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Convergence of an iterative algorithm for systems of variational inequalities and nonexpansive mappings with applications

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ABSTRACT

In this paper, we consider the problem of convergence of an iterative algorithm for a system of generalized variational inequalities and a nonexpansive mapping. Strong convergence theorems are established in the framework of real Banach spaces.

1. Introduction

Variational inequalities introduced by Stampacchia [1] in the early sixties have had a great impact and influence in the development of almost all branches of pure and applied sciences and have witnessed an explosive growth in theoretical advances, algorithmic development, etc; see for e.g. [1–18] and the references therein.

Let C be a nonempty closed convex subset of a real Hilbert space H and P_C the metric projection of H onto C. Recall that a mapping $S : C \to C$ is said to be nonexpansive if

$$\|Sx - Sy\| \leq \|x - y\|, \quad \forall x, y \in C.$$

In this paper, we use $F(S)$ to denote the fixed point set of the mapping S.

Let $A : C \to H$ be a mapping. Recall the following definitions.

(a) A is said to be monotone if

$$\langle Ax - Ay, x - y\rangle \geq 0, \quad \forall x, y \in C.$$

(b) A is said to be α-strongly monotone if there exists a positive real number α such that

$$\langle Ax - Ay, x - y\rangle \geq \alpha\|x - y\|^2, \quad \forall x, y \in C.$$

(a) A is said to be α-inverse-strongly monotone if there exists a positive real number α such that

$$\langle Ax - Ay, x - y\rangle \geq \alpha\|Ax - Ay\|^2, \quad \forall x, y \in C.$$

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Recall that the classical variational inequality, denoted by $VI(C, A)$, is to find $u \in C$ such that

$$\langle Au, v - u\rangle \geq 0, \quad \forall v \in C. \tag{1.1}$$

For given $z \in H$ and $u \in C$, we see that the following inequality holds

$$\langle u - z, v - u\rangle \geq 0, \quad \forall v \in C,$$

if and only if $u = P_C z$. It is known that projection operator P_C is nonexpansive. It is also known that P_C satisfies

$$\langle x - y, P_C x - P_C y\rangle \geq \|P_C x - P_C y\|^2, \quad \forall x, y \in H.$$

One can see that the variational inequality (1.1) is equivalent to a fixed point problem. An element $x^* \in C$ is a solution of the variational inequality (1.1) if and only if $x^* \in C$ is a fixed point of the mapping $P_C(I - \lambda A)$, where I is the identity mapping and $\lambda > 0$ is a constant. This alternative equivalent formulation has played a significant role in the studies of the variational inequalities and related optimization problems.

For a monotone mapping $A : C \to H$, Verma [14–17] studied the following problem of finding $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle \lambda Ay^* + x^* - y^*, x - x^*\rangle \geq 0, \quad \forall x \in C, \\ \langle \mu Ax^* + y^* - x^*, x - y^*\rangle \geq 0, \quad \forall x \in C, \end{cases} \tag{1.2}$$

where $\lambda, \mu > 0$ are constant. If we add up the requirement that $x^* = y^*$, then the problem (1.2) is reduced to the classical variational inequality (1.1). Further, the problem (1.2) is equivalent to the following projection formulas

$$\begin{cases} x^* = P_C(I - \lambda A)y^*, \\ y^* = P_C(I - \mu A)x^*. \end{cases}$$

The problem of finding solutions of (1.2) by using iterative methods has been studied by many authors, see [4,6,8,11,14–17] and the references therein.

Recently, many authors also studied the problem of finding a common element of the fixed point set of nonexpansive mappings and the solution set of variational inequalities for α-inverse-strongly monotone mappings in the framework of real Hilbert spaces. Iiduka and Takahashi [9] introduced an iterative method for finding a common element of the fixed point set of a single nonexpansive mapping and the solution set of variational inequalities for an α-inverse-strongly monotone mapping. To be more precise, they proved the following theorem.

Theorem IT. *Let C be a closed convex subset of a real Hilbert space H. Let A be an α-inverse-strongly monotone mapping of C into H and S a nonexpansive mapping of C into itself such that $F(S) \cap VI(C, A) \neq \emptyset$. Suppose that $x_1 = x \in C$ and $\{x_n\}$ is given by*

$$x_{n+1} = \alpha_n x + (1 - \alpha_n)SP_C(x_n - \lambda_n Ax_n)$$

for every $n = 1, 2, \ldots$, where $\{\alpha_n\}$ is a sequence in $[0, 1)$ and $\{\lambda_n\}$ is a sequence in $[a, b]$. If $\{\alpha_n\}$ and $\{\lambda_n\}$ are chosen so that $\{\lambda_n\} \subset [a, b]$ for some a, b with $0 < a < b < 2\alpha$,

$$\lim_{n\to\infty} \alpha_n = 0, \qquad \sum_{n=1}^{\infty} \alpha_n = \infty, \qquad \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty \quad \text{and} \quad \sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty,$$

then $\{x_n\}$ converges strongly to $P_{F(S)\cap VI(C,A)}x$.

Recently, Yao and Yao [18] further studied the problem of finding a common element in fixed point set of a nonexpansive mapping and solution set of a classical variational inequality for a inverse-strongly monotone mapping by considering a relaxed extra-gradient methods. More precisely, they proved the following theorem.

