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The Channel Assignment Problem for Mutually Adjacent Sites

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Fix a finite interference set T of nonnegative integers, $0 \in T$. A *T-coloring* of a simple graph $G = (V, E)$ is a function $f\colon V \to \{0, 1, 2, \ldots\}$ such that for $\{u, v\} \in E(G)$, $|f(u) - f(v)| \notin T$. Let σ_n denote the optimal span of the T-colorings f of the complete graph K_n, that is, $\sigma_n = \min_f\{\max_{u, v \in V}|f(u) - f(v)|\}$. It was shown by Rabinowitz and Proulx that the asymptotic coloring efficiency $rt(T) := \lim_{n \to \infty}(n/\sigma_n)$ exists for every set T. We prove the stronger result that the difference sequence $\{\sigma_{n+1} - \sigma_n\}_{n=1}^{\infty}$ is eventually periodic for any T. The entire sequence $\sigma := (\sigma_n)_{n=1}^{\infty}$ is determined by a finite number of coloring strategies. The greedy (first-fit) T-coloring of K_n also leads to an eventually periodic sequence. We prove these results by studying a special directed graph $D(T)$. Earlier work of Cantor and Gordon on sequences with missing differences in T is discussed in connection with T-coloring.

1. INTRODUCTION

In the channel assignment or T-coloring problem introduced by Hale [4], an integer broadcast channel is assigned to each of several locations so that interference between nearby locations is avoided. The interference is modeled by a finite set T of nonnegative integers, including 0, of forbid-

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den channel differences. Let the simple graph $G = (V, E)$ be formed by the channel location set V with edge set E containing nearby (interfering) pairs of locations. Then a valid channel assignment, called a *T-coloring*, is a function f: $V(G) \to \{0, 1, \dots\}$, such that for $\{u, v\} \in E(G)$, we have $|f(u) - f(v)| \notin T$. The *T-span* of f is the difference of the largest and smallest colors used in $f(V)$. The *T-span* of G, denoted $sp_T(G)$, is the minimum span of any T-coloring f of G. To compute the T-span of a graph, we need only consider T-colorings f that use 0, and for these f the T-span is simply $\max_v f(v)$. Conventional vertex coloring is the case $T = \{0\}$, where $sp_T(G) = \chi(G) - 1$.

Early results on T-colorings were obtained by Cozzens and Roberts [2] (also see [11]). Among their findings is that for all graphs G with largest clique size $\omega(G)$,

$$sp_T(K_{\omega(G)}) \le sp_T(G) \le sp_T(K_{\chi(G)}).$$

In particular, if $\chi(G) = \omega(G)$, then $sp_T(G) = sp_T(K_{\chi(G)})$. Many examples of sets T have been found for which every graph G satisfies $sp_T(G) = sp_T(K_{\chi(G)})$, so that the chromatic number determines the T-span [2, 6, 7, 10]. It is clear then that the computation of the T-spans of cliques plays a fundamental role in the theory of T-colorings.

In this article we study the infinite *optimal sequence*

$$\sigma := (\sigma_n)_{n=1}^{\infty}, \qquad \text{where } \sigma_n := sp_T(K_n),$$

and its *difference sequence*

$$\Delta\sigma = (\Delta\sigma_n)_{n=2}^{\infty}, \qquad \text{where } \Delta\sigma_n = \sigma_n - \sigma_{n-1}.$$

The sequence σ clearly begins with 0 and is strictly increasing. In the literature on T-coloring, Rabinowitz and Proulx [9] introduced the asymptotic coloring efficiency $rt(T) := \lim_{n \to \infty}(n/\sigma_n)$, which measures the proportion of all available integers that are used asymptotically in optimal T-colorings of K_n. They proved that the limit exists and is a rational number at most $\frac{1}{2}$, except it is 1 when $T = \{0\}$.

In Section 2 we introduce our main tool, which is a special directed graph $D(T)$ with arclengths such that the optimal T-colorings are equivalent to shortest walks in $D(T)$. Several revealing examples of optimal sequences are presented. The graphs $D(T)$ are used in Section 3 to derive the existence and rationality of $rt(T)$. We find it more natural to work with its reciprocal, $R = R(T) := \lim_{n\to\infty}(\sigma_n/n)$, which has a nice interpretation as the minimum average step length of any cycle in $D(T)$. Included among some other remarks about $rt(T)$ is the fact that $rt(T) \geq 1/|T|$, which follows from the simple new bound, $\sigma_n \leq (n-1)|T|$.

Since T-colorings of K_n correspond to sequences of n integers, it is natural to consider increasing *infinite* sequences S of nonnegative integers such that no two terms differ by an element of T. In this form the problem was already studied by number theorists, a fact which was not realized by graph theorists until this work was recently brought to our attention. Motzkin [8] proposed studying the supremum $\mu(T)$ of the asymptotic upper densities of these sequences S. Let $\tau = t(T) := \max\{t : t \in T\}$. Cantor and Gordon [1] proved in 1973 that there exists a periodic T-coloring sequence S^{CG}, with period at most 2^τ, such that S^{CG} has density $\mu(T)$. Our description of $rt(T)$ in terms of the average step length allows us to conclude what one would expect, that $rt(T)$ and $\mu(T)$ are the same. So results on sequences with missing differences can be applied to T-colorings.

