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ON THE CONNECTION BETWEEN REAL AND COMPLEX INTERPOLATION OF QUASI-BANACH SPACES (*)

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ABSTRACT. – We describe a new approach to interpolate by the complex method quasi-Banach couples formed by real-intermediate spaces. End-point cases are also considered, and applications are given to function spaces and to operator spaces. © Elsevier, Paris.

0. Introduction

Among the known relationships between real and complex interpolation methods, the following formula due to LIONS, BERGH and KARADZOV

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(*see* [1], Theorem 4.7.2) is specially useful in characterizing complex interpolation spaces

$$(1) \qquad [(A_0, A_1)_{\theta_0, q_0}, (A_0, A_1)_{\theta_1, q_1}]_\lambda = (A_0, A_1)_{\theta, q}.$$

Here (A_0, A_1) is any Banach couple, $0 < \theta_0, \theta_1, \lambda < 1$, $\theta = (1-\lambda)\theta_0 + \lambda\theta_1$, $1 \leq q_0, q_1 \leq \infty$, assuming however that q_0 and q_1 are not both equal to ∞, and $1/q = (1-\lambda)/q_0 + \lambda/q_1$.

In this paper, we investigate the validity of (1) and its end-point versions for quasi-Banach couples. This question has already been studied by several authors (*see*, for example, [4], [5] and [11]) and the outcome has found important applications in determining complex interpolation spaces between quasi-Banach couples (for instance, H_p-spaces with $p < 1$).

The usual approach to this problem in the literature is based on the definition of the complex method. However, our techniques rest on the construction of the real interpolation space. So, we prove the left side inclusion in (1) by realizing $(A_0, A_1)_{\theta, q}$ as a J-space. This gives a new proof for a result in [4]. For the right side embedding we use the description of $(A_0, A_1)_{\theta, q}$ by means of the K-functional.

This new approach also works in the end-point cases, that is to say, when we take A_1 instead of $(A_0, A_1)_{\theta_1, q_1}$. In particular, we show that for all $a \in A_0 \cap A_1$ we have

$$\|a\|_{(A_0, A_1)_{(1-\lambda)\theta+\lambda, q/(1-\lambda)}} \leq M \|a\|_{[(A_0, A_1)_{\theta, q}, A_1]_\lambda}$$

where $0 < \theta, \lambda < 1$, $0 < q < \infty$ and (A_0, A_1) is a quasi-Banach couple satisfying a quite natural hypothesis (*see* condition $(\mathbf{h})$ in the next section). For the special case $q = 1$, this formula was obtained by PISIER [11], page 115, under a rather restrictive assumption on the couple (A_0, A_1).

Ideas similar to those developed here have been used in [5] and [3] to investigate the connection between real and complex interpolation spaces for some particular Banach couples.

In the last section of the paper, applications are given to function spaces and to operator spaces. We also show that complex reiteration formula is valid for quasi-Banach couples formed by real interpolation from a couple satisfying $(\mathbf{h})$.

1. Interpolation of quasi-Banach couples

We start by reviewing the interpolation methods that we shall deal with. Let (A_0, A_1) be a quasi-Banach couple, let $0 < \theta < 1$ and $0 < q \leq \infty$. The real interpolation space $(A_0, A_1)_{\theta, q}$ consists of all elements $a \in A_0 + A_1$ having a finite quasi-norm

$$\|a\|_{\theta, q} = \left(\sum_{\nu \in \mathbb{Z}} (2^{-\theta\nu} K(2^\nu, a))^q\right)^{1/q}$$

(the sum should be replaced by the supremum if $q = \infty$). Here, for $0 < t < \infty$, we put

$$K(t, a) = K(t, a; A_0, A_1) = \inf\{\|a_0\|_{A_0} + t\|a_1\|_{A_1} ; a = a_0 + a_1, a_j \in A_j\}$$

The space $(A_0, A_1)_{\theta, q}$ can also be described by means of the J-functional

$$J(t, a) = J(t, a; A_0, A_1) = \max\{\|a\|_{A_0}, t\|a\|_{A_1}\}.$$

It turns out that $a \in (A_0, A_1)_{\theta, q}$ if and only if a can be represented in the form

$$a = \sum_{\nu=-\infty}^{\infty} u_\nu \quad \text{(convergence in } A_0 + A_1)$$

with $(u_\nu) \subset A_0 \cap A_1$ and

$$\left(\sum_{\nu=-\infty}^{\infty} (2^{-\theta\nu} J(2^\nu, u_\nu))^q\right)^{1/q} < \infty. \tag{2}$$

Moreover, taking the infimum of the values of the sum (2) over all representations of a as above, we get a quasi-norm equivalent to $\|\cdot\|_{\theta, q}$. In the sequel, we denote by $\|\cdot\|_{\theta, q}$ any of these two quasi-norms. This however will not cause any confusion. *See* [1] and [14] for full details on this construction.

In order to introduce the complex interpolation space $[A_0, A_1]_\theta$, consider the open strip $S = \{z \in \mathbb{C};\ 0 < \operatorname{Re} z < 1\}$ and let $\mathcal{F}(S)$ be the space of all scalar valued functions f continuous and bounded on $\bar{S} = \{z \in \mathbb{C};\ 0 \leq \operatorname{Re} z \leq 1\}$ and analytic on S. Denote by $\mathcal{F}(A_0, A_1)$ the collection of all functions f that can be written as a finite sum $f(z) = \sum_{k=1}^{N} f_k(z) a_k$ where $f_k \in \mathcal{F}(S)$ and $a_k \in A_0 \cap A_1$. We put

$$\|f\|_{\mathcal{F}} = \max\{\sup_{s \in \mathbb{R}} \|f(is)\|_{A_0}, \sup_{s \in \mathbb{R}} \|f(1 + is)\|_{A_1}\}$$

and for all $a \in A_0 \cap A_1$ let

$$\|a\|_{[\theta]} = \inf\{\|f\|_{\mathcal{F}} : f(\theta) = a, \ f \in \mathcal{F}(A_0, A_1)\}.$$

Since the maximum principle may fail for functions taking values in a quasi-Banach space, the functional $\|\cdot\|_{[\theta]}$ is, in general, only a semi-quasi-norm. Let

$$N = \{a \in A_0 \cap A_1; \ \|a\|_{[\theta]} = 0\},$$

then $(A_0 \cap A_1/N, \|\cdot\|_{[\theta]})$ is a quasi-normed space. We define $[A_0, A_1]_\theta$ as the completion of $(A_0 \cap A_1/N, \|\cdot\|_{[\theta]})$ (*see* [4]).