Theorem YY. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let A be an α-inverse-strongly monotone mapping of C into H and S a nonexpansive mapping of C into itself such that $F(S) \cap \Omega \neq \emptyset$, where Ω denotes the set of solutions of a variational inequality for the α-inverse-strongly monotone mapping. Suppose that $x_1 = u \in C$ and $\{x_n\}$, $\{y_n\}$ are given by*

$$\begin{cases} x_1 = u \in C, \\ y_n = P_C(x_n - \lambda_n Ax_n), \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n SP_C(I - \lambda_n A)y_n, \quad n \geq 1, \end{cases}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are three sequences in $[0, 1]$ and $\{\lambda_n\}$ is a sequence in $[0, 2a]$. If $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\lambda_n\}$ are chosen so that $\{\lambda_n\} \subset [a, b]$ for some a, b with $0 < a < b < 2\alpha$ and

(a) $\alpha_n + \beta_n + \gamma_n = 1, \forall n \geq 1$;
(b) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$;
(c) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;
(d) $\lim_{n\to\infty}(\lambda_{n+1} - \lambda_n) = 0$,

then the sequence $\{x_n\}$ defined by the above iterative algorithm converges strongly to $P_{F(S)\cap\Omega}u$.

In this paper, we consider the problem of convergence of an iterative algorithm for a system of generalized variational inequalities and a nonexpansive mapping. Strong convergence theorems of common elements are established in the framework of real Banach spaces. Note that no Banach space is q-uniformly smooth for $q > 2$; see [19] for more details. We prove the strong convergence of the purposed iterative scheme in uniformly convex and 2-uniformly smooth Banach spaces. The results presented in this paper improve and extend the corresponding results announced by Aoyama et al. [2], Iiduka and Takahashi [9], Yao and Yao [18] and some others.

2. Preliminaries

Let C be a nonempty closed convex subset of a Banach space E and E^* the dual space of E. Let $\langle\cdot,\cdot\rangle$ denote the pairing between E and E^*. For $q > 1$, the generalized duality mapping $J_q : E \to 2^{E^*}$ is defined by

$$J_q(x) = \{f \in E^* : \langle x, f\rangle = \|x\|^q,\ \|f\| = \|x\|^{q-1}\}, \quad \forall x \in E.$$

In particular, $J = J_2$ is called the normalized duality mapping. It is known that $J_q(x) = \|x\|^{q-2}J(x)$ for all $x \in E$. If E is a Hilbert space, then $J = I$, the identity mapping. Further, we have the following properties of the generalized duality mapping J_q:

(a) $J_q(x) = \|x\|^{q-2}J_2(x)$ for all $x \in E$ with $x \neq 0$;
(b) $J_q(tx) = t^{q-1}J_q(x)$ for all $x \in E$ and $t \in [0,\infty)$;
(c) $J_q(-x) = -J_q(x)$ for all $x \in E$.

Let $B = \{x \in E : \|x\| = 1\}$. E is said to be uniformly convex if, for any $\epsilon \in (0, 2]$, there exists $\delta > 0$ such that, for any $x, y \in B$,

$$\|x-y\| \geq \epsilon \quad \text{implies} \quad \left\|\frac{x+y}{2}\right\| \leq 1-\delta.$$

It is known that a uniformly convex Banach space is reflexive and strictly convex. E is said to be Gâteaux differentiable if the limit

$$\lim_{t\to 0}\frac{\|x+ty\|-\|x\|}{t} \tag{Δ}$$

exists for each $x, y \in B$. In this case, E is said to be smooth. The norm of E is said to be uniformly Gâteaux differentiable if for each $y \in B$, the limit (Δ) is attained uniformly for $x \in B$. The norm of E is said to be Fréchet differentiable, if for each $x \in B$, the limit (Δ) is attained uniformly for $y \in B$. The norm of E is said to be uniformly Fréchet differentiable, if the limit (Δ) is attained uniformly for $x, y \in B$. It is well-known that (uniform) Fréchet differentiability of the norm of E implies (uniform) Gâteaux differentiability of the norm of E.

The modulus of smoothness of E is defined by

$$\rho(t) = \sup\left\{\frac{1}{2}(\|x+y\|+\|x-y\|)-1 : x, y \in E,\ \|x\| = 1,\ \|y\| \leq t\right\}.$$

A Banach space E is said to be uniformly smooth if $\lim_{t\to 0}\frac{\rho(t)}{t} = 0$. Let $q > 1$. A Banach space E is said to be q-uniformly smooth if there exists a fixed constant $c > 0$ such that $\rho(t) \leq ct^q$. It is well-known that E is uniformly smooth if and only if the norm of E is uniformly Fréchet differentiable. If E is q-uniformly smooth, then $q \leq 2$ and E is uniformly smooth, and hence the norm of E is uniformly Fréchet differentiable, in particular, the norm of E is Fréchet differentiable. Note that

(a) E is a uniformly smooth Banach space if and only if J is single-valued and uniformly continuous on any bounded subset of E.
(b) All Hilbert spaces, L^p (or l^p) spaces ($p \geq 2$) and the Sobolev spaces W^p_m ($p \geq 2$) are 2-uniformly smooth, while L^p (or l^p) and W^p_m spaces ($1 < p \leq 2$) are p-uniformly smooth.
(c) Typical examples of both uniformly convex and uniformly smooth Banach spaces are L^p, where $p > 1$. More precisely, L^p is $\min\{p, 2\}$-uniformly smooth for every $p > 1$.

