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A generalized flow splitting model for day-to-day traffic assignment

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Abstract

The splitting rate model proposed by Smith and Mounce (2011) establishes a traffic evolution process on a link-node network representation, which overcomes the difficulties in applying traditional path-based models and provides the ease of implementing controls at nodes. While their model offers a new method for modeling traffic evolution, it contains an ad-hoc step of flow adjustment to preserve the flow conservation. This flow adjustment step leads to difficulties in analyzing the system properties. This paper proposes a generalized flow splitting model for day-to-day traffic assignment based on the concept of splitting flow at nodes. The proposed model preserves the flow conservation endogenously by introducing the inflow variable into the formulation. The generalized formulation provides the ease to construct a variety of day-to-day traffic assignment models, and serves as a framework for analyzing the models' properties, such as the invariance property and the preservation of the Lipschitz continuity and strong monotonicity. Specifically, a proportional-adjustment model and a projection-type model are developed based on the proposed generalized formulation. A numerical example demonstrates the ease of implementing the proposed generalized model, as well as its convergence to user equilibrium.

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Keywords: day-to-day traffic assignment; flow splitting; flow conservation; user equilibrium.

1. Introduction

With the advancement of intelligent transportation systems, travelers have the capability of accessing historical and real-time traffic information, thereby adjusting their path choices in accordance with their day-to-day experience and information provision. Understanding and modeling travelers' day-to-day choice adjustment plays an important

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role in developing effective advance traffic management systems. Many day-to-day traffic assignment models have been proposed and used in day-to-day traffic control strategies such as dynamic congestion pricing (Friesz et al. 2004; Yang and Szeto 2006; Yang et al. 2007; Guo 2013; Guo et al. 2013), signal timing (Hu and Mahmassani 1997; Smith and Mounce 2011), and transit operator strategies (Cantarella et al. 2013).

In the literature, most deterministic day-to-day traffic assignment models are path-based models, i.e., they all explicitly use path flow variables and provide explicit path flow evolution trajectories. Yang and Zhang (2009) classify these path-based models into four categories: the proportional-switch adjustment process (Smith, 1984; Smith and Wisten 1995; Peeta and Yang 2003), network tâtonnement adjustment (Friesz et al. 1994), projected dynamical system (Zhang and Nagurney 1996; Nagurney and Zhang 1997) and evolutionary traffic dynamics (Sandholm 2001; Yang 2005). Since these models involve a differential equation for each possible path, the problem becomes intractable when the network size increases.

He et al. (2010) discuss the shortcomings of path-based models and propose a model built upon link flows to eliminate the realism issues inherent in path-based models. Since its state variables contain link flows only, the link-based model (LBM) has a tractable problem size that is suitable for analyzing the traffic evolution for a large-scale network, e.g., the traffic evolution after the I-35W Bridge collapse in Minneapolis, Minnesota (He and Liu 2012). Han and Du (2012) extend the LBM to model traffic evolution for networks with non-separable link cost functions. Guo et al. (2013) develop a generalized link-based model to cover a variety of discrete-time day-to-day traffic evolution processes.

Another way to overcome the shortcomings in path-based models is to construct a traffic evolution model on a link-node network representation. Smith and Mounce (2011) propose a splitting rate model (SRM), which adjusts link flow splitting rates at each node. Formulating a day-to-day traffic assignment model as a SRM has two main advantages. First, as the SRM is constructed on a node-link network representation, its problem size is tractable, and the path-non-uniqueness and path-overlapping problems identified in He et al. (2010) can be overcome. Second, various control strategies can be applied on nodes and easily combined into the model, since the flow adjustment is directly formulated at nodes.

However, maintaining the flow conservation is not an easy task in the absence of path flow variables. The LBM in He et al. (2010) essentially embeds a traffic assignment sub-problem to ensure that the link flows satisfy the conservation law. For a large network, an efficient path generation algorithm is needed to solve the traffic assignment sub-problem in the LBM. The SRM in Smith and Mounce (2011) requires an additional computational adjustment of link flows to maintain flow conservation at nodes. This ad-hoc process induces difficulties in analyzing the system properties, especially when the link cost function is non-separable.

This paper proposes a general formulation for link-based deterministic day-to-day traffic assignment models. The proposed generalized flow splitting model not only enables the computation of the link flow trajectories by circumventing the realism issues of path-based models, but also maintains flow conservation endogenously. Since it is developed on the link-node network representation, path variables are completely absent in the formulation, while link flows are updated through a demand distribution matrix to guarantee the flow conservation.

The proposed general formulation can provide modeling flexibility and serve as a unified umbrella for establishing a variety of day-to-day traffic assignment models based on different assumptions on travelers' path swapping decision at nodes. The representation of flow splitting rates can vary to accommodate different routing policies and control strategies in the real world. The realization of the splitting rate may follow either the proportional adjustment used in the SRM, or a projection operator similar to the LBM. Any specified model can directly inherit the analytical properties of the general framework. Thus, the cumbersome analysis due to the exogenous flow adjustment process in the SRM could be avoided.

This remainder of the paper is organized as follows. The next section reviews the SRM proposed in Smith and Mounce (2011) and discusses its inherent problems. Section 3 presents the mathematical formulation of the proposed flow splitting modeling framework, which is established on an acyclic sub-network. The relationship between the modeling framework and existing link-based models is also discussed. Section 4 provides mathematical properties of the proposed flow splitting modeling framework. Section 5 demonstrates the applicability of the proposed modeling framework using a numerical example constructed on a small network. The final section provides some concluding comments.

2. Preliminaries

This section introduces the notation associated with the modeling of the day-to-day traffic evolution process, and a brief review of the SRM.

2.1. Notation

Let the traffic network $\mathcal{G}(\mathcal{N},\mathcal{L})$ be represented by a directed graph $\mathcal{G}$ with node set $\mathcal{N}$ and link set $\mathcal{L}$. Assume that this graph is strongly connected, i.e., for any pair of nodes (i,j) there exists at least a directed path connecting them. Let $\mathcal{S}\subseteq\mathcal{N}$ be the set of trip destinations in the network. For each node $i\in\mathcal{N}$, the set of its outgoing (downstream) links is denoted by $\mathcal{L}_i^+\subseteq\mathcal{L}$ and its incoming (upstream) links is denoted by $\mathcal{L}_i^-\subseteq\mathcal{L}$.

Let x_a^s denote the flow on link $a\in\mathcal{L}$ traveling toward destination $s\in\mathcal{S}$. Link flow x_a is the aggregation of the flows towards different destinations, i.e., $x_a=\sum_a x_a^s$, for all $a\in\mathcal{L}$. Denote link flow vector $\mathbf{x}=[x_a]$. The link cost function of link $a\in\mathcal{L}$ is denoted by $\tau_a(\mathbf{x})$ which is a continuous function of link flow vector $\mathbf{x}$. Denote d_{rs} as the travel demand from origin $r\in\mathcal{N}$ to destination $s\in\mathcal{S}$, $r\neq s$. Additional notation will be introduced when necessary.

2.2. Splitting rate traffic rerouting model

The SRM is a deterministic day-to-day traffic evolution model developed by Smith and Mounce (2011) to describe the traffic rerouting dynamics at nodes. The SRM is built upon the path choice behavior assumption similar to the path-based proportional flow swapping model developed by Smith (1984). The SRM can be presented in a concise form as:

$$\mathbf{x}(t+1)-\mathbf{x}(t)=\gamma\mathbf{\Delta}(\mathbf{x}(t))\mathbf{x}(t) \tag{1}$$

where $\mathbf{\Delta}(\mathbf{x})$ is a block diagonal matrix with $|\mathcal{L}|*|\mathcal{S}|$ rows and $|\mathcal{L}|*|\mathcal{S}|$ columns defined by:

$$\mathbf{\Delta}(\mathbf{x})=diag\{\mathbf{\Delta}^s(\mathbf{x})\}. \tag{2}$$

For each destination $s\in\mathcal{S}$, $\mathbf{\Delta}^s(\mathbf{x})$ is a $|\mathcal{L}|$ by $|\mathcal{L}|$ square matrix. Each element $\delta_{ab}^s(\mathbf{x})$ in matrix $\mathbf{\Delta}^s(\mathbf{x})$ is defined by:

$$\delta_{ab}^s(\mathbf{x})=\begin{cases}-\sum_{e\in\mathcal{L}_i^+,\,e\neq a}\left[c_a^s(\mathbf{x})-c_e^s(\mathbf{x})\right]_+ & \text{if } a=b,\\ \left[c_b^s(\mathbf{x})-c_a^s(\mathbf{x})\right]_+ & \text{if } b\in\mathcal{L}_i^+,\text{ and } a\neq b,\\ 0 & \text{otherwise}\end{cases} \tag{3}$$

where $x_+=\max\{0,x\}$ denotes a non-negativity projection. The SRM is formulated based on the *flow-weighted path cost* c_a^s which measures a cost from the tail node of link a to destination s.