Observe that for Banach couples we recover the usual complex interpolation method (*see* [1] and [14]). In the present quasi-Banach context, the presence of the quotient $A_0 \cap A_1/N$ and the subsequent completion produce important obstructions in developing the theory. We shall discuss this matter a little later in this section.

Proceedings as in the Banach case, one can check that if (B_0, B_1) is another quasi-Banach couple and $T : A_j \to B_j$ is a linear operator with $\|Ta_j\|_{B_j} \leq M_j\|a_j\|_{A_j}$ $(j = 0, 1)$, then for $0 < \theta < 1$ and $a \in A_0 \cap A_1$ we have

$$\|Ta\|_{[B_0, B_1]_\theta} \leq M_0^{1-\theta} M_1^\theta \|a\|_{[A_0, A_1]_\theta}.$$

We can then extend $T : (A_0 \cap A_1, \|\cdot\|_{[\theta]}) \to [B_0, B_1]_\theta$ to an operator acting from $[A_0, A_1]_\theta$ into $[B_0, B_1]_\theta$ that we still denote by the same letter T. Subsequently, we refer to this result as the interpolation theorem. Due to the quotient and extension involved in it, we must be more careful when using the interpolation theorem than in the case of Banach spaces.

The next property can be also established as in the Banach case: Let $0 < \theta < 1$ and $f \in \mathcal{F}(A_0, A_1)$, then

$$\text{(3)} \quad \log\|f(\theta)\|_{[\theta]} \leq \int_{-\infty}^{\infty} \log\|f(is)\|_{A_0} P_0(\theta, s)\, ds + \int_{-\infty}^{\infty} \log\|f(1+is)\|_{A_1} P_1(\theta, s)\, ds.$$

Here $P_j(\theta, s)$ are the Poisson kernels for S. Thus

$$\int_{-\infty}^{\infty} P_0(\theta, s)\, ds = 1 - \theta, \qquad \int_{-\infty}^{\infty} P_1(\theta, s)\, ds = \theta.$$

Take now $\alpha,\ \beta > 0$. Since the functions $e^{(1-\theta)\alpha s}$ and $e^{\theta\beta s}$ are convex, using [12], Lemma 2.1.1 (or [2], Prop. 3) it follows from inequality (3) that

$$(4)\quad \|f(\theta)\|_{[\theta]} \le \left(\frac{1}{1-\theta}\int_{-\infty}^{\infty}\|f(is)\|_{A_0}^{(1-\theta)\alpha}P_0(\theta,\,s)\,ds\right)^{1/\alpha} \cdot\left(\frac{1}{\theta}\int_{-\infty}^{\infty}\|f(1+is)\|_{A_1}^{\theta\beta}P_1(\theta,\,s)\,ds\right)^{1/\beta}.$$

Let us analyze the relationship between $[A_0,\ A_1]_\theta$ and spaces $A_0\cap A_1$, A_0+A_1. Here we are going to exhibit important differences to the Banach case.

If $a\in A_0\cap A_1$, it is easy to see that

$$(5)\qquad \|a\|_{[\theta]} \le \max\{\|a\|_{A_0},\ \|a\|_{A_1}\} := \|a\|_{A_0\cap A_1}\,.$$

But the quasi-norm of $[A_0,\ A_1]_\theta$ may be identically zero (*see* [13], § 3). If for a given couple $(A_0,\ A_1)$ it turns out that $\|\cdot\|_{[\theta]}$ is a quasi-norm, then we have the continuous (and dense) inclusion

$$A_0\cap A_1 \hookrightarrow [A_0,\ A_1]_\theta.$$

On the other hand, $[A_0,\ A_1]_\theta$ may not be identifiable to a subspace of A_0+A_1. This problem may arise even if $(A_0\cap A_1,\ \|\cdot\|_{[\theta]})$ is continuously embedded in A_0+A_1 because the extension of this map to $[A_0,\ A_1]_\theta$ may fail to be one to one.

One has some control on $[A_0,\ A_1]_\theta$ if the couple $(A_0,\ A_1)$ satisfies the following condition (*see* [9], § 5):

(**h**) There exists $0<\theta_0<1$ and $C>0$ such that for all $f\in\tilde{\mathcal{F}}(A_0,\ A_1)$ and all $0<t<\infty$

$$K(t,\ f(\theta_0)) \le C\,\sup\nolimits_{z\in\partial S}\{K(t,\ f(z))\}.$$

Here ∂S stands for the boundary of S and $\tilde{\mathcal{F}}(A_0,\ A_1)$ is defined similarly to $\mathcal{F}(A_0,\ A_1)$ but allowing now the vectors a_k to belong to A_0+A_1.

Indeed, let us first show that if (**h**) holds for some $\theta_0\in(0,\ 1)$ then it holds for every $\theta\in(0,\ 1)$ and with the same constant: Take any $f\in\tilde{\mathcal{F}}(A_0,\ A_1)$ and let g be a conformal map of S onto itself such that $g(\theta_0)=\theta$. Then $f(\theta)=f(g(\theta_0))$ and $f\circ g$ belongs to $\tilde{\mathcal{F}}(A_0,\ A_1)$ as well. Where

$$K(t,\ f(\theta)) = K(t,\ f(g(\theta_0))) \le C\,\sup\nolimits_{z\in\partial S}\{K(t,\ f(g(z)))\} = C\,\sup\nolimits_{w\in\partial S}\{K(t,\ f(w))\}.$$

Now, given any $a \in A_0 \cap A_1$, we can check that

$$\|a\|_{A_0+A_1} := K(1, a) \le C\|a\|_{[\theta]}$$

or even the stronger inequality

$$\|a\|_{\theta,\infty} = \sup_{\nu\in\mathbb{Z}}(2^{-\theta\nu}K(2^\nu, a)) \le C\|a\|_{[\theta]}$$

by using the same arguments as in the Banach case (*see* [1], Theorem 4.7.1). The constant C in these inequalities is the same as the one in $(\mathbf{h})$. As a consequence, we see that $\|\cdot\|_{[\theta]}$ is a quasi-norm in this case and so $A_0 \cap A_1$ is densely and continously embedded in $[A_0, A_1]_\theta$.

