On the $s$th Laplacian eigenvalue of trees of order $st + 1$∗

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Abstract

Let $\lambda_1(\mathcal{G}) \geq \lambda_2(\mathcal{G}) \geq \cdots \geq \lambda_n(\mathcal{G}) = 0$ be the Laplacian eigenvalues of a simple undirected graph $\mathcal{G}$. Let $s \geq 2$ and $t \geq 2$ be integers and let $T_{s,t}$ be the rooted tree of three levels and order $st + 1$ such that the vertex root has degree $s$, the vertices in level 2 have degree $t$ and the $s(t - 1)$ pendants vertices are in level 3. We prove that

$$\lambda_s(T_{s,t}) = \max\{\lambda_s(\mathcal{T}) : \mathcal{T} \text{ is a tree of order } st + 1\} = \frac{1}{2} \left( t + 1 + \sqrt{t^2 + 2t - 3} \right).$$


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1. Preliminaries

Let $\mathcal{G} = (V, E)$ be a simple undirected graph on $n$ vertices. The Laplacian matrix of $\mathcal{G}$ is the $n \times n$ matrix $L(\mathcal{G}) = D(\mathcal{G}) - A(\mathcal{G})$ where $A(\mathcal{G})$ is the adjacency matrix and $D(\mathcal{G})$ is the diagonal matrix of vertex degrees. It is well known that $L(\mathcal{G})$ is a positive semidefinite matrix and that $(0, e)$ is an eigenpair of $L(\mathcal{G})$ where $e$ is the all ones vector. Let us denote the eigenvalues of $L(\mathcal{G})$ by $0 = \lambda_n(\mathcal{G}) \leq \lambda_{n-1}(\mathcal{G}) \leq \cdots \leq \lambda_2(\mathcal{G}) \leq \lambda_1(\mathcal{G})$.

In [5], some of the many results known for Laplacian matrices are given. Fiedler [2] proved that $\mathcal{G}$ is a connected graph if and only if $\lambda_{n-1}(\mathcal{G}) > 0$. This eigenvalue is called the algebraic connectivity of $\mathcal{G}$.

We recall that a tree is a connected acyclic graph.

Let $T$ be an unweighted rooted tree of $k$ levels such that in each level the vertices have equal degree. We agree that the vertex root is at level 1. For $j = 1, 2, 3, \ldots, k$, let $d_{k-j+1}$ and $n_{k-j+1}$ be the degree of the vertices and the number of them in level $j$. Observe that $d_k$ is the degree of the vertex root, $n_1 = 1$ and $n_1$ is the number of vertices in level $k$ (the number of pendant vertices). We assume $d_k > 1$. Let $\Omega = \{j : 1 \leq j \leq k - 1, n_j > n_{j+1}\}$ and let $\sigma(A)$ denotes the set of eigenvalues of a matrix $A$. In [6, 2005], we characterized completely the eigenvalues of $L(T)$. They are the eigenvalues of leading principal submatrices of a nonnegative symmetric tridiagonal matrix of order $k \times k$. More precisely

**Theorem 1** [6, Theorem 4]. If $T_j$ is the $j \times j$ leading principal submatrix of the $k \times k$ symmetric tridiagonal matrix

$$
T_k = \begin{bmatrix}
1 & \sqrt{d_2-1} & \sqrt{d_3-1} & \cdots & \sqrt{d_{k-1}-1} & \sqrt{d_k} \\
\sqrt{d_2-1} & d_2 & \sqrt{d_3-1} & \cdots & \sqrt{d_{k-1}-1} & \sqrt{d_k} \\
\sqrt{d_3-1} & d_3 & \sqrt{d_4-1} & \cdots & \sqrt{d_{k-1}-1} & \sqrt{d_k} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\sqrt{d_{k-1}-1} & d_{k-1} & \sqrt{d_{k-2}-1} & \cdots & \sqrt{d_2-1} & \sqrt{d_1} \\
\sqrt{d_k} & \sqrt{d_k} & \sqrt{d_k} & \cdots & \sqrt{d_2-1} & d_1
\end{bmatrix},
$$

then

(a) $\sigma(L(T)) = (\cup_{j \in \Omega} \sigma(T_j)) \cup \sigma(T_k)$

and

(b) the multiplicity of each eigenvalue of the matrix $T_j$, as an eigenvalue of $L(T)$, is $n_j - n_{j+1}$ for $j \in \Omega$, and the eigenvalues of $T_k$, as eigenvalues of $L(T)$, are simple.