The formulation of the flow-weighted path cost plays the critical role in the SRM and can be presented as follows. Assume we focus on the calculation of the flow-weighted path cost c_a^s of link a connecting to downstream node i in a network. The link and node locations and associated attributes are illustrated in Fig. 1.

As shown in Fig. 1, each link in the network is associated with a flow proportion p_b which is defined in Smith and Mounce (2011) as:

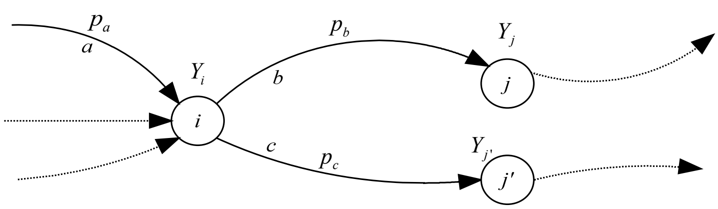

Fig. 1. Calculation of the generalized cost.

$$p_b^s = \frac{x_b^s}{\sum_{c \in \mathcal{L}_i^+} x_c^s} . \tag{4}$$

In equation (4), the numerator represents the flow on link b that is incident from node i while the denominator represents the summation of all outgoing flows from node i toward destination s. Therefore, equation (4) prescribes how traffic flow splits (in terms of proportion) at node i. Each node i in the network is associated with a node potential to represent the flow-weighted cost from node i to destination node s through links with tail node i. Particularly, let $Y_s^s = 0$ for destination node s. The flow-weighted node potential is formulated as:

$$Y_i^s = \sum_{b=(i,j) \in \mathcal{L}_i^+} \left(\tau_b(\mathbf{x}) + Y_j^s \right) \cdot p_b^s . \tag{5}$$

After the flow-weighted node potential Y_i^s is computed, the generalized cost c_a^s for link $a \in \mathcal{L}_i^-$ in equation (3) is defined by:

$$c_a^s = \tau_a(\mathbf{x}) + Y_i^s . \tag{6}$$

As $\sum_{b \in \mathcal{L}_i^+} p_b^s = 1$, equation (5) represents that the node potential Y_i^s is a normalized linear combination of the generalized costs c_b^s on all downstream links with weights p_b^s. Therefore, it has a concise form:

$$Y_i^s = \sum_{b=(i,j) \in \mathcal{L}_i^+} p_b^s c_b^s . \tag{7}$$

2.3. Problem statement

One of the main challenges of applying dynamical system (1) is to maintain the flow conservation at the nodes. According to formulation (1), for each node the total exit flow is conserved with the value of total inflow in previous time step. Considering that the upstream inflow rates may change in the current time step, flow conservation is lost at the node level.

To maintain the flow conservation, Smith and Mounce (2011) suggest an additional link flow adjustment process after applying dynamical system (1) to compute the link flow pattern. The flow adjustment process updates link flows by an iterative approach:

$$x_b^s(k+1) = \frac{x_b^s(k)}{\sum_{e \in \mathcal{L}_i^+} x_e^s(k)} \cdot \sum_{a \in \mathcal{L}_i^-} x_a^s(k) , \ \forall b \in \mathcal{L}_i^+ \tag{8}$$

where k denotes the iteration number. As shown by (8), the outgoing link flow from node i is adjusted proportional to the total inflow through upstream links. Smith and Mounce (2011) show that when the network is acyclic (i.e., loop free), the iterative process converges in a finite number of iterations and the flow conservation is maintained.

Although adding the link flow adjustment process (8) ensures that the link flows generated by day-to-day traffic evolution model (1) maintains the conservation law, such an ad-hoc process induces difficulties in analyzing the system properties, such as its asymptotic stability and attraction domain (Bie and Lo 2010). More importantly, as the value of total inflow appears as the denominator in the link flow adjustment process (8), this process is not well defined when the total inflow of node i is zero. Because of this, Smith and Mounce (2011) assume that each non-destination node has a positive demand to ensure the denominator in the right-hand-side of equation (8) is always positive. Nevertheless, this is a strong assumption for practical applications.

Another challenge of applying dynamical system (1) is to have a well-defined formulation of the node potential Y_i^s in constructing the flow swapping matrix $\mathbf{\Delta}(\mathbf{x})$. As shown by (4), the flow proportion p_b^s is not well defined if the total outflow at node i is zero. The assumption of positive demand at each non-destination node resolves this issue. As the same assumption is used for the flow adjustment process (8), the SRM (1) would have limited application for large-scale networks. Hence, there is a research need to develop a well-defined formulation for the node potential and generalized cost for the wider application of the SRM.

Based on the aforementioned challenges of applying the SRM, this paper proposes a generalized formulation for modeling day-to-day traffic assignment, which follows the link flow splitting rate structure and introduces an inflow variable at each node. The proposed general formulation not only allows the computation of the link flow evolution without the realism issues of path-based models, but also maintains flow conservation endogenously. As in the SRM, path flow variables are completely absent in the proposed formulation, while the existence of the inflow variable helps in maintaining the flow conservation law. A well-defined generalized cost formulation is developed based on the properties of the acyclic network structure. In addition, the generalized modeling framework can provide flexibility in modeling the splitting rates. Various routing policies and traffic control strategies can be accommodated easily by the general framework. A specific realization of splitting rate may follow either the proportional adjustment form of the SRM or a projection mapping form of the LBM. The detailed formulation of the proposed model is presented in the next section.

3. Generalized flow splitting model

This section first discusses the assumptions imposed on the network structure that help in formulating the generalized flow splitting model and deriving its analytical properties. Then, the detailed mathematical formulation of the proposed model is presented. To illustrate how to apply the proposed model, the flow splitting model is specified by adopting the path choice behavior assumptions from the SRM (1) and the LBM.

3.1. Network structure

The generalized flow splitting model is inspired by the origin-based algorithm (Bar-Gera 2002; Dial 2006; Nie 2010) for the static traffic assignment problem, where origin-destination (OD) demands are distributed onto sub-networks decomposed by origin or destination. A set of sub-networks needs to be constructed before the analytical formulation of the proposed model is presented.

For the given traffic network $\mathcal{G}(\mathcal{N},\mathcal{L})$, all paths connecting all origins to the same particular destination can form a sub-network for that destination. Denote $\mathcal{G}_s$ as the sub-network for destination s. It is assumed that each sub-network $\mathcal{G}_s$ satisfies the following assumptions.

1. $\mathcal{G}_s$ contains no loops, i.e., $\mathcal{G}_s$ is acyclic;
2. $\mathcal{G}_s$ is strongly connected, i.e., there exists at least a path in $\mathcal{G}_s$ connecting every non-destination node to destination s.

Note that the first assumption is a strong one that may not be satisfied by general networks. However, acyclic sub-networks can be constructed for each destination based on reasonable paths, which exclude links that take the traveler back toward the origin (Sheffi 1985). If only reasonable paths are used by travelers, the link flow pattern can be analyzed on the corresponding acyclic network.

As sub-network $\mathcal{G}_s$ is acyclic, we can construct a topological ordering of nodes and a topological ordering of links (Chapter 3 in Ahuja et al. 1993) such that for every link $(i.j)\in\mathcal{L}$, order $(i)<$ order (j). For a specific sub-network $\mathcal{G}_s$, the topological ordering can be constructed as follows. First, as sub-network $\mathcal{G}_s$ is acyclic, there exists at least one node that does not have a predecessor. Put all nodes without predecessors into an order by their node potentials defined by (5). For each ordered node, we can rank its downstream links (directly incident from the node) by their generalized travel costs to destination s. After these nodes and links are ranked, they can be temporarily removed from $\mathcal{G}_s$ and the rest of the sub-network remains acyclic and strongly connected. This step can be done repeatedly until all nodes and links of sub-network $\mathcal{G}_s$ are ranked, and the results are the topological orderings of nodes and links. The topological ordering concept is important in constructing the generalized flow splitting model and analyzing its properties.

