Journal of Multivariate Analysis 101 (2010) 2571–2586

Contents lists available at ScienceDirect

Journal of Multivariate Analysis

journal homepage: www.elsevier.com/locate/jmva

A multivariate version of Hoeffding's Phi-Square

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ARTICLE INFO

Article history:
Received 23 October 2009
Available online 23 July 2010

AMS 2000 subject class:
Primary 62H20
60F05
62G20
Secondary 60G15
62G10

Keywords:
Multivariate measure of association
Copula
Nonparametric estimation
Empirical copula process
Weak convergence
Nonparametric bootstrap
Strong mixing

ABSTRACT

A multivariate measure of association is proposed, which extends the bivariate copula-based measure Phi-Square introduced by Hoeffding [22]. We discuss its analytical properties and calculate its explicit value for some copulas of simple form; a simulation procedure to approximate its value is provided otherwise. A nonparametric estimator for multivariate Phi-Square is derived and its asymptotic behavior is established based on the weak convergence of the empirical copula process both in the case of independent observations and dependent observations from strictly stationary strong mixing sequences. The asymptotic variance of the estimator can be estimated by means of nonparametric bootstrap methods. For illustration, the theoretical results are applied to financial asset return data.

1. Introduction

Measuring the degree of association between the components of a d-dimensional ($d > 2$) random vector $\mathbf{X} = (X_1, \ldots, X_d)$ has attracted much interest among scientists and practitioners in recent years. Naturally, such measures of multivariate association are based on the copula of the random vector $\mathbf{X}$, i.e., they are invariant with respect to the marginal distributions of the random variables $X_i, i = 1, \ldots, d$. Wolff [49] introduces a class of multivariate measures of association which is based on the L^1- and L^∞-norms of the difference between the copula and the independence copula (see [16] for various extensions). Other authors generalize existing bivariate measures of association to the multivariate case. For example, various multivariate extensions of Spearman's Rho are considered by Nelsen [33] and Schmid and Schmidt [38–40]. Blomqvist's Beta is generalized by Úbeda-Flores [47] and Schmid and Schmidt [41], whereas a multivariate version of Gini's Gamma is proposed by Behboodian et al. [2]. Further, Joe [27] and Nelsen [33] discuss multivariate generalizations of Kendall's Tau. A multivariate version of Spearman's footrule is considered by Genest et al. [19]. Joe [26,25] investigates another type of multivariate measures which is based on the Kullback–Leibler mutual information. General considerations of multivariate measures of association, in particular of measures of concordance, are discussed in [49,27,11,45]. For a survey of copula-based measures of multivariate association, we refer to [42]. Despite their common basis on the copula of a random vector, the aforementioned measures generally differ with regard to their analytical properties. We do not know of a measure that is superior in every respect and the choice of an appropriate measure depends essentially on the type of application.

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0047-259X/$ – see front matter
doi:10.1016/j.jmva.2010.07.006

In the present paper, we propose a multivariate extension of the bivariate measure of association which was suggested by Wassily Hoeffding in his seminal doctoral dissertation (see [22,17]). Multivariate Hoeffding's Phi-Square (which we denote by Φ^2) is based on a Cramér–von Mises functional and can be interpreted as the (normalized) average squared difference between the copula of a random vector and the independence copula. It offers a set of properties which are advantageous for many applications. For example, the measure is zero if and only if the components of $\mathbf{X}$ are stochastically independent. This leads to the construction of statistical tests for stochastic independence based on Hoeffding's Phi-Square (cf. [20]). Multivariate Hoeffding's Φ^2 is based on a L^2-type distance between the copula C of a random vector and the independence copula and attains its maximal value in the case of comonotonicity, i.e., when the copula equals the upper Fréchet–Hoeffding bound. The concept of comonotonicity is of interest, e.g., in actuarial science or finance (see [9,10]). Further, multivariate Hoeffding's Phi-Square can be estimated with low computational complexity, even for large dimension d, and nonparametric statistical inference for the estimator can be established based on the empirical copula. The measure can thus be used to quantify the degree of association of multivariate empirical data (e.g., financial asset returns).

The paper is organized as follows. Section 2 introduces relevant definitions and notation. Multivariate Hoeffding's Φ^2 is introduced in Section 3 and some of its analytical properties are investigated. We calculate the explicit value of multivariate Hoeffding's Φ^2 for some copulas of simple form and describe a simulation algorithm to approximate its value in cases where the copula is of a more complicated form. In Section 4, a nonparametric estimator $\widehat{\Phi}_n^2$ for Φ^2 based on the empirical copula is derived. We establish its asymptotic behavior both in the case of independent observations and dependent observations from strictly stationary strong mixing sequences. In general, two cases need to be distinguished: If $\Phi^2 > 0$, asymptotic normality of $\sqrt{n}(\widehat{\Phi}_n^2 - \Phi^2)$ can be shown using the functional Delta-method. The asymptotic variance can consistently be estimated by means of a nonparametric (moving block) bootstrap method. When $\Phi^2 = 0$, weak convergence of $n\widehat{\Phi}_n^2$ to a non-normal distribution follows. We show how the estimator can be adapted to account for small sample sizes. Section 5 presents an empirical study of financial contagion related to the bankruptcy of Lehman Brothers Inc. in September 2008 using multivariate Hoeffding's Φ^2. A brief conclusion is given in Section 6.

2. Notation and definitions

Let $\mathbf{X} = (X_1, \ldots, X_d)$ be a d-dimensional random vector ($d \geq 2$), defined on some probability space $(\Omega, \mathcal{F}, P)$, with distribution function $F(\mathbf{x}) = P(X_1 \leq x_1, \ldots, X_d \leq x_d)$ for $\mathbf{x} = (x_1, \ldots, x_d) \in \mathbb{R}^d$ and marginal distribution functions $F_i(x) = P(X_i \leq x)$ whose survival functions are denoted by $\bar{F}_i(x) = P(X_i > x)$ for $x \in \mathbb{R}$ and $i = 1, \ldots, d$. If not stated otherwise, we always assume that the F_i are continuous functions. According to Sklar's theorem [44], there exists a unique copula $C : [0, 1]^d \to [0, 1]$ such that

$$F(\mathbf{x}) = C(F_1(x_1), \ldots, F_d(x_d)) \quad \text{for all } \mathbf{x} \in \mathbb{R}^d.$$

The copula C is also referred to as the copula of the random vector $\mathbf{X}$ (if an explicit reference to the random vector is required, we denote its copula by $C_{\mathbf{X}}$). It represents the joint distribution function of the random variables $U_i = F_i(X_i)$, $i = 1, \ldots, d$, i.e., $C(u_1, \ldots, u_d) = P(U_1 \leq u_1, \ldots, U_d \leq u_d) = P(\mathbf{U} \leq \mathbf{u})$ for all $\mathbf{u} = (u_1, \ldots, u_d) \in [0, 1]^d$ and random vector $\mathbf{U} = (U_1, \ldots, U_d)$. Moreover, $C(\mathbf{u}) = F(F_1^{-1}(u_1), \ldots, F_d^{-1}(u_d))$ for all $\mathbf{u} \in [0, 1]^d$. The generalized inverse function G^{-1} is defined by $G^{-1}(u) := \inf\{x \in \mathbb{R} \cup \{\infty\} | G(x) \geq u\}$ for all $u \in (0, 1]$ and $G^{-1}(0) := \sup\{x \in \mathbb{R} \cup \{-\infty\} | G(x) = 0\}$.

The survival copula $\breve{C}$ of C is given by $\breve{C}(\mathbf{u}) = P(\mathbf{U} > \mathbf{1} - \mathbf{u})$ where $\mathbf{1} - \mathbf{u} = (1 - u_1, \ldots, 1 - u_d)$. It represents the copula of the random vector $-\mathbf{X} = (-X_1, \ldots, -X_d)$. The random vector $\mathbf{X} = (X_1, \ldots, X_d)$ is said to be marginally symmetric about $\mathbf{a} \in \mathbb{R}^d$ if $X_1, \ldots, X_d$ are symmetric about $a_1, \ldots, a_d$, respectively (i.e., $F_k(a_k + x) = \bar{F}_k(a_k - x)$ for all $x \in \mathbb{R}$, $k = 1, \ldots, d$). Radial symmetry of $\mathbf{X}$ about $\mathbf{a} \in \mathbb{R}^d$ is given if, and only if, the random vector $\mathbf{X}$ is marginally symmetric and the copula $C_{\mathbf{X}}$ of $\mathbf{X}$ equals its survival copula $\breve{C}_{\mathbf{X}}$, that is $C_{\mathbf{X}}(\mathbf{u}) = P(\mathbf{U} \leq \mathbf{u}) = P(\mathbf{U} > \mathbf{1} - \mathbf{u}) = \breve{C}_{\mathbf{X}}(\mathbf{u})$ for all $\mathbf{u} \in [0, 1]^d$. Let α_k be a strictly decreasing transformation of the kth component X_k of $\mathbf{X}$, defined on the range of X_k, and denote the transformed random vector by $\boldsymbol{\alpha}_k(\mathbf{X}) = (X_1, \ldots, X_{k-1}, \alpha_k(X_k), X_{k+1}, \ldots, X_d)$, $k = 1, \ldots, d$. The random vector $\mathbf{X}$ is then said to be jointly symmetric about $\mathbf{a} \in \mathbb{R}^d$ if, and only if, $\mathbf{X}$ is marginally symmetric and its copula $C_{\mathbf{X}}$ equals the copula $C_{\alpha_k(\mathbf{X})}$ of $\boldsymbol{\alpha}_k(\mathbf{X})$, $k = 1, \ldots, d$. Note that joint symmetry implies radial symmetry; see [34], Chapter 2.7.

Every copula C is bounded in the following sense:

$$\begin{aligned} W(\mathbf{u}) &:= \max\{u_1 + \cdots + u_d - (d-1), 0\} \\ &\leq C(\mathbf{u}) \leq \min\{u_1, \ldots, u_d\} =: M(\mathbf{u}) \quad \text{for all } \mathbf{u} \in [0, 1]^d, \end{aligned}$$

where M and W are called the upper and lower Fréchet–Hoeffding bounds, respectively. The upper bound M is a copula itself and is also known as the comonotonic copula. It represents the copula of $X_1, \ldots, X_d$ if $F_1(X_1) = \cdots = F_d(X_d)$ with probability one, i.e., if there is (with probability one) a strictly increasing functional relationship between X_i and $X_j (i \neq j)$. By contrast, the lower bound W is a copula only for dimension $d = 2$. Moreover, the independence copula Π with

$$\Pi(\mathbf{u}) := \prod_{i=1}^{d} u_i, \quad \mathbf{u} \in [0, 1]^d,$$

describes the dependence structure of stochastically independent random variables $X_1, \ldots, X_d$. For a detailed treatment of copulas, we refer to [34,28]. Further, the space $\ell^\infty([0, 1]^d)$ is the collection of all uniformly bounded real-valued functions defined on $[0, 1]^d$. It is equipped with the uniform metric $m(f_1, f_2) = \sup_{\mathbf{t} \in [0,1]^d} |f_1(\mathbf{t}) - f_2(\mathbf{t})|$.