Next, we always assume that E is a smooth Banach space. Let C be a nonempty closed convex subset of E. Recall that an operator A of C into E is said to be accretive if

$$\langle Ax - Ay, J(x-y)\rangle \geq 0, \quad \forall x, y \in c.$$

A is said to be α-inverse-strongly accretive if there exists a constant $\alpha > 0$ such that

$$\langle Ax - Ay, J(x-y)\rangle \geq \alpha\|Ax - Ay\|^2, \quad \forall x, y \in C.$$

Let D be a subset of C and Q be a mapping of C into D. Then Q is said to be sunny if

$$Q(Qx + t(x - Qx)) = Qx,$$

whenever $Qx + t(x - Qx) \in C$ for $x \in C$ and $t \geq 0$. A mapping Q of C into itself is called a retraction if $Q^2 = Q$. If a mapping Q of C into itself is a retraction, then $Qz = z$ for all $z \in R(Q)$, where $R(Q)$ is the range of Q. A subset D of C is called a sunny nonexpansive retract of C if there exists a sunny nonexpansive retraction from C onto D.

The following result describes a characterization of sunny nonexpansive retractions on a smooth Banach space.

Proposition 2.1 (Reich [20]). *Let E be a smooth Banach space and C a nonempty subset of E. Let $Q : E \to C$ be a retraction. Then the following are equivalent:*

(a) *Q is sunny and nonexpansive;*
(b) $\|Qx - Qy\|^2 \leq \langle x - y, J(Qx - Qy)\rangle$, $\forall x, y \in E$;
(c) $\langle x - Qx, J(y - Qx)\rangle \leq 0$, $\forall x \in E, y \in C$.

Proposition 2.2 (Kitahara and Takahashi [21]). *Let C be a nonempty closed convex subset of a uniformly convex and uniformly smooth Banach space E and S a nonexpansive mapping of C into itself with $F(S) \neq \emptyset$. Then the set $F(S)$ is a sunny nonexpansive retract of C.*

For the class of nonexpansive mappings, one classical way to study nonexpansive mappings is to use contractions to approximate a nonexpansive mapping [22,23]. More precisely, take $t \in (0, 1)$ and define a contraction $S_t : C \to C$ by

$$S_t x = tu + (1 - t)Sx, \quad \forall x \in C,$$

where $u \in C$ is a fixed point. Banach's contraction mapping principle guarantees that S_t has a unique fixed point x_t in C. That is,

$$x_t = tu + (1 - t)Sx_t.$$

It is unclear, in general, what the behavior of x_t is as $t \to 0$, even if S has a fixed point. However, in the case of S having a fixed point, Browder [22] proved that if E is a Hilbert space, then x_t converges strongly to a fixed point of S. Reich [23] extended Browder's result to the setting of Banach spaces.

Reich [23] showed that if E is uniformly smooth and if D is the fixed point set of a nonexpansive mapping from C into itself, then there is a unique sunny nonexpansive retraction from C onto D and it can be constructed as follows.

Proposition 2.3. *Let E be a uniformly smooth Banach space and $S : C \to C$ a nonexpansive mapping with a fixed point. For each fixed $u \in C$ and every $t \in (0, 1)$, the unique fixed point $x_t \in C$ of the contraction $C \ni x \mapsto tu + (1 - t)Sx$ converges strongly as $t \to 0$ to a fixed point of S. Define $Q : C \to D$ by $Qu = s - \lim_{t\to 0} x_t$. Then Q is the unique sunny nonexpansive retract from C onto D; that is, Q satisfies the property:*

$$\langle u - Qu, J(y - Qu)\rangle \leq 0, \quad \forall u \in C, y \in D.$$

Recently, Aoyama et al. [2] first considered the following generalized variational inequality problem in a smooth Banach space E.

Let C be a nonempty closed convex subset of E and A an accretive operator of C into E. Find a point $u \in C$ such that

$$\langle Au, J(v - u)\rangle \geq 0, \quad \forall v \in C. \tag{2.1}$$

Aoyama et al. [2] proved that the variational inequality (2.1) is equivalent to a fixed point problem. An element $x^* \in C$ is a solution of the variational inequality (2.1) if and only if $x^* \in C$ is a fixed point of the mapping $Q_C(I - \lambda A)$, where I is the identity mapping, $\lambda > 0$ is a constant and Q_C is a sunny nonexpansive retraction from E onto C, see [2] for more details.

Motivated by Aoyama et al. [2], we consider the following general system of variational inequalities.

Let $A : C \to E$ be an α-inverse-strongly accretive mapping. Find $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle \lambda Ay^* + x^* - y^*, J(x - x^*)\rangle \geq 0, & \forall x \in C, \\ \langle \mu Ax^* + y^* - x^*, J(x - y^*)\rangle \geq 0, & \forall x \in C. \end{cases} \tag{2.2}$$

If we add up the requirement that $x^* = y^*$, then the problem (2.2) is reduced to the generalized variational inequality (2.1). In a real Hilbert space, the system (2.2) is reduced to (1.2).

In order to prove our main results, we also need the following lemmas.

Lemma 2.1 (Browder [24]). *Let E be a uniformly convex Banach space, C a nonempty closed convex subset of E and $S : C \to C$ a nonexpansive mapping. Then $I - S$ is demi-closed at zero.*

The following lemma is a corollary of Bruck's results in [25].

Lemma 2.2. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let S_1 and S_2 be two nonexpansive mappings from C into itself with a common fixed point. Define a mapping $S : C \to C$ by*

$$Sx = \delta S_1 x + (1 - \delta)S_2 x, \quad \forall x \in C,$$

where δ is a constant in $(0, 1)$. Then S is nonexpansive and $F(S) = F(S_1) \cap F(S_2)$.