3.2. Model formulation

As in the SRM, the generalized flow splitting model is constructed on a single-destination sub-network $\mathcal{G}_s$. Thus, the model formulation and its properties are considered on the sub-network $\mathcal{G}_s$. Further, define variable λ_i associated with node i that measures the total inflow at node i heading to destination s. Mathematically, it can be represented as:

$$\lambda_i^s = d_i^s + \sum_{a\in\mathcal{L}_i^-} x_a^s \tag{9}$$

where $d_i^s \geq 0$ denotes the OD demand between node i and destination s. Note that, differing from the SRM, d_i^s may be zero in our model. Let $\mathbf{x}^{i,s}$ denote the outflow vector at node i, i.e., $\mathbf{x}^{i,s} = \left[x_a^{i,s}\right],\ a \in \mathcal{L}_i^+$.

An outflow distribution function $F^i\left(\lambda_i, \mathbf{x}^i\right): R_+^{n_i+1} \mapsto R_+^{n_i}$ is defined as follows.

Definition 1. The *outflow distribution function* $F^i\left(\lambda_i^s, \mathbf{x}^{i,s}\right)$ is a continuous function, satisfying the following assumptions:

(a) $F^i\left(\lambda_i^s, \mathbf{x}^{i,s}\right)$ is negatively monotone with respect to outflows, i.e., for any feasible outflows $\mathbf{x}_1^{i,s}$ and $\mathbf{x}_2^{i,s}$,

$$\left[F^i\left(\lambda_i^s, \mathbf{x}_1^{i,s}\right) - F^i\left(\lambda_i^s, \mathbf{x}_2^{i,s}\right)\right]^T \left(\mathbf{x}_1^{i,s} - \mathbf{x}_2^{i,s}\right) < 0; \tag{10}$$

(b) For every nonempty subset $\mathcal{E} \subset \mathcal{L}_i^+$, if $c_a \to +\infty$ for all $a \in \mathcal{L}_i^+ \setminus \mathcal{E}$, and $x_e^{i,s} \to x_e^{\mathcal{E}}$ for all $e \in \mathcal{E}$, then:

$$F_a^i\left(\lambda_i^s, \mathbf{x}^{i,s}\right) \to 0 \ \ \forall a \in \mathcal{L}_i^+ \setminus \mathcal{E} \text{ and } F_e^i\left(\lambda_i^s, \mathbf{x}^{i,s}\right) \to F_e^i\left(\lambda_i^s, \mathbf{x}^{\mathcal{E}}\right) \ \ \forall e \in \mathcal{E}; \tag{11}$$

(c) The mapping result satisfies:

$$\sum_{a\in\mathcal{L}_i^+} F_a^i\left(\lambda_i^s, \mathbf{x}^{i,s}\right) = \lambda_i^s, \text{ and } F_a^i\left(\lambda_i^s, \mathbf{x}^{i,s}\right) \geq 0 \ \forall a \in \mathcal{L}_i^+. \tag{12}$$

In the definition, assumption (a) implies that if flow on link a increases, then the resulting distributed outflows on other links with the same tail node will tend to increase, namely, $\partial F_e^i\left(\lambda_i^s, \mathbf{x}^{i,s}\right) / \partial x_a^{i,s} > 0$, for all $a, e \in \mathcal{L}_i^+$, and $a \neq e$. Assumption (b) ensures that when a link's cost increases to infinity, its flow tends to zero and the resulting outflows converge to a stable pattern. Assumption (c) ensures the link flow conservation.

Following the formulation proposed in Smith and Mounce (2011), the outflow dynamics at node i can be represented by:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x}^{i,s} = \varphi\left[F^i\left(\lambda_i^s, \mathbf{x}^{i,s}\right) - \mathbf{x}^{i,s}\right] \tag{13}$$

where parameter $0 < \varphi \leq 1$ represents the link flow change rate. Let the cardinality $\left|\mathcal{L}_i^+\right| = n_i$ denote the number of links with tail node i. Dynamical system (13) is constructed for describing link flow evolution in a continuous-time space. When it is applied to discrete-time day-to-day traffic assignment, this system provides a link flow update process as:

$$\mathbf{x}^{i,s}(t+1) = \mathbf{x}^{i,s}(t) + \varphi\left[F^i\left(\lambda_i^s(t+1), \mathbf{x}^{i,s}(t)\right) - \mathbf{x}^{i,s}(t)\right], \ \ \forall i \in \mathcal{N}. \tag{14}$$

Note that on the right-hand-side of Eq. (14), the inflow variable $\lambda_i^s(t+1)$ is a result of the link flow pattern $\mathbf{x}_i^s(t+1)$ on "day" $t+1$ to ensure the flow conservation, i.e.:

$$\lambda_i^s(t+1) = d_i^s + \sum_{a\in\mathcal{L}_i^-} x_a^s(t+1), \ \ \forall i \in \mathcal{N}. \tag{15}$$

Note that the inflow variable $\lambda_i^s(t+1)$ is a function of the result of updated link flow pattern $\mathbf{x}(t+1)$. Therefore, the day-to-day flow update is an implicit nonlinear equation system that seems difficult to solve. However, with the acyclic network representation and topological orderings of nodes and links, the values $\lambda_i(t+1)$ can be determined by the Gauss-Seidel method.

Equation system (14) and (15) defines the generalized flow splitting model as it determines how link flows split at the node level. The inflow variables $\lambda_i^s(t+1)$ in the model play the key role in maintaining the flow conservation, and the outflow distribution function $F^i\left(\lambda_i^s, \mathbf{x}^{i,s}\right)$ characterizes travelers' path decision behavior at nodes. When the outflow distribution function is specified based on different path choice decision assumptions, a variety of day-to-day traffic assignment models can be developed. We illustrate this process in the next section.

3.3. Applications of the generalized flow splitting model

The proposed flow splitting model (14) and (15) provides a general structure to describe link flow evolution. The model can have various presentations when the flow distribution function $F^i\left(\lambda_i^s, \mathbf{x}^{i,s}\right)$ is specified differently. Applying the behavioral assumptions used in the SRM and the LBM, two different flow splitting models can be developed. It also illustrates that the proposed model can serve as a bridge connecting the SRM and the LBM.

3.3.1. Proportional-adjustment flow splitting model

To illustrate how to develop a day-to-day traffic assignment model by using the generalized flow splitting model (14) and (15), the first example uses the same path choice behavior assumption as the SRM, i.e., traffic flow swaps from one outgoing link to another at a rate which is proportional to the generalized cost difference between these two links. Let:

$$F^i\left(\lambda_i^s(t+1),\mathbf{x}^{i,s}(t)\right)=\mathbf{x}^{i,s}(t)+\gamma\Delta_i^s(\mathbf{x})\mathbf{x}^{i,s}(t),\tag{16}$$

where $\mathbf{\Delta}_i^s(\mathbf{x})$ is defined by (3). Let $\varphi=1$ in equation (14), then we have the same SRM proposed by Smith and Mounce (2011). Note that the formulation of flow distribution function (16) does not involve the inflow variable $\lambda_i(t+1)$. Therefore,

$$\sum_{e\in\mathcal{L}_i^+}x_e^{i,s}(t+1)=\sum_{e\in\mathcal{L}_i^+}x_e^{i,s}(t)=\lambda_i^s(t)\neq\lambda_i^s(t+1)=d_i^s+\sum_{e\in\mathcal{L}_i^-}x_e^{i,s}(t+1),\tag{17}$$

if total inflow at node i is changed. Flow conservation cannot be maintained and the ad-hoc flow adjustment process (8) is still required.

To ensure the flow conservation, flow distribution function (16) can be revised by including the inflow variable as follows:

$$F^i\left(\lambda_i^s(t+1),\mathbf{x}^{i,s}(t)\right)=\frac{\lambda_i^s(t+1)}{\lambda_i^s(t)}\left[\mathbf{x}^{i,s}(t)+\gamma\Delta_i^s(\mathbf{x})\mathbf{x}^{i,s}(t)\right].\tag{18}$$

Based on equality (18), summing up all outflows from node i shows that:

$$\sum_{e\in\mathcal{L}_i^+}x_e^{i,s}(t+1)=\frac{\lambda_i^s(t+1)}{\lambda_i^s(t)}\cdot\sum_{e\in\mathcal{L}_i^+}x_e^{i,s}(t)=\lambda_i^s(t+1)=d_i^s+\sum_{e\in\mathcal{L}_i^-}x_e^{i,s}(t+1).\tag{19}$$

Therefore, the proportional-adjustment flow splitting model (18) maintains flow conservation. Although equation (18) contains an implicit variable $\lambda_i(t+1)$, it can be solved by the Gauss-Seidel method when the sub-network is acyclic. However, as the inflow variable $\lambda_i(t)$ appears as a denominator, the flow splitting model (18) is not well defined if $\lambda_i(t)=0$. Therefore, the assumption that each non-destination node has a positive demand is needed. In the next section, another flow splitting model is developed by applying the path choice behavior assumption used in the LMB (He et al. 2010), where the assumption of positive demand on the non-destination nodes is no longer needed.