The results mentioned above concern the asymptotic behavior of the optimal sequences σ. Our main result deals with σ_n in general. In Section 4 it is shown that the difference sequence $\Delta\sigma$ is *eventually periodic*, which means that after some number of terms, the sequence $\Delta\sigma$ consists of a repeating pattern. Further, the period and the initial segment length can be bounded in terms of τ.

We compare the optimal T-span σ_n to the *greedy* (first-fit) span γ_n of K_n, which is obtained by coloring the vertices sequentially, always using the lowest nonnegative integer that will not create a difference in T. We form the sequence $\gamma := (\gamma_n)_{n=1}^{\infty}$. In general, greedy coloring is worse than optimal coloring, even in the asymptotic sense. Using $D(T)$, we see easily that the difference sequence for greedy colorings, $\Delta\gamma = (\Delta\gamma_n)_{n=2}^{\infty}$, where $\Delta\gamma_n = \gamma_n - \gamma_{n-1}$, is eventually periodic. It is comparatively easy to show this, since the greedy T-coloring of K_n begins with the greedy T-coloring of K_{n-1}. This is not the case for the optimal sequence σ, which does not itself correspond to a valid T-coloring sequence. We present an example in which no single coloring strategy is optimal for all n or even for all sufficiently large n.

We show in Section 5 that for any T there is a finite set of coloring strategies such that for all n, σ_n is attained by some strategy in the set. The strategies are eventually periodic T-colorings. This generalizes our main result, that $\Delta\sigma$ is eventually periodic, but unlike before, this approach does not yield a bound on the period and when it starts.

The paper concludes with some directions for future research.

2. T-Sequences and Directed T-Graphs

Consider any set T with $\tau = \tau(T) > 0$. We construct a directed graph, denoted by $D(T)$, such that directed walks starting from a certain vertex correspond to good T-colorings of complete graphs.

A *T-sequence of order* $n \geq 1$ is a sequence of n positive integers, $s = (a_1, a_2, a_3, \ldots, a_n)$, such that $a_i \leq \tau + 1$ for all i and for any $1 \leq m' \leq m \leq n$, $\sum_{i=m'}^{m} a_i \notin T$. We consider the empty sequence, denoted by Λ, to be a *T-sequence of order* 0. Let $sum(s)$ denote the sum of the terms of sequence s. For example, if $T = \{0, 1, 4, 5\}$, then $(3, 3, 3, 3)$ is a T-sequence (with order 4 and sum 12) but $(2, 2, 6, 2)$ is not.

A *good* T-coloring of K_n is a T-coloring that begins with 0 and has no jump between consecutive terms exceeding $\tau(T) + 1$. Good T-colorings of K_n correspond to T-sequences of order $n - 1$ and vice versa, by taking the elements of the sequence to be the jumps between consecutive labels in the coloring. The T-span of a good T-coloring is the sum of its corresponding T-sequence. For example, when $T = \{0, 1, 4, 5\}$, the T-coloring $\{5, 8, 17, 19\}$ of K_4 can be translated and its jump of 9 can be compressed to yield the good T-coloring $\{0, 3, 9, 11\}$, which corresponds to the T-sequence $(3, 6, 2)$. This coloring has span 11, which is still not optimal for K_4.

Clearly, every optimal T-coloring for K_n using 0 is good, so it suffices to consider only good T-colorings of K_n. Therefore, we see that for $n \geq 0$, σ_{n+1} is the minimum sum of any T-sequence of order n.

Let $s = (a_1, \ldots, a_n)$ be a T-sequence. Every T-sequence $s' = (a_1, \ldots, a_{n+1})$ consists of s together with $a_{n+1} > 0$ such that for all q with $1 \leq q \leq n + 1$, we have $\sum_{i=q}^{n+1} a_i \notin T$. We define $p(s)$ to be the subsequence $(a_q, a_{q+1}, \ldots, a_n)$ where q is the smallest value such that $\sum_{i=q}^{n} a_i < \tau$. So $p(s)$ consists of precisely the terms in s that affect the possible values for a_{n+1} in s'.

We are now ready to define the directed graph $D(T)$ together with its associated arclength function. Its vertex set $V(D)$ consists of all T-sequences s with $sum(s) < \tau$. Its arc set $E(D)$ consists of an arc $(u, p(s))$ of length b from each $u = (a_1, \ldots, a_n) \in V(D)$ to $p(s)$ for each T-sequence s of the form $(a_1, \ldots, a_n, b)$, where $1 \leq b \leq \tau + 1$. Note that there is an arc from each vertex to Λ of length $\tau + 1$, including a loop at Λ. In fact, all arcs entering a single vertex v have the same length, which we denote by $l(v)$.