3. A multivariate version of Hoeffding's Phi-Square

In his seminal dissertation, Hoeffding [22] suggests the following measure Φ^2 to quantify the amount of association between the components of the two-dimensional random vector $\mathbf{X}$ with copula C

$$\Phi^2 = 90 \int_{[0,1]^2} \{C(u_1, u_2) - \Pi(u_1, u_2)\}^2 \, du_1 du_2. \tag{1}$$

For the d-dimensional random vector $\mathbf{X}$ with copula C, we define a multivariate version of Φ^2 by

$$\Phi^2 := \Phi^2(C) = h(d) \int_{[0,1]^d} \{C(\mathbf{u}) - \Pi(\mathbf{u})\}^2 \, d\mathbf{u}, \tag{2}$$

with normalization constant $h(d) = [\int_{[0,1]^d} \{M(\mathbf{u}) - \Pi(\mathbf{u})\}^2 \, d\mathbf{u}]^{-1}$, whose explicit form is calculated later. In particular, Φ^2 can be regarded as a continuous functional on the subset of $l^\infty([0, 1]^d)$ for which the integral on the right-hand side of Eq. (2) is well-defined. This relationship is used in Section 4 where the statistical properties of Φ^2 are derived. An alternative multivariate measure of association can be defined by

$$\Phi := \Phi(C) = +\sqrt{\Phi^2(C)}.$$

This measure can be interpreted as the normalized average distance between the copula C and the independence copula Π with respect to the L^2-norm. Bivariate measures of association of this form based on the L^1- and L^∞-norms are considered by Schweizer and Wolff [43]. For the related multivariate case, we refer to [49,16]. If C is the copula of the random vector $\mathbf{X}$, we also refer to Φ^2 and Φ as $\Phi^2_{\mathbf{X}}$ and $\Phi_{\mathbf{X}}$, respectively. Various analytical properties of Φ^2 are discussed next. They can also be established for Φ.

Normalization: An important property of Φ^2 is that

$$\Phi^2 = 0 \quad \text{if and only if } C = \Pi.$$

In order to motivate the specific form of the normalization factor $h(d)$ in Eq. (2), we calculate the value of the defining integral for the lower and upper Fréchet–Hoeffding bounds, respectively. For $C = M$, we have

$$h(d)^{-1} = \int_{[0,1]^d} \{M(\mathbf{u}) - \Pi(\mathbf{u})\}^2 \, d\mathbf{u} = \frac{2}{(d+1)(d+2)} - \frac{1}{2^d} \frac{d!}{\prod_{i=0}^{d} \left(i + \frac{1}{2}\right)} + \left(\frac{1}{3}\right)^d, \tag{3}$$

and for $C = W$, we obtain

$$g(d)^{-1} = \int_{[0,1]^d} \{W(\mathbf{u}) - \Pi(\mathbf{u})\}^2 \, d\mathbf{u} = \frac{2}{(d+2)!} - 2\sum_{d}^{i=0} \binom{d}{i} (-1)^i \frac{1}{(d+1+i)!} + \left(\frac{1}{3}\right)^d. \tag{4}$$

The calculations are outlined in Appendix A. Both expressions converge to zero as the dimension d tends to infinity. In particular, direct calculations yield that

$$h(d)^{-1} \geq g(d)^{-1} \quad \text{for all } d \geq 2,$$

such that the range of Φ^2 as defined in (2) is restricted to the interval $[0, 1]$, i.e.,

$$0 \leq \Phi^2 \leq 1 \quad \text{and} \quad \Phi^2 = 1 \quad \text{iff} \quad C = M \quad \text{for } d \geq 3. \tag{5}$$

Note that for dimension $d = 2$, it holds that $\Phi^2 = 1$ iff $C = M$ or $C = W$ since $g(2) = h(2)$.

Remark. In the bivariate case, Hoeffding's Phi-Square thus represents a measure for strictly monotone functional dependence. In consequence, a value of one of Hoeffding's Phi-Square also implies that the random variables X_1 and X_2 are completely dependent, i.e., that there exists a one-to-one function ψ (which is not necessarily monotone) such that $P(X_2 = \psi(X_1)) = 1$ (cf. [23,30]). However, the converse does not hold. For example, two random variables X_1 and X_2 are completely dependent if their copula is a shuffle of M. According to Nelsen [34], Theorem 3.2.2, we find shuffles of M which (uniformly) approximate the independence copula arbitrarily closely (see also [32]). Hence, the value of bivariate Hoeffding's Phi-Square can be made arbitrarily small for completely dependent random variables.

Invariance with respect to permutations: For every permutation π of the components of $\mathbf{X}$ we have $\Phi^2_{\mathbf{X}} = \Phi^2_{\pi(\mathbf{X})}$ according to Fubini's Theorem.

Monotonicity: For copulas C_1 and C_2 with $\Pi(\mathbf{u}) \leq C_1(\mathbf{u}) \leq C_2(\mathbf{u}) \leq M(\mathbf{u})$ for all $\mathbf{u} \in [0, 1]^d$, we have $\Phi^2(C_1) \leq \Phi^2(C_2)$. For copulas C_3 and C_4 such that $W(\mathbf{u}) \leq C_3(\mathbf{u}) \leq C_4(\mathbf{u}) \leq \Pi(\mathbf{u})$ for all $\mathbf{u} \in [0, 1]^d$, it follows that $\Phi^2(C_3) \geq \Phi^2(C_4)$. This property generalizes a result of Yanagimoto [50] to the multivariate case.

Invariance under strictly monotonic transformations: The behavior of Φ^2 with respect to strictly monotonic transformations is given in the next proposition.

Proposition 1. *Let* $\mathbf{X}$ *be a d-dimensional random vector with copula C.*

(i) *For dimension* $d \geq 2$, $\Phi^2_{\mathbf{X}}$ *is invariant with regard to strictly increasing transformations of one or several components of* $\mathbf{X}$.

(ii) *For dimension* $d = 2$, $\Phi^2_{\mathbf{X}}$ *is invariant under strictly decreasing transformations of one or both components of* $\mathbf{X}$.
For $d \geq 3$, *let* α_k *be a strictly decreasing transformation of the kth component* X_k *of* $\mathbf{X}$, $k \in \{1, \ldots, d\}$, *and let* $\boldsymbol{\alpha}_k(\mathbf{X}) = (X_1, \ldots, X_{k-1}, \alpha_k(X_k), X_{k+1}, \ldots, X_d)$. *Then,* $\Phi^2_{\mathbf{X}} = \Phi^2_{\boldsymbol{\alpha}_k(\mathbf{X})}$ *if one of the following three conditions holds:*

- $X_1, \ldots, X_d$ *are jointly symmetric about* $\mathbf{a} \in \mathbb{R}^d$, *or*
- $(X_1, \ldots, X_{k-1}, X_{k+1}, \ldots, X_d)$ *is stochastically independent of* X_k, *or*
- $X_1, \ldots, X_{k-1}, X_{k+1}, \ldots, X_d$ *are mutually stochastically independent.*

The *proof* is outlined in Appendix C. If part (ii) of the above proposition is not satisfied, equality of $\Phi^2_{\mathbf{X}}$ and $\Phi^2_{\boldsymbol{\alpha}_k(\mathbf{X})}$ does not hold in general. For example, let the copula C of $\mathbf{X}$ be the comonotonic copula, i.e., $C = M$, and let α_1 be a strictly decreasing transformation of the first component X_1 of $\mathbf{X}$. Then, $C_{\boldsymbol{\alpha}_1(\mathbf{X})}(\mathbf{u}) \neq M(\mathbf{u})$ for all $\mathbf{u} \in [0, 1]^d$, implying that $\Phi^2_{\boldsymbol{\alpha}_1(\mathbf{X})} < \Phi^2_{\mathbf{X}}$. However, when applying the (strictly decreasing) inverse function α_1^{-1} of α_1 to the first component of $\boldsymbol{\alpha}_1(\mathbf{X})$, we obtain that $\Phi^2_{\boldsymbol{\alpha}_1^{-1}(\boldsymbol{\alpha}_1(\mathbf{X}))} = \Phi^2_{\mathbf{X}} > \Phi^2_{\boldsymbol{\alpha}_1(\mathbf{X})}$.

Duality: For dimension $d = 2$, Hoeffding's Phi-Square satisfies duality, i.e. $\Phi^2_{\mathbf{X}} = \Phi^2_{-\mathbf{X}}$ or equivalently $\Phi^2(C) = \Phi^2(\breve{C})$, since Φ^2 is invariant under strictly decreasing transformations of all components of $\mathbf{X}$ according to Proposition 1, part (ii). For dimension $d \geq 3$, duality does not hold in general except in the case that the random vector $\mathbf{X}$ is radially symmetric about $\mathbf{a} \in \mathbb{R}^d$. Taylor [45] discusses duality in the context of multivariate measures of concordance.

Continuity: If $\{C_m\}_{m\in\mathbb{N}}$ is a sequence of copulas such that $C_m(\mathbf{u}) \to C(\mathbf{u})$ for all $\mathbf{u} \in [0, 1]^d$, then $\Phi^2(C_m) \to \Phi^2(C)$ as a direct consequence of the dominated convergence theorem.

Examples. (i) Let C be the d-dimensional Farlie–Gumbel–Morgenstern copula defined by $C(u_1, \ldots, u_d) = \prod_{i=1}^d u_i + \theta \prod_{i=1}^d u_i(1 - u_i)$, $|\theta| \leq 1$. Then, Hoeffding's Phi-Square is given by

$$\Phi^2 = h(d)\theta^2 \left(\frac{1}{30}\right)^d, \quad d \geq 2.$$

It follows that $\Phi^2 \leq 1/10$ for $d \geq 2$. This illustrates the restricted range of dependence modeled by the family of Farlie–Gumbel–Morgenstern copulas.

(ii) Let $C(\mathbf{u}) = \theta M(\mathbf{u}) + (1 - \theta)\Pi(\mathbf{u})$ with $0 \leqslant \theta \leqslant 1$. Then

$$\Phi^2 = \theta^2, \qquad d \geq 2.$$

Note that for this family of copulas, the value of Φ^2 does not depend on the dimension d.

It is difficult to derive an explicit expression for Φ^2 if C is of a more complicated structure than in examples (i) and (ii). In this case, the value of Φ^2 needs to be determined by simulation. The following equivalent representation of Φ^2 is useful for this purpose:

$$\Phi^2 = h(d) \int_{[0,1]^d} \{C(\mathbf{u}) - \Pi(\mathbf{u})\}^2 \, \mathrm{d}\mathbf{u} = h(d) E_\Pi \left[\{C(\mathbf{U}) - \Pi(\mathbf{U})\}^2\right], \tag{6}$$

where the random vector $\mathbf{U} = (U_1, \ldots, U_d)$ is uniformly distributed on $[0, 1]^d$ with stochastically independent components U_i, $i = 1, \ldots, d$ (which is indicated by the subscript Π). Thus, an approximation of Φ^2 is obtained by estimating the expectation on the right-hand side of Eq. (6) consistently as follows:

$$\hat{E}_\Pi \left[\{C(\mathbf{U}) - \Pi(\mathbf{U})\}^2\right] = \frac{1}{n} \sum_{i=1}^n \{C(\mathbf{U}_i) - \Pi(\mathbf{U}_i)\}^2, \tag{7}$$

with $\mathbf{U}_1, \ldots, \mathbf{U}_n$ being independent and identically distributed Monte Carlo replications from $\mathbf{U}$. For illustration we compute the approximated value of Φ^2 for the equi-correlated Gaussian copula, given by

$$C_\rho(u_1, \ldots, u_d) = \Phi_\rho \left(\Phi^{-1}(u_1), \ldots, \Phi^{-1}(u_d)\right) \tag{8}$$

with

$$\Phi_\rho(x_1, \ldots, x_d) = \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_d} (2\pi)^{-\frac{d}{2}} \left|\Sigma_\rho\right|^{-\frac{1}{2}} \exp\left(-\frac{1}{2}\mathbf{x}^\top \Sigma_\rho^{-1} \mathbf{x}\right) \mathrm{d}x_d \cdots \mathrm{d}x_1$$

and $\Sigma_\rho = \rho\left(\mathbf{1}\mathbf{1}^\top\right) + (1 - \rho)\mathbf{I}$ with $-\frac{1}{d-1} < \rho < 1$. The approximated values of Φ^2 and Φ for different choices of the parameter ρ and for dimensions $d = 2$, $d = 5$, and $d = 10$ are displayed in Fig. 1.

The values of Φ^2 form a parabolic curve; see [22] for a power series representation of Φ^2 for the bivariate Gaussian copula.