3.3.2. Projection-type flow splitting model

Although the LBM is constructed on a link-path network representation, its modeling concept can be used in developing a flow splitting model following the structure of (14) and (15). The basic idea is to consider each node as an origin and its outgoing links as "paths" connecting to the destination. Each node has the "demand" as its current inflow $\lambda_i^s(t)$ defined by (15).

The LBM in He et al. (2010) is formulated as:

$$\mathbf{x}(t+1)=\mathbf{x}(t)+\varphi\left[\mathbf{y}(\mathbf{x}(t))-\mathbf{x}(t)\right],\tag{20}$$

where $\mathbf{y}$ represents the "target flow" determined by the minimization problem:

$$\min_{\mathbf{y}\in\Pi}\beta\boldsymbol{\tau}(\mathbf{x}(t))^T\mathbf{y}+(1-\beta)\left\|\mathbf{y}-\mathbf{x}(t)\right\|^2\tag{21}$$

and the feasible set $\Pi=\{\mathbf{y}\,|\,\mathbf{Af}=\mathbf{y},\Phi\mathbf{f}=\mathbf{d},\mathbf{f}\geq 0\}$ represents the link flow pattern that satisfies flow conservation in the relationship with the path flow pattern, and the relationship between path flow and OD demand. As discussed by Han and Du (2012), the "target flow" $\mathbf{y}$ can be represented by using the projection operator as:

$$\mathbf{y}(\mathbf{x}(t))=\mathrm{Pr}_{\Pi}\left(\mathbf{x}(t)-\gamma\boldsymbol{\tau}(\mathbf{x}(t))\right),\tag{22}$$

where $\gamma=\beta/2(1-\beta)$. Then, the flow conservation is guaranteed by projecting the flow pattern onto the link flow feasible set Π. The same concept can be used to develop an alternative flow splitting model based on (14).

In the flow splitting model, specify the outflow distribution function by using the projection operator as:

$$F^i\left(\lambda_i^s(t+1),\mathbf{x}^{i,s}(t)\right)=\mathrm{Pr}_{\Omega^{i,s}(t+1)}\left[\mathbf{x}^{i,s}(t)-\gamma\mathbf{c}^{i,s}(t)\right].\tag{23}$$

where $\mathbf{c}^{i,s}(t)$ is the flow-weighted path cost vector on day t. Each entry of the vector, i.e. $c_a^{i,s}(t), \forall a \in \mathcal{L}_i^+$ is defined by (5) and (6). The feasible outflow set $\Omega^{i,s}(t+1)$ for node i on day $t+1$ is defined by:

$$\Omega^{i,s}(t+1) = \left\{ \mathbf{y}^{i,s} \middle| \mathbf{y}^{i,s} = \left[y_e^s \right] \in R_+^{n_i}, \sum_{e \in \mathcal{L}_i^+} y_e^s = \lambda_i^s(t+1), y_e^s \geq 0, \forall e \in \mathcal{L}_i^+ \right\}. \tag{24}$$

As the feasible outflow set $\Omega^{i,s}(t+1)$ involves the total inflow $\lambda_i^s(t+1)$, it is defined implicitly on day $t+1$. Similar to the determination of $\lambda_i^s(t+1)$, $\Omega^{i,s}(t+1)$ can be constructed by using the Gauss-Seidel method due to the acyclic network structure. With the specified outflow distribution function (23), a projection-type flow splitting model can be formulated as:

$$\mathbf{x}^{i,s}(t+1) = \mathbf{x}^{i,s}(t) + \varphi\left[\Pr_{\Omega^{i,s}(t+1)}\left[\mathbf{x}^{i,s}(t) - \gamma \mathbf{c}^{i,s}(t) \right] - \mathbf{x}^{i,s}(t) \right], \ \forall i \in \mathcal{N}. \tag{25}$$

In the projection-type flow splitting model (25), a sequence of quadratic programming problems (21) needs to be solved to redistribute the outflows at each node. Note that the number of downstream links of each node i, i.e., n_i, is a small number. Therefore, solving the quadratic programming problem (21) is easy when the inflow $\lambda_i(t+1)$ is determined by the Gauss-Seidel method.

Let Ω denote a feasible link flow set that includes all link flows. Any $x_e^{i,s} \in \Omega$, $i \in \mathcal{N}, a \in \mathcal{L}_i^+$, must satisfy $x_e^{i,s} \geq 0$ and the following conditions:

$$\sum_{e \in \mathcal{L}_i^-} x_e^{i,s} - \sum_{e \in \mathcal{L}_i^+} x_e^{i,s} = \begin{cases} \sum_r d_{rs}, & \text{if } i = s \\ -d_{rs}, & \text{if } i = r \\ 0, & \text{otherwise} \end{cases} \quad \forall i \in \mathcal{N}. \tag{26}$$

As the inflow variable is defined as $\lambda_i^s = \sum_{e \in \mathcal{L}_i^-} x_e^{i,s}$, we can see that when a link flow pattern $\mathbf{x} = \{x_a^i\}$ satisfies the feasible link flow set $\Omega^{i,s}$ defined by (24), it automatically satisfies the link flow feasible set Ω defined by (26). The definition of feasible set Ω helps in analyzing the properties of the day-to-day traffic assignment model (25). One main property of the projection-type flow splitting model is presented as follows.

Proposition 1. A stable point of the day-to-day model (25) solves the variational inequality problem:

$$\left(\mathbf{x} - \mathbf{x}^*\right)^T \mathbf{c}\left(\mathbf{x}^*\right) \geq 0, \quad \forall \mathbf{x} \in \Omega. \tag{27}$$

Proof. If $\mathbf{x}^*$ is a stable point of the day-to-day model (25), then:

$$\Pr_{\Omega^{i,s*}}\left[\mathbf{x}^{i,s*} - \gamma \mathbf{c}\left(\mathbf{x}^{i,s*}\right) \right] = \mathbf{x}^{i,s*} \quad \forall i \in \mathcal{N} \text{ and } \forall s \tag{28}$$

which implies that $\Pr_\Omega\left[\mathbf{x}^* - \gamma \mathbf{c}\left(\mathbf{x}^*\right) \right] - \mathbf{x}^* = 0$. Based on the work of Gowda and Pang (1993), $\mathbf{x}^*$ solves a variational inequality problem as shown in (27). ■

As $\mathbf{c}\left(\mathbf{x}^*\right)$ represents a generalized link cost defined by (5) and (6), the variational inequality (27) is not a typical formulation for solving deterministic user equilibrium (UE). Whether the stable point of the day-to-day model (25) satisfies UE depends on the properties of the generalized cost function $\mathbf{c}\left(\mathbf{x}^*\right)$.

4. Model properties

Smith and Mounce (2011) analyze the properties of the SRM (1) based on the assumption of separable link cost functions. Here, we derive properties of the generalized flow splitting model, which assist in further analyzing the properties of any specific day-to-day traffic assignment model developed based upon the general modeling framework (13). These properties do not rely on the separability of link cost functions.

This section first focuses on the flow conservation of the proposed generalized flow splitting model, since this property is difficult to maintain when path flow variables are absent. Comparing the proposed flow splitting model (13) with the SRM (1), the major difference is the existence of the inflow variable λ_i defined by (9), which essentially assists in preserving the flow conservation at every node. The flow conservation of the generalized flow splitting model (14) is summarized into the following proposition.

Proposition 2. The generalized flow splitting model, defined by (14), maintains the link flow feasibility, i.e., $\mathbf{x}(t) \in \Omega$ for all t, as long as the initial link flow pattern $\mathbf{x}(0)$ is feasible.