Lemma 2.3. *For given $(x^*, y^*) \in C \times C$, where $y^* = Q_C(x^* - \mu Ax^*)$, (x^*, y^*) is a solution of problem (2.2) if and only if x^* is a fixed point of the mapping $D : C \to C$ defined by*

$$D(x) = Q_C[Q_C(x - \mu Ax) - \lambda AQ_C(x - \mu Ax)],$$

where $\lambda, \mu > 0$ are constants and Q_C is a sunny nonexpansive retraction from E onto C.

Proof.

$$\begin{cases}\langle \lambda Ay^* + x^* - y^*, J(x - x^*)\rangle \geq 0, & \forall x \in C,\\ \langle \mu Ax^* + y^* - x^*, J(x - y^*)\rangle \geq 0, & \forall x \in C.\end{cases}$$

$$\Longleftrightarrow$$

$$\begin{cases}x^* = Q_C(y^* - \lambda Ay^*),\\ y^* = Q_C(x^* - \mu Ax^*).\end{cases}$$

$$\Longleftrightarrow$$

$$x^* = Q_C[Q_C(x^* - \mu Ax^*) - \lambda AQ_C(x^* - \mu Ax^*)].$$

This completes the proof. □

Lemma 2.4 (*Suzuki [26]*). *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space E and $\{\beta_n\}$ a sequence in $[0, 1]$ with*

$$0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1.$$

Suppose that $x_{n+1} = (1 - \beta_n)y_n + \beta_n x_n$ for all integers $n \geq 0$ and

$$\limsup_{n\to\infty}\big(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|\big) \leq 0.$$

Then $\lim_{n\to\infty} \|y_n - x_n\| = 0$.

Lemma 2.5 (*Xu [19]*). *Let E be a real 2-uniformly smooth Banach space with the best smooth constant K. Then the following inequality holds:*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, Jx\rangle + 2\|Ky\|^2, \quad \forall x, y \in E.$$

Lemma 2.6 (*Xu [27]*). *Assume that $\{\alpha_n\}$ is a sequence of nonnegative real numbers such that*

$$\alpha_{n+1} \leq (1 - \gamma_n)\alpha_n + \delta_n,$$

where $\{\gamma_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence such that

(a) $\sum_{n=1}^{\infty} \gamma_n = \infty$;
(b) $\limsup_{n\to\infty} \delta_n/\gamma_n \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} \alpha_n = 0$.

3. Main results

Theorem 3.1. *Let E be a uniformly convex and 2-uniformly smooth Banach space with the best smooth constant K, C a nonempty closed convex subset of E and Q_C a sunny nonexpansive retraction from E onto C. Let $A : C \to E$ be an α-inverse-strongly accretive mapping and $S : C \to C$ a nonexpansive mapping with a fixed point. Assume that $\mathcal{F} := F(S) \cap F(D) \neq \emptyset$, where D is defined as Lemma 2.3. Let $\{x_n\}$ be a sequence generated in the following manner*

$$\begin{cases}x_1 = u \in C,\\ y_n = Q_C(x_n - \mu Ax_n),\\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n[\delta Sx_n + (1 - \delta)Q_C(y_n - \lambda Ay_n)], \quad n \geq 1,\end{cases} \tag{Φ}$$

where $\delta \in (0, 1)$, $\lambda, \mu \in (0, \alpha/K^2)$ and $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $[0, 1]$ such that

(C1) $\alpha_n + \beta_n + \gamma_n = 1, \forall n \geq 1$;
(C2) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$;
(C3) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.

Then the sequence $\{x_n\}$ defined by (Φ) converges strongly to $\bar{x} = Q_{\mathcal{F}}u$ and $(\bar{x}, \bar{y})$, where $\bar{y} = Q_C(\bar{x} - \mu A\bar{x})$ and $Q_{\mathcal{F}}$ is a sunny nonexpansive retraction of C onto $\mathcal{F}$, is a solution of the problem (2.2).

Proof. First, we show that $\mathcal{F}$ is closed and convex. We know that $F(S)$ is closed and convex. Next, we show that $F(D)$ is closed and convex. For any $\lambda, \mu \in (0, \alpha/K^2)$, we have that the mappings $I - \lambda A$ and $I - \mu A$ are nonexpansive. Indeed, from the Lemma 2.5, for all $x, y \in C$, we have

$$\begin{aligned}\|(I - \lambda A)x - (I - \lambda A)y\|^2 &= \|(x - y) - \lambda(Ax - Ay)\|^2\\ &\leq \|x - y\|^2 - 2\lambda\langle Ax - Ay, J(x - y)\rangle + 2K^2\lambda^2\|Ax - Ay\|^2\\ &\leq \|x - y\|^2 - 2\lambda\alpha\|Ax - Ay\|^2 + 2K^2\lambda^2\|Ax - Ay\|^2\\ &= \|x - y\|^2 + 2\lambda(\lambda K^2 - \alpha)\|Ax - Ay\|^2\\ &\leq \|x - y\|^2.\end{aligned}$$