Let $s \geq 2$ and $t \geq 2$ be given integers. We denote by $T_{s,t}$ the rooted tree of three levels and order $st + 1$ such that the vertex root has degree $s$, the vertices in level 2 have degree $t$ and the $s(t - 1)$ pendant vertices are in level 3. Let us illustrate the above notations and Theorem 1 with the tree $T_{3,5}$.

**Example 1.** For the tree $T_{3,5}$ we have $s = 3, t = 5, n_1 = (t - 1)s = 12, d_1 = 1, n_2 = s = 3, d_2 = t = 5, n_3 = 1, d_3 = s = 3$ and $\Omega = \{1, 2\}$. From Theorem 1
\[ \sigma(L(T_{s,t})) = \sigma(T_1) \cup \sigma(T_2) \cup \sigma(T_3), \]

where

\[ T_1 = [1], \quad T_2 = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} \quad \text{and} \quad T_3 = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 5 & \sqrt{3} \\ 0 & \sqrt{3} & 3 \end{bmatrix}. \]

**Lemma 1** [3]. If \( A \) is an \( m \times m \) symmetric tridiagonal matrix with nonzero codiagonal entries then the eigenvalues of any \( (m-1) \times (m-1) \) principal submatrix strictly interlace the eigenvalues of \( A \). In particular, the eigenvalues of such a matrix \( A \) are simple.

We are ready to characterize the eigenvalues of \( L(T_{s,t}) \) and their multiplicities.

**Theorem 2.** If

\[ T_1 = [1], \quad T_2 = \begin{bmatrix} 1 & \sqrt{t-1} \\ \sqrt{t-1} & t \end{bmatrix} \quad \text{and} \quad T_3 = \begin{bmatrix} 1 & \sqrt{t-1} & 0 \\ \sqrt{t-1} & t & \sqrt{3} \\ 0 & \sqrt{3} & s \end{bmatrix}, \]

then

\[ \sigma(L(T_{s,2})) = \sigma(T_2) \cup \sigma(T_3) \quad (1) \]
\[ \sigma(L(T_{s,1})) = \sigma(T_1) \cup \sigma(T_2) \cup \sigma(T_3) \quad \text{if} \ t > 2. \quad (2) \]

(b) The largest eigenvalue of \( T_3 \) is the largest eigenvalue of \( L(T_{s,1}) \) and the largest eigenvalue of \( T_2 \) is the second largest eigenvalue of \( L(T_{s,1}) \).

(c) The following table gives the eigenvalues of \( L(T_{s,t}) \) together with their corresponding multiplicities which are indicated in the last column:

| \( \lambda_1(T_{s,t}) = \frac{1}{2} \left( t + 1 \pm \sqrt{t^2 + 2t - 3} \right) \) | \( \lambda_2(T_{s,t}) = \frac{s(t - 2)}{s - 1} \) |
| \( \lambda_3(T_{s,t}) = \cdots = \lambda_s(T_{s,t}) = \frac{1}{2} \left( t + 1 + \sqrt{t^2 + 2t - 3} \right) \) | 1 |

**Proof.** (a) If \( t = 2 \) then \( n_1 = n_2 = s \) and thus \( \Omega = \{2\} \). If \( t > 2 \) then \( n_1 = s(t - 1) > n_2 = s \) and thus \( \Omega = \{1, 2\} \). We now apply Theorem 1, part (a), to obtain (1) and (2).

(b) From Lemma 1, we have that the eigenvalue of \( T_1 \) strictly interlaces the eigenvalues of \( T_2 \) and the eigenvalues of \( T_2 \) strictly interlace the eigenvalues of \( T_3 \). Thus (b) is proved.

(c) The eigenvalues of \( L(T_{s,t}) \) are easily obtained solving the characteristic equations of \( T_1, T_2 \) and \( T_3 \). Finally, from Theorem 1, part (b), and the fact that the characteristic equations of \( T_1, T_2 \) and \( T_3 \) do not have common roots, (c) is proved. \( \square \)