Proof. Assume the day-to-day link flow pattern $\mathbf{x}(t)$ follows the generalized flow splitting model (14). We need to prove that if $\mathbf{x}(t) \in \Omega$, then $\mathbf{x}(t+1) \in \Omega$. Assume that the feasible link flow pattern on day t, $\mathbf{x}(t)$, has been decomposed by destination s at each node i denoted by $\mathbf{x}^{i,s}(t)$. As $\mathbf{x}(t) \in \Omega$, it implies that:

$$d_i^s + \sum_{a \in \mathcal{L}_i^-} x_a^{i,s}(t) = \lambda_i^s(t) = \sum_{a \in \mathcal{L}_i^+} x_a^{i,s}(t), \ \forall i \in \mathcal{N}. \tag{29}$$

As all nodes in the network $\mathcal{G}_s$ have a topological order, the intermediate inflow $\tilde{\lambda}_i^s$ on day $t+1$ can be updated in the topological order using definition (9). Define the intermediate exit flows $\tilde{\mathbf{x}}^{i,s}(t+1) = F^i\left(\lambda_i^s(t+1), \mathbf{x}^{i,s}(t)\right)$, $\tilde{\mathbf{x}}^{i,s}(t+1) \geq 0$. By assumption (c) in the definition of the flow distribution function, it satisfies:

$$d_i^s + \sum_{a \in \mathcal{L}_i^-} \tilde{x}_a^{i,s}(t+1) = \lambda_i^s(t+1) = \sum_{a \in \mathcal{L}_i^+} \tilde{x}_a^{i,s}(t+1), \ \forall i \in \mathcal{N}. \tag{30}$$

Let $0 < \varphi \leq 1$. Adding (29) to (30) with weights φ and $1-\varphi$ provides:

$$d_i^s + \sum_{a \in \mathcal{L}_i^-} \left[(1-\varphi) x_a^{i,s}(t) + \varphi \tilde{x}_a^{i,s}(t+1)\right] = \sum_{a \in \mathcal{L}_i^+} \left[(1-\varphi) x_a^{i,s}(t) + \varphi \tilde{x}_a^{i,s}(t+1)\right], \ \forall i \in \mathcal{N}. \tag{31}$$

According to the flow splitting model (14), Eq. (31) is equivalent to:

$$d_i^s + \sum_{a \in \mathcal{L}_i^-} x_a^{i,s}(t+1) = \sum_{a \in \mathcal{L}_i^+} x_a^{i,s}(t+1), \ \forall i \in \mathcal{N}. \tag{32}$$

The above equation shows that $\mathbf{x}^{i,s}(t+1)$ satisfies flow conservation and $x_a^{i,s}(t+1) \geq 0$. Thus, $\mathbf{x}(t+1) \in \Omega$. This proposition is proved by mathematical induction. ■

Proposition 2 is also labeled as the invariance property in Han and Du (2012) and Guo et al. (2013). Through the proof, it can be seen that assumption (c) in the definition of the outflow distribution function plays the critical role in maintaining the flow pattern's feasibility. In the proportional-adjustment flow splitting model, assumption (c) is satisfied by the introduction of the normalized term $\lambda_i(t+1)/\lambda_i(t)$ in (18). In the projection-type flow splitting model, assumption (c) is satisfied by the projection operator in (25).

We now focus on analyzing the structure of the generalized cost function $\mathbf{c}(\mathbf{x})$ used to specify the outflow distribution function. Note that each entry of the cost vector $\mathbf{c}(\mathbf{x})$, i.e. $c_a^s(\mathbf{x})$ represents a flow-weighted path cost from the tail node of link a to destination s, as defined by (4), (5) and (6). The cost vector $\mathbf{c}(\mathbf{x})$ is used in both the proportional-adjustment flow splitting model and the projection-type flow splitting model. Its value directly impacts the splitting rates in the model formulation. Analyzing the structure of the flow-weighted path cost $\mathbf{c}(\mathbf{x})$ assists in deriving the properties of the generalized flow splitting model and constructing day-to-day traffic assignment models.

Note that the flow splitting proportion p_b^s defined by Eq. (4) is not well defined when the tail node of link b has zero total inflow. Here, we redefine the flow splitting proportion p_b as:

$$p_b^s = \begin{cases} x_b^s / \lambda_i^s, & \text{if } \lambda_i^s > 0 \\ 1/n_i, & \text{otherwise} \end{cases}, \ b \in \mathcal{L}_i^+. \tag{33}$$

The main difference between Eq. (4) and Eq. (33) is that p_b^s is defined as $1/n_i$ when the total inflow is zero. Under this definition, the node potential Y_i^s is an arithmetic average of the generalized costs of its downstream links when the inflow is zero. With a well-defined flow splitting proportion (33), the assumption of positive demand at each non-destination node is no longer needed.

Proposition 3. The generalized link cost $\mathbf{c}(\mathbf{x})$, defined by (5), (6), and (33) has a concise representation:

$$\mathbf{c}(\mathbf{x}) = \mathbf{Q}(\mathbf{x}) \boldsymbol{\tau}(\mathbf{x}), \tag{34}$$

where $\mathbf{Q}(\mathbf{x})$ is a positive definite matrix for all feasible link flows $\mathbf{x} \in \Omega$.

Proof. For each acyclic sub-network $\mathcal{G}_s$, define a *flow transfer matrix* $\mathbf{P} = (P_{ab}^s)$, which is a $|\mathcal{L}| \times |\mathcal{L}|$ square matrix. Each element of the flow transfer matrix $\mathbf{P}$ indicates how much flow, in terms of proportion, will transfer from link a to link b. According to (33), if a and b are adjacent links, i.e., $a \in \mathcal{L}_i^-, b \in \mathcal{L}_i^+$ of a node $i \in \mathcal{N}$, then element P_{ab} in the flow transfer matrix can be defined by:

$$P_{ab}^s = p_b^s = \begin{cases} x_b^s / \lambda_i^s, & \text{if } \lambda_i^s > 0 \\ 1/n_i, & \text{otherwise} \end{cases}. \tag{35}$$

If a and b are not adjacent, then $P_{ab}^s = 0$. As sub-network $\mathcal{G}_s$ is acyclic, all links in the sub-network have a topological ordering. Thus, the flow transfer matrix $\mathbf{P}$ is *strictly* upper triangular, i.e., its diagonal and lower triangular elements are all zeros.

Similar to the determination of the weight matrix in solving stochastic user equilibrium problem in Bell (1995), each element of the product $\mathbf{PP} = \mathbf{P}^2$ indicates the flow transfer proportion from link a to link b through a path containing exactly two nodes; and each element of matrix $\mathbf{PP}^2 = \mathbf{P}^3$ indicates the flow transfer proportion from link a to link b through a path containing exactly three nodes; and so forth. As each decomposed sub-network is acyclic, there exists a path $\vec{r}$ in the sub-network $\mathcal{G}_s$ that contains the largest number of nodes. Denote the number of nodes on path $\vec{r}$ as K. Then, for any integer $n > K$, $\mathbf{P}^n = 0$.

Based on the definition equations (5) and (6) for $\mathbf{c}(\mathbf{x})$, i.e.:

$$c_a^s(\mathbf{x}) = \tau_a(\mathbf{x}) + Y_i^s \text{ and } Y_i^s = \sum\nolimits_{b=(i,j)\in\mathcal{L}_i^+} \left(\tau_b(\mathbf{x}) + Y_j^s\right)\cdot p_b^s, \tag{36}$$

the relationship between $\mathbf{c}(\mathbf{x})$ and $\boldsymbol{\tau}(\mathbf{x})$ can be derived as:

$$\mathbf{c}(\mathbf{x}) = \left[\mathbf{I} + \mathbf{P} + \mathbf{P}^2 + \ldots + \mathbf{P}^K\right] * \boldsymbol{\tau}(\mathbf{x}), \tag{37}$$

where $\mathbf{I}$ represents the identity matrix. Denote $\mathbf{Q} = \mathbf{I} + \mathbf{P} + \mathbf{P}^2 + \ldots + \mathbf{P}^K$, then we have the relationship (34).

As the sub-network is acyclic, matrix $\mathbf{P}^n$ is strictly upper triangular with all elements smaller than one, for all positive integers n. Therefore, matrix $\mathbf{Q}(\mathbf{x})$ is an upper triangular matrix, whose diagonal elements are ones. It implies that $\mathbf{Q}(\mathbf{x})$ is positive definite with eigenvalues all equivalent to one for any flow pattern $\mathbf{x}$. This completes the proof. ■

Remark 1. For a large network, it is difficult to determine the value of K to compute $\mathbf{Q}(\mathbf{x})$, as finding the longest path is not easy. However, as discussed by Bell (1995), the expansion

$$\left(\mathbf{I} - \mathbf{P}\right)^{-1} = \mathbf{I} + \mathbf{P} + \mathbf{P}^2 + \ldots \tag{38}$$

provides an easy way to compute $\mathbf{Q}(\mathbf{x})$, as $\mathbf{P}^n = 0$ when $n > K$. Therefore, finding K is not necessary, and $\mathbf{Q}(\mathbf{x})$ can be determined by $\mathbf{Q}(\mathbf{x}) = \left(\mathbf{I} - \mathbf{P}(\mathbf{x})\right)^{-1}$. We use a simple example to illustrate the key idea shown in Proposition 3. Suppose we have an acyclic network illustrated by Fig. 2. The flow splitting proportion p_b^s defined by (33) ensures that link flow conservation at node i, i.e., $\sum_{a\in\mathcal{L}_i^+} p_a^s = 1$. The update of link flow may follow the proportional-adjustment flow splitting model or the projection-type flow splitting model.