Each T-sequence of order $n \geq 0$ corresponds to one and only one directed n-step walk (i.e., a walk with n arcs) starting from Λ in $D(T)$. The sum of the T-sequence is the total length (sum of arclengths) of the corresponding walk W in $D(T)$, denoted $l(W)$.

PROPOSITION 1. *Given T and $n \geq 0$,*

$$\sigma_{n+1} = \min\{l(W_n) : W_n \text{ is an } n\text{-step directed walk starting at } \Lambda\}. \blacksquare$$

EXAMPLE 1. Recall the standard example $T = \{0, 1, 4, 5\}$ (Hale [4]). The graph $D(T)$ is shown in Fig. 1, including its arclengths. It is routine to verify that the shortest walk in $D(T)$ from Λ with n steps is $[\Lambda(2)]$ for $n = 1$ and $[\Lambda(3)(3)\ldots(3)]$ for $n > 1$. The corresponding T-sequences are (2) and $(3, 3, \ldots)$, which yield the optimal sequence and difference sequence

$$\sigma = (0, 2, 6, 9, 12, \ldots)$$
$$\Delta\sigma = (2, 4, 3, 3, 3, \ldots).$$

For comparison, note that the greedy coloring gives

$$\gamma = (0, 2, 8, 10, 16, 18, \ldots)$$
$$\Delta\gamma = (2, 6, 2, 6, 2, \ldots).$$

EXAMPLE 2. Consider $T = \{0, 4\}$. Figure 2 shows $D(T)$. One can calculate from $D(T)$ that the greedy coloring is always optimal:

$$\sigma = \gamma = (0, 1, 2, 3, 8, 9, 10, 11, 16, \ldots)$$
$$\Delta\sigma = \Delta\gamma = (1, 1, 1, 5, 1, 1, 1, 5, \ldots).$$

Indeed, for any τ, the set $T = \{0, \tau\}$ has been shown to have the property that the greedy coloring is optimal (Raychaudhuri [10]). The optimal sequence repeats the pattern of $\tau - 1$ 1's followed by $\tau + 1$. Even with the assistance of the graph $D(T)$, this observation is not immediate for general τ: One must carefully check the walks, which is in fact equivalent

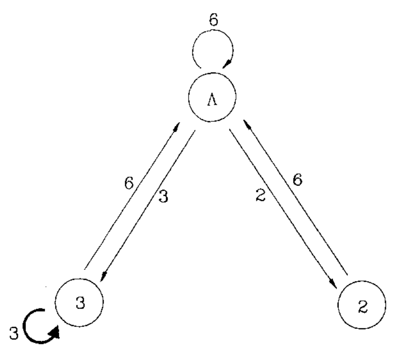

FIG. 1. The graph $D(T)$ for $T = \{0, 1, 4, 5\}$. The optimal cycle is indicated with a bold arc

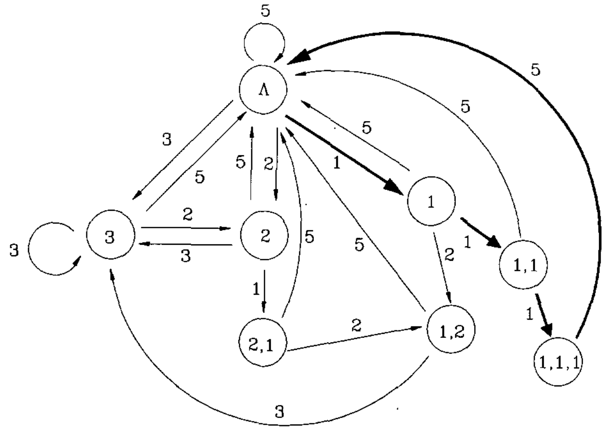

FIG. 2. The graph $D(T)$ for $T = \{0, 4\}$.

to doing the original T-colorings. Liu [6, 7] gave a shorter proof using graph homomorphisms.

EXAMPLE 3. Let $T = \{0, 2, 4, 6, 7, 9, 12\}$. Greedy T-coloring in this case obtains

$$\gamma = (0, 1, 11, 14, 19, 22, 27, 30, \dots)$$
$$\Delta\gamma = (1, 10, 3, 5, 3, 5, \dots).$$

Another T-coloring in this case is

$$\alpha = (0, 3, 8, 11, 16, 19, 24, 27, \dots)$$
$$\Delta\alpha = (3, 5, 3, 5, 3, 5, \dots).$$

By the T-graph shown in Fig. 3, one can obtain the optimal T-sequence:

$$\sigma = (0, 1, 8, 11, 16, 19, 24, 27, \dots)$$
$$\Delta\sigma = (1, 7, 3, 5, 3, 5, 3, 5, \dots).$$

EXAMPLE 4. Let $T = [0, 15] - \{4, 5, 8, 11\}$, with $D(T)$ as shown in Fig. 4. The greedy T-coloring is easily found to be

$$\gamma = (0, 4, 8, 24, 28, 32, 48, 52, 56, 72, \dots)$$
$$\Delta\gamma = (4, 4, 16, 4, 4, 16, 4, 4, 16, \dots).$$