**Remark 1.** The eigenvalues of \( L(T_{s,t}) \) in decreasing order are

\[ \lambda_1(T_{s,t}) = \frac{1}{2} \left( s + t + 1 + \sqrt{(s + t + 1)^2 - 4(st + 1)} \right), \]
\[ \lambda_2(T_{s,t}) = \lambda_3(T_{s,t}) = \cdots = \lambda_s(T_{s,t}) = \frac{1}{2} \left( t + 1 + \sqrt{t^2 + 2t - 3} \right). \]
\[
\lambda_{s+1}(\mathcal{F}_{s,t}) = \lambda_{s+2}(\mathcal{F}_{s,t}) = \cdots = \lambda_{s(t-1)}(\mathcal{F}_{s,t}) = 1,
\]
\[
\lambda_{s(t-1)+1}(\mathcal{F}_{s,t}) = \lambda_{s(t-1)+2}(\mathcal{F}_{s,t}) = \cdots = \lambda_{st}(\mathcal{F}_{s,t}) = \frac{1}{2} \left( t + 1 - \sqrt{t^2 + 2t - 3} \right),
\]
\[
\lambda_{st+1}(\mathcal{F}_{s,t}) = 0.
\]

Observe that
\[
\lambda_s(\mathcal{F}_{s,t}) = \frac{1}{2} \left( t + 1 + \sqrt{t^2 + 2t - 3} \right)
\]
for all \( s \geq 2 \).

The case \( s = 2 \) and \( t \geq 4 \) is studied by Shao et al. in [8]. They prove the following theorem.

**Theorem 3** [8, Theorem 2.1(a)]. If \( \mathcal{F} \) is a tree of order \( 2t + 1 \) with \( t \geq 4 \) then
\[
\lambda_2(\mathcal{F}) \leq \frac{1}{2} \left( t + 1 + \sqrt{t^2 + 2t - 3} \right),
\]
with equality if and only if \( \mathcal{F} = \mathcal{F}_{2,t} \).

**Remark 2.** A direct computation proves that (4) is also true for trees of order 5 \((t = 2)\) and order 7 \((t = 3)\) with equality if and only if \( \mathcal{F} = \mathcal{F}_{2,2} \) and \( \mathcal{F} = \mathcal{F}_{2,3} \) respectively. Therefore, if \( \mathcal{F} \) is a tree of order \( 2t + 1 \) with \( t \geq 2 \) then
\[
\lambda_2(\mathcal{F}_{2,t}) = \max\{\lambda_2(\mathcal{F}) : \mathcal{F} \text{ is a tree of order } 2t + 1\}.
\]

In [8] the above mentioned authors propose the following conjecture.

**Conjecture 1.** Let \( t, s \) be positive integers with \( s \geq 3 \) and \( t \geq 2 \). Then
\[
\lambda_s(\mathcal{F}_{s,t}) = \max\{\lambda_s(\mathcal{F}) : \mathcal{F} \text{ is a tree of order } st + 1\}.
\]

In this paper, we prove that this conjecture is true.

**2. The largest \( s \)th Laplacian eigenvalue of trees of order \( st + 1 \)**

We recall the following facts.

Let \( \mathcal{G} \) be a graph and let \( \mathcal{G}' = \mathcal{G} + e \) be the graph obtained from \( \mathcal{G} \) by inserting a new edge \( e \) into \( \mathcal{G} \).

**Lemma 2** [1, Theorem 2.1]. The Laplacian eigenvalues of \( \mathcal{G} \) interlace the Laplacian eigenvalues of \( \mathcal{G}' \):
\[
0 = \lambda_{n+1}(\mathcal{G}') = \lambda_n(\mathcal{G}) \leq \lambda_n(\mathcal{G}') \leq \cdots \leq \lambda_2(\mathcal{G}') \leq \lambda_2(\mathcal{G}) \leq \cdots \leq \lambda_1(\mathcal{G}) \leq \lambda_1(\mathcal{G}').
\]

From Lemma 2, we immediately have the following corollary.

**Corollary 1.** If \( \mathcal{G}_1 \) is a subgraph of order \( m \) of the graph \( \mathcal{G} \) then \( \lambda_k(\mathcal{G}_1) \leq \lambda_k(\mathcal{G}) \) for \( k = 1, 2, \ldots, m \).

**Lemma 3** [4]. If \( A \) is an \( n \times n \) Hermitian matrix with eigenvalues \( \lambda_n \leq \lambda_{n-1} \leq \cdots \leq \lambda_1 \) and \( B \) is an \( m \times m \) principal submatrix of \( A \) with eigenvalues \( \mu_m \leq \mu_{m-1} \leq \cdots \leq \mu_1 \) then
\[
\lambda_{n-m+i} \leq \mu_i \leq \lambda_i \\
\text{for } i = 1, 2, \ldots, m. \text{ If } m = n - 1 \text{ then} \\
\lambda_{i+1} \leq \mu_i \leq \lambda_i \\
\text{for } i = 1, 2, \ldots, n - 1.
\]

**Lemma 4** [7]. Let \( T \) be a tree of order \( n \). For any positive integer \( a \) there exists a vertex \( v \) of \( T \) such that there is one component of \( T - v \) with order not exceeding \( \max\{n - a - 1, a\} \) and all the other components of \( T - v \) have orders not exceeding \( a \).