For the acyclic network shown in Fig. 2, the flow transfer matrix $\mathbf{P}$ can be constructed to establish the relationship between incoming and outgoing flows of a node, as:

$$\mathbf{P} = \begin{bmatrix} 0 & p_2 & p_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & p_4 & p_5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & p_6 & p_7 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{39}$$

As previously discussed, the flow transfer matrix $\mathbf{P}$ is upper triangular. The first row of matrix $\mathbf{P}$ shows that if link 1 carries one unit of flow, p_2 and p_3 units of flow transfer from link 1 to links 2 and 3, respectively.

For this example, we can directly observe that the longest path in the network contains five nodes. Therefore, the matrix $\mathbf{Q} = \mathbf{I} + \mathbf{P} + \mathbf{P}^2 + \mathbf{P}^3 + \mathbf{P}^4 + \mathbf{P}^5$. Or, we can compute $\mathbf{Q}$ by $\mathbf{Q} = \left(\mathbf{I} - \mathbf{P}\right)^{-1}$. As matrix $\mathbf{I} - \mathbf{P}$ is upper triangular and sparse, its inverse is easy to compute. Matrix $\mathbf{Q}$ can be computed using either method, as:

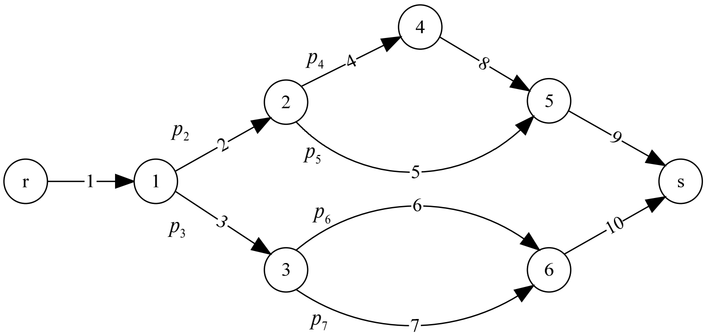

Fig. 2. A ten-link network.

$$\mathbf{Q}=\begin{bmatrix} 1 & p_2 & p_3 & p_2p_4 & p_2p_5 & p_3p_6 & p_3p_7 & p_2p_4 & p_2 & p_3 \\ 0 & 1 & 0 & p_4 & p_5 & 0 & 0 & p_4 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & p_6 & p_7 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad (40)$$

Matrix $\mathbf{Q}$ shows the current demand distribution information and is called the *demand distribution matrix* in this study. The i th row of matrix $\mathbf{Q}$ indicates how flow on link i will be distributed on downstream links. For example, if one unit of flow departs from r, p_2 of it traverses through link 2, and p_2p_4 of it traverses through link 4. Similarly, if one unit flow is on link 3, then p_6 of it traverses through link 6, and all of it traverses through link 10. Each column j of $\mathbf{Q}$ indicates that the current flow on link j is from which upstream links. For example, if we look at the 8th column of $\mathbf{Q}$, it shows that p_2p_4 of flow on link 1 traverses to link 8, p_4 of flow on link 2 traverses to link 8, and all flow on link 4 traverses to link 8.

Remark 2. As the demand distribution matrix is derived by considering all possible paths connecting from each node to the destination, the product of $\mathbf{Qd}$ preserves flow conservation automatically. Thereby, using the product $\mathbf{Qd}$ to formulate the outflow distribution function F does not require the implicit inflow variable $\lambda_i^s(t+1)$. This attractive property enables the construction of day-to-day traffic assignment models with a simple representation.

The ad-hoc flow adjustment (8) used in the SRM is to update the outflow distribution at each node by using the transfer matrix $\mathbf{P}$. As the longest path contains a finite number of links, the update process can be terminated in a finite number of steps. The outputs of the process (8) are the elements in the demand distribution matrix $\mathbf{Q}$ when all non-destination nodes have positive demand.

With the relationship between the generalized link cost $\mathbf{c}(\mathbf{x})$ and link cost function $\boldsymbol{\tau}(\mathbf{x})$ defined by (34), some important properties of the link cost function $\boldsymbol{\tau}(\mathbf{x})$ can be maintained. This is important in applying the generalized link cost $\mathbf{c}(\mathbf{x})$ to develop flow splitting models and analyzing the properties of the model. We summarize these properties as follows.

Proposition 4. If performance function $\boldsymbol{\tau}(\mathbf{x})$ is Lipschitz continuous and strongly monotone, then the generalized link cost $\mathbf{c}(\mathbf{x})$, defined by (5), (6), and (33), is Lipschitz continuous and strongly monotone as well.

Proof. We first prove the Lipschitz continuity of $c_a(\mathbf{x})$. Let $q_{ae}(\mathbf{x})$ denote the element in matrix $\mathbf{Q}(\mathbf{x})$ indicating the flow distribution from link a to link e. Accroding to Proposition3, $\mathbf{c}(\mathbf{x})=\mathbf{Q}(\mathbf{x})\boldsymbol{\tau}(\mathbf{x})$. Therefore:

$$c_a^s(\mathbf{x}) = \sum_{e\in\mathcal{L}} q_{ae}^s(\mathbf{x})\tau_e(\mathbf{x}). \tag{41}$$

where $0 \le q_{ae}^s(\mathbf{x}) \le 1$ for all $\mathbf{x}$. Note that a linear combination of Lipschitz continuous functions is Lipschitz continuous. Let $L_e < \infty$ be the Lipschitz constant for the link cost function $\tau_e(\mathbf{x})$, $\forall e \in \mathcal{L}$. Then, $c_a^s(\mathbf{x})$ are Lipschitz continuous with the Lipschitz constant $L_a = \sum_{e\in\mathcal{L}} q_{ae}^s(\mathbf{x}) L_e$, $\forall a \in \mathcal{L}$. As the number of links in the network is finite and $q_{ae}^s(\mathbf{x}) \le 1$, $L_a < \infty$.

Next, we prove the strong monotonicity of $\mathbf{c}(\mathbf{x})$. By the chain rule, the Jacobian of the generalized link cost $\mathbf{c}(\mathbf{x})$ can be derived as:

$$\nabla\mathbf{c}(\mathbf{x}) = \mathbf{Q}(\mathbf{x})\nabla\boldsymbol{\tau}(\mathbf{x}) + \nabla\mathbf{Q}(\mathbf{x})\boldsymbol{\tau}(\mathbf{x}). \tag{42}$$

Note that $\mathbf{Q}(\mathbf{x}) = (\mathbf{I} - \mathbf{P}(\mathbf{x}))^{-1}$, thus $\mathbf{Q}(\mathbf{x})(\mathbf{I} - \mathbf{P}(\mathbf{x})) = \mathbf{I}$, or $\mathbf{Q}(\mathbf{x}) - \mathbf{Q}(\mathbf{x})\mathbf{P}(\mathbf{x}) = \mathbf{I}$. By the chain rule:

$$\nabla\mathbf{Q}(\mathbf{x}) - \nabla\mathbf{Q}(\mathbf{x})\mathbf{P}(\mathbf{x}) - \mathbf{Q}(\mathbf{x})\nabla\mathbf{P}(\mathbf{x}) = \mathbf{0}. \tag{43}$$

It implies that $\nabla\mathbf{Q}(\mathbf{x}) = \mathbf{Q}(\mathbf{x})\nabla\mathbf{P}(\mathbf{x})(\mathbf{I} - \mathbf{P}(\mathbf{x}))^{-1}$. Replace $(\mathbf{I} - \mathbf{P}(\mathbf{x}))^{-1}$ by $\mathbf{Q}(\mathbf{x})$, $\nabla\mathbf{Q}(\mathbf{x}) = \mathbf{Q}(\mathbf{x})\nabla\mathbf{P}(\mathbf{x})\mathbf{Q}(\mathbf{x})$.