This shows that $I-\lambda A$ is a nonexpansive mapping, so is $I-\mu A$. On the other hand, from Lemma 2.3 we can see that

$$D=Q_C[Q_C(I-\mu A)-\lambda AQ_C(I-\mu A)]=Q_C(I-\lambda A)Q_C(I-\mu A).$$

That is, D is nonexpansive. This shows that $\mathcal{F}=F(S)\cap F(D)$ is closed and convex. Letting $x^*\in\mathcal{F}=F(S)\cap F(D)$, we from Lemma 2.3 obtain that

$$x^*=Q_C[Q_C(x^*-\mu Ax^*)-\lambda AQ_C(x^*-\mu Ax^*)].$$

Putting $y^*=Q_C(x^*-\mu Ax^*)$, we see that

$$x^*=Q_C(y^*-\lambda Ay^*).$$

Putting $e_n=\delta Sx_n+(1-\delta)Q_C(y_n-\lambda Ay_n)$ for each $n\geq 1$, we arrive at

$$\begin{aligned}\|e_n-x^*\| &= \|\delta Sx_n+(1-\delta)Q_C(y_n-\lambda Ay_n)-x^*\|\\ &\leq \delta\|Sx_n-x^*\|+(1-\delta)\|Q_C(y_n-\lambda Ay_n)-x^*\|\\ &\leq \delta\|x_n-x^*\|+(1-\delta)\|Q_C(y_n-\lambda Ay_n)-Q_C(y^*-\lambda Ay^*)\|\\ &\leq \delta\|x_n-x^*\|+(1-\delta)\|y_n-y^*\|\\ &= \delta\|x_n-x^*\|+(1-\delta)\|Q_C(x_n-\mu Ax_n)-Q_C(x^*-\mu Ax^*)\|\\ &\leq \|x_n-x^*\|.\end{aligned}$$

It follows that

$$\begin{aligned}\|x_{n+1}-x^*\| &= \|\alpha_n u+\beta_n x_n+\gamma_n e_n-x^*\|\\ &\leq \alpha_n\|u-x^*\|+\beta_n\|x_n-x^*\|+\gamma_n\|e_n-x^*\|\\ &\leq \alpha_n\|u-x^*\|+\beta_n\|x_n-x^*\|+\gamma_n\|x_n-x^*\|\\ &\leq \alpha_n\|u-x^*\|+(1-\alpha_n)\|x_n-x^*\|\\ &\leq \max\{\|u-x^*\|,\|x_1-x^*\|\}\\ &= \|u-x^*\|,\end{aligned}$$

which implies that the sequence $\{x_n\}$ is bounded, so are $\{y_n\}$ and $\{e_n\}$.

On the other hand, we have

$$\begin{aligned}\|e_{n+1}-e_n\| &= \|\delta Sx_{n+1}+(1-\delta)Q_C(y_{n+1}-\lambda Ay_{n+1})-[\delta Sx_n+(1-\delta)Q_C(y_n-\lambda Ay_n)]\|\\ &\leq \delta\|Sx_{n+1}-Sx_n\|+(1-\delta)\|Q_C(y_{n+1}-\lambda Ay_{n+1})-Q_C(y_n-\lambda Ay_n)\|\\ &\leq \delta\|x_{n+1}-x_n\|+(1-\delta)\|y_{n+1}-y_n\|\\ &= \delta\|x_{n+1}-x_n\|+(1-\delta)\|Q_C(x_{n+1}-\mu Ax_{n+1})-Q_C(x_n-\mu Ax_n)\|\\ &\leq \delta\|x_{n+1}-x_n\|+(1-\delta)\|x_{n+1}-x_n\|\\ &= \|x_{n+1}-x_n\|.\end{aligned}\tag{3.1}$$

Next, we claim that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0.\tag{3.2}$$

Putting $t_n=\frac{x_{n+1}-\beta_n x_n}{1-\beta_n}$ for each $n\geq 1$, we see that

$$x_{n+1}=(1-\beta_n)t_n+\beta_n x_n,\quad \forall n\geq 1.\tag{3.3}$$

Now, we compute $\|t_{n+1}-t_n\|$. From

$$\begin{aligned}t_{n+1}-t_n &= \frac{\alpha_{n+1}u+\gamma_{n+1}e_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_n u+\gamma_n e_n}{1-\beta_n}\\ &= \frac{\alpha_{n+1}}{1-\beta_{n+1}}u+\frac{1-\beta_{n+1}-\alpha_{n+1}}{1-\beta_{n+1}}e_{n+1}-\frac{\alpha_n}{1-\beta_n}u-\frac{1-\beta_n-\alpha_n}{1-\beta_n}e_n\\ &= \frac{\alpha_{n+1}}{1-\beta_{n+1}}(u-e_{n+1})+\frac{\alpha_n}{1-\beta_n}(e_n-u)+e_{n+1}-e_n,\end{aligned}$$

we have

$$\|t_{n+1}-t_n\|\leq\frac{\alpha_{n+1}}{1-\beta_{n+1}}\|u-e_{n+1}\|+\frac{\alpha_n}{1-\beta_n}\|e_n-u\|+\|e_{n+1}-e_n\|.\tag{3.4}$$

Substituting (3.1) into (3.4), we arrive at

$$\|t_{n+1} - t_n\| - \|x_{n+1} - x_n\| \leq \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|u - e_{n+1}\| + \frac{\alpha_n}{1-\beta_n}\|e_n - u\|.$$

It follows from the conditions (C2) and (C3) that

$$\limsup_{n\to\infty}\left(\|t_{n+1} - t_n\| - \|x_{n+1} - x_{n+1}\|\right) < 0.$$

From Lemma 2.4, we obtain that

$$\lim_{n\to\infty}\|t_n - x_n\| = 0. \tag{3.5}$$

Thanks to (3.3), we see that

$$x_{n+1} - x_n = (1-\beta_n)(t_n - x_n),$$

which combines with (3.5) yields that (3.2) holds. Noting that

$$x_{n+1} - x_n = \alpha_n(u - x_n) + \gamma_n(e_n - x_n),$$

and the conditions (C2) and (C3), we obtain that

$$\lim_{n\to\infty}\|e_n - x_n\| = 0. \tag{3.6}$$

Next, we prove that

$$\limsup_{n\to\infty}\langle u - \bar{x}, J(x_n - \bar{x})\rangle \leq 0, \tag{3.7}$$

where $\bar{x} = Q_{\mathcal{F}}u$. Define a mapping $G : C \to C$ by

$$Gx = \delta Sx + (1-\delta)Q_C(I - \lambda A)Q_C(I - \mu A)x, \quad \forall x \in C.$$

