|_{L^\infty}\|\nabla_h\bar\theta\|^2_{L^2},$$

where the constant C depends on λ, $\|u^{\mathrm{int}}\|_{L^\infty}$ and $\|\theta_1\|_{H^3}$.

Assume that

$$\left\|\nabla_h u_h^{\mathrm{int}}\right\|_{L^\infty(\omega)}\leqslant\frac{\lambda}{2};$$

this assumption is discussed in Remark 1.9 following Proposition 1.8. Then

$$\int_\omega|\partial_z\nabla_h\bar\theta|^2+\int_\omega|\nabla_h\bar\theta|^2\leqslant C,$$

where the constant C depends on λ, θ_1 and u^{int}. These estimates easily lead to the existence of a solution $\bar\theta$ of Eq. (1.18); the uniqueness of $\bar\theta$ follows from the estimates above with $\theta_1=0$. The same method also shows that under condition (1.15) on $\nabla_h u_h^{\mathrm{int}}$, $D_h^2\bar\theta$ is bounded in $L^2(\omega_h,H^1([0,1]))$. Plugging this estimate back into (6.6), we deduce that $\nabla_h\bar\theta\in L^2(\omega_h,H^2[0,1])$, and thus that $\nabla_h\bar\theta$ is bounded in $L^2(\omega_h,W^{1,\infty}([0,1]))$.

Concerning the function θ^{BL}, the existence and uniqueness are obvious; indeed, we recall that θ^{BL} is a solution of:

$$-\lambda\partial_{\zeta\zeta}\theta^{\mathrm{BL}}(x_h,\zeta)+\epsilon U_h^{\mathrm{BL}}(x_h,\zeta)\cdot\nabla_h\theta_1=0,$$
$$\theta^{\mathrm{BL}}(x_h,\zeta)\underset{\zeta\to\infty}{\longrightarrow}0,$$

where the function U^{BL} was computed in Section 2. Thus we have merely,

$$\theta^{\mathrm{BL}}(x_h,\zeta)=\frac{1}{2\lambda}\nabla_h\theta_1\cdot\sum_\pm\left(\sigma\pm i\sigma^\perp\right)\frac{\exp(-\lambda^\pm(x_h)\zeta)}{(\lambda^\pm(x_h))^3}.$$

Remark 6.1. The same kind of arguments also show that there exists a unique solution θ of Eq. (1.16). However, the estimates on θ blow up as ϵ vanishes because

$$\left\|u^{\mathrm{stat}}\right\|_{L^2}=O\left(\epsilon^{-1/2}\right).$$

• **Proof of convergence:**

We construct an approximate solution of (1.16) as follows: we set

$$\theta^{\mathrm{app}}(x_h,z)=\bar\theta(x_h,z)+\epsilon\theta^{\mathrm{BL}}\left(x_h,\frac{1-z}{\epsilon}\right)+\epsilon\tilde\theta(x_h,z),$$

where $\tilde\theta$ is any bounded and smooth function such that θ^{app} satisfies the boundary conditions (1.17). We choose for instance

$$\tilde\theta(x_h,z)=\frac{z-1}{\epsilon}\partial_\zeta\theta^{\mathrm{BL}}_{|\zeta=\frac{1}{\epsilon}}(x_h)-\theta^{\mathrm{BL}}_{|\zeta=0}(x_h).$$

Notice that by construction,

$$\partial_z\theta^{\mathrm{app}}_{|z=0}=0,\qquad \theta^{\mathrm{app}}_{|z=1}=\theta_1.$$

Moreover, using the definition of θ^{BL}, it is easily proved that $\tilde\theta=O(1)$ in $W^{2,\infty}(\omega)$ (provided the stress σ is smooth and vanishes at a sufficiently high order near $y=0$).

Consequently,

$$\begin{aligned}&-\lambda\partial_{zz}\theta^{\mathrm{app}}-\lambda\epsilon^2\Delta_h\theta^{\mathrm{app}}+u^{\mathrm{stat}}\cdot\nabla\theta^{\mathrm{app}}\\&\quad=u_h^{\mathrm{BL}}\cdot\nabla_h(\bar\theta-\theta_1)-\lambda\epsilon^2\Delta_h\bar\theta+u_3^{\mathrm{BL}}\partial_z\bar\theta+v^{\mathrm{int}}\cdot\nabla\bar\theta-\lambda\epsilon^3\Delta_h\theta^{\mathrm{BL}}\left(x_h,\frac{1-z}{\epsilon}\right)\\&\qquad+\epsilon u^{\mathrm{stat}}\cdot\nabla\theta^{\mathrm{BL}}\left(x_h,\frac{1-z}{\epsilon}\right)-\lambda\epsilon^3\Delta_h\tilde\theta+\epsilon u^{\mathrm{stat}}\cdot\nabla\tilde\theta.\end{aligned}\tag{6.7}$$

According to the results of Sections 2 and 3, we have,

$$\begin{gathered}\left\|u_3^{\mathrm{BL}}\right\|_{L^2}=O(\sqrt{\epsilon}),\qquad \left\|v^{\mathrm{int}}\right\|_{L^2}=o(\epsilon),\\ \epsilon\left\|u_h^{\mathrm{stat}}\right\|_{L^\infty},\left\|u_3^{\mathrm{stat}}\right\|_{L^\infty}=O(1),\qquad \epsilon\left\|u^{\mathrm{stat}}\right\|_{L^2}=O(\sqrt{\epsilon}).\end{gathered}$$

These estimates, together with the ones derived above on $\bar\theta$, enable us to bound all the terms in the right-hand side of (6.7), except for the first one. Using Hardy's inequality, we have:

$$\begin{aligned}\left\|u_h^{\mathrm{BL}}\cdot\nabla_h(\bar\theta-\theta_1)\right\|_{L^2(\omega)}&\leqslant\left\|(1-z)u_h^{\mathrm{BL}}\right\|_{L^\infty(\omega_h,L^2([0,1]))}\left\|(1-z)^{-1}\nabla_h(\bar\theta-\theta_1)\right\|_{L^2(\omega_h,L^\infty([0,1]))}\\&\leqslant C\sqrt{\epsilon}\left\|\partial_z\nabla_h(\bar\theta-\theta_1)\right\|_{L^2(\omega_h,L^\infty([0,1]))}.\end{aligned}$$

Thus θ^{app} is an approximate solution of (1.16), with an error term $o(1)$ in $L^2(\omega)$. As a consequence, $\theta-\theta^{\mathrm{app}}$ satisfies:

$$\begin{gathered}-\lambda\partial_{zz}\big(\theta-\theta^{\mathrm{app}}\big)-\lambda\epsilon^2\Delta_h\big(\theta-\theta^{\mathrm{app}}\big)+u^{\mathrm{stat}}\cdot\nabla\big(\theta-\theta^{\mathrm{app}}\big)=o(1),\\ \big(\theta-\theta^{\mathrm{app}}\big)_{|z=1}=0,\qquad \partial_z\big(\theta-\theta^{\mathrm{app}}\big)_{|z=0}=0.\end{gathered}$$

Multiplying the above equation by $\theta-\theta^{\mathrm{app}}$ and using the Poincaré inequality, we prove that

$$\left\|\partial_z\big(\theta-\theta^{\mathrm{app}}\big)\right\|_{L^2(\omega)}=o(1),$$

and thus the proposition is proved.

Acknowledgements

This work received the support of the Agence Nationale de la Recherche (project ANR-08-BLAN-0301-01).

The authors are very grateful to David Gérard-Varet for his careful reading of the manuscript.

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