According to the definition of $\mathbf{P}(\mathbf{x})$, each non-zero element in row i carries flow from link i. Therefore, $\mathbf{Q}(\mathbf{x})\nabla\mathbf{P}(\mathbf{x}) = \nabla\mathbf{P}(\mathbf{x})$. In addition, note that $\nabla\mathbf{Q}(\mathbf{x})$ is a three-dimensional matrix. Let $\nabla\mathbf{Q}^i(\mathbf{x})$ denote the Jacobian of the *i*th row of $\nabla\mathbf{Q}(\mathbf{x})$. Note that $\nabla\mathbf{Q}(\mathbf{x}) = \mathbf{Q}(\mathbf{x})\nabla\mathbf{P}(\mathbf{x})\mathbf{Q}(\mathbf{x}) = \nabla\mathbf{P}(\mathbf{x})\mathbf{Q}(\mathbf{x})$. Then the first i rows of $\nabla\mathbf{Q}(\mathbf{x})$ are all zeros. $\nabla\mathbf{Q}(\mathbf{x})\boldsymbol{\tau}(\mathbf{x})$ is a strictly upper triangular matrix with all elements positive. Therefore, $\nabla\mathbf{Q}(\mathbf{x})\boldsymbol{\tau}(\mathbf{x})$ is a positive semi-definite matrix for all $\mathbf{x}$.

A link cost function $\boldsymbol{\tau}(\mathbf{x})$ is strongly monotone if and only if $\nabla\boldsymbol{\tau}(\mathbf{x})$ is uniformly positive definite for all $\mathbf{x}$, i.e., $\mathbf{x}^T\nabla\boldsymbol{\tau}(\mathbf{x})\mathbf{x} > \rho\|\mathbf{x}\|^2$ for all $\mathbf{x}$ (Facchinei and Pang, 2003). As shown by Proposition 3, $\mathbf{Q}(\mathbf{x})$ is positive definite for all $\mathbf{x}$. As $\nabla\boldsymbol{\tau}(\mathbf{x})$ is positive definite, $\mathbf{Q}(\mathbf{x})\nabla\boldsymbol{\tau}(\mathbf{x})$ is positive definite.

Due to (42), $\nabla\mathbf{c}(\mathbf{x})$ is a combination of a positive definite matrix $\mathbf{Q}(\mathbf{x})\nabla\boldsymbol{\tau}(\mathbf{x})$ plus a positive semi-definite matrix $\nabla\mathbf{Q}(\mathbf{x})\boldsymbol{\tau}(\mathbf{x})$. Therefore, $\nabla\mathbf{c}(\mathbf{x})$ is a positive definite matrix and $\mathbf{c}(\mathbf{x})$ is strongly monotone for all $\mathbf{x}$. ■

Corollary 1. If link cost function $\boldsymbol{\tau}(\mathbf{x})$ is strictly monotone, then the generalized link cost $\mathbf{c}(\mathbf{x})$, defined by (5), (6), and (33), is strictly monotone.

Proof. This corollary holds readily as strong monotonicity implies strict monotonicity. ■

The Lipschitz continuity and strong monotonicity presented in Proposition 4 are important properties in analyzing the uniqueness of a stable point, the system stability and convergence to user equilibrium, for example, in the studies of Nagurney and Zhang (1997) and He and Liu (2012). The propositions presented in this section can serve as preliminaries for deriving the properties of the stable point and asymptotic stability property for the generalized flow splitting model in the future. The relationship between the flow transfer matrix $\mathbf{P}$ and demand distribution matrix $\mathbf{Q}$, as discussed in Remarks 1 and 2, offers an easy way to implement the flow splitting model. We use a numerical example to demonstrate this in the next section.

5. Numerical example

By applying the relationships derived in Proposition 3, Remark 1 and Remark 2, a day-to-day traffic assignment process built upon the generalized flow splitting model (14) and (15) contains the following three main steps:

Step 1: Based on the current link flows $\mathbf{x}(t)$, compute the link cost function $\boldsymbol{\tau}(\mathbf{x}(t))$. Compute the current flow transfer matrix $\mathbf{P}(\mathbf{x}(t))$ and the demand distribution matrix $\mathbf{Q}(\mathbf{x}(t)) = (\mathbf{I} - \mathbf{P}(\mathbf{x}(t)))^{-1}$. Update the generalized link cost function $\mathbf{c}(\mathbf{x}(t))$ as $\mathbf{c}(\mathbf{x}(t)) = \mathbf{Q}(\mathbf{x}(t))\boldsymbol{\tau}(\mathbf{x}(t))$;

Step 2: Compute the "target flow" $\mathbf{y}(t+1)$ using the outflow distribution function, $\mathbf{y}(t+1) = F(\lambda(t+1), \mathbf{x}(t))$;

Step 3: Update link flow $\mathbf{x}(t+1) = (1-\varphi)\mathbf{x}(t) + \mathbf{y}(t+1)$; go to Step 1.

The main step in this day-to-day traffic assignment process is Step 2, where a set of outflow distribution problems needs to be solved. Note that though we separate the update of the "target flow" $\mathbf{y}(t+1)$ and the updated link flow $\mathbf{x}(t+1)$ into two steps, both of them can be solved simultaneously by following the Gauss-Seidel method due to the acyclic network structure, i.e., alternatively updating $\mathbf{y}^i(t+1)$ and $\mathbf{x}^i(t+1)$ in the sequence of topological ordering of nodes.

In what follows, we demonstrate how to construct a new day-to-day traffic assignment process based on the proposed general model. The main task is to specify the outflow distribution function in Step 2. Here, we adopt the same path choice behavior assumption used in Smith and Mounce (2011), i.e., travelers' path choice at node i depends on the generalized cost $c(\mathbf{x}(t))$ on the outgoing links of node i.

In Step 2, the flow swapping matrix $\boldsymbol{\Delta}$ can be first computed based on the current link flow pattern $\mathbf{x}(t)$ and corresponding generalized link cost pattern $\mathbf{c}(\mathbf{x}(t)) = \mathbf{Q}(\mathbf{x})\boldsymbol{\tau}(\mathbf{x})$. An intermediate flow pattern $\hat{\mathbf{y}}(t+1)$ can be computed by:

$$\hat{\mathbf{y}}(t+1) = \mathbf{x}(t) + \gamma\boldsymbol{\Delta}(\mathbf{x})\mathbf{x}. \tag{44}$$

Note that the flow swapping matrix $\boldsymbol{\Delta}$ is computed based on $\mathbf{c}(\mathbf{x}(t)) = \mathbf{Q}(\mathbf{x})\boldsymbol{\tau}(\mathbf{x})$ with p_a defined by (33) instead of (4). Therefore, it is well defined under the condition that total the inflow of a node is zero. As $\hat{\mathbf{y}}(t+1)$ does not consider upstream link flow changes, it does not preserve flow conservation. However, the intermediate flow pattern $\hat{\mathbf{y}}(t+1)$ provides the information on the tendency of travelers' path swapping behavior at each individual node. The intermediate flow pattern $\hat{\mathbf{y}}(t+1)$ helps in constructing the outflow distribution function F.

Using the intermediate flow pattern $\hat{\mathbf{y}}(t+1)$ and the definition of splitting rate (35), the flow transfer matrix $\hat{\mathbf{P}}(\hat{\mathbf{y}}(t+1))$ can be computed, which summarizes the travelers' path swapping behavior at nodes. Then, the demand distribution matrix $\hat{\mathbf{Q}}$ can be computed by $\hat{\mathbf{Q}}(\hat{\mathbf{y}}(t+1)) = \left[\mathbf{I} - \hat{\mathbf{P}}(\hat{\mathbf{y}}(t+1))\right]^{-1}$ according to Remark 1. The "target flow" $\mathbf{y}(t+1)$ is determined by the flow distribution function:

$$\mathbf{y}(t+1) = F(\boldsymbol{\lambda}(t+1), \mathbf{x}(t)) = \hat{\mathbf{Q}}(\hat{\mathbf{y}}(t+1))\mathbf{d}. \tag{45}$$

where $\mathbf{d}$ represents the demand vector. Note that the flow distribution function F in (45) does not contain the inflow variables λ explicitly to ensure the flow conservation. This is because the demand distribution matrix $\hat{\mathbf{Q}}$ implicitly covers all paths connecting any node pairs, as discussed in Remark 2. After we have the "target flow" $\mathbf{y}(t+1)$, the link flow pattern on day $t+1$ is updated by:

$$\mathbf{x}(t+1) = (1-\varphi)\mathbf{x}(t) + \varphi\mathbf{y}(t+1). \tag{46}$$

Next, the ten-link network shown in **Error! Reference source not found.** is used to illustrate the applicability of the new proportional-adjustment flow splitting model based on Eq. (45). Assume only one OD pair (r,s) in the network with one unit travel demand. The free flow travel time is set as one unit of time for all links. Link capacities are set as one unit. The link cost function is assumed to follow the BPR function:

$$\tau_a(x_a) = \tau_a^0\left[1 + 0.15\left(\frac{x_a}{C_a}\right)^4\right], \tag{47}$$

where τ_a^0 represents free flow travel time and C_a represents link capacity.