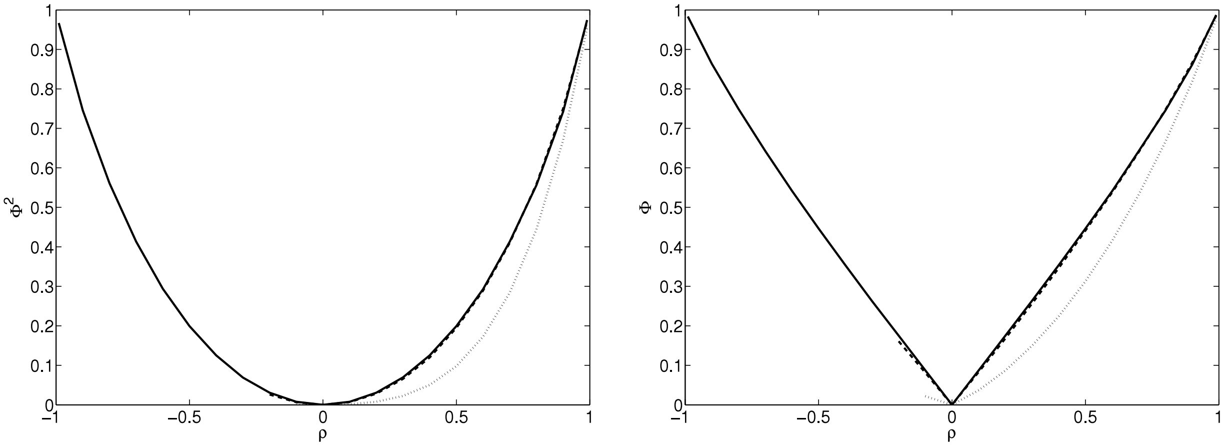

Fig. 1. *Gaussian copula.* Approximated values of Φ^2 (left panel) and Φ (right panel) in the case of a d-dimensional equi-correlated Gaussian copula with parameter ρ for dimension $d = 2$ (solid line), $d = 5$ (dashed line), and $d = 10$ (dotted line); calculations are based on $n = 100,000$ Monte Carlo replications.

4. Statistical inference for multivariate Hoeffding's Phi-Square

Statistical inference for multivariate Hoeffding's Phi-Square as introduced in formula (2) is based on the empirical copula which was first discussed by Rüschendorf [37] and Deheuvels [8]. We derive a nonparametric estimator for multivariate Hoeffding's Phi-Square and establish its asymptotic behavior based on the weak convergence of the empirical copula process. After illustrating our approach on the basis of independent observations, we generalize it to the case of dependent observations from strictly stationary strong mixing sequences.

4.1. Nonparametric estimation

Let $\mathbf{X}_1, \ldots, \mathbf{X}_n$ be an (i.i.d.) random sample from the d-dimensional random vector $\mathbf{X}$ with distribution function F and copula C. We assume that both F and C as well as the marginal distribution functions F_i, $i = 1, \ldots, d$, are completely unknown. The copula C is estimated by the empirical copula $\widehat{C}_n$, which is defined as

$$\widehat{C}_n(\mathbf{u}) = \frac{1}{n}\sum_{j=1}^{n}\prod_{i=1}^{d}\mathbf{1}_{\{\widehat{U}_{ij,n}\le u_i\}} \quad \text{for } \mathbf{u} \in [0,1]^d \tag{9}$$

with pseudo-observations $\widehat{U}_{ij,n} = \widehat{F}_{i,n}(X_{ij})$ for $i = 1, \ldots, d$ and $j = 1, \ldots, n$, and $\widehat{F}_{i,n}(x) = \frac{1}{n}\sum_{j=1}^{n}\mathbf{1}_{\{X_{ij}\le x\}}$ for $x \in \mathbb{R}$. As $\widehat{U}_{ij,n} = \frac{1}{n}$ (rank of X_{ij} in $X_{i1}, \ldots, X_{in}$), statistical inference is based on the ranks of the observations. For fixed n, we suppress the subindex and refer to the pseudo-observations as $\widehat{U}_{ij}$ if it is clear from the context.

A nonparametric estimator for Φ^2 is then obtained by replacing the copula C in formula (2) by the empirical copula $\widehat{C}_n$, i.e.,

$$\widehat{\Phi}_n^2 := \Phi^2(\widehat{C}_n) = h(d)\int_{[0,1]^d}\left\{\widehat{C}_n(\mathbf{u}) - \Pi(\mathbf{u})\right\}^2 \mathrm{d}\mathbf{u}.$$

The estimator is based on a Cramér–von Mises statistic and can explicitly be determined by

$$\widehat{\Phi}_n^2 = h(d)\left\{\left(\frac{1}{n}\right)^2\sum_{j=1}^{n}\sum_{k=1}^{n}\prod_{i=1}^{d}(1-\max\{\widehat{U}_{ij},\widehat{U}_{ik}\}) - \frac{2}{n}\left(\frac{1}{2}\right)^d\sum_{j=1}^{n}\prod_{i=1}^{d}(1-\widehat{U}_{ij}^2) + \left(\frac{1}{3}\right)^d\right\}. \tag{10}$$

The derivation is outlined in Appendix B. Obviously, an estimator for the alternative measure Φ is given by $\widehat{\Phi}_n = +\sqrt{\widehat{\Phi}_n^2}$. The asymptotic distributions of $\widehat{\Phi}_n^2$ and $\widehat{\Phi}_n$ can be deduced from the asymptotic behavior of the empirical copula process which has been discussed, e.g., by Rüschendorf [37], Gänßler and Stute [18], Van der Vaart and Wellner [48], and Tsukahara [46]. The following result is shown in [14]:

Proposition 2. *Let $\mathbf{X}_1, \ldots, \mathbf{X}_n$ be a random sample from the d-dimensional random vector $\mathbf{X}$ with joint distribution function F and copula C. If the ith partial derivatives $D_iC(\mathbf{u})$ of C exist and are continuous for $i = 1, \ldots, d$, the empirical process $\sqrt{n}(\widehat{C}_n - C)$ converges weakly in $\ell^\infty([0,1]^d)$ to the process $\mathbb{G}_C$ which takes the form*

$$\mathbb{G}_C(\mathbf{u}) = \mathbb{B}_C(\mathbf{u}) - \sum_{i=1}^{d} D_i C(\mathbf{u}) \mathbb{B}_C(\mathbf{u}^{(i)}). \tag{11}$$

The process $\mathbb{B}_C$ is a d-dimensional Brownian bridge on $[0, 1]^d$ with covariance function $E\{\mathbb{B}_C(\mathbf{u})\mathbb{B}_C(\mathbf{v})\} = C(\mathbf{u} \wedge \mathbf{v}) - C(\mathbf{u})C(\mathbf{v})$. The vector $\mathbf{u}^{(i)}$ corresponds to the vector where all coordinates, except the ith coordinate of $\mathbf{u}$, are replaced by 1.

Then, asymptotic normality of $\widehat{\Phi}_n^2$ and $\widehat{\Phi}_n$ can be established.

Theorem 3. *Under the assumptions of Proposition 2 and if $C \neq \Pi$, it follows that*

$$\sqrt{n}(\widehat{\Phi}_n^2 - \Phi^2) \xrightarrow{\mathrm{d}} Z_{\Phi^2} \tag{12}$$

where $Z_{\Phi^2} \sim N(0, \sigma_{\Phi^2}^2)$ with

$$\sigma_{\Phi^2}^2 = \{2h(d)\}^2 \int_{[0,1]^d} \int_{[0,1]^d} \{C(\mathbf{u}) - \Pi(\mathbf{u})\} E\{\mathbb{G}_C(\mathbf{u})\mathbb{G}_C(\mathbf{v})\}\{C(\mathbf{v}) - \Pi(\mathbf{v})\} \mathrm{d}\mathbf{u}\mathrm{d}\mathbf{v}. \tag{13}$$

Regarding the alternative measure Φ, we have

$$\sqrt{n}(\widehat{\Phi}_n - \Phi) \xrightarrow{\mathrm{d}} Z_{\Phi}$$

with $Z_{\Phi} \sim N(0, \sigma_{\Phi}^2)$ and

$$\sigma_{\Phi}^2 = \frac{\sigma_{\Phi^2}^2}{4\Phi^2} = h(d) \frac{\int_{[0,1]^d} \int_{[0,1]^d} \{C(\mathbf{u}) - \Pi(\mathbf{u})\} E\{\mathbb{G}_C(\mathbf{u})\mathbb{G}_C(\mathbf{v})\}\{C(\mathbf{v}) - \Pi(\mathbf{v})\} \mathrm{d}\mathbf{u}\mathrm{d}\mathbf{v}}{\int_{[0,1]^d} \{C(\mathbf{u}) - \Pi(\mathbf{u})\}^2 \mathrm{d}\mathbf{u}}.$$

The *proof* is given in Appendix C. Note that the assumption $C \neq \Pi$ guarantees that the limiting random variable is non-degenerated as implied by the form of the variance $\sigma_{\Phi^2}^2$. However, if $C = \Pi$, Proposition 2 and an application of the continuous mapping theorem yield

$$n\widehat{\Phi}_n^2 \xrightarrow{\mathrm{d}} h(d) \int_{[0,1]^d} \{\mathbb{G}_\Pi(\mathbf{u})\}^2 \mathrm{d}\mathbf{u}, \quad \text{as } n \to \infty, \tag{14}$$

with

$$E\left[h(d) \int_{[0,1]^d} \{\mathbb{G}_\Pi(\mathbf{u})\}^2 \mathrm{d}\mathbf{u}\right] = h(d) \left\{ \left(\frac{1}{2}\right)^d - \left(\frac{1}{3}\right)^d - \frac{d}{6}\left(\frac{1}{3}\right)^{d-1} \right\}.$$

The asymptotic distribution of $\widehat{\Phi}_n^2$ when $C = \Pi$ is important for the construction of tests for stochastic independence between the components of a multivariate random vector based on Hoeffding's Phi-Square. In the bivariate setting, such tests have been studied by Hoeffding [24] and Blum et al. [3]. Regarding the multivariate case, we mention Genest and Rémillard [21] and Genest et al. [20] who consider various combinations of Cramér–von Mises statistics with special regard to their asymptotic local efficiency. In our setting, a hypothesis test for $H_0 : C = \Pi$ against $H_1 : C \neq \Pi$ is performed by rejecting H_0 if the value of $n\widehat{\Phi}_n^2$ exceeds the $(1 - \alpha)$-quantile of the limiting distribution in Eq. (14). The latter can be determined by simulation; approximate critical values for the test statistic $\{h(d)\}^{-1} n\widehat{\Phi}_n^2$ are also provided in [20].