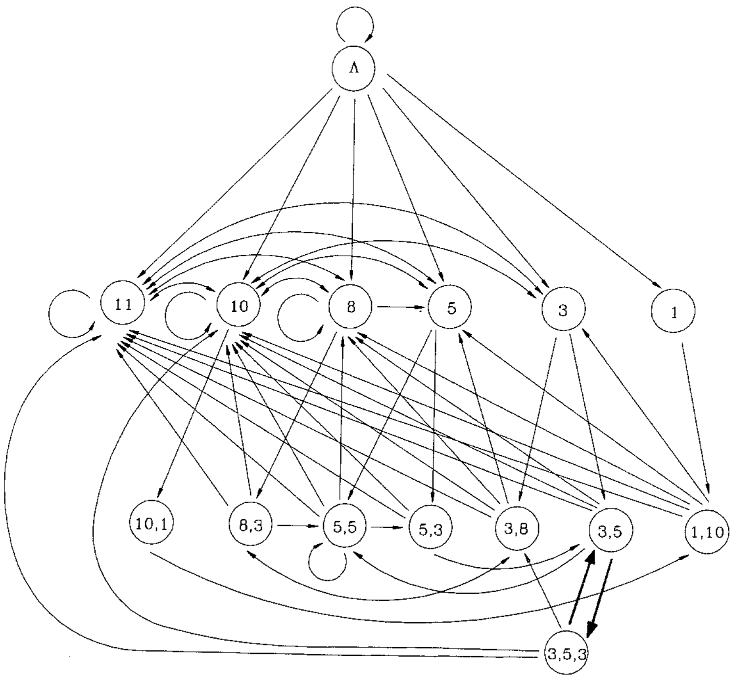

FIG. 3. The graph $D(T)$ for $T = \{0, 2, 4, 6, 7, 9, 12\}$ with arclengths omitted and arcs into vertex Λ not shown.

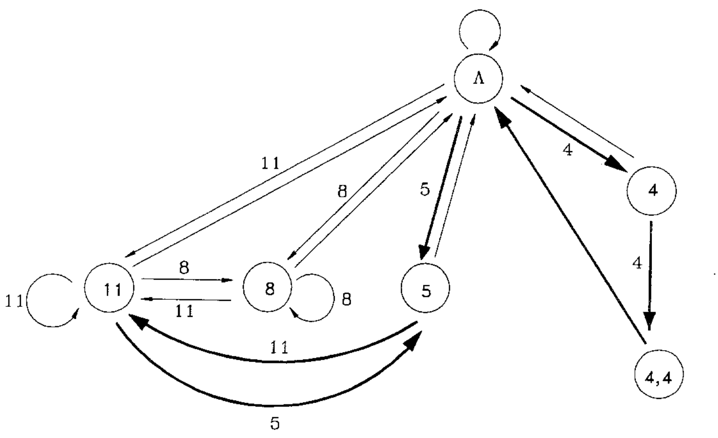

FIG. 4. The graph $D(T)$ for $T = [0, 15] - \{4, 5, 8, 11\}$. Arcs into Λ have length 16.

Two more T-colorings are given by α and β below:

$$\begin{aligned}
\alpha &= (0, 5, 16, 21, 32, 37, 48, 53, 64, 69, \dots) \\
\Delta\alpha &= (5, 11, 5, 11, 5, 11, 5, 11, \dots) \\
\beta &= (0, 4, 8, 24, 29, 40, 45, 56, 61, 72, 77, \dots) \\
\Delta\beta &= (4, 4, 16, 5, 11, 5, 11, 5, 11, 5, 11, \dots).
\end{aligned}$$

One can check that the optimal sequence is derived from the combination of the three sequences above.

$$\begin{aligned}
\sigma &= (0, 4, 8, 21, 28, 32, 45, 52, 56, 69, \dots) \\
\Delta\sigma &= (4, 4, 13, 7, 4, 13, 7, 4, 13, \dots).
\end{aligned}$$

No single strategy is optimal for all n, nor even for all sufficiently large n. Another feature of this example is that most terms of $\Delta\sigma$ belong to the set T of forbidden differences.

3. The Asymptotic Coloring Efficiency

Rabinowitz and Proulx [9] discovered the existence and rationality of the asymptotic coloring efficiency $rt(T)$. There is a nice interpretation for $rt(T)$ in terms of the graph $D(T)$ that leads to an easier direct proof. Our proof involves searching for a *cycle* (closed directed walk with no repeated vertices except its ends) of minimum average length per step. The proof in [9] also uses cycles, but our graph and arguments (derived independently) are considerably simpler. We find it more convenient to work with the reciprocal limit, $R(T) := \lim_{n\to\infty}(\sigma_n/n)$. For any walk W in $D(T)$, let $l(W)$ and $s(W)$ denote its length and number of steps (arcs). Let its average length be $\bar{l}(W) := l(W)/s(W)$. The walk is *simple* if no vertices are repeated.