From Lemma 2.2, we see that G is a nonexpansive mapping with

$$\begin{aligned} F(G) &= F(S) \cap F(Q_C(I - \lambda A)Q_C(I - \mu A)) \\ &= F(S) \cap F(D) \\ &= \mathcal{F}. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} \|x_n - Gx_n\| &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - Gx_n\| \\ &\leq \|x_n - x_{n+1}\| + \alpha_n\|u - Gx_n\| + \beta_n\|x_n - Gx_n\|. \end{aligned}$$

This implies that

$$(1-\beta_n)\|x_n - Gx_n\| \leq \|x_n - x_{n+1}\| + \alpha_n\|u - Gx_n\|.$$

It follows from the conditions (C2), (C3) and (3.2) that

$$\lim_{n\to\infty}\|x_n - Gx_n\| = 0. \tag{3.8}$$

Let z_t be the fixed point of the contraction $z \mapsto tu + (1-t)Gz$, where $t \in (0, 1)$. That is,

$$z_t = tu + (1-t)Gz_t.$$

It follows that

$$\|z_t - x_n\| = \|(1-t)(Gz_t - x_n) + t(u - x_n)\|.$$

On the other hand, for any $t \in (0, 1)$, we see that

$$\begin{aligned} \|z_t - x_n\|^2 &= (1-t)\langle Gz_t - x_n, J(z_t - x_n)\rangle + t\langle u - x_n, J(z_t - x_n)\rangle \\ &= (1-t)\left(\langle Gz_t - Gx_n, J(z_t - x_n)\rangle + \langle Gx_n - x_n, J(z_t - x_n)\rangle\right) \\ &\quad + t\langle u - z_t, J(z_t - x_n)\rangle + t\langle z_t - x_n, J(z_t - x_n)\rangle \\ &\leq (1-t)\left(\|z_t - x_n\|^2 + \|Gx_n - x_n\|\|z_t - x_n\|\right) + t\langle u - z_t, J(z_t - x_n)\rangle + t\|z_t - x_n\|^2 \\ &\leq \|z_t - x_n\|^2 + \|Gx_n - x_n\|\|z_t - x_n\| + t\langle u - z_t, J(z_t - x_n)\rangle. \end{aligned}$$

It follows that

$$\langle z_t - u, J(z_t - x_n)\rangle \leq \frac{1}{t}\|Gx_n - x_n\|\|z_t - x_n\| \quad \forall t \in (0, 1).$$

In view of (3.8), we see that

$$\limsup_{n\to\infty}\langle z_t - u, J(z_t - x_n)\rangle \leq 0. \tag{3.9}$$

On the other hand, we see that $Q_{F(G)}u = \lim_{t\to 0} z_t$ and $F(G) = \mathcal{F}$. It follows that $z_t \to \bar{x} = Q_{\mathcal{F}}u$ as $t \to 0$. Since the fact that J is strong to weak* uniformly continuous on bounded subsets of E, we see that

$$\begin{aligned}
&|\langle u - \bar{x}, J(x_n - \bar{x})\rangle - \langle z_t - u, J(z_t - x_n)\rangle| \\
&\quad \leq |\langle u - \bar{x}, J(x_n - \bar{x})\rangle - \langle u - \bar{x}, J(x_n - z_t)\rangle| + |\langle u - \bar{x}, J(x_n - z_t)\rangle - \langle z_t - \bar{x}, J(z_t - x_n)\rangle| \\
&\quad \leq |\langle \bar{x} - q, J(x_n - q) - J(x_n - z_t)\rangle| + |\langle z_t - \bar{x}, J(x_n - z_t)\rangle| \\
&\quad \leq \|u - \bar{x}\|\|J(x_n - \bar{x}) - J(x_n - z_t)\| + \|z_t - \bar{x}\|\|x_n - z_t\| \to 0, \quad \text{as } t \to 0.
\end{aligned}$$

Hence, for any $\epsilon > 0$, there exists $\delta > 0$ such that $\forall t \in (0, \delta)$ the following inequality holds

$$\langle u - \bar{x}, J(x_n - \bar{x})\rangle \leq \langle z_t - u, J(z_t - x_n)\rangle + \epsilon.$$

This implies that

$$\limsup_{n\to\infty}\langle u - \bar{x}, J(x_n - \bar{x})\rangle \leq \limsup_{n\to\infty}\langle z_t - u, J(z_t - x_n)\rangle + \epsilon.$$

Since ϵ is arbitrary and (3.9), we see that $\limsup_{n\to\infty}\langle u - \bar{x}, J(x_n - \bar{x})\rangle \leq 0$. That is,

$$\limsup_{n\to\infty}\langle u - \bar{x}, J(x_{n+1} - \bar{x})\rangle \leq 0. \tag{3.10}$$