Before to state the next lemma, let us denote by \( L_v(G) \) the principal submatrix obtained by deleting from the Laplacian matrix \( L(G) \) the row and column corresponding to the vertex \( v \) of \( G \).

**Lemma 5** [8]. Let \( T = (V, E) \). Let \( v \in V \). Let \( T_1, T_2, \ldots, T_p \) be all the connected components of \( T - v \). For \( j = 1, 2, \ldots, p \), let \( v_j \) be the unique vertex in \( T_j \) such that \( vv_j \in E \) and let \( T_j' \) be the tree obtained from \( T_j \) by adding the vertex \( v \) and the edge \( vv_j \) to \( T_j \). Then, labeling the vertices of \( T \) such that \( v \) is the first one and, for \( 1 \leq i < j \leq l \), any vertex of \( T_i \) preceedes any vertex of \( T_j \), we have

\[
L_v(T) = L_v(T_1') \oplus L_v(T_2') \oplus \cdots \oplus L_v(T_p') \tag{5}
\]

From now on, let

\[
b(t) = \frac{1}{2} \left( t + 1 + \sqrt{t^2 + 2t - 3} \right).
\]

Clearly, for \( t \geq 2 \), \( b(t) \) is a strictly increasing function and \( t < b(t) \). From (3), we recall that

\[
\lambda_s(T_{s,t}) = \frac{1}{2} \left( t + 1 + \sqrt{t^2 + 2t - 3} \right)
\]

for all \( s \geq 2 \).

Let us denote by

\[
\lambda_n(A) \leq \lambda_{n-1}(A) \leq \cdots \leq \lambda_2(A) \leq \lambda_1(A),
\]

the eigenvalues of an \( n \times n \) matrix \( A \) with only real eigenvalues.

**Lemma 6.** Let \( T \) be a tree of order \( r \) with \( r \leq t \) and let \( T' \) be the tree of order \( r + 1 \) obtained from \( T \) by adding a new vertex \( v \) to \( T \) and a new edge \( uv \) between \( v \) and some vertex \( u \) of \( T \). Then

\[
\lambda_1(L_v(T')) \leq b(t).
\]

**Proof.** Take two copies of the tree \( T' \). Let \( \mathcal{D} \) be the tree of order \( 2r + 1 \) obtained by identifying the two vertices \( v \) in the two copies of \( T' \). Then

\[
L_v(\mathcal{D}) = L_v(T') \oplus L_v(T').
\]

From Lemma 3, we have

\[
\lambda_1(L_v(T')) = \lambda_2(L_v(\mathcal{D})) \leq \lambda_2(L_v(\mathcal{D})) \leq \lambda_1(L_v(\mathcal{D})) = \lambda_1(L_v(T')).
\]
Hence
\[ \lambda_2(\mathcal{G}) = \lambda_2(L(\mathcal{G})) = \lambda_1(L_v(\mathcal{T}')). \]
Thus, by Theorem 3, we have
\[ \lambda_1(L_v(\mathcal{T}')) = \lambda_2(\mathcal{G}) \leq b(r) \leq b(t), \]
and the proof is complete. □

We are ready to prove our main result in this paper.

**Theorem 4.** If \( s \geq 2 \) and \( t \geq 2 \) are given integers then
\[ \lambda_s(\mathcal{T}_{st}) = \max\{\lambda_s(\mathcal{T}) : \mathcal{T} \text{ is a tree of order } st + 1\}. \]

**Proof.** By induction on \( s \). From Theorem 3 and Remark 2, the result is true for \( s = 2 \). Let \( s \geq 3 \).

We assume that
\[ \lambda_{s-1}(\mathcal{T}_{s-1,r}) = \max\{\lambda_{s-1}(\mathcal{T}) : \mathcal{T} \text{ is a tree of order } (s-1)t + 1\}. \]

Let \( \mathcal{G} \) be a tree of order not exceeding \((s-1)t + 1\). Let \( \mathcal{T} \) be a tree of order \((s-1)t + 1\) obtained from \( \mathcal{G} \) by inserting new edges. From Corollary 1 and the hypothesis of induction, it follows that
\[ \lambda_{s-1}(\mathcal{G}) \leq \lambda_{s-1}(\mathcal{T}) \leq \lambda_{s-1}(\mathcal{T}_{s-1,r}). \]

Since \( \lambda_{s-1}(\mathcal{T}_{s-1,r}) = b(t) \), we have
\[ \lambda_{s-1}(\mathcal{G}) \leq b(t) \tag{6} \]
for all tree \( \mathcal{G} \) of order not exceeding \((s-1)t + 1\). Let \( \mathcal{T} \) be a tree of order \( st + 1 \). Let \( a = t \).