Remark. When the univariate marginal distribution functions F_i are known, Hoeffding's Phi-Square can also be estimated using the theory of U-statistics. Consider the random variables $U_{ij} = F_i(X_{ij}), i = 1, \ldots, d, j = 1, \ldots, n$ with $\mathbf{U}_j = (U_{1j}, \ldots, U_{dj})$ having distribution function C. Since

$$\begin{aligned}\Phi^2 &= h(d) \int_{[0,1]^d} \{C(\mathbf{u}) - \Pi(\mathbf{u})\}^2 \,\mathrm{d}\mathbf{u} \\ &= \int_{[0,1]^d} \int_{[0,1]^d} \int_{[0,1]^d} h(d) \left(\prod_{i=1}^{d} \mathbf{1}_{\{x_i \le u_i\}} - \prod_{i=1}^{d} u_i\right) \left(\prod_{i=1}^{d} \mathbf{1}_{\{y_i \le u_i\}} - \prod_{i=1}^{d} u_i\right) \mathrm{d}\mathbf{u}\mathrm{d}C(\mathbf{x})\mathrm{d}C(\mathbf{y}),\end{aligned}$$

an unbiased estimator of the latter based on the random sample $\mathbf{U}_1, \ldots, \mathbf{U}_n$ is given by the U-statistic

$$U_n(\psi) = \binom{n}{2}^{-1} \sum_{1 \le j < k \le n} \psi(\mathbf{U_j}, \mathbf{U_k})$$

with kernel ψ of degree 2, defined by

$$\psi(\mathbf{x},\mathbf{y}) = h(d)\int_{[0,1,]^d}\left(\prod_{i=1}^{d}\mathbf{1}_{\{x_i\le u_i\}}-\prod_{i=1}^{d}u_i\right)\left(\prod_{i=1}^{d}\mathbf{1}_{\{y_i\le u_i\}}-\prod_{i=1}^{d}u_i\right)\mathrm{d}\mathbf{u},\quad \mathbf{x},\mathbf{y}\in[0,1]^d.$$

Results from the theory of U-statistics (see, e.g., Chapter 3 in [31]) and standard calculations yield that $\sqrt{n}\{U_n(\psi)-\Phi^2\}$ is asymptotically normally distributed with mean zero and variance

$$\begin{aligned}\sigma_U^2 &= \mathrm{Var}\{\psi_1(\mathbf{X})\}\\ &= 4\{h(d)\}^2\int_{[0,1]^d}\int_{[0,1]^d}\{C(\mathbf{u})-\Pi(\mathbf{u})\}\{C(\mathbf{u}\wedge\mathbf{v})-C(\mathbf{u})C(\mathbf{v})\}\{C(\mathbf{v})-\Pi(\mathbf{v})\}\mathrm{d}\mathbf{u}\mathrm{d}\mathbf{v},\end{aligned}$$

where $\psi_1(\mathbf{x}) = E\{\psi(\mathbf{X},\mathbf{Y})|\mathbf{X}=\mathbf{x}\}$ with independent random vectors $\mathbf{X}$, $\mathbf{Y}$ having distribution function C. The asymptotic variance coincides with the asymptotic variance of $\sqrt{n}(\widehat{\Phi}_n^2-\Phi^2)$ for known marginal distribution functions (cf. Eq. (13)). In particular, $U_n(\psi)$ is degenerate when $C=\Pi$. The fact that both estimators have the same asymptotic distribution in the case of known margins follows also from the relationship

$$\sqrt{n}(\widehat{\Phi}_n^2-\Phi^2) = \frac{1}{n^{3/2}}\sum_{j=1}^{n}\psi(\mathbf{U_j},\mathbf{U_j})+\sqrt{n}\left\{\frac{n-1}{n}\cdot U_n(\psi)-\Phi^2\right\}$$

where the first term in the right equation converges to zero in probability for $n\to\infty$. In the case of unknown marginal distribution functions, the estimation of Φ^2 by means of U-statistics is more involved in comparison to the above approach based on the empirical copula.

Now let $\{\mathbf{X}_j = (X_{1j},\dots,X_{dj})\}_{j\in\mathbb{Z}}$ be a strictly stationary sequence of d-dimensional random vectors, being defined on a probability space $(\Omega,\mathcal{F},P)$, with distribution function F, continuous marginal distribution functions F_i, $i=1,\dots,d$, and copula C. In order to describe temporal dependence between the observations we use the concept of α-mixing or strong mixing. Suppose $\mathcal{A}$ and $\mathcal{B}$ are two σ-fields included in $\mathcal{F}$ and define

$$\alpha(\mathcal{A},\mathcal{B}) = \sup_{A\in\mathcal{A},B\in\mathcal{B}}|P(A\cap B)-P(A)P(B)|.$$

The mixing coefficient $\alpha_{\mathbf{X}}$ associated with the sequence $\{\mathbf{X}_j\}_{j\in\mathbb{Z}}$ is given by $\alpha_{\mathbf{X}}(r)=\sup_{s\ge0}\alpha(\mathcal{F}_s,\mathcal{F}^{s+r})$ where $\mathcal{F}_t=\sigma\{\mathbf{X}_j, j\le t\}$ and $\mathcal{F}^t=\sigma\{\mathbf{X}_j, j\ge t\}$ denote the σ-fields generated by $\mathbf{X}_j, j\le t$, and $\mathbf{X}_j, j\ge t$, respectively. The process $\{\mathbf{X}_j\}_{j\in\mathbb{Z}}$ is said to be strong mixing if

$$\alpha_{\mathbf{X}}(r)\to 0\quad\text{for } r\to\infty.$$

Assume that our observations are realizations of the sample $\mathbf{X}_1,\dots,\mathbf{X}_n$ and denote by $\widehat{\Phi}_n^2$ the corresponding estimator for Hoeffding's Phi-Square calculated according to (10). The asymptotic behavior of $\widehat{\Phi}_n^2$ is given next.

Theorem 4. *Let $\mathbf{X}_1,\dots,\mathbf{X}_n$ be observations from the strictly stationary strong mixing sequence $\{\mathbf{X}_j\}_{j\in\mathbb{Z}}$ with coefficient $\alpha_{\mathbf{X}}(r)$ satisfying $\alpha_{\mathbf{X}}(r)=O(r^{-a})$ for some $a>1$. If the ith partial derivatives $D_iC(\mathbf{u})$ of C exist and are continuous for $i=1,\dots,d$, and $C\ne\Pi$, we have $\sqrt{n}(\widehat{\Phi}_n^2-\Phi^2)\xrightarrow{d} Z_{\Phi^2}\sim N(0,\sigma^2_{\Phi^2})$ with*

$$\sigma^2_{\Phi^2} = \{2h(d)\}^2\int_{[0,1]^d}\int_{[0,1]^d}E\left[\{C(\mathbf{u})-\Pi(\mathbf{u})\}\mathbb{G}^*(\mathbf{u})\mathbb{G}^*(\mathbf{v})\{C(\mathbf{v})-\Pi(\mathbf{v})\}\right]\mathrm{d}\mathbf{u}\mathrm{d}\mathbf{v}. \tag{15}$$

The process $\mathbb{G}^*$ has the same form as in Eq. (11) with process $\mathbb{B}_C$ being replaced by the centered Gaussian process $\mathbb{B}^*$ in $[0,1]^d$ having covariance function

$$E\{\mathbb{B}^*(\mathbf{u})\mathbb{B}^*(\mathbf{v})\} = \sum_{j\in\mathbb{Z}}E\left[\{\mathbf{1}_{\{\mathbf{U}_0\le\mathbf{u}\}}-C(\mathbf{u})\}\{\mathbf{1}_{\{\mathbf{U}_j\le\mathbf{v}\}}-C(\mathbf{v})\}\right],$$

where $\mathbf{U}_j=(F_1(X_{1j}),\dots,F_d(X_{dj})), j\in\mathbb{Z}$.

The *proof* is outlined in Appendix C. The result is based on the weak convergence of the empirical copula process to the Gaussian process $\mathbb{G}^*$ for strictly stationary strong mixing sequences; cf. also [15] for the asymptotic properties of the smoothed empirical copula process in this setting. Note that the asymptotic variance in Eq. (15) depends not only on the copula C as in the case of independent observations (cf. Eq. (13)) but also on the joint distribution of $\mathbf{U}_0$ and $\mathbf{U}_j$, $j\in\mathbb{Z}$. Theorem 4 can be transferred to sequences with temporal dependence structures other than strong mixing; we refer, for example, to [1,7,12].

Even though the asymptotic variance has a closed-form expression, it cannot be calculated explicitly in most cases but has to be estimated adequately. For dependent observations, a bootstrap method, the (moving) block bootstrap, has been proposed by Künsch [29], which is briefly described in the following. Given the sample $\mathbf{X}_1, \ldots, \mathbf{X}_n$, we define blocks of size l, $l(n) = o(n)$, of consecutive observations by

$$B_{s,l} = \{\mathbf{X}_{s+1}, \ldots, \mathbf{X}_{s+l}\}, \quad s = 0, \ldots, n-l.$$

The block bootstrap draws with replacement k blocks from the blocks $B_{s,l}$, $s = 0, \ldots, n-l$ where we assume that $n = kl$ (otherwise the last block is shortened). With $S_1, \ldots, S_k$ being independent and uniformly distributed random variables on $\{0, \ldots, n-l\}$, the bootstrap sample thus comprises those observations from $\mathbf{X}_1, \ldots, \mathbf{X}_n$ which are among the k blocks $B_{S_1,l}, \ldots, B_{S_k,l}$, i.e.,

$$\mathbf{X}_1^B = \mathbf{X}_{S_1+1}, \ldots, \mathbf{X}_l^B = \mathbf{X}_{S_1+l}, \qquad \mathbf{X}_{l+1}^B = \mathbf{X}_{S_2+1}, \ldots, \mathbf{X}_n^B = \mathbf{X}_{S_k+l}.$$

The block length l is a function of n, i.e., $l = l(n)$ with $l(n) \to \infty$ as $n \to \infty$. For a discussion regarding the choice of $l(n)$; see [29,5]. Denote by $\widehat{C}_n^B$ and $\widehat{F}_n^B$ the empirical copula and the empirical distribution function of the block bootstrap sample $\mathbf{X}_1^B, \ldots, \mathbf{X}_n^B$, respectively, and let $\widehat{\Phi}_n^{2,B}$ be the corresponding estimator for Hoeffding's Phi-Square. It follows that the block bootstrap can be applied to estimate the asymptotic variance of $\sqrt{n}(\widehat{\Phi}^2 - \Phi^2)$.

Proposition 5. *Let $(\mathbf{X}_j^B)_{j=1,\ldots,n}$ be the block bootstrap sample from $(\mathbf{X}_j)_{j=1,\ldots,n}$, which are observations of a strictly stationary, strong mixing sequence $\{\mathbf{X}_j\}_{j\in\mathbb{Z}}$ of d-dimensional random vectors with distribution function F and copula C whose partial derivatives exist and are continuous. Suppose further that $\sqrt{n}(\widehat{C}_n^B - \widehat{C}_n)$ converges weakly in $\ell^\infty([0,1]^d)$ to the same Gaussian limit as $\sqrt{n}(\widehat{C}_n - C)$ in probability. If $C \neq \Pi$, the sequences $\sqrt{n}(\widehat{\Phi}_n^2 - \Phi^2)$ and $\sqrt{n}(\widehat{\Phi}_n^{2,B} - \widehat{\Phi}_n^2)$ converge weakly to the same Gaussian limit in probability.*

The sequence $\sqrt{n}(\widehat{C}_n^B - \widehat{C}_n)$ converges weakly in probability to the same Gaussian limit as $\sqrt{n}(\widehat{C}_n - C)$ if the (uniform) empirical process $\sqrt{n}(\widehat{F}_n^B - \widehat{F}_n)$ converges weakly in probability to the same Gaussian limit as $\sqrt{n}(\widehat{F}_n - F)$, provided that all partial derivatives of the copula exist and are continuous. This can be shown analogously to the proof of Theorem 3.9.11 in [48] using the functional Delta-method (cf. also proof of Theorem 4). The block bootstrap for empirical processes has been discussed in various settings and for different dependence structures; for an overview see [35] and references therein. The following sufficient conditions for $\sqrt{n}(\widehat{F}_n^B - \widehat{F}_n)$ to converge weakly (in the space $D([0,1]^d)$) in probability to the appropriate Gaussian process for strong mixing sequences are derived in [4]:

$$\sum_{r=0}^{\infty} (r+1)^{16(d+1)} \alpha_{\mathbf{X}}^{1/2}(r) < \infty \quad \text{and} \quad \text{block length } l(n) = O(n^{1/2-\varepsilon}), \quad \varepsilon > 0.$$

The results of a simulation study, which assesses the performance of the bootstrap variance estimator, are presented in Section 4.2. Note that in the case of independent observations, the standard bootstrap, which draws with replacement n single observations from the sample $\mathbf{X}_1, \ldots, \mathbf{X}_n$, can be used to estimate the asymptotic variance of $\sqrt{n}(\widehat{\Phi}_n^2 - \Phi^2)$.