For $\nu \in \mathbb{Z}$, put

$$F_\nu = (A_0 + A_1, K(2^\nu, \cdot)$$

and consider the couple (F_ν, F_ν). The K-functional for (F_ν, F_ν) is related to the quasi-norm $\|\cdot\|_{F_\nu} = K(2^\nu, \cdot; A_0, A_1)$. Namely,

$$K(t, a; F_\nu, F_\nu) \le \min(1, t)\|a\|_{F_\nu} \le cK(t, a; F_\nu, F_\nu).$$

Here c is the constant in the quasi-triangle inequality for $\|\cdot\|_{F_\nu}$ (*i.e.* $c = \max\{c_{A_0}, c_{A_1}\}$ where c_{A_j} is the corresponding constant for A_j). It is clear that if (A_0, A_1) satisfies $(\mathbf{h})$ with constant C then (F_ν, F_ν) also satisfies $(\mathbf{h})$ now with the constant $M = cC$. Our previous considerations show that for any $a \in F_\nu$ it holds

$$\|a\|_{F_\nu} \le c\|a\|_{F_\nu+F_\nu} \le cM\|a\|_{[F_\nu, F_\nu]_\theta} \le cM\|a\|_{F_\nu\cap F_\nu} = cM\|a\|_{F_\nu}.$$

Hence, for every $\theta \in (0, 1)$, we have

$$[F_\nu, F_\nu]_\theta = F_\nu. \tag{6}$$

with equivalence of quasi-norms. Furthermore, composing functions of $\mathcal{F}(F_\nu, F_\nu)$ with conformal mappings of S onto itself one can check that

$$[F_\nu, F_\nu]_\theta = [F_\nu, F_\nu]_\eta$$

with equality of quasi-norms for any $\theta, \eta \in (0, 1)$.

For later use we calculate now the space $[2^{-\nu s_0}F_\nu, 2^{-\nu s_1}F_\nu]_\theta$. Here $s_0, s_1 \in \mathbb{R}$ and by $2^{-\nu s_j}F_\nu$ we mean the space

$$(A_0 + A_1, 2^{-\nu s_j}K(2^\nu, \cdot; A_0, A_1)).$$

First note that if (A_0, A_1) satisfies $(\mathbf{h})$ then the couple $(2^{-\nu s_0} F_\nu, 2^{-\nu s_1} F_\nu)$ also satisfies $(\mathbf{h})$, because this couple has the same constant in $(\mathbf{h})$ as the couple (F_ν, F_ν), this again in view of the formula

$$K(t, a; 2^{-\nu s_0} F_\nu, 2^{-\nu s_1} F_\nu) = 2^{-\nu s_0} K(t\, 2^{-\nu (s_1 - s_0)}, a; F_\nu, F_\nu).$$

Since the transformation $f(z) \to 2^{-\nu s_0 (1-z) - \nu s_1 z} f(z)$ is an isometry between $\mathcal{F}(2^{-\nu s_0} F_\nu, 2^{-\nu s_1} F_\nu)$ and $\mathcal{F}(F_\nu, F_\nu)$, for every $a \in F_\nu$ we have

$$\|a\|_{[2^{-\nu s_0} F_\nu, 2^{-\nu s_1} F_\nu]} = 2^{-\nu s} \|a\|_{[F_\nu, F_\nu]_\theta} \sim 2^{-\nu s} \|a\|_{F_\nu} .$$

Here $s = (1-\theta) s_0 + \theta s_1$ and, as we showed in (6), the constants in the equivalence $\sim$ only depend on the couple (A_0, A_1). Therefore

$$[2^{-\nu s_0} F_\nu, 2^{-\nu s_1} F_\nu]_\theta = 2^{-\nu s} F_\nu. \tag{7}$$

Examples of couples satisfying hypothesis $(\mathbf{h})$ are (L_{p_0}, L_{p_1}) and (H_{p_0}, H_{p_1}) (*see* [9], § 5).

For more information on the complex method for quasi-Banach couples we refer to [4], [12], [7], [13], [2].

2. Connections between real and complex interpolation

Let $(A_\nu)_{\nu \in \mathbb{Z}}$ be a sequence of quasi-Banach spaces, let c_{A_ν} be the constant in the quasi-triangle inequality of $\| \cdot \|_{A_\nu}$ and suppose that

$$\sup_{\nu \in \mathbb{Z}} \{c_{A_\nu}\} < \infty.$$

For $0 < q \leq \infty$, we denote by $\ell_q(A_\nu)$ the usual vector-valued ℓ_q-space defined by means of the sequence (A_ν), that is to say,

$$\ell_q(A_\nu) = \{(a_\nu);\ a_\nu \in A_\nu \text{ and } \|(a_\nu)\|_{\ell_q(A_\nu)} < \infty\}$$

where

$$\|(a_\nu)\|_{\ell_q(A_\nu)} = \Big(\sum_{\nu=-\infty}^{\infty} \|a_\nu\|_{A_\nu}^q\Big)^{1/q}$$

(with the usual supremum interpretation for the case $q = \infty$).

The following result will be important in our later considerations. The proof is similar to the one in the case for Banach couples, but in the present situation some new difficulties arise in carrying over the arguments.

LEMMA 1. – *Set* $\{(A_\nu, B_\nu)\}_{\nu\in\mathbb{Z}}$ *be a sequence of quasi-Banach couples such that*

(8) $$\sup_{\nu\in\mathbb{Z}}\{c_{A_\nu}, c_{B_\nu}\} < \infty$$

let $0 < \theta < 1$, $0 < q_0, q_1 \le \infty$, *assuming however that* q_0 *and* q_1 *are not both equal to* ∞, *and let* $1/q = (1-\theta)/q_0 + \theta/q_1$.

Denote by D *the set of all sequences* $a = (a_\nu)$ *having only a finite number of coordinates* $a_\nu \neq 0$ *and with* $a_\nu \in A_\nu \cap B_\nu$ *or every* $\nu \in \mathbb{Z}$. *Given any* $a \in D$, *we have*

(9) $$\|a\|_{[\ell_{q_0}(A_\nu),\, \ell_{q_1}(B_\nu)]_\theta} \le \|a\|_{\ell_q([A_\nu,\, B_\nu]_\theta)}\,.$$

Moreover, if for each couple (A_ν, B_ν) *the functional* $\|\cdot\|_{[\theta]}$ *is a quasi-norm in* $A_\nu \cap B_\nu$, *then*

(10) $$\ell_q([A_\nu, B_\nu]_\theta) = [\ell_{q_0}(A_\nu), \ell_{q_1}(B_\nu)]_\theta.$$

Proof. – Assumption (8) gives that

$$\sup_{\nu\in\mathbb{Z}}\{c_{[A_\nu,\, B_\nu]_\theta}\} < \infty$$

so the quasi-Banach space $\ell_q([A_\nu, B_\nu]_\theta)$ is well-defined.