Theorem 2. *For any T, the limit $R(T) = \lim_{n\to\infty}(\sigma_n/n)$ exists and equals $\min_C \bar{l}(C)$, the minimum taken over all cycles C in $D(T)$. For $T = \{0\}$, $R(T) = 1$; otherwise $R(T) \geq 2$.*

Proof. If $T = \{0\}$, then $\sigma_n = n - 1$, so that $R(T) = 1$. In this case, $D(T)$ consists of a loop at Λ with length 1, and we have $\min_C \bar{l}(C) = 1 = R(T)$.

Suppose now that $\tau(T) \geq 1$. Let $\bar{l}^* := \min_C \bar{l}(C)$, the minimum taken over the (finite) collection of nonempty cycles in $D(T)$, and let C^* be a cycle attaining the minimum. For $n \geq |V(D)|$, we can construct a walk W_n on n steps as follows: Take a simple walk P from Λ to some vertex v of

C^*, with P otherwise avoiding C^*. Then cycle around C^* until n steps have been taken. If we combine any incomplete trip around C^* at the end of P_n with P, we obtain a simple walk Q. This gives

$$\sigma_n \le l(W_n) = l(Q) + (n - s(Q))\bar{l}^*,$$

so that for all n, we have $\sigma_n \le c_1 + n\bar{l}^*$, where $c_1 = c_1(T)$ is constant.

On the other hand, consider a shortest walk on n steps starting at A, call it $Q_n = [Av_1v_2\dots v_n]$. We can decompose Q_n into cycles C^i and a simple walk Q', so that $E(Q_n) = E(Q') \cup E(C^1) \cup E(C^2) \cup \cdots \cup E(C^m)$. This gives $\sigma_n = l(Q_n) \ge \Sigma_{i=1}^m l(C^i) \ge (n - s(Q'))\bar{l}^*$. Thus, for all n we have $\sigma_n \ge c_2 + n\bar{l}^*$, where $c_2 = c_2(T)$ is a constant.

We have shown that $|\sigma_n - n\bar{l}^*|$ is bounded, and hence $R(T) = \lim_{n\to\infty}(\sigma_n/n) = \bar{l}^*$. Let $T' := \{0, \tau\}$, where $\tau = \tau(T)$. Then for all m, we see that $sp_T(K_m) \ge sp_{T'}(K_m)$. From Example 2 we calculate that $R(T') = 2$, so that $R(T) \ge 2$. ∎

As noted in the Introduction, Cantor and Gordon [1] proved that for any set T, there is a periodic T-sequence $S^{\text{CG}} = \{\alpha_n^{\text{CG}}\}_{n=0}^{\infty}$, with asymptotic density, denoted by $\delta(S^{\text{CG}})$, equal to $\mu(T)$.

Corollary 3. *For any T, $\mu(T) = rt(T) = 1/R(T)$.*

Proof. Referring to the proof of Theorem 2, let $S^R = \{\alpha_n^R\}_{n=0}^{\infty}$, where $\alpha_n^R = l(W_n)$. Then $rt(T) = 1/\bar{l}(C^*) = \delta(S^R)$, which is at most the supremum $\mu(T)$ over all sequences. Now $\mu(T) = \delta(S^{\text{CG}}) = \lim_{n\to\infty}(n/\alpha_n^{\text{CG}})$, which is at most $\lim_{n\to\infty}(n/\sigma_n) = rt(T)$, since the optimal span $\sigma_n \le \alpha_n^{\text{CG}}$. ∎

We next sketch a very short proof of the existence of the limit $R(T) = 1/rt(T)$ due to Michael Filaseta (private communication). Observe that the sequence (σ_n/n) is bounded since for all n, $0 \le \sigma_n/n \le \tau + 1$. Then if the sequence (σ_n/n) does not have a limit as $n \to \infty$, it must have at least two accumulation points. But one can show that this cannot happen by properly applying the elementary fact that $\sigma_{m+n} > \sigma_m + \sigma_n$ for all $m, n \ge 1$. Thus $R(T) = \lim_{n\to\infty}(\sigma_n/n)$ exists.

Cantor and Gordon [1] determined $\mu(T) = rt(T)$ for $|T| \le 3$, which was independently rediscovered by different methods by Rabinowitz and Proulx [9]. Haralambis [5] and Rabinowitz and Proulx [9] obtained partial results for $|T| = 4$. Tesman [13] (cf. Liu [7]) discovered that the set $T = \{0\} \cup [a, b]$, where $1 \le a \le b$, has $R(T) = (a + b)/a$, so all rationals $x \ge 2$ are achieved by appropriate sets T.