Theorem 4 together with Proposition 5 enables the calculation of an asymptotic $(1-\alpha)$-confidence interval for Hoeffding's Phi-Square $\Phi^2 \in (0,1)$, given by

$$\widehat{\Phi}_n^2 \pm \Phi^{-1}\left(1 - \frac{\alpha}{2}\right) \hat{\sigma}_{\Phi_n^2}^B / \sqrt{n}.$$

Further, an asymptotic hypothesis test for $\mathrm{H}_0 : \Phi^2 = \Phi_0^2$ against $\mathrm{H}_1 : \Phi^2 \neq \Phi_0^2$ with $\Phi_0^2 \in (0,1)$ can be constructed by rejecting the null hypothesis at the confidence level α if

$$\left| \sqrt{n} \frac{(\hat{\Phi}_n^2 - \Phi_0^2)}{\hat{\sigma}_{\Phi_n^2}^B} \right| > \Phi^{-1}\left(1 - \frac{\alpha}{2}\right).$$

Here, $(\hat{\sigma}_{\Phi_n^2}^B)^2$ denotes the consistent bootstrap variance estimator for $\sigma_{\Phi^2}^2$, obtained by the block bootstrap. Note that in the case $\Phi_0^2 = 1$, the copula corresponds to the upper Fréchet–Hoeffding bound M which does not possess continuous first partial derivatives.

The above results can be extended to statistically analyze the difference of two Hoeffding's Phi-Squares. In a financial context, this may be of interest for assessing whether Hoeffding's Phi-Square of one portfolio of financial assets significantly differs from that of another portfolio (cf. Section 5). Suppose $\Phi_{\mathbf{X}}^2$ and $\Phi_{\mathbf{Y}}^2$ are multivariate Hoeffding's Phi-Squares associated with the strictly stationary sequences $\{\mathbf{X}_j\}_{j\in\mathbb{Z}}$ and $\{\mathbf{Y}_j\}_{j\in\mathbb{Z}}$ of d-dimensional random vectors with distribution functions $F_{\mathbf{X}}$ and $F_{\mathbf{Y}}$, continuous marginal distribution functions, and copulas $C_{\mathbf{X}}$ and $C_{\mathbf{Y}}$, respectively. Since the two sequences do not have to be necessarily independent, consider the sequence $\{\mathbf{Z}_j = (\mathbf{X}_j, \mathbf{Y}_j)\}_{j\in\mathbb{Z}}$ of $2d$-dimensional random vectors with joint distribution function $F_{\mathbf{Z}}$, continuous marginal distribution functions $F_{\mathbf{Z},i}$, $i = 1, \ldots, 2d$, and copula $C_{\mathbf{Z}}$ such that $C_{\mathbf{Z}}(\mathbf{u}, 1, \ldots, 1) = C_{\mathbf{X}}(\mathbf{u})$ and $C_{\mathbf{Z}}(1, \ldots, 1, \mathbf{v}) = C_{\mathbf{Y}}(\mathbf{v})$ for all $\mathbf{u}, \mathbf{v} \in [0,1]^d$.

Theorem 6. *Let $\mathbf{Z}_1 = (\mathbf{X}_1, \mathbf{Y}_1), \ldots, \mathbf{Z}_n = (\mathbf{X}_n, \mathbf{Y}_n)$ be observations of the strictly stationary, strong mixing sequence $\{\mathbf{Z}_j = (\mathbf{X}_j, \mathbf{Y}_j)\}_{j\in\mathbb{Z}}$ with strong mixing coefficient $\alpha_{\mathbf{Z}}$ satisfying $\alpha_{\mathbf{Z}}(r) = O(r^{-a})$ for some $a > 1$. If the ith partial derivatives of $C_{\mathbf{Z}}$ exist and are continuous for $i = 1, \ldots, 2d$, and $C_{\mathbf{X}}, C_{\mathbf{Y}} \neq \Pi$, we have*

$$\sqrt{n}\left\{\hat{\Phi}_{\mathbf{X}}^2-\Phi_{\mathbf{X}}^2-(\hat{\Phi}_{\mathbf{Y}}^2-\Phi_{\mathbf{Y}}^2)\right\} \xrightarrow{\text{d}} W \sim N(0,\sigma^2) \quad \text{as } n\to\infty,$$

where $\sigma^2=\sigma^2_{\Phi^2_{\mathbf{X}}}+\sigma^2_{\Phi^2_{\mathbf{Y}}}-2\sigma_{\Phi^2_{\mathbf{X}},\Phi^2_{\mathbf{Y}}}$ *with*

$$\sigma_{\Phi^2_{\mathbf{X}},\Phi^2_{\mathbf{Y}}}=\{2h(d)\}^2\int_{[0,1]^d}\int_{[0,1]^d}E\left[\{C_{\mathbf{X}}(\mathbf{u})-\Pi(\mathbf{u})\}\mathbb{G}^*_{\mathbf{X}}(\mathbf{u})\mathbb{G}^*_{\mathbf{Y}}(\mathbf{v})\{C_{\mathbf{Y}}(\mathbf{v})-\Pi(\mathbf{v})\}\right]\mathrm{d}\mathbf{u}\mathrm{d}\mathbf{v},$$

and $\sigma^2_{\Phi^2_{\mathbf{X}}}=\sigma_{\Phi^2_{\mathbf{X}},\Phi^2_{\mathbf{X}}}$ *and* $\sigma^2_{\Phi^2_{\mathbf{Y}}}=\sigma_{\Phi^2_{\mathbf{Y}},\Phi^2_{\mathbf{Y}}}$ *(cf. Eq. (15)). The processes* $\mathbb{G}^*_{\mathbf{X}}$ *and* $\mathbb{G}^*_{\mathbf{Y}}$ *are Gaussian processes on* $[0,1]^d$ *with covariance structure as given in Theorem 4.*

The *proof* is given in Appendix C. Analogously to the discussion prior to Theorem 6, an asymptotic confidence interval or a statistical hypothesis test for the difference of two Hoeffding's Phi-Squares can be formulated (cf. Section 5).

4.2. Small sample adjustments

In order to reduce bias in finite samples, the independence copula Π in the definition of $\widehat{\Phi}^2_n$ in Eq. (10) can be replaced by its discrete counterpart $\prod_{i=1}^d U_n(u_i)$ where U_n denotes the (univariate) distribution function of a random variable uniformly distributed on the set $\left\{\frac{1}{n},\dots,\frac{n}{n}\right\}$. This has also been proposed by Genest et al. [20] in the context of tests for stochastic independence (cf. Section 4.1). In order to ensure the normalization property of the estimator in small samples, we introduce the normalization factor $h(d,n)$ depending on both dimension d and sample size n. It is obtained by replacing the upper Fréchet–Hoeffding bound M with its discrete counterpart $M_n(\mathbf{u}):=\min\{U_n(u_1),\dots,U_n(u_d)\}$, which represents an adequate upper bound of the empirical copula for given sample size n. A small sample estimator for Φ^2 thus has the form

$$\tilde{\Phi}^2_n=h(d,n)\int_{[0,1]^d}\left\{\widehat{C}_n(\mathbf{u})-\prod_{i=1}^d U_n(u_i)\right\}^2\mathrm{d}\mathbf{u} \tag{16}$$

with

$$h(d,n)^{-1}=\int_{[0,1]^d}\left\{M_n(\mathbf{u})-\prod_{i=1}^d U_n(u_i)\right\}^2\mathrm{d}\mathbf{u}.$$

We obtain

$$\tilde{\Phi}^2_n = h(d,n)\left\{\left(\frac{1}{n}\right)^2\sum_{j=1}^n\sum_{k=1}^n\prod_{i=1}^d(1-\max\{\widehat{U}_{ij},\widehat{U}_{ik}\}) - \frac{2}{n}\left(\frac{1}{2}\right)^d\sum_{j=1}^n\prod_{i=1}^d\left\{1-\widehat{U}_{ij}^2-\frac{1-\widehat{U}_{ij}}{n}\right\}+\left(\frac{1}{3}\right)^d\left\{\frac{(n-1)(2n-1)}{2n^2}\right\}^d\right\},$$

with

$$h(d,n)^{-1}=\left(\frac{1}{n}\right)^2\sum_{j=1}^n\sum_{k=1}^n\left(1-\max\left\{\frac{j}{n},\frac{k}{n}\right\}\right)^d-\frac{2}{n}\sum_{j=1}^n\left\{\frac{n(n-1)-j(j-1)}{2n^2}\right\}^d+\left(\frac{1}{3}\right)^d\left\{\frac{(n-1)(2n-1)}{2n^2}\right\}^d.$$

Note that $\tilde{\Phi}^2_n$ and $\widehat{\Phi}^2_n$ have the same asymptotic distribution, i.e. under the assumptions of Theorem 4

$$\sqrt{n}(\tilde{\Phi}^2_n-\Phi^2)\xrightarrow{\text{d}} Z_{\Phi^2}\sim N(0,\sigma^2_{\Phi^2}).$$

This can be shown analogously to the proof of Theorem 4 using the fact that $\lim_{n\to\infty}\sqrt{n}\{h(d,n)-h(d)\}=0$. Accordingly it follows that the bootstrap to estimate the asymptotic variance of $\sqrt{n}(\tilde{\Phi}^2_n-\Phi^2)$ works (cf. Proposition 5).

A simulation study is carried out in order to investigate the finite-sample performance of the block bootstrap estimator for the asymptotic standard deviation σ_{Φ^2} of $\sqrt{n}(\tilde{\Phi}^2_n-\Phi^2)$.

For comparison, we additionally provide the corresponding simulation results when using a nonparametric jackknife method to estimate the unknown standard deviation. For dependent observations, Künsch [29] introduces the delete-l jackknife which, in contrast to the block bootstrap described in the previous section, is based on systematically deleting one block $B_{s,l}$ of l consecutive observations each time from the original sample, $s=0,\dots,n-l$. Let $\tilde{\Phi}^{2,(s)}_n$ denote the estimator of Hoeffding's Phi-Square calculated from the original sample when we have deleted block $B_{s,l}$, $s=0,\dots,n-l$ and define $\tilde{\Phi}^{2,(.)}_n=(n-l+1)^{-1}\sum_{s=0}^{n-l}\tilde{\Phi}^{2,(s)}_n$. The jackknife estimate of the standard deviation is then given by

$$\hat{\sigma}^J=\sqrt{\frac{(n-l)^2}{nl(n-l+1)}\sum_{s=0}^{n-l}\left(\tilde{\Phi}^{2,(s)}_n-\tilde{\Phi}^{2,(.)}_n\right)^2}.$$

Table 1

Gaussian copula (independent observations). Simulation results for estimating the asymptotic standard deviation of $\sqrt{n}(\tilde{\Phi}_n^2 - \Phi^2)$ by means of the nonparametric block bootstrap with block length l and the delete-l jackknife (for $l = 5$): The table shows the empirical means $m(\cdot)$ and the empirical standard deviations $s(\cdot)$ of the respective estimates, which are calculated based on 1000 Monte Carlo simulations of sample size n of a d-dimensional equi-correlated Gaussian copula with parameter ρ and 250 bootstrap samples. The bootstrap estimates are labeled by the superscript B, jackknife estimates by J.