Inequality (9) follows by the same argument as in [14], Theorem 1.18.1/Step 2.

In order to establish (10), let us first show that for any $a = (a_\nu) \in \ell_{q_0}(A_\nu) \cap \ell_{q_1}(B_\nu)$ we have

(11) $$\|a\|_{\ell_q([A_\nu,\, B_\nu]_\theta)} \le \|a\|_{[\ell_{q_0}(A_\nu),\, \ell_{q_1}(B_\nu)]_\theta}\,.$$

Let $f = (f_\nu) \in \mathcal{F}(\ell_{q_0}(A_\nu), \ell_{q_1}(B_\nu))$ with $f_\nu \in \mathcal{F}(A_\nu, B_\nu)$ and $f_\nu(\theta) = a_\nu$ for every $\nu \in \mathbb{Z}$. The choice $\alpha = q_0/(1-\theta)$, $\beta = q_1/\theta$ in (4) yields

$$\|f_\nu(\theta)\|_{[A_\nu,\, B_\nu]_\theta} \le \left(\frac{1}{1-\theta}\int_{-\infty}^{\infty}\|f_\nu(is)\|_{A_\nu}^{q_0} P_0(\theta, s)\, ds\right)^{(1-\theta)/q_0} \cdot \left(\frac{1}{\theta}\int_{-\infty}^{\infty}\|f_\nu(1+is)\|_{B_\nu}^{q_1} P_1(\theta, s)\, ds\right)^{\theta/q_1}.$$

Hence, using Hölder's inequality with $(1-\theta)\, q/q_0 + \theta\, q/q_1 = 1$, we derive

$$\begin{aligned}
\|a\|_{\ell_q([A_\nu,B_\nu]_\theta)} &\le \Bigg[\sum_{\nu=-\infty}^{\infty}\left(\frac{1}{1-\theta}\int_{-\infty}^{\infty}\|f_\nu(is)\|_{A_\nu}^{q_0} P_0(\theta,s)ds\right)^{(1-\theta)\,q/q_0} \\
&\qquad \cdot\left(\frac{1}{\theta}\int_{-\infty}^{\infty}\|f_\nu(1+is)\|_{B_\nu}^{q_1} P_1(\theta,\, s)\, ds\right)^{\theta q/q_1}\Bigg]^{1/q} \\
&\le \left(\frac{1}{1-\theta}\int_{-\infty}^{\infty}\sum_{\nu=-\infty}^{\infty}\|f_\nu(is)\|_{A_\nu}^{q_0} P_0(\theta,\, s)\, ds\right)^{(1-\theta)/q_0} \\
&\qquad \cdot\left(\frac{1}{\theta}\int_{-\infty}^{\infty}\sum_{\nu=-\infty}^{\infty}\|f_\nu(1+is)\|_{B_\nu}^{q_1} P_1(\theta,\, s)\, ds\right)^{\theta/q_1} \\
&\le \left(\sup\nolimits_{s\in\mathbb{R}}\|f\,(is)\|_{\ell_{q_0}(A_\nu)}\right)^{1-\theta}\left(\sup\nolimits_{s\in\mathbb{R}}\|f\,(1+is)\|_{\ell_{q_1}(B_\nu)}\right)^{\theta} \\
&\le \|f\|_{\mathcal{F}(\ell_{q_0}(A_\nu),\,\ell_{q_1}(B_\nu))}.
\end{aligned}$$

Taking the infimum (11) follows.

By assumption, for each couple $(A_\nu,\, B_\nu)$ the functional $\|\cdot\|_{[\theta]}$ is a quasi-norm in $A_\nu \cap B_\nu$. This implies that $A_\nu \cap B_\nu$ is dense in $[A_\nu,\, B_\nu]_\theta$ for each $\nu \in \mathbb{Z}$ and so D is dense in $\ell_q([A_\nu, B_\nu]_\theta)$.

On the other hand, as a consequence of (11), we find that $\|\cdot\|_{[\ell_{q_0}(A_\nu),\,\ell_{q_1}(B_\nu)]_\theta}$ is a quasi-norm in $\ell_{q_0}(A_\nu) \cap \ell_{q_1}(B_\nu)$. Whence $\ell_{q_0}(A_\nu) \cap \ell_{q_1}(B_\nu)$ is dense in $[\ell_{q_0}(A_\nu),\, \ell_{q_1}(B_\nu)]_\theta$. Next we show that any element of $\ell_{q_0}(A_\nu) \cap \ell_{q_1}(B_\nu)$ can be approximated by elements of D, which implies that D is also dense in $[\ell_{q_0}(A_\nu),\, \ell_{q_1}(B_\nu)]_\theta$.

Given $a = (a_\nu) \in \ell_{q_0}(A_\nu) \cap \ell_{q_1}(B_\nu)$, let us associate to a the sequence (a^n) of elements of D defined by

$$a^n = (a_\nu^n) \qquad \text{with} \quad a_\nu^n = \begin{cases} a_\nu & \text{if} \quad |\nu| \le n \\ 0 & \text{otherwise} \end{cases}.$$

We know that at least one of the q_j is finite. Let it be q_0. Then

$$\|a - a^n\|_{\ell_{q_0}(A_\nu)} \to 0 \quad \text{as } n \to \infty.$$

Moreover

$$\|a - a^n\|_{\ell_{q_1}(B_\nu)} \le \|a\|_{\ell_{q_1}(B_\nu)}.$$

Hence

$$\|a - a''\|_{[\ell_{q_0}(A_\nu), \ell_{q_1}(B_\nu)]_\theta} \leq \|a - a''\|^{1-\theta}_{\ell_{q_0}(A_\nu)} \|a - a''\|^{\theta}_{\ell_{q_1}(B_\nu)}$$
$$\leq \|a - a''\|^{1-\theta}_{\ell_{q_0}(A_\nu)} \|a\|^{\theta}_{\ell_{q_1}(B_\nu)} \to 0 \quad \text{as } n \to \infty.$$

The density of D is $[\ell_{q_0}(A_\nu), \ell_{q_1}(B_\nu)]_\theta$ and $\ell_q([A_\nu, B_\nu]_\theta)$ together with (9) and (11) give equality (10). The proof is complete. ■