Some results of Cantor and Gordon [1] apply to infinite as well as finite coloring sets T. We cite one such result here because of its possible connection to other results in the T-coloring literature: If $T = \{0, t_1, t_2, \dots\}$

and d is a positive integer, then $\mu(T) = \mu(dT)$, where $dT := \{0, dt_1, dt_2, \dots\}$. Stewart and Tijdeman [12] also consider Motzkin's problem when the forbidden difference set is infinite. They give conditions on *infinite* sets T sufficient for $\mu(T) > 0$.

We next present a bound on the asymptotic coloring efficiency in terms of $|T|$.

THEOREM 4. *For any T and n, $\sigma_n \leq \gamma_n \leq (n-1)|T|$. Hence, $rt(T) \geq 1/|T|$, which is attained by $T = \{0, 1, \dots, r\}$.*

Proof. Fix the set T and n. When K_n is greedily T-colored, some label in the interval $[0, (n-1)|T|]$ is available for the nth vertex, since for each $i = 1, 2, \dots, n-1$, precisely $|T|$ integers differ from the label at vertex i by an element from the forbidden difference set T. Thus, $\gamma_n \leq (n-1)|T|$. Since an optimal coloring is no worse than the greedy one, $\sigma_n \leq \gamma_n$. For $T = \{0, 1, \dots, r\}$, we have $\sigma_n = (n-1)(r+1) = (n-1)|T|$. ∎

Since any graph G satisfies $sp_T(G) \leq sp_T(K_{\chi(G)})$, it follows immediately from Theorem 4 that $sp_T(G) \leq |T|(\chi(G) - 1)$, a result of Tesman [13, p. 23] that generalizes the σ_n bound in Theorem 4. Tesman's proof depended on several other results. The bound on γ_n in Theorem 4 is simple, yet apparently new. We now refer to the examples from Section 2 and their corresponding figures. In Example 1, we easily find that the loop at 3, which has $\bar{l} = 3$, is an optimal cycle. We see from the optimal sequence σ that $R(T) = 3$. What is particularly interesting about this example is that greedy is not asymptotically optimal: it grows at the rate $(2+6)/2 = 4$.

One can calculate directly from the optimal sequence for Example 2 that $R = 2$. The optimal cycle is the one used by greedy coloring, $[\Lambda(1)(1,1)(1,1,1)\Lambda]$, with $\bar{l} = (1+1+1+5)/4 = 2$.

The optimal cycle in Example 3 is $[(3,5)(3,5,3)(3,5)]$, so the minimum average length in this case is $(3+5)/2 = 4 = R(T)$. The cycles $[\Lambda(4)(4,4)\Lambda]$ and $[(5)(5,11)(5)]$ attain the optimal value of $(4+4+16)/3 = (5+11)/2 = 8 = R(T)$ in Example 4.

4. Eventual Periodicity

In all of the examples above, the optimal difference sequence $\Delta\sigma$ is periodic or quickly becomes periodic. We prove the eventual periodicity for general T, our main result, in Theorem 5 below by using the graphs $D(T)$. The examples also suggest that the greedy difference sequence $\Delta\gamma$ is eventually periodic, which we prove for general T in Theorem 6 below.

THEOREM 5. *Given T, the sequence $\Delta\sigma$ describing the optimal T-coloring of complete graphs is eventually periodic.*

Proof. If $T = \{0\}$, then $\sigma_n = n - 1$, and $\Delta\sigma = (1, 1, 1, \dots)$ is periodic. For the rest of the proof suppose that $\tau = \tau(T) \geq 1$. Let $c = c(T)$ be the maximum order of a vertex (T-sequence) in $V(D)$, and let $(b_1, b_2, \dots, b_c)$ be such a vertex. Note that $0 \leq c \leq \tau - 1$. We require the following fact.

CLAIM. *For any two vertices u, v in $V(D)$, there is a walk from u to v of exactly $c + 1$ steps with length $\leq 4\tau$.*

Proof of Claim. Let $v = (a_1, \dots, a_r)$, where r is the order of v. If $r = c$, then

$$[uA(a_1)(a_1, a_2) \dots (a_1, a_2, \dots, a_{c-1})v]$$

is a walk with $c + 1$ steps and length $\leq (\tau + 1) + \tau - 1 = 2\tau$. If v has order $r \leq c - 1$ then the walk

$$[uA(b_1)(b_1, b_2) \dots (b_1, b_2, \dots, b_{c-r-1})A(a_1)(a_1, a_2) \dots v]$$

consists of $c + 1$ steps and has length $\leq 4\tau$, since steps into A have length $\tau + 1$ and paths from A out to vertices have length at most $\tau - 1$. ∎

We now show that $\Delta\sigma$ is eventually periodic. For any $v \in V(D)$ and $n \geq 1$, define

$$f_n(v) := \min\{l(W) : W \text{ is a walk from } A \text{ to } v \text{ of } n - 1 \text{ steps}\}.$$

When no such walk W exists, $f_n(v)$ is taken to be ∞. By Proposition 1, we get

$$\sigma_n = sp_T(K_n) = \min_{v \in V(D)} f_n(v). \tag{1}$$

We then define

$$g_n(v) := f_n(v) - \sigma_n, \qquad \text{which is } \geq 0 \text{ for all } v. \tag{2}$$