ρ	n	Φ^2	$m(\tilde{\Phi}_n^2)$	$s(\tilde{\Phi}_n^2)$	$m(\hat{\sigma}^B)$	$m(\hat{\sigma}^J)$	$s(\hat{\sigma}^B)$	$s(\hat{\sigma}^J)$
Dimension $d = 2$								
0.2	50	0.032	0.077	0.047	0.055	0.049	0.019	0.026
	100	0.032	0.054	0.034	0.035	0.033	0.012	0.015
	500	0.032	0.035	0.015	0.015	0.015	0.003	0.003
0.5	50	0.197	0.231	0.095	0.089	0.094	0.017	0.022
	100	0.197	0.218	0.070	0.067	0.069	0.010	0.011
	500	0.197	0.202	0.032	0.031	0.032	0.003	0.002
−0.1	50	0.008	0.056	0.035	0.047	0.037	0.015	0.021
	100	0.008	0.032	0.021	0.026	0.022	0.010	0.012
	500	0.008	0.013	0.008	0.008	0.008	0.003	0.003
Dimension $d = 5$								
0.2	50	0.028	0.044	0.023	0.026	0.022	0.010	0.010
	100	0.028	0.036	0.016	0.017	0.015	0.005	0.005
	500	0.028	0.030	0.007	0.007	0.007	0.001	0.001
0.5	50	0.191	0.208	0.065	0.063	0.062	0.013	0.014
	100	0.191	0.202	0.048	0.045	0.046	0.007	0.008
	500	0.191	0.196	0.022	0.021	0.021	0.002	0.002
−0.1	50	0.007	0.015	0.004	0.005	0.004	0.001	0.002
	100	0.007	0.011	0.004	0.003	0.003	0.001	0.001
	500	0.007	0.007	0.002	0.002	0.002	0.000	0.000
Dimension $d = 10$								
0.2	50	0.007	0.014	0.009	0.012	0.008	0.007	0.005
	100	0.007	0.011	0.005	0.007	0.005	0.003	0.003
	500	0.007	0.008	0.002	0.002	0.002	0.001	0.001
0.5	50	0.098	0.111	0.046	0.049	0.043	0.017	0.016
	100	0.098	0.107	0.033	0.035	0.033	0.009	0.009
	500	0.098	0.100	0.015	0.015	0.015	0.002	0.002
−0.1	50	0.001	0.001	0.000	0.000	0.000	0.000	0.000
	100	0.001	0.001	0.000	0.000	0.000	0.000	0.000
	500	0.001	0.001	0.000	0.000	0.000	0.000	0.000

We consider observations from an $AR(1)$-process with autoregressive coefficient β (cf. Table 2) based on the equi-correlated Gaussian copula as defined in (8). To generate these observations, we proceed as follows: Simulate n independent d-dimensional random variates $\mathbf{U}_j = (U_{j1}, \dots, U_{jd}), j = 1, \dots, n$, from the equi-correlated Gaussian copula with parameter ρ. Set $\boldsymbol{\varepsilon}_j = (\Phi^{-1}(U_{j1}), \dots, \Phi^{-1}(U_{jd})), j = 1, \dots, n$. A sample $(\mathbf{X}_j)_{j=1,\dots,n}$ of the $AR(1)$-process is then obtained by setting $\mathbf{X}_1 = \boldsymbol{\varepsilon}_1$ and completing the recursion $\mathbf{X}_j = \beta \mathbf{X}_{j-1} + \boldsymbol{\varepsilon}_j, j = 2, \dots, n$. Additionally, we consider the case of independent observations from the equi-correlated Gaussian copula (cf. Table 1). To ease comparison, the block bootstrap is used in this case, too.

Tables 1 and 2 outline the simulation results for dimensions $d = 2, 5$, and 10, sample sizes $n = 50, 100$, and 500, and different choices of the copula parameter ρ. Their calculation is based on 1000 Monte Carlo simulations of size n and 250 bootstrap replications, respectively. For simplicity, we set the block length $l = 5$ in all simulations. The autoregressive coefficient β of the $AR(1)$-process equals 0.5. The third column of Tables 1 and 2 shows an approximation to the true value of Φ^2, which is calculated from a sample of size 100,000. Comparing the latter to $m(\tilde{\Phi}_n^2)$ (column 4), we observe a finite-sample bias which strongly depends on the dimension d and the parameter choices, and which decreases with increasing sample size. The standard deviation estimations $s(\tilde{\Phi}_n^2)$ and the empirical means of the block bootstrap estimations, $m(\hat{\sigma}^B)$, as well as the delete-l jackknife estimations, $m(\hat{\sigma}^J)$, for the standard deviation are given in columns 5, 6, and 7. There is a good agreement between their values, especially for the sample sizes $n = 100$ and $n = 500$, implying that the bootstrap and the jackknife procedure to estimate the asymptotic standard deviation of $\sqrt{n}(\tilde{\Phi}_n^2 - \Phi^2)$ perform well for the considered Gaussian copula models. Further, the standard error s of the bootstrap standard deviation estimations is quite small (column 8) and slightly smaller than the obtained jackknife estimates (column 9) in lower dimensions. For large sample size n, however, the jackknife is of higher computational complexity.

5. Empirical study

For illustration purposes, we apply the theoretical results of Section 4 to empirical financial data. We analyze the association between the daily (log-)returns of major S&P global sector indices before and after the bankruptcy of Lehman

Table 2

Gaussian copula (*dependent* $AR(1)$ *observations*). Simulation results for estimating the asymptotic standard deviation of $\sqrt{n}(\tilde{\Phi}_n^2 - \Phi^2)$ by means of the nonparametric block bootstrap with block length l and the delete-l jackknife (for $l = 5$): The table shows the empirical means $m(\cdot)$ and the empirical standard deviations $s(\cdot)$ of the respective estimates, which are calculated based on 1000 Monte Carlo simulations of sample size n of a d-dimensional equi-correlated Gaussian copula with parameter ρ, $AR(1)$-processes with standard normal residuals in each margin (with coefficient $\beta = 0.5$ for the first lag) and 250 bootstrap samples. The bootstrap estimates are labeled by the superscript B, jackknife estimates by J.

ρ	n	Φ^2	$m(\tilde{\Phi}_n^2)$	$s(\tilde{\Phi}_n^2)$	$m(\hat{\sigma}^B)$	$m(\hat{\sigma}^J)$	$s(\hat{\sigma}^B)$	$s(\hat{\sigma}^J)$
Dimension $d = 2$								
0.2	50	0.032	0.086	0.059	0.065	0.060	0.022	0.033
	100	0.032	0.059	0.043	0.042	0.040	0.016	0.021
	500	0.032	0.036	0.018	0.017	0.017	0.005	0.005
0.5	50	0.200	0.242	0.114	0.101	0.110	0.022	0.031
	100	0.200	0.222	0.086	0.076	0.081	0.013	0.015
	500	0.200	0.203	0.039	0.037	0.037	0.003	0.003
−0.1	50	0.008	0.067	0.045	0.058	0.050	0.019	0.029
	100	0.008	0.037	0.025	0.032	0.028	0.012	0.016
	500	0.008	0.014	0.010	0.010	0.009	0.004	0.005
Dimension $d = 5$								
0.2	50	0.028	0.047	0.031	0.030	0.026	0.014	0.016
	100	0.028	0.039	0.020	0.020	0.019	0.008	0.008
	500	0.028	0.031	0.009	0.008	0.008	0.002	0.002
0.5	50	0.192	0.212	0.082	0.072	0.074	0.019	0.023
	100	0.192	0.205	0.057	0.053	0.054	0.010	0.011
	500	0.192	0.196	0.026	0.025	0.025	0.003	0.002
−0.1	50	0.007	0.015	0.005	0.006	0.005	0.002	0.002
	100	0.007	0.011	0.004	0.004	0.004	0.001	0.001
	500	0.007	0.008	0.002	0.002	0.002	0.000	0.000
Dimension $d = 10$								
0.2	50	0.007	0.015	0.011	0.013	0.009	0.009	0.008
	100	0.007	0.012	0.007	0.008	0.006	0.005	0.004
	500	0.007	0.008	0.002	0.003	0.002	0.001	0.001
0.5	50	0.099	0.111	0.055	0.053	0.049	0.020	0.023
	100	0.099	0.109	0.042	0.039	0.037	0.014	0.014
	500	0.099	0.100	0.018	0.017	0.017	0.003	0.003
−0.1	50	0.001	0.001	0.000	0.000	0.000	0.000	0.000
	100	0.001	0.001	0.000	0.000	0.000	0.000	0.000
	500	0.001	0.001	0.000	0.000	0.000	0.000	0.000

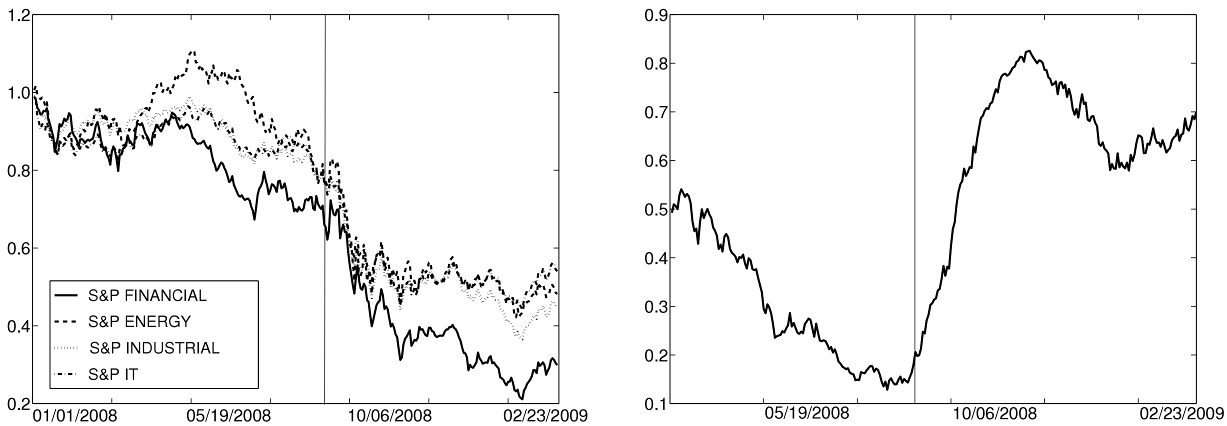

Fig. 2. Evolution of the S&P global sector indices Financials, Energy, Industrials, and IT with respect to their value on January 1, 2008 (left panel). Multivariate Hoeffding's Phi-Square $\tilde{\Phi}^2$ of the four indices' returns series, where the estimation is based on a moving window approach with window size 50 (right panel). The vertical line indicates the 15th of September 2008, the day of the bankruptcy of Lehman Brothers Inc.

Brothers Inc. using multivariate Hoeffding's Phi-Square. The major S&P global sector indices considered are Financials, Energy, Industrials, and IT during the period from 1st January 2008 to 8th April 2009 (331 observations).

Fig. 2 (left panel) shows the evolution of the four indices over the considered time horizon. To ease comparison, all series are plotted with respect to their value on January 1, 2008. The vertical line indicates the 15th of September 2008, the day of the bankruptcy of Lehman Brothers Inc. All series decrease in mid 2008 in the course of deteriorating financial markets; they decline especially sharply after the bankruptcy of Lehman Brothers Inc. Table 3 reports the first four moments of the

Table 3
First four moments (in %) and results of the Jarque–Bera (JB) test, calculated for the returns of the S&P indices, as well as results of the Ljung–Box (LB) Q-statistics, calculated up to lag twenty from the squared returns.

	Financials	Energy	Industrials	IT
Mean	−0.3140	−0.1732	−0.2213	−0.1637
Standard deviation	3.1696	3.0553	2.1199	2.1037
Skewness	0.1490	−0.2462	−0.1139	0.1148
Kurtosis	5.2523	6.6540	4.8467	5.2024
JB statistics	71.1875	187.4846	47.7495	67.6229
JB p-values	0.0000	0.0000	0.0001	0.0000
LB Q-statistics	141.8456	447.6130	386.8856	266.6008
LB p-values	0.0000	0.0000	0.0000	0.0000

daily returns of the four indices as well as the related results of the Jarque–Bera (JB) test, calculated over the entire time horizon. In addition, the last two rows of the table display the results of the Ljung–Box (LB) Q-statistics, computed from the squared returns of the indices up to lag twenty. All return series show skewness and excess kurtosis. The Jarque–Bera (JB) test rejects the null hypothesis of normality at all standard levels of significance. Further, all squared returns show significant serial correlation as indicated by the Ljung–Box (LB) test, which rejects the null hypothesis of no serial correlation. We fit an ARMA–GARCH model to each return series (which cannot be rejected by common goodness-of-fit tests). The estimated parameters are consistent with the assumption of strong mixing series (cf. [6]).