Next, consider a quasi-Banach couple (A_0, A_1) fulfilling hypothesis $(\mathbf{h})$ and, for $\nu \in \mathbb{Z}$ and $s_0, s_1 \in \mathbb{R}$, put

$$2^{-s_j\nu} F_\nu = (A_0 + A_1, 2^{-s_j\nu} K(2^\nu, \cdot)).$$

As we checked in Section 1, in this hypothesis, for each couple $(2^{-s_0\nu} F_\nu, 2^{-s_1\nu} F_\nu)$ the functional $\| \cdot \|_{[\theta]}$ is a quasi-norm in $2^{-s_0\nu} F_\nu \cap 2^{-s_1\nu} F_\nu$. Besides, the constants in the quasi-triangle inequality satisfy

$$\sup_{\nu \in \mathbb{Z}} \{c_{2^{-s_0\nu} F_\nu}, c_{2^{-s_1\nu} F_\nu}\} \leq \max\{c_{A_0}, c_{A_1}\} < \infty.$$

Hence, as a direct application of (7) and of Lemma 1 we obtain

COROLLARY 2. – *Let $0 < q_0, q_1 < \infty$ or $0 < q_0 < \infty$ and $q_1 = \infty$, let $-\infty < s_0, s_1 < \infty$, $0 < \theta < 1$, $1/q = (1-\theta)/q_0 + \theta/q_1$ and $s = (1-\theta) s_0 + \theta s_1$. Then*

(i) $$[\ell_{q_0}(2^{-s_0\nu} F_\nu), \ell_{q_1}(2^{-s_1\nu} F_\nu)]_\theta = \ell_q(2^{-s\nu} F_\nu).$$

In particular, for $A_0 = A_1 = \mathbb{C}$, we get with equality of quasi-norms

(ii) $$[\ell_{q_0}(2^{-s_0\nu}), \ell_{q_1}(2^{-s_1\nu})]_\theta = \ell_q(2^{-s\nu}).$$

Now we are in a position to establish

THEOREM 3. – *Let (A_0, A_1) be a quasi-Banach couple satisfying $(\mathbf{h})$. Let $0 < \theta_0, \theta_1, \lambda < 1$, $0 < q_0, q_1 < \infty$ or $0 < q_0 < \infty$ and $q_1 = \infty$. Put $\theta = (1-\lambda)\theta_0 + \lambda\theta_1$, $1/q = (1-\lambda)/q_0 + \lambda/q_1$ and $E_j = (A_0, A_1)_{\theta_j, q_j}$. Then there exists a constant M such that for all $a \in E_0 \cap E_1$ we have*

$$\|a\|_{(A_0, A_1)_{\theta, q}} \leq \|a\|_{[E_0, E_1]_\lambda}.$$

Proof. – Write as before

$$2^{-\nu s_j} F_\nu = (A_0 + A_1,\ 2^{-\nu s_j} K(2^\nu, \cdot)).$$

Realizing $E_j = (A_0, A_1)_{\theta_j, q_j}$ as a K-space, we see that the operator R associating to each $a \in E_j$ the constant sequence $R(a) = (\dots, a, a, , \dots)$ is bounded from

$$E_0 \to \ell_{q_0}(2^{-\theta_0 \nu} F_\nu) \qquad \text{and} \qquad E_1 \to \ell_{q_1}(2^{-\theta_1 \nu} F_\nu)$$

and its quasi-norm is $\lesssim 1$ in both cases. Therefore, interpolating by the complex method, we obtain that

$$R : [E_0, E_1]_\lambda \to [\ell_{q_0}(2^{-\theta_0 \nu} F_\nu),\ \ell_{q_1}(2^{-\theta_1 \nu} F_\nu)]_\lambda$$

is bounded as well. We can identify the last space by using Corollary 2/(i). Hence

$$R : [E_0, E_1]_\lambda \to \ell_q(2^{-\theta \nu} F_\nu)$$

is bounded. In other words, there is a constant $M > 0$ such that for all $a \in E_0 \cap E_1$ we have

$$\|a\|_{(A_0, A_1)_{\theta, q}} = \|R(a)\|_{\ell_q(2^{-\theta\nu} F_\nu)} \leq M \|a\|_{[E_0, E_1]_\lambda}. \quad \blacksquare$$

The same method works in the end-point case where we replace $(A_0, A_1)_{\theta_1, q_1}$ by A_1:

THEOREM 4. – *Let (B, A_1) be a quasi-Banach couple fulfilling condition* $(\mathbf{h})$. *Let $0 < \eta < 1$, $0 < q < \infty$ and put*

$$A_0 = (B, A_1)_{\eta, q}.$$

If $0 < \theta < 1$ and $1/p = (1 - \theta)/q$ then there is a constant M such that for all $a \in A_0 \cap A_1$ we have

$$\|a\|_{(A_0, A_1)_{\theta, p}} \leq M \|a\|_{[A_0, A_1]_\theta}.$$

Proof. – Given $\lambda > 0$, we now denote by λF_ν the space $B + A_1$ quasi-normed by $\lambda K(2^\nu, \cdot; B, A_1)$. We get again that the operator

$$R : A_0 \to \ell_q(2^{-\eta\nu} F_\nu)$$

is bounded. Moreover, a direct estimate shows that

$$R : A_1 \to \ell_\infty(2^{-\nu} F_\nu)$$

is bounded. So, interpolating and using Corollary 2/(i) we derive the boundedness of

$$R : [A_0, A_1]_\theta \to [\ell_q(2^{-\eta\nu} F_\nu), \ell_\infty(2^{-\nu} F_\nu)]_\theta = \ell_p(2^{-((1-\theta)\eta+\theta)\nu} F_\nu).$$

Whence there is a constant M such that for all $a \in A_0 \cap A_1$ we have

$$\|a\|_{(B, A_1)_{(1-\theta)\eta+\theta, p}} \leq M \|a\|_{[A_0, A_1]_\theta}.$$

On the other hand, according to the reiteration theorem for the real method ([1], Theorem 3.11.5), we have

$$(B, A_1)_{(1-\theta)\eta+\theta, p} = ((B, A_1)_{\eta, q}, A_1)_{\theta, p} = (A_0, A_1)_{\theta, p}$$

which establishes the result. ■

In the special case $q = 1$, Theorem 4 was proved by PISIER [11], p. 115-116, under rather restrictive assumptions on the spaces involved.

As CWIKEL, MILMAN and SAGHER showed in [4], Theorem 3, the converse inequality of Theorem 3 holds without requiring any condition on the couple (A_0, A_1). Next we indicate another approach to Cwikel-Milman-Sagher result. This time the proof is based on the description of the real interpolation space as a J-space, but however we still need the first step of the proof in [4]. Our approach is "dual" of the one developed in Theorem 3 and it also works in the end-point case.