Let $A(v)$ denote the set $\{w\colon (w, v) \in E(D)\}$. We recall that every arc into v has the same length $l(v)$, so that we can recursively compute the values of f_n:

$$\begin{aligned} f_{n+1}(v) &= l(v) + \min_{w \in A(v)} f_n(w), \\ &= l(v) + \min_{w \in A(v)} g_n(w) + \sigma_n. \end{aligned} \tag{3}$$

Hence,

$$\begin{aligned} \sigma_{n+1} &= \min_v f_{n+1}(v) \\ &= \min_v \left\{ l(v) + \min_{w \in A(v)} g_n(w) + \sigma_n \right\}, \end{aligned} \tag{4}$$

$$\Delta\sigma_{n+1} = \sigma_{n+1} - \sigma_n = \min_v \left\{ l(v) + \min_{w \in A(v)} g_n(w) \right\}. \tag{5}$$

By the claim, for any $w, v \in V(D)$,

$$f_{n+c+1}(w) \leq f_n(v) + 4\tau,$$

which implies by minimizing over v that for all w,

$$f_{n+c+1}(w) \leq \sigma_n + 4\tau.$$

Also, for all w',

$$\sigma_n + c + 1 \leq \sigma_{n+c+1} \leq f_{n+c+1}(w').$$

Therefore, for any $w, w' \in V(D)$,

$$|f_{n+c+1}(w) - f_{n+c+1}(w')| \leq 4\tau - (c+1) \leq 4\tau - 1.$$

Since $g_{n+c+1}(w') = 0$ for some w', it follows that for all w, $0 \leq g_{n+c+1}(w) \leq 4\tau - 1$. So there exist n_1, n_2 with $c + 1 \leq n_1 < n_2 \leq (4\tau)^{|V(D)|} + c + 1$, such that $g_{n_1}(w) = g_{n_2}(w)$, for all $v \in V(D)$. By Eq. (5), we get $\Delta\sigma_{n_1+1} = \Delta\sigma_{n_2+1}$. By Eqs. (2) and (3), we obtain for all v and n,

$$g_{n+1}(v) = l(v) + \min_{w \in A(v)} g_n(w) - \Delta\sigma_{n+1}.$$

This implies $g_{n_1+1}(w) = g_{n_2+1}(w)$, for all $w \in V(D)$. Repeating the above procedure, we obtain, for all $k \geq 1$, $\Delta\sigma_{n_1+k} = \Delta\sigma_{n_2+k}$. Therefore, $\Delta\sigma$ is eventually periodic with period at most $(n_2 - n_1) \leq (4\tau)^{|V(D)|}$, and its first period is completed after at most its first $n_2 \leq (4\tau)^{|V(D)|+\tau}$ terms. ▌

THEOREM 6. *Given T, the sequence $\Delta\gamma$ describing the greedy T-coloring of complete graphs is eventually periodic.*

Proof. The greedy T-coloring of K_n uses label 0 at the first vertex and never jumps more than $\tau + 1$, so it corresponds to a walk $[Av_1 \ldots v_{n-1}]$ of $n - 1$ steps in $D(T)$. When K_n is T-colored, the lowest available color is used for the next vertex. In $D(T)$ this corresponds to proceeding to the vertex v_n that is closest to v_{n-1}, i.e., along the shortest outgoing arc. When $n > |V(D)|$, there exist $i < j \leq |V(D)|$ such that $v_i = v_j$. Leaving either v_i or v_j, greedy goes to the same vertex, so that $v_{i+1} = v_{j+1}$, and by induction, greedy cycles with period $j - i$. ▌

5. OPTIMAL STRATEGIES

The examples in Section 2 suggest that a finite number of periodic coloring techniques suffice to achieve the optimum σ_n for all n. We now verify this.

We define a *strategy* $S = (W, C)$ to be a walk W in $D(T)$ starting at A together with an optimal cycle C (thus $\bar{l}(C) = \bar{l}^* = R(T)$) that starts and ends at a vertex v on W. Note that the vertex of attachment v need not be at the end of W. The strategy then specifies, for each n, a walk of n steps as follows: Given n, let j be the smallest integer ≥ 0 such that $n \leq s(W) + j(s(C))$. Form walk W_n by taking the first n steps along the walk consisting of W with j repetitions of cycle C inserted at the first occurrence of v. The strategy S generates a coloring sequence $\alpha(S) = (\alpha_0(S), \alpha_1(S), \dots)$, where $\alpha_n(S) = l(W_n)$. Clearly $\alpha(S)$ is eventually periodic with period $s(C)$ and $\alpha_n(S) \sim n\bar{l}^* = nR(T)$, as $n \to \infty$. A strange thing about our definition of strategies is that the sequence $\alpha(S)$ will not be increasing in general, since we may have $\alpha_{n+1}(S) < \alpha_n(S)$ when $n = s(W) + j(s(C))$.