In the newly constructed flow splitting model (45) and (46), parameter φ is set as 1 and parameter γ is set as 0.4. As only one unit of demand is generated from origin r, the demand vector $\mathbf{d} = [1,0,0,0,0,0,0,0,0,0]^T$. Let the initial link flows on the first day be $x_1 = x_2 = x_4 = x_8 = x_9 = 1$ and all other link flows be zero. Note that, in the SRM, the node potentials of Y_3 and Y_6 are not well defined due to zero outflows at nodes 3 and 6 and Eq. (4). By contrast, in the proposed generalized splitting model, the flow transfer matrix is defined by (35) which is well defined for nodes with zero outflows.

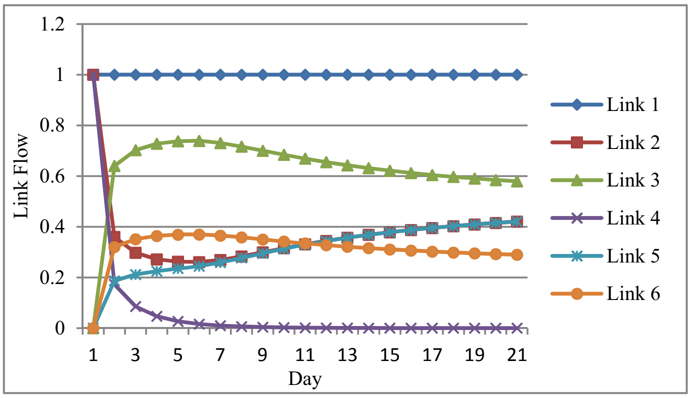

Fig. 3. Link flow evolution.

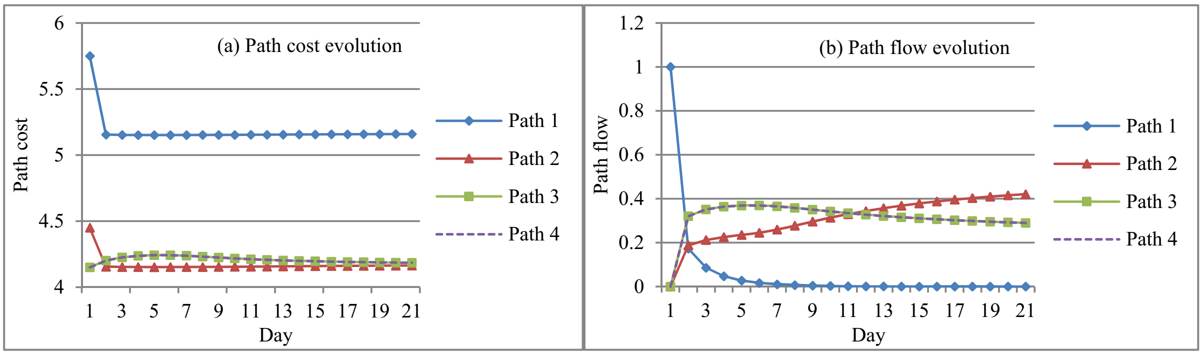

Fig. 4. (a) Path cost evolution; (b) Path flow evolution.

The link flow day-to-day evolution results based on the flow splitting model (45) and (46) are summarized in Fig. 3. Note that the network is symmetric, namely, $x_7 = x_6(t)$, $x_8(t) = x_4(t)$, $x_9(t) = x_2(t)$, and $x_{10}(t) = x_3(t)$. Hence, Fig. 3 only presents the link flow evolution on links 1, 2, 3, 4, 5, and 6.

As shown in Fig. 3, the flow on link 4 decreases from one day to the next since the path cost through link 4 remains the highest throughout the day-to-day traffic assignment process. On the contrary, the flow on link 5 keeps increasing as the path cost through link 5 remains the lowest.

As $\varphi = 1$, $\mathbf{x}(t+1) = \mathbf{y}(t+1) = \hat{\mathbf{Q}}(\hat{\mathbf{y}}(t+1))\mathbf{d}$. In addition, because $\mathbf{d} = [1,0,0,0,0,0,0,0,0,0]^T$, the link flow on each day is characterized by the first row of $\hat{\mathbf{Q}}(\hat{\mathbf{y}}(t+1))$. For this example, the first row of $\hat{\mathbf{Q}}(\hat{\mathbf{y}}(t+1))$ has the specific form: $\mathbf{q} = [q_1, q_2, q_3, q_4, q_5, q_6, q_7, q_8, q_9, q_{10}] = [1, \hat{p}_2, \hat{p}_3, \hat{p}_2\hat{p}_4, \hat{p}_2\hat{p}_5, \hat{p}_3\hat{p}_6, \hat{p}_3\hat{p}_7, \hat{p}_2\hat{p}_4, \hat{p}_2, \hat{p}_3]$ according to (40). Note that $\hat{p}_2 + \hat{p}_3 = 1$, $\hat{p}_4 + \hat{p}_5 = 1$, and $\hat{p}_6 + \hat{p}_7 = 1$. Therefore, $q_2 + q_3 = \hat{p}_2 + \hat{p}_3 = 1 = q_1$, $q_4 + q_5 = \hat{p}_2\hat{p}_4 + \hat{p}_2\hat{p}_5 = \hat{p}_2 = q_2$, $q_6 + q_7 = \hat{p}_3\hat{p}_6 + \hat{p}_3\hat{p}_7 = \hat{p}_3 = q_3$, $q_9 + q_{10} = \hat{p}_2 + \hat{p}_3 = 1 = q_1$. In summary, the link flow pattern on each day maintains flow conservation.

To evaluate the convergence of the day-to-day traffic assignment, the evolutions of path flow and path cost are evaluated. Denote Path 1 as the link sequence $\{1, 2, 4, 8, 9\}$, Path 2 as the link sequence $\{1, 2, 5, 9\}$, Path 3 as the link sequence $\{1, 3, 6, 10\}$, and Path 4 as the link sequence $\{1, 3, 7, 10\}$. As illustrated in **Error! Reference source not found.**, the cost of Path 1 remains the highest while its flow keeps reducing to zero. Flow on Path 2 keeps increasing as its cost remains the lowest. Flows on paths 3 and 4 gradually reduce after peaking in the early assignments. The path cost differences between Paths 2, 3 and 4 keep shrinking from day to day, indicating that the traffic flow pattern evolves toward user equilibrium. As the model (45) and (46) is built upon the same behavioral assumption as the SRM using proportional adjustment defined by $\boldsymbol{\Delta}(\mathbf{x})$, its convergence to user equilibrium has been proved by Smith and Mounce (2011).

6. Concluding remarks

In this paper, a generalized flow splitting model is developed, which can serve as a modeling framework for discrete-time day-to-day traffic assignment. The representation of the generalized flow splitting model relies on the specification of outflow distribution function that can be specified by adopting various path choice behavior assumptions. We provide two examples where the outflow distribution function is specified using the concept of proportional adjustment and projection operator, respectively. As these models are defined on decomposed acyclic sub-networks, the orderings of nodes and links provide attractive mathematical properties in updating the day-to-day link flows. Particularly, the invariance property of the proposed model, and the Lipschitz continuity and strong monotonicity of the generalized cost function are proved rigorously. A numerical example illustrates the ease of developing the flow splitting model and its convergence to user equilibrium flow pattern.

In future research, the sufficient conditions can be derived for the asymptotic stability of the generalized flow splitting model, i.e., what kinds of outflow distribution functions can preserve system stability? By doing so, we can ignore the tedious stability analysis as long as the distribution function satisfies the sufficient conditions. In addition, efficient algorithms for determining an acyclic sub-network should be considered before applying the proposed model to day-to-day traffic assignment on large-scale networks. Finally, routing and control variables can be introduced into the formulation of the generalized flow splitting model.

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