THEOREM 5. – *Let (A_0, A_1) be a quasi-Banach couple. Let*

$0 < \theta_0, \theta_1, \lambda < 1$, $0 < q_0, q_1 < \infty$, or $0 < q_0 < \infty$ and $q_1 = \infty$. Put $\theta = (1-\lambda)\theta_0 + \lambda\theta_1$, $1/q = (1-\lambda)/q_0 + \lambda/q_1$ and $E_j = (A_0, A_1)_{\theta_j, q_j}$. Then there exists a constant C such that for all $a \in A_0 \cap A_1$ we have

$$\|a\|_{[E_0, E_1]_\lambda} \leq C \|a\|_{(A_0, A_1)_{\theta, q}}.$$

Proof. – We start by observing that there is a constant C depending only on the couple (A_0, A_1) such that for each $a \in A_0 \cap A_1$ there is a sequence $(u_\nu) \subset A_0 \cap A_1$ $(\subset E_0 \cap E_1)$ with only a finite number of terms different from zero, such that

$$a = \sum_{\nu=-\infty}^{\infty} u_\nu$$

and

$$\left(\sum_{\nu=-\infty}^{\infty} (2^{-\theta\nu} J(2^\nu, u_\nu))^q\right)^{1/q} \leq C\|a\|_{(A_0, A_1)_{\theta, q}}$$

where

$$J(2^\nu,\ u_\nu) = J(2^\nu,\ u_\nu;\ A_0,\ A_1).$$

Indeed, choose $N \in \mathbb{N}$ sufficiently large so that

$$2^{-\theta N}\|a\|_{A_0} \leq \|a\|_{(A_0,\, A_1)_{\theta,\, q}}$$

and

$$2^{-(1-\theta)(N-1)}\|a\|_{A_1} \leq \|a\|_{(A_0,\, A_1)_{\theta,\, q}}.$$

For $|\nu| < N$ we can find a decomposition $a = a_{0,\nu} + a_{1,\nu}$ such that

$$\|a_{0,\nu}\|_{A_0} + 2^\nu\|a_{1,\nu}\|_{A_1} \leq 2\,K(2^\nu,\ a)$$

while if $|\nu| \geq N$ we write

$$a_{0,\nu} = \begin{cases} a & \text{if} \quad \nu \geq N \\ 0 & \text{if} \quad \nu \leq -N \end{cases}, \qquad a_{1,\nu} = \begin{cases} 0 & \text{if} \quad \nu \geq N \\ a & \text{if} \quad \nu \leq -N \end{cases}.$$

Let

$$u_\nu = a_{0,\nu} - a_{0,\nu-1} = a_{1,\nu-1} - a_{1,\nu}, \qquad \nu \in \mathbb{Z}.$$

Then $(u_\nu) \subset A_0 \cap A_1$ and

$$a - \sum_{\nu=s}^{r} u_\nu = 0$$

for $s \leq -N < N \leq r$. Moreover, if $-N + 2 \leq \nu \leq N - 1$ we see that

$$\begin{aligned} &J(2^\nu,\ u_\nu) \\ &\leq \max\{c_{A_0}(\|a_{0,\nu}\|_{A_0} + \|a_{0,\nu-1}\|_{A_0}),\ 2^\nu c_{A_1}(\|a_{1,\nu}\|_{A_1} + \|a_{1,\nu-1}\|_{A_1})\} \\ &\leq \max\{c_{A_0},\ c_{A_1}\}(2\,K(2^\nu,\ a) + 4\,K(2^{\nu-1},\ a)) \\ &\leq 6\,\max\{c_{A_0},\ c_{A_1}\}\,K(2^\nu,\ a), \end{aligned}$$

while

$$\begin{aligned} &J(2^N,\ u_N) \leq c_{A_0}(\|a\|_{A_0} + 4\,K(2^N,\ a)), \\ &\qquad J(2^{-N+1},\ u_{-N+1}) \leq c_{A_1}(2\,K(2^{-N+1},\ a) + 2^{-N+1}\|a\|_{A_1}), \end{aligned}$$

and

$$J(2^\nu,\ u_\nu) = 0 \qquad \text{if} \quad \nu \geq N+1 \quad \text{or} \quad \nu \leq -N.$$

Hence

$$\begin{aligned}
&\sum_{\nu=-\infty}^{\infty} (2^{-\theta\nu} J(2^\nu, u_\nu))^q \\
&\quad \le (6 \max\{c_{A_0}, c_{A_1}\})^q \sum_{\nu=-N+1}^{N} (2^{-\theta\nu} K(2^\nu, a))^q \\
&\qquad + c_{A_0}^q (2^{-\theta N} \|a_0\|_{A_0})^q + c_{A_1}^q (2^{-(1-\theta)(N-1)} \|a_1\|_{A_1})^q \\
&\quad \le C^q \|a\|^q_{(A_0, A_1)_{\theta, q}}.
\end{aligned}$$

Given any $a \in A_0 \cap A_1$ choose a representation $a = \sum_{\nu=-\infty}^{\infty} u_\nu$ as above and consider the operator T associating to any sequence of scalars (λ_ν) the vector of $A_0 + A_1$ defined by

$$T(\lambda_\nu) = \sum_{\nu=-\infty}^{\infty} \frac{\lambda_\nu u_\nu}{J(2^\nu, u_\nu)}$$

where it is understood that the ν-th term is omitted if $u_\nu = 0$. The description of $E_j = (A_0, A_1)_{\theta, q_j}$ as a J-space gives that

$$\begin{aligned}
\Big\| \sum_{\nu=-\infty}^{\infty} \frac{\lambda_\nu u_\nu}{J(2^\nu, u_\nu)} \Big\|_{E_j} &\le \left(\sum_{\nu=-\infty}^{\infty} \left(2^{-\theta_j \nu} J\Big(2^\nu, \frac{\lambda_\nu u_\nu}{J(2^\nu, u_\nu)}\Big) \right)^{q_j} \right)^{1/q_j} \\
&\le \left(\sum_{\nu=-\infty}^{\infty} \left(2^{-\theta_j \nu} |\lambda_\nu| \right)^{q_j} \right)^{1/q_j} = \|(\lambda_\nu)\|_{\ell_{q_j}(2^{-\theta_j \nu})}.
\end{aligned}$$