Finally, we show that $x_n \to \bar{x}$ as $n \to \infty$. Observe that

$$\begin{aligned}
\|x_{n+1} - \bar{x}\|^2 &= \langle \alpha_n u + \beta_n x_n + \gamma_n e_n - \bar{x}, J(x_{n+1} - \bar{x})\rangle \\
&= \alpha_n\langle u - \bar{x}, J(x_{n+1} - \bar{x})\rangle + \beta_n\langle x_n - \bar{x}, J(x_{n+1} - \bar{x})\rangle + \gamma_n\langle e_n - \bar{x}, J(x_{n+1} - \bar{x})\rangle \\
&\leq \alpha_n\langle u - \bar{x}, J(x_{n+1} - \bar{x})\rangle + \beta_n\|x_n - \bar{x}\|\|x_{n+1} - \bar{x}\| + \gamma_n\|e_n - \bar{x}\|\|x_{n+1} - \bar{x}\| \\
&\leq \alpha_n\langle u - \bar{x}, J(x_{n+1} - \bar{x})\rangle + (1 - \alpha_n)\|x_n - \bar{x}\|\|x_{n+1} - \bar{x}\| \\
&\leq \alpha_n\langle u - \bar{x}, J(x_{n+1} - \bar{x})\rangle + \frac{1 - \alpha_n}{2}(\|x_n - \bar{x}\|^2 + \|x_{n+1} - \bar{x}\|^2),
\end{aligned}$$

which implies that

$$\|x_{n+1} - \bar{x}\|^2 \leq (1 - \alpha_n)\|x_n - \bar{x}\|^2 + 2\alpha_n\langle u - \bar{x}, J(x_{n+1} - \bar{x})\rangle. \tag{3.11}$$

From the condition (C2), (3.10) and applying Lemma 2.6 to (3.11), we obtain that

$$\lim_{n\to\infty}\|x_n - \bar{x}\| = 0.$$

This completes the proof. □

Remark 3.2. Since L^p for all $p \geq 2$ is uniformly convex and 2-uniformly smooth, we see that Theorem 3.1 is applicable to L^p for all $p \geq 2$.

Remark 3.3. There are a number of sequences satisfying the restrictions (C1)–(C3), for example $\alpha_n = \frac{1}{n+1}$, $\beta_n = \frac{n}{2n+1}$ and $\gamma_n = \frac{n^2}{2n^2+3n+1}$ for each $n \geq 1$.

4. Applications

In real Hilbert spaces, Lemma 2.3 is reduced to the following.

Lemma 4.1. *For given $(x^*, y^*) \in C$, where $y^* = P_C(x^* - \mu Ax^*)$, (x^*, y^*) is a solution of problem (1.2) if and only if x^* is a fixed point of the mapping $D' : C \to C$ defined by*

$$D'(x) = P_C[P_C(x - \mu Ax) - \lambda AP_C(x - \mu Ax)],$$

where P_C is a metric projection H onto C.

It is well known that the smooth constant $K = \frac{\sqrt{2}}{2}$ in Hilbert spaces. From Theorem 3.1, we can obtain the following result immediately.

Theorem 4.1. *Let H be a real Hilbert space, C a nonempty closed convex subset of H. Let $A : C \to H$ be an α-inverse-strongly monotone mapping and $S : C \to C$ a nonexpansive mapping with a fixed point. Assume that $F := F(S) \cap F(D') \neq \emptyset$, where D' is defined as Lemma 4.1. Suppose that $\{x_n\}$ is generated by*

$$\begin{cases} x_1 = u \in C, \\ y_n = P_C(x_n - \mu A x_n), \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n[\delta S x_n + (1-\delta)P_C(y_n - \lambda A y_n)], \quad n \geq 1, \end{cases} \tag{4.1}$$

where $\delta \in (0,1)$, $\lambda, \mu \in (0, 2\alpha)$ and $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $[0,1]$ such that

(C1) $\alpha_n + \beta_n + \gamma_n = 1, \forall n \geq 1$;
(C2) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(C3) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.

Then the sequence $\{x_n\}$ defined by (4.1) converges strongly to $\bar{x} = P_{\mathcal{F}} u$ and $(\bar{x}, \bar{y})$ is a solution of the problem (1.2), where $\bar{y} = P_C(\bar{x} - \mu A \bar{x})$.

Next, we always assume that E is a uniformly convex and 2-uniformly smooth Banach space.

Recall that an operator B with domain $D(B)$ and range $R(B)$ in E is accretive, if for each $x_i \in D(B)$ and $y_i \in Bx_i (i = 1, 2)$,

$$\langle y_2 - y_1, J(x_2 - x_1) \rangle \geq 0,$$

Observe that x is a zero of an accretive mapping B if and only if it is a fixed point of the pseudo-contractive mapping $T := I - B$. An accretive operator B is m-accretive if $R(I + rB) = E$ for each $r > 0$. Next, we assume that B is m-accretive and has a zero (i.e., the inclusion $0 \in B(z)$ is solvable). The set of zeros of B is denoted by Ω. Hence,

$$\Omega = \{z \in D(B) : 0 \in B(z)\} = B^{-1}(0).$$

For each $r > 0$, we denote by J_r the resolvent of B, i.e., $J_r^B = (I + rB)^{-1}$. Note that if B is m-accretive, then $J_r^B : E \to E$ is nonexpansive and $F(J_r^B) = \Omega$ for all $r > 0$.

From Theorem 3.1, we can conclude the following result immediately.