THEOREM 7. *For any set T, there is a finite collection of T-coloring sequences, $\{\alpha(i) : i \in I\}$, where each $\Delta\alpha(i)$ is eventually periodic, such that for all n, the optimal value $\sigma_n = sp_T(K_n)$ is $\min_{i \in I} \alpha_n(i)$. Further, we may assume for all i that the average of the terms in one period of $\Delta\alpha(i)$ is $R(T) = \bar{l}^*$.*

Proof. Fix the set T and n. We show that there is a shortest walk of n steps for $D(T)$ that is specified by a strategy (W, C), with $s(W)$ bounded in terms of T independently of n. This will imply the theorem since the number of strategies (W, C) with bounded $s(W)$ is finite for given T.

Let Q_n be a shortest walk of n steps for $D(T)$. As in the proof of Theorem 2, Q_n decomposes into a simple walk Q' and cycles C_i which begin and end at their vertices of attachment. Form a walk W_1 using the edges from Q' and from one copy of each distinct cycle appearing in the C_i's. The walk W_1 reaches every vertex in Q_n.

Suppose there is an optimal cycle C among the C_i's, and let its vertex of attachment be v. Then no non-optimal cycle D appears more than $s(C)$ times among the C_i's, or else we could replace $s(C)$ copies of D in the list of cycles in Q_n not used for W_1 by $s(D)$ copies of C. Then reattaching the cycles in this list to W_1 would create a walk on n steps with length less than $l(Q_n)$, contradicting the optimality of Q_n. Similarly, we can repeatedly replace $s(C)$ appearances of the optimal cycle D in the list by $s(D)$ appearances of C. This shows that there is an optimal walk R_n in which no cycle besides C appears in the decomposition more than $s(C)$ times. Reattaching all cycles besides copies of C, if any, gives a decomposition of R_n into a walk W_2, with $s(W_2)$ bounded in terms of C, and copies of C. Attaching the copies of C at the first occurrence of v produces an optimal walk of n steps specified by the strategy (W_2, C).

There remains the case in which no cycle C_i for Q_n is optimal. Let C be any optimal cycle for $D(T)$. We can view Q_n as specified trivially by the strategy (Q_n, C). It then suffices to show that $s(Q_n) = n$ is bounded in terms of T. Let D be any C_i and suppose it appears N times in the decomposition after W_1 is removed. If N is not too small, we may replace the N copies of D by some number of copies of C, together with at most 2τ arcs to go from W_1 to C and back and at most $s(C)$ extra arcs (e.g., loops at Λ) to produce a walk S_n of exactly n steps. Because $\bar{l}(D) > \bar{l}(C)$ and because $l(S_n) \geq l(Q_n)$, N is bounded in terms of C, D, τ. Doing this for all D, we see that $s(Q_n)$ is bounded for given C, τ, i.e., in terms of T alone. ▌

6. Directions for Future Research

Given a cycle C^* in the graph $D(T)$ with the smallest average step length, a sensible approach for T-coloring large complete graphs is to take a path from Λ over to C^* and wind around C^* repeatedly. In fact, starting anywhere in C^* and winding around repeatedly corresponds to a periodic T-coloring that achieves $R(T)$ and so differs from σ by at most a constant for all n. Cantor and Gordon [1] showed how to reduce such a periodic T-coloring to one that still achieves $R(T)$ and has period at most 2^τ. However, since $V(D)$ can be very large in general, it may not be practical to actually locate an optimal cycle in $D(T)$.

Our proof of Theorem 6 shows that the period for greedy coloring and the number of terms before periodicity begins are bounded above by $|V(D)|$, which in turn is bounded for all T with given largest element τ. In the worst case, when $T = \{0, \tau\}$, $|V(D)|$ is the number of compositions of integers $< \tau$, a very large number. For the optimal sequence $\Delta\sigma$, the period and the number of terms before it starts are bounded in the proof of Theorem 5 by $(4\tau)^{|V(D)|}$. In all examples we have constructed, $\Delta\gamma$ and $\Delta\sigma$ become periodic very quickly, after a number of terms on the order of τ. It is important then to try to improve our discouragingly large bounds on the period and when it begins.

The proof of eventual periodicity in Theorem 5 depends on a repetition of all values of the function g. However, one does not have to compute all of these values. One can recognize the repeating pattern and guarantee its repetition, looking only at σ, after seeing only a bounded number of initial terms of $\Delta\sigma$, where the bound depends on T.

Considerable effort [3, 6, 7, 10, 13] has been devoted to studying sets T such that greedy is always optimal, i.e., $\gamma = \sigma$. In both Example 3 and Example 4 the greedy coloring, while not optimal, is asymptotically optimal, i.e., $\gamma_n \sim \sigma_n$ as $n \to \infty$. This occurs precisely when the greedy algorithm acting on $D(T)$ arrives at a cycle that has optimal average step length. Whenever this occurs, $\gamma_n - \sigma_n$ is bounded independently of n. It would be interesting to determine the sets T for which greedy is asymptotically optimal.

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