Thus $T : \ell_{q_j}(2^{-\theta_j \nu}) \to E_j$ is bounded with quasi-norm ≤ 1. Interpolating by the complex method, we find that

$$T : [\ell_{q_0}(2^{-\theta_0 \nu}), \ell_{q_1}(2^{-\theta_1 \nu})]_\lambda \to [E_0, E_1]_\lambda$$

is bounded as well. It follows then from the equality

$$\ell_q(2^{-\theta\nu}) = [\ell_{q_0}(2^{-\theta_0 \nu}), \ell_{q_1}(2^{-\theta_1 \nu})]_\lambda \quad \text{(Corollary 2/(ii))}$$

that the operator

$$T : \ell_q(2^{-\theta\nu}) \to [E_0, E_1]_\lambda$$

is bounded. Since $(J(2^\nu, u_\nu))$ has only a finite number of terms $J(2^\nu, u_\nu) \ne 0$, it is clear that

$$T(J(2^\nu, u_\nu)) = \sum_{\nu=-\infty}^{\infty} u_\nu = a$$

and therefore we conclude that

$$\|a\|_{[E_0, E_1]_\lambda} \leq \|(J(2^\nu, u_\nu))\|_{\ell_q(2^{-\theta\nu})} \leq C\|a\|_{(A_0, A_1)_{\theta, q}}. \quad \blacksquare$$

In the end-point case we have

THEOREM 6. – *Let (B, A_1) be a quasi-Banach couple, let $0 < \eta, \theta < 1$, $0 < q \leq \infty$ and put*

$$A_0 = (B, A_1)_{\eta, q}.$$

If A_1 is r-normed and $1/p = (1-\theta)/q + \theta/r$, then there exists a constant C such that for every $a \in B \cap A_1$, we have

$$\|a\|_{[A_0, A_1]_\theta} \leq C\|a\|_{(A_0, A_1)_{\theta, p}}.$$

Proof. – The reiteration theorem yields

$$(A_0, A_1)_{\theta, p} = ((B, A_1)_{\eta, q}, A_1)_{\theta, p} = (B, A_1)_{(1-\theta)\eta+\theta, p}.$$

So our aim is to prove that

$$\|a\|_{[A_0, A_1]_\theta} \leq C\|a\|_{(B, A_1)_{(1-\theta)\eta+\theta, p}}. \tag{12}$$

We know that there is a constant C such that for any $a \in B \cap A_1$ there exists a sequence $(u_\nu) \subset B \cap A_1$ $(\subset A_0 \cap A_1)$ with only a finite number of non-zero terms such that

$$a = \sum_{\nu=-\infty}^{\infty} u_\nu$$

and

$$\Big(\sum_{\nu=-\infty}^{\infty} (2^{-((1-\theta)\eta+\theta)\nu} J(2^\nu, u_\nu))^p\Big)^{1/p} \leq C\|a\|_{(B, A_1)_{(1-\theta)\eta+\theta, p}}$$

where $J(2^\nu, u_\nu) = J(2^\nu, u_\nu; B, A_1)$.

Given any $a \in B \cap A_1$ and any representation $a = \sum_{\nu=-\infty}^{\infty} u_\nu$ as above, consider the operator T defined by means of this representation

$$T(\lambda_\nu) = \sum_{\nu=-\infty}^{\infty} \frac{\lambda_\nu u_\nu}{J(2^\nu, u_\nu)}$$

where it is again understood that the ν-th term is omitted if $u_\nu = 0$. We see that T is bounded with quasi-norm ≤ 1 acting from $\ell_q(2^{-\eta\nu})$ into

$A_0 = (B, A_1)_{\eta, q}$. On the other hand, taking into account that A_1 is r-normed we obtain

$$\left\| \sum_{\nu=-\infty}^{\infty} \frac{\lambda_\nu u_\nu}{J(2^\nu, u_\nu)} \right\|_{A_1} \le \left(\sum_{\nu=-\infty}^{\infty} \left\| \frac{\lambda_\nu u_\nu}{J(2^\nu, u_\nu)} \right\|_{A_1}^r \right)^{1/r} \le \left(\sum_{\nu=-\infty}^{\infty} (2^{-\nu} |\lambda_\nu|)^r \right)^{1/r}$$

Hence

$$T : 1_r(2^{-\nu}) \to A_1$$

is bounded, having quasi-norm ≤ 1. Interpolating by the complex method

$$T : \ell_p(2^{-((1-\theta)\eta+\theta)\nu}) = [\ell_q(2^{-\eta\nu}), \ell_r(2^{-\nu})]_\theta \to [A_0, A_1]_\theta$$

is also bounded. Then inequality (12) follows from the fact that $(J(2^\nu, u_\nu))$ has only a finite number of terms different from zero and therefore

$$T(J(2^\nu, u_\nu)) = \sum_{\nu=-\infty}^{\infty} u_\nu = a. \quad \blacksquare$$

3. Applications

Theorems 3 and 6 allow us to determine complex interpolation spaces as soon as the corresponding real interpolation spaces are known.

As a first application we show that the reiteration formula is valid for quasi-Banach couples formed by real interpolation from a couple satisfying (**h**).

COROLLARY 7. – *Let (A_0, A_1) be a quasi-Banach couple satisfying* (**h**). *Let $0 < \eta_0, \eta_1 < 1$, $0 < q_0 < \infty$, $0 < q_1 \le \infty$ and put*

$$E_j = (A_0, A_1)_{\eta_j, q_j}.$$

If $0 < \theta_0, \theta_1, \lambda < 1$ and $\theta = (1-\lambda)\theta_0 + \lambda\theta_1$, then

$$[[E_0, E_1]_{\theta_0}, [E_0, E_1]_{\theta_1}]_\lambda = [E_0, E_1]_\theta.$$

Proof. – Put

$$\eta = (1-\theta)\eta_0 + \theta\eta_1, \qquad 1/q = (1-\theta)/q_0 + \theta/q_1.$$

Since $q < \infty$, $A_0 \cap A_1$ is dense in $(A_0, A_1)_{\eta, q}$. We claim that $A_0 \cap A_1$ is also dense in $[E_0, E_1]_\theta$.