Then, \( \max\{st + 1 - t - 1, t\} = (s-1)t \). From Lemma 4, there exists a vertex \( v \) of \( \mathcal{T} \) such that there is one component of \( \mathcal{T} - v \), say \( \mathcal{T}_1 \), with order not exceeding \((s-1)t \) and such that all the other components of \( \mathcal{T} - v \), say \( \mathcal{T}_2, \ldots, \mathcal{T}_p \), have orders not exceeding \( t \). Let \( \mathcal{T}_1', \mathcal{T}_2', \ldots, \mathcal{T}_p' \) as in Lemma 5. Thus, \( \mathcal{T}_1' \) has order not exceeding \((s-1)t + 1 \) and \( \mathcal{T}_2', \ldots, \mathcal{T}_p' \) have orders not exceeding \( t + 1 \).

We apply (6) to the tree \( \mathcal{T}_1' \) to obtain
\[ \lambda_{s-1}(L_v(\mathcal{T}_1')) \leq \lambda_{s-1}(\mathcal{T}_1') \leq b(t). \tag{7} \]
We now apply Lemma 6 to the trees \( \mathcal{T}_2', \ldots, \mathcal{T}_p' \) to get
\[ \lambda_1(L_v(\mathcal{T}_j')) \leq b(t) \quad \text{for } j = 2, 3, \ldots, p. \tag{8} \]
We claim that
\[ \lambda_{s-1}(L_v(\mathcal{T})) \leq \max\{\lambda_{s-1}(L_v(\mathcal{T}_1')), \lambda_1(L_v(\mathcal{T}_2')), \ldots, \lambda_1(L_v(\mathcal{T}_p'))\}. \tag{9} \]
Suppose that
\[ \lambda_{s-1}(L_v(\mathcal{T})) > \max\{\lambda_{s-1}(L_v(\mathcal{T}_1')), \lambda_1(L_v(\mathcal{T}_2')), \ldots, \lambda_1(L_v(\mathcal{T}_p'))\}. \]
From this assumption and (5), we obtain that there are at most \( s - 2 \) many eigenvalues of \( L_v(\mathcal{T}) \) which can be greater than or equal to \( \lambda_{s-1}(L_v(\mathcal{T})) \). Clearly, this is a contradiction. Thus the inequality (9) is proved. From (9), (7) and (8), it follows that
\[ \lambda_{s-1}(L_v(\mathcal{T})) \leq b(t). \]
Finally, from this inequality and the fact that \( \lambda_s(\mathcal{T}) \leq \lambda_{s-1}(L_v(\mathcal{T})) \), we have
\[ \lambda_s(\mathcal{T}) \leq b(t) = \lambda_s(\mathcal{T}_{st}), \]
which completes the proof. □
Example 2. Let $n = 31$. Then

\[
\begin{align*}
\max\{\lambda_2(\mathcal{T}) : \mathcal{T} \text{ is a tree of order } 31\} &= \lambda_2(\mathcal{T}_{2,15}) = 15.9073 = b(15), \\
\max\{\lambda_3(\mathcal{T}) : \mathcal{T} \text{ is a tree of order } 31\} &= \lambda_3(\mathcal{T}_{3,10}) = 10.9083 = b(10), \\
\max\{\lambda_5(\mathcal{T}) : \mathcal{T} \text{ is a tree of order } 31\} &= \lambda_5(\mathcal{T}_{5,6}) = 6.8541 = b(6), \\
\max\{\lambda_6(\mathcal{T}) : \mathcal{T} \text{ is a tree of order } 31\} &= \lambda_6(\mathcal{T}_{6,5}) = 5.8284 = b(5), \\
\max\{\lambda_{10}(\mathcal{T}) : \mathcal{T} \text{ is a tree of order } 31\} &= \lambda_{10}(\mathcal{T}_{10,3}) = 3.7321 = b(3)
\end{align*}
\]

and

\[
\begin{align*}
\max\{\lambda_{15}(\mathcal{T}) : \mathcal{T} \text{ is a tree of order } 31\} &= \lambda_{15}(\mathcal{T}_{15,2}) = 2.6180 = b(2), \\
\end{align*}
\]

rounded to 4 decimal places.

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