Fig. 2 (right panel) displays the evolution of multivariate Hoeffding's Phi-Square of the indices' returns, estimated on the basis of a moving window approach with window size 50. Again, the vertical line indicates the day of Lehman's bankruptcy. We observe a sharp increase of Hoeffding's Phi-Square after this date and, hence, an increase of the association between the indices' returns. In order to verify whether this increase is statistically significant, we compare Hoeffding's Phi-Square over the two distinct time periods before and after this date using the results on the statistical properties of the difference of two Hoeffding's Phi-Squares discussed in Section 4.1. Note that the test by Genest and Rémillard [21] (cf. Section 4.1) rejects the null hypothesis of stochastic independence (i.e., $C = \Pi$) with a p-value of 0.0005 such that Theorem 6 can be applied. We calculate the estimated values (based on 250 bootstrap samples with block length $l = 5$) of Hoeffding's Phi-Square and the asymptotic variances and covariance as stated in Theorem 6 for both time periods which comprise $n = 100$ observations each:

$$\begin{matrix} \tilde{\Phi}^2_{\text{before}} = 0.1982, & (\tilde{\sigma}^B_{\text{before}})^2 = 0.1663, & \\ \tilde{\Phi}^2_{\text{after}} = 0.7437, & (\tilde{\sigma}^B_{\text{after}})^2 = 0.2064, & \end{matrix} \quad \tilde{\sigma}^B_{\text{before,after}} = -0.0287.$$

The choice of the block length $l = 5$ is motivated by the results of the simulation study in Section 4.2. The corresponding test statistic has the form

$$\left| \sqrt{n} \frac{(\tilde{\Phi}^2_{\text{after}} - \tilde{\Phi}^2_{\text{before}})}{\sqrt{(\tilde{\sigma}^B_{\text{before}})^2 + (\tilde{\sigma}^B_{\text{after}})^2 - 2\tilde{\sigma}^B_{\text{before,after}}}} \right| = 8.3190,$$

with corresponding p-value 0.0000. Hence, we conclude that there has been a significant increase in association between the returns of the four indices after the bankruptcy of Lehman Brothers Inc.

6. Conclusion

A multivariate version is proposed for Hoeffding's bivariate measure of association Phi-Square. A nonparametric estimator for the proposed measure is obtained on the basis of the empirical copula process. Its asymptotic distribution is established for the cases of independent observations as well as of dependent observations from a strictly stationary strong mixing sequence. The asymptotic distribution can be approximated by nonparametric bootstrap methods. This allows for the calculation of asymptotic confidence intervals for Hoeffding's Phi-Square and the construction of hypothesis tests. The results are derived under the general assumption of continuous marginal distributions.

Acknowledgments

We thank the associate editor and three anonymous referees for their comments which led to an improvement of the paper. Moreover, we are grateful to the Regional Computing Center at the University of Cologne for providing the computational resources required. Funding in support of the second author's work was provided by the Deutsche Forschungsgemeinschaft (DFG).

Appendix A. Derivation of the functions $h(\boldsymbol{d})^{-1}$ and $g(\boldsymbol{d})^{-1}$

We calculate the explicit form of the functions $h(d)^{-1}$ and $g(d)^{-1}$, as stated in Eqs. (3) and (4). Regarding the function $h(d)^{-1}$, we have

$$h(d)^{-1} = \int_{[0,1]^d} \{M(\mathbf{u}) - \Pi(\mathbf{u})\}^2 \, d\mathbf{u}$$
$$= \int_{[0,1]^d} \{M(\mathbf{u})\}^2 \, d\mathbf{u} - 2 \int_{[0,1]^d} M(\mathbf{u})\Pi(\mathbf{u}) d\mathbf{u} + \int_{[0,1]^d} \{\Pi(\mathbf{u})\}^2 \, d\mathbf{u}.$$

The first summand on the left-hand side of the above equation can be written as

$$\int_{[0,1]^d} \{M(\mathbf{u})\}^2 \, d\mathbf{u} = E\left([\min\{U_1, \dots, U_d\}]^2\right) = E\left(X^2\right)$$

where $U_1, \dots, U_d$ are i.i.d. from $U \sim R[0, 1]$ and $X = \min\{U_1, \dots, U_d\}$. Therefore,

$$E\left(X^2\right) = d \int_0^1 x^2 (1-x)^{d-1} dx = \frac{2}{(d+1)(d+2)}. \tag{A.1}$$

For the second summand, we obtain

$$\int_{[0,1]^d} M(\mathbf{u})\Pi(\mathbf{u}) d\mathbf{u} = \frac{1}{2^d} \int_{[0,1]^d} \min\{u_1, \dots, u_d\} \prod_{i=1}^d 2u_i d\mathbf{u} = \frac{1}{2^d} E\left(\min\{V_1, \dots, V_d\}\right) = \frac{1}{2^d} E(Y)$$

where $V_1, V_2, \dots, V_d$ are i.i.d. from V, which has density $f_V(v) = 2v$ for $0 \leqslant v \leqslant 1$ and $Y = \min\{V_1, V_2, \dots, V_d\}$. Thus,

$$\frac{1}{2^d} E(Y) = \frac{1}{2^d} \int_0^1 x d(1-x^2)^{d-1} 2x dx = \frac{1}{2^d} \int_0^1 (1-x^2)^d dx = \frac{1}{2^d} \frac{1}{2} \frac{\Gamma(d+1)\sqrt{\pi}}{\Gamma\left(d+1+\frac{1}{2}\right)}$$
$$= \frac{1}{2^d} \frac{d!}{\prod_{i=0}^d \left(i + \frac{1}{2}\right)}. \tag{A.2}$$

Combining Eqs. (A.1) and (A.2) and using that $\int_{[0,1]^d} \{\Pi(\mathbf{u})\}^2 d\mathbf{u} = \left(\frac{1}{3}\right)^d$ yields the asserted form of $h(d)^{-1}$.

Regarding the function $g(d)^{-1}$ as defined in Eq. (4), we have

$$g(d)^{-1} = \int_{[0,1]^d} \{W(\mathbf{u}) - \Pi(\mathbf{u})\}^2 \, d\mathbf{u}$$
$$= \int_{[0,1]^d} \{W(\mathbf{u})\}^2 \, d\mathbf{u} - 2 \int_{[0,1]^d} W(\mathbf{u})\Pi(\mathbf{u}) d\mathbf{u} + \int_{[0,1]^d} \{\Pi(\mathbf{u})\}^2 \, d\mathbf{u}.$$

For the first summand, it follows that

$$\int_{[0,1]^d} \{W(\mathbf{u})\}^2 \, d\mathbf{u} = \int_0^1 \cdots \int_{d-2-\sum_{i=1}^{d-2} u_i}^1 \int_{d-1-\sum_{i=1}^{d-1} u_i}^1 \left(\sum_{i=1}^d u_i - d + 1\right)^2 du_d \cdots du_2 du_1 = \frac{2}{(d+2)!}. \tag{A.3}$$

Partial integration of the second term further yields

$$\int_{[0,1]^d} W(\mathbf{u})\Pi(\mathbf{u}) d\mathbf{u} = \int_0^1 \cdots \int_{d-2-\sum_{i=1}^{d-2} u_i}^1 u_{d-1} \int_{d-1-\sum_{i=1}^{d-1} u_i}^1 u_d \left(\sum_{i=1}^d u_i - d - 1\right) du_d \cdots du_2 du_1$$
$$= \sum_{i=1}^d \binom{d}{i} (-1)^i \frac{1}{(d+1+i)!}. \tag{A.4}$$

Again, by combining Eqs. (A.3) and (A.4) and using that $\int_{[0,1]^d} \{\Pi(\mathbf{u})\}^2 d\mathbf{u} = \left(\frac{1}{3}\right)^d$, we obtain the asserted form of $g(d)^{-1}$.

Appendix B. Derivation of the estimator $\widehat{\Phi}_n^2$

We outline the derivation of the estimator $\widehat{\Phi}_n^2$ as given in (10).

$$\{h(d)\}^{-1} \widehat{\Phi}_n^2 = \int_{[0,1]^d} \left\{ \widehat{C}_n(\mathbf{u}) - \prod_{i=1}^d u_i \right\}^2 d\mathbf{u} = \int_{[0,1]^d} \left\{ \frac{1}{n} \sum_{j=1}^n \left(\prod_{i=1}^d \mathbf{1}_{\{\widehat{U}_{ij} \leq u_i\}} - \prod_{i=1}^d u_i \right) \right\}^2 d\mathbf{u}$$
$$= \left(\frac{1}{n}\right)^2 \sum_{j=1}^n \sum_{k=1}^n \int_{[0,1]^d} \left(\prod_{i=1}^d \mathbf{1}_{\{\widehat{U}_{ij} \leq u_i\}} - \prod_{i=1}^d u_i \right) \left(\prod_{i=1}^d \mathbf{1}_{\{\widehat{U}_{ik} \leq u_i\}} - \prod_{i=1}^d u_i \right) d\mathbf{u}$$

$$= \left(\frac{1}{n}\right)^2 \sum_{j=1}^{n}\sum_{k=1}^{n}\int_{[0,1]^d}\left(\prod_{i=1}^{d}\mathbf{1}_{\{\max\{\widehat{U}_{ij},\widehat{U}_{ik}\}\le u_i\}} + \prod_{i=1}^{d}u_i^2 - \prod_{i=1}^{d}u_i\mathbf{1}_{\{\widehat{U}_{ij}\le u_i\}} - \prod_{i=1}^{d}u_i\mathbf{1}_{\{\widehat{U}_{ik}\le u_i\}}\right)\mathrm{d}\mathbf{u}$$
$$= \left(\frac{1}{n}\right)^2 \sum_{j=1}^{n}\sum_{k=1}^{n}\prod_{i=1}^{d}(1-\max\{\widehat{U}_{ij},\widehat{U}_{ik}\}) - \frac{2}{n}\left(\frac{1}{2}\right)^d\sum_{j=1}^{n}\prod_{i=1}^{d}(1-\widehat{U}_{ij}^2) + \left(\frac{1}{3}\right)^d.$$

Appendix C. Proofs

Proof of Proposition 1. Consider the d-dimensional random vector $\mathbf{X}$ with copula C.

(i) For all $d \geq 2$, the copula C is invariant under strictly increasing transformations of one or several components of $\mathbf{X}$ (see, e.g., [13, Theorem 2.6]). As direct functional of the copula, $\Phi^2_{\mathbf{X}}$ thus inherits this property.

(ii) Let α_k be a strictly decreasing transformation of the kth component X_k of $\mathbf{X}$, $k \in \{1,\dots,d\}$, defined on the range of X_k. For dimension $d = 2$, we have

$$\begin{aligned}
h(2)^{-1}\Phi^2_{(\alpha_1(X_1),\alpha_2(X_2))} &= \int_{[0,1]^2}\left\{C_{(\alpha_1(X_1),\alpha_2(X_2))}(u_1,u_2) - \Pi(u_1,u_2)\right\}^2 \mathrm{d}u_1\mathrm{d}u_2\\
&= \int_{[0,1]^2}\left\{C_{(X_1,X_2)}(1,u_2) - C_{(X_1,\alpha_2(X_2))}(1-u_1,u_2) - u_1u_2\right\}^2 \mathrm{d}u_1\mathrm{d}u_2\\
&= \int_{[0,1]^2}\left\{u_1+u_2-1-C_{(X_1,X_2)}(1-u_1,1-u_2) - u_1u_2\right\}^2 \mathrm{d}u_1\mathrm{d}u_2\\
&= \int_{[0,1]^2}\left\{1-x+1-y-1+C_{(X_1,X_2)}(x,y) - (1-x)(1-y)\right\}^2 \mathrm{d}x\mathrm{d}y\\
&= \int_{[0,1]^2}\left\{C_{(X_1,X_2)}(x,y) - xy\right\}^2 \mathrm{d}x\mathrm{d}y = h(2)^{-1}\Phi^2_{\mathbf{X}},
\end{aligned}$$

see e.g. [13], Theorem 2.7, for the second equation. For dimension $d \geq 3$, let $\boldsymbol{\alpha}_k(\mathbf{X}) = (X_1,\dots,X_{k-1},\alpha_k(X_k),X_{k+1},\dots,X_d)$ denote the random vector where the kth component of $\mathbf{X}$ is transformed by the function α_k. Without loss of generality set $k = 1$. If a random vector $\mathbf{X}$ is jointly symmetric about $\mathbf{a} \in \mathbb{R}^d$ then its copula is invariant with respect to a strictly decreasing transformation of any component. Hence, Φ^2 is invariant as well. For non-jointly symmetric random vectors, consider