Theorem 4.2. *Let E be a uniformly convex and 2-uniformly smooth Banach space with the best smooth constant K, A an α-inverse-strongly accretive mapping and B an m-accretive mapping. Assume that $\mathcal{F} := B^{-1}(0) \cap A^{-1}(0) \neq \emptyset$. Suppose that $\{x_n\}$ is generated by*

$$\begin{cases} x_1 = u \in E, \\ y_n = x_n - \mu A x_n, \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n[\delta J_r^B x_n + (1-\delta)(y_n - \lambda A y_n)], \quad n \geq 1, \end{cases} \tag{4.2}$$

where $\delta \in (0,1)$, $\lambda, \mu \in (0, \alpha/K^2)$ and $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $[0,1]$ such that

(C1) $\alpha_n + \beta_n + \gamma_n = 1, \forall n \geq 1$;
(C2) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(C3) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.

Then the sequence $\{x_n\}$ defined by (4.2) converges strongly to $\bar{x} = Q_{\mathcal{F}} u$, where $Q_{\mathcal{F}}$ is a sunny nonexpansive retraction of E onto $\mathcal{F}$.

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References

[1] G. Stampacchia, Formes bilineaires coercivites sur les ensembles convexes, CR Acad. Sci. Paris 258 (1964) 4413–4416.
[2] K. Aoyama, H. Iiduka, W. Takahashi, Weak convergence of an iterative sequence for accretive operators in Banach spaces, Fixed Point Theory Appl. 2006 (2006) 35390.
[3] Y.J. Cho, Y. Yao, H. Zhou, Strong convergence of an iterative algorithm for accretive operators in Banach spaces, J. Comput. Anal. Appl. 10 (2008) 113–125.
[4] S.S. Chang, H.W.J. Lee, C.K. Chan, Generalized system for relaxed cocoercive variational inequalities in Hilbert spaces, Appl. Math. Lett. 20 (2007) 329–334.
[5] L.C. Ceng, J.C. Yao, An extragradient-like approximation method for variational inequality problems and fixed point problems, Appl. Math. Comput. 190 (2007) 205–215.

[6] L.C. Ceng, C.Y. Wang, J.C. Yao, Strong convergence theorems by a relaxed extragradient method for a general system of variational inequalities, Math. Meth. Oper. Res. 67 (2008) 375–390.
[7] I. Cioranescu, Geometry of Banach Spaces, Duality Mappings and Nonlinear Problems, Kluwer Academic Publishers, Dordrecht, 1990.
[8] Z. Huang, M.A. Noor, An explicit projection method for a system of nonlinear variational inequalities with different (γ, r)-cocoercive mappings, Appl. Math. Comput. 190 (2007) 356–361.
[9] H. Iiduka, W. Takahashi, Strong convergence theorems for nonexpansive mappings and inverse-strongly monotone mappings, Nonlinear Anal. 61 (2005) 341–350.
[10] M. Aslam Noor, Some developments in general variational inequalities, Appl. Math. Comput. 152 (2004) 199–277.
[11] X. Qin, S.M. Kang, M. Shang, Generalized system for relaxed cocoercive variational inequalities in Hilbert spaces, Appl. Anal. 87 (2008) 421–430.
[12] X. Qin, M. Shang, H. Zhou, Strong convergence of a general iterative method for variational inequality problems and fixed point problems in Hilbert spaces, Appl. Math. Comput. 200 (2008) 242–253.
[13] X. Qin, M.A. Noor, General Wiener–Hopf equation technique for nonexpansive mappings and general variational inequalities in Hilbert spaces, Appl. Math. Comput. 201 (2008) 716–722.
[14] R.U. Verma, Generalized system for relaxed cocoercive variational inequalities and its projection methods, J. Optim. Theory Appl. 121 (2004) 203–210.
[15] R.U. Verma, Generalized class of partial relaxed monotonicity and its connections, Adv. Nonlinear Var. Inequal. 7 (2004) 155–164.
[16] R.U. Verma, General convergence analysis for two-step projection methods and applications to variational problems, Appl. Math. Lett. 18 (2005) 1286–1292.
[17] R.U. Verma, On a new system of nonlinear variational inequalities and associated iterative algorithms, Math. Sci. Res. Hot-Line 3 (1999) 65–68.
[18] Y. Yao, J.C. Yao, On modified iterative method for nonexpansive mappings and monotone mappings, Appl. Math. Comput. 186 (2007) 1551–1558.
[19] H.K. Xu, Inequalities in Banach spaces with applications, Nonlinear Anal. 16 (1991) 1127–1138.
[20] S. Reich, Asymptotic behavior of contractions in Banach spaces, J. Math. Anal. Appl. 44 (1973) 57–70.
[21] S. Kitahara, W. Takahashi, Image recovery by convex combinations of sunny nonexpansive retractions, Topol. Meth. Nonlinear Anal. 2 (1993) 333–342.
[22] F.E. Browder, Fixed point theorems for noncompact mappings in Hilbert spaces, Proc. Natl. Acad. Sci. USA 53 (1965) 1272–1276.
[23] S. Reich, Strong convergence theorems for resolvents of accretive operators in Banach spaces, J. Math. Anal. Appl. 75 (1980) 287–292.
[24] F.E. Browder, Nonlinear operators and nonlinear equations of evolution in Banach spaces, Proc. Symp. Pure. Math. 18 (1976) 78–81.
[25] R.E. Bruck, Properties of fixed point sets of nonexpansive mappings in Banach spaces, Tras. Amer. Math. Soc. 179 (1973) 251–262.
[26] T. Suzuki, Strong convergence of Krasnoselskii and Mann's type sequences for one-parameter nonexpansive semigroups without Bochne integrals, J. Math. Anal. Appl. 305 (2005) 227–239.
[27] H.K. Xu, Iterative algorithms for nonlinear operators, J. London Math. Soc. 66 (2002) 240–256.