$$\begin{aligned}
h(d)^{-1}\Phi^2_{\boldsymbol{\alpha}_1(\mathbf{X})} &= \int_{[0,1]^d}\left\{C_{\boldsymbol{\alpha}_1(\mathbf{X})}(u_1,u_2,\dots,u_d) - \Pi(u_1,u_2,\dots,u_d)\right\}^2 \mathrm{d}\mathbf{u}\\
&= \int_{[0,1]^d}\{C_{\mathbf{X}}(1,u_2,\dots,u_d) - C_{\mathbf{X}}(1-u_1,u_2,\dots,u_d) - u_1u_2\cdot\dots\cdot u_d\}^2 \mathrm{d}\mathbf{u}\\
&= \int_{[0,1]^d}[\{C_{\mathbf{X}}(1,u_2,\dots,u_d) - u_2\cdot\dots\cdot u_d\} - \{C_{\mathbf{X}}(x,u_2,\dots,u_d) - xu_2\cdot\dots\cdot u_d\}]^2 \mathrm{d}x\mathrm{d}\mathbf{u}'\\
&\quad \text{where } \mathbf{u}' = (u_2,\dots,u_d)\\
&= \int_{[0,1]^{d-1}}\{C_{\mathbf{X}}(1,u_2,\dots,u_d) - u_2\cdot\dots\cdot u_d\}^2 \mathrm{d}\mathbf{u}' + \int_{[0,1]^d}\{C_{\mathbf{X}}(x,u_2,\dots,u_d) - xu_2\cdot\dots\cdot u_d\}^2 \mathrm{d}x\mathrm{d}\mathbf{u}'\\
&\quad - \int_{[0,1]^{d-1}}\{C_{\mathbf{X}}(1,u_2,\dots,u_d) - u_2\cdot\dots\cdot u_d\}\left\{2\int_{[0,1]}C_{\mathbf{X}}(x,u_2,\dots,u_d)\mathrm{d}x - u_2\cdot\dots\cdot u_d\right\}\mathrm{d}\mathbf{u}'. \quad \text{(C.1)}
\end{aligned}$$

According to Eq. (C.1), it holds that $\Phi^2_{\boldsymbol{\alpha}_1(\mathbf{X})} = \Phi^2_{\mathbf{X}}$ if either

- $C_{\mathbf{X}}(1,u_2,\dots,u_d) = u_2\cdot\dots\cdot u_d$, meaning that $X_2,\dots,X_d$ are independent or
- $C_{\mathbf{X}}(1,u_2,\dots,u_d) = 2\int_{[0,1]}C_{\mathbf{X}}(x,u_2,\dots,u_d)\mathrm{d}x$. This condition is fulfilled if X_1 is independent of $(X_2,\dots,X_d)$ since $C_{\mathbf{X}}(x,u_2,\dots,u_d) = xC_{\mathbf{X}}(1,u_2,\dots,u_d)$ in this case. □

Proof of Theorem 3. Note that $\Phi^2 = \varphi(C)$ represents a Hadamard-differentiable map φ on $\ell^\infty([0,1]^d)$ of the copula C (see [14,48] for the relevant definitions and background reading). Its derivative φ'_C at $C \in \ell^\infty([0,1]^d)$, a continuous linear map on $\ell^\infty([0,1]^d)$, is given by

$$\varphi'_C(D) = 2h(d)\int_{[0,1]^d}\{C(\mathbf{u}) - \Pi(\mathbf{u})\}D(\mathbf{u})\mathrm{d}\mathbf{u},$$

which can be shown as follows: For all converging sequences $t_n \to 0$ and $D_n \to D$ such that $C + t_nD_n \in \ell^\infty([0,1]^d)$ for every n, we have

$$\begin{aligned}\frac{\varphi(C+t_nD_n)-\varphi(C)}{t_n} &= \frac{h(d)\int_{[0,1]^d}\{C(\mathbf{u})-\Pi(\mathbf{u})+t_nD_n(\mathbf{u})\}^2\mathrm{d}\mathbf{u}}{t_n}-\frac{h(d)\int_{[0,1]^d}\{C(\mathbf{u})-\Pi(\mathbf{u})\}^2\mathrm{d}\mathbf{u}}{t_n}\\ &= \frac{2h(d)t_n\int_{[0,1]^d}\{C(\mathbf{u})-\Pi(\mathbf{u})\}D_n(\mathbf{u})\mathrm{d}\mathbf{u}+t_n^2\int_{[0,1]^d}D_n^2(\mathbf{u})\mathrm{d}\mathbf{u}}{t_n}\\ &\to 2h(d)\int_{[0,1]^d}\{C(\mathbf{u})-\Pi(\mathbf{u})\}D(\mathbf{u})\mathrm{d}\mathbf{u},\end{aligned} \tag{C.2}$$

for $n\to\infty$, since the second integral in Eq. (C.2) is bounded for all D_n. An application of the functional Delta-method given in Theorem 3.9.4 in [48] together with Proposition 2 then implies

$$\sqrt{n}(\widehat{\Phi}^2-\Phi^2)=\sqrt{n}\{\varphi(\widehat{C}_n)-\varphi(C)\}\xrightarrow{\mathrm{d}}\varphi_C'(\mathbb{G}_C), \tag{C.3}$$

where $\varphi_C'(\mathbb{G}_C)=2h(d)\int_{[0,1]^d}\{C(\mathbf{u})-\Pi(\mathbf{u})\}\mathbb{G}_C(\mathbf{u})\mathrm{d}\mathbf{u}$. Using the fact that $\mathbb{G}_C(\mathbf{u})$ is a tight Gaussian process, Lemma 3.9.8 in [48], p. 377, implies that $Z_{\Phi^2}=\varphi_C'(\mathbb{G}_C)$ is normally distributed with mean zero and variance $\sigma_{\Phi^2}^2$ as stated in the theorem. Another application of the Delta-method to (C.3) yields the weak convergence of $\sqrt{n}\{\Phi(\widehat{C}_n)-\Phi(C)\}$ to the random variable $Z_\Phi\sim N(0,\sigma_\Phi^2)$. □

Proof of Theorem 4. Let $\widehat{C}_n$ denote the empirical copula based on the sample $\mathbf{X_1},\ldots,\mathbf{X_n}$. Given the weak convergence of the empirical copula $\sqrt{n}(\widehat{C}_n-C)$ process to the Gaussian process $\mathbb{G}^*$, the asymptotic behavior of $\sqrt{n}(\widehat{\Phi}_n^2-\Phi^2)$ as stated in the theorem follows by mimicking the proof of Theorem 3. Weak convergence of $\sqrt{n}(\widehat{C}_n-C)$ can be established analogously as in [14] (see also [7]) and we outline the single steps in the following. First, by considering the transformed random variables $U_{ij}=F_i(X_{ij}), i=1,\ldots,d, j=1,\ldots,n$, it is possible to confine the analysis to the case where the marginal distributions F_i of $F, i=1,\ldots,d$, are uniform distributions on $[0,1]$ and thus, F has compact support $[0,1]^d$. Second, the functional Delta-method is applied which is based on the following representation of the copula C as a map of its distribution function F (cf. [48]):

$$C(\mathbf{u})=\phi(F)(\mathbf{u})=F(F_1^{-1}(u_1),\ldots,F_d^{-1}(u_d)),\quad \mathbf{u}\in[0,1]^d$$

with map $\phi:D([0,1]^d)\to\ell^\infty([0,1]^d)$. Here, the space $D([0,1]^d)$ comprises all real-valued cadlag functions and $C([0,1]^d)$ the space of all continuous real-valued functions defined on $[0,1]^d$, both equipped with the uniform metric m. If all partial derivatives of the copula exist and are continuous, the map ϕ is Hadamard-differentiable at F as a map from $D([0,1]^d)$ (tangentially to $C([0,1]^d)$, cf. Lemma 2 in [14]) with derivative ϕ_F'. Further, Rio [36] shows that under the above assumptions on the mixing coefficient $\alpha(r)$, it holds that

$$\sqrt{n}\{\widehat{F}_n(\mathbf{u})-F(\mathbf{u})\}\xrightarrow{w}\mathbb{B}^*(\mathbf{u})$$

in $\ell^\infty([0,1]^d)$ with Gaussian process $\mathbb{B}^*$ as defined in the theorem. Hence, an application of the functional Delta-method yields the weak convergence of $\sqrt{n}\{\phi(\widehat{F}_n)-\phi(F)\}$ to the process $\phi_F'(\mathbb{B}^*)=\mathbb{G}^*$. Since

$$\sup_{\mathbf{u}\in[0,1]^d}|\phi(\widehat{F}_n)(\mathbf{u})-\widehat{C}_n(\mathbf{u})|=O\left(\frac{1}{n}\right)$$

cf. [14], the assertion follows by an application of Slutsky's theorem. □

Proof of Theorem 6. Let $\widehat{C}_{\mathbf{Z},n}$ denote the empirical copula based on the sample $\mathbf{Z_1},\ldots,\mathbf{Z_n}$. Under the assumption of the theorem, weak convergence of the empirical copula process $\sqrt{n}\{\widehat{C}_{\mathbf{Z},n}-C_{\mathbf{Z}}\}$ to the Gaussian process $\mathbb{G}_{\mathbf{Z}}^*$ in $\ell^\infty([0,1]^{2d})$ follows (cf. proof of Theorem 4). Since

$$\begin{pmatrix}\Phi_{\mathbf{X}}^2\\ \Phi_{\mathbf{Y}}^2\end{pmatrix}=\begin{pmatrix}\Phi^2\{C_{\mathbf{Z}}(\mathbf{u},1,\ldots,1)\}\\ \Phi^2\{C_{\mathbf{Z}}(1,\ldots,1,\mathbf{v})\}\end{pmatrix}=g(C_{\mathbf{Z}}),$$

the asymptotic behavior of $(\widehat{\Phi}_{\mathbf{X}}^2,\widehat{\Phi}_{\mathbf{Y}}^2)^\top$ can be established analogously as in the proof of Theorem 3 using the Hadamard differentiability of the map g at $C_{\mathbf{Z}}$ whose derivative is denoted by $g_{C_{\mathbf{Z}}}'$. Hence, $\sqrt{n}\{(\widehat{\Phi}_{\mathbf{X}}^2,\widehat{\Phi}_{\mathbf{Y}}^2)^\top-(\Phi_{\mathbf{X}}^2,\Phi_{\mathbf{Y}}^2)^\top\}$ converges in distribution to the multivariate normally distributed random vector $g_{C_{\mathbf{Z}}}'(\mathbb{G}_{\mathbf{Z}}^*)$ given by

$$g_{C_{\mathbf{Z}}}'(\mathbb{G}_{\mathbf{Z}}^*)=\begin{pmatrix}\int_{[0,1]^d}\{C_{\mathbf{X}}(\mathbf{u})-\Pi(\mathbf{u})\}\mathbb{G}_{\mathbf{Z}}^*(\mathbf{u},1\ldots,1)\mathrm{d}\mathbf{u}\\ \int_{[0,1]^d}\{C_{\mathbf{Y}}(\mathbf{v})-\Pi(\mathbf{v})\}\mathbb{G}_{\mathbf{Z}}^*(1,\ldots,1,\mathbf{v})\mathrm{d}\mathbf{v}\end{pmatrix}.$$

With $\mathbb{G}_{\mathbf{X}}^*(\mathbf{u})=\mathbb{G}_{\mathbf{Z}}^*(\mathbf{u},1,\ldots,1)$ and $\mathbb{G}_{\mathbf{Y}}^*(\mathbf{v})=\mathbb{G}_{\mathbf{Z}}^*(1,\ldots,1,\mathbf{v})$, apply the continuous mapping theorem to conclude the proof. □

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