Indeed, according to Theorem 3, $\| \cdot \|_{[\theta]}$ is a quasi-norm in $E_0 \cap E_1$. Thus $E_0 \cap E_1$ is dense in $[E_0, E_1]_\theta$. Hence we only need to show that $A_0 \cap A_1$ is dense in $E_0 \cap E_1$ for $\| \cdot \|_{[\theta]}$. Using the fundamental lemma [1], Lemma 3.3.2, given any

$$a \in E_0 \cap E_1 = (A_0, A_1)_{\eta_0, q_0} \cap (A_0, A_1)_{\eta_1, q_1},$$

we can find $(v_n) \subset A_0 \cap A_1$ such that

$$\|v_n\|_{E_1} = \|v_n\|_{(A_0, A_1)_{\theta_1, q_1; J}} \leq 4\|a\|_{(A_0, A_1)_{\theta_1, q_1; K}}$$

and

$$\|a - v_n\|_{E_0} \leq 4 \big(\sum_{|\nu| > n} (2^{-\eta_0 \nu} K(2^\nu, a))^{q_0}\big)^{1/q_0} \to 0 \quad \text{as } n \to \infty.$$

Since

$$\|a - v_n\|_{[E_0, E_1]_\theta} \leq \|a - v_n\|_{E_0}^{1-\theta} \, \|a - v_n\|_{E_1}^{\theta}$$

density of $A_0 \cap A_1$ in $E_0 \cap E_1$ follows.

By Theorems 3 and 5, we have then

$$[E_0, E_1]_\theta = [(A_0, A_1)_{\eta_0, q_0}, (A_0, A_1)_{\eta_1, q_1}]_\theta = (A_0, A_1)_{\eta, q}.$$

Another application of Theorems 3 and 5 yields

$$[E_0, E_1]_{\theta_j} = (A_0, A_1)_{\mu_j, p_j}$$

where

$$\mu_j = (1 - \theta_j)\,\eta_0 + \theta_j\,\eta_1 \quad \text{and} \quad 1/p_j = (1 - \theta_j)/q_0 + \theta_j/q_1 \ (j = 0, 1).$$

Finally, using for the third time Theorems 3 and 5, we conclude

$$\begin{aligned}[][E_0, E_1]_{\theta_0}, [E_0, E_1]_{\theta_1}]_\lambda &= [(A_0, A_1)_{\mu_0, p_0}, (A_0, A_1)_{\mu_1, p_1}]_\lambda \\ &= (A_0, A_1)_{\eta, p} = [E_0, E_1]_\theta. \quad \blacksquare\end{aligned}$$

Next we consider applications to some concrete spaces.

As we said in Section 1, the couples (L_{p_0}, L_{p_1}) and (H_{p_0}, H_{p_1}) satisfy condition $(\mathbf{h})$. Since the real interpolation spaces for these couples are known (*see* [1] and [6]), by using Theorem 3 and 5 we may conclude that

$$(13) \qquad [L_{p_0 q_0}, L_{p_1 q_1}]_\theta = L_{p,q}$$

and

$$(14) \qquad [H_{p_0 q_0}, H_{p_1 q_1}]_\theta = H_{p,q}$$

provided that $0 < p_0, p_1, q_0 < \infty$, $0 < q_1 \leq \infty$, $0 < \theta < 1$, $1/p = (1-\theta)/p_0 + \theta/p_1$ and $1/q = (1-\theta)/q_0 + \theta/q_1$. Here $L_{p,q}$ is the Lorentz function space and $H_{p,q}$ consists of all tempered distributions f on $\mathbb{R}^n$ such that

$$\sup_{t>0} \{t^{-n} |\phi_t \star f|\} \in L_{p,q}$$

where ϕ is a sufficiently regular function with $\int_{\mathbb{R}^n} \phi(x)\,dx \neq 0$ and $\phi_t(x) = \phi(x/t)$ (*see* [6]). For $p = q$ these are the usual H_p classes.

In order to derive an interpolation result for Morrey type spaces, we first prove the following auxiliary lemma. Its validity in the Banach case was pointed out in [8], (99). We require here the extension to quasi-Banach couples.

LEMMA 8. – *Let X be any set, let w_0, w_1 be positive weights on X and let (A_0, A_1) be a quasi-Banach couple. Denote by $L^\infty_{w_j}(A_j)$ the collection of all functions $a : X \to A_j$ such that*

$$\|a\|_{L^\infty_{w_j}(A_j)} = \sup_{x \in X} \{w_j(x) \|a(x)\|_{A_j}\} < \infty.$$

Then, if $0 < \theta < 1$ and $a \in L^\infty_{w_0}(A_0) \cap L^\infty_{w_1}(A_1)$, we have

$$\|a\|_{L^\infty_{w_0^{1-\theta} w_1^\theta}([A_0, A_1]_\theta)} \leq \|a\|_{[L^\infty_{w_0}(A_0), L^\infty_{w_1}(A_1)]_\theta}.$$

Proof. – Given $a \in L^\infty_{w_0}(A_0) \cap L^\infty_{w_1}(A_1)$ take any $f \in \mathcal{F}(L^\infty_{w_0}(A_0), L^\infty_{w_1}(A_1))$ with $f(\theta, x) = a(x)$. Using (4) with $\alpha = 1/(1-\theta)$ and $\beta = 1/\theta$, we have

$$
\begin{aligned}
\|a\|_{L^{\infty}_{w_0^{1-\theta} w_1^{\theta}}([A_0, A_1]_\theta)} &= \sup_{x \in X} \{w_0^{1-\theta}(x)\, w_1^{\theta}(x) \|a(x)\|_{[A_0, A_1]_\theta}\} \\
&\leq \sup_{x \in X} \left\{ \left(\frac{1}{1-\theta} \int_{-\infty}^{\infty} w_0(x) \|f(is, x)\|_{A_0} P_0(\theta, s)\, ds \right)^{1-\theta} \right. \\
&\qquad \left. \cdot \left(\frac{1}{\theta} \int_{-\infty}^{\infty} w_1(x) \|f(1+is, x)\|_{A_1} P_1(\theta, s)\, ds \right)^{\theta} \right\} \\
&\leq \left(\frac{1}{1-\theta} \int_{-\infty}^{\infty} \|f(is)\|_{L^{\infty}_{w_0}(A_0)} P_0(\theta, s)\, ds \right)^{1-\theta} \\
&\qquad \cdot \left(\frac{1}{\theta} \int_{-\infty}^{\infty} \|f(1+is)\|_{L^{\infty}_{w_1}(A_1)} P_1(\theta, s)\, ds \right)^{\theta} \\
&\leq \left(\sup_{s \in \mathbb{R}} \|f(is)\|_{L^{\infty}_{w_0}(A_0)} \right)^{1-\theta} \left(\sup_{s \in \mathbb{R}} \|f(1+is)\|_{L^{\infty}_{w_1}(A_1)} \right)^{\theta} \leq \|f\|_{\mathcal{F}}.
\end{aligned}
$$

This gives the result. ■

Let now C_0 be any open subset of $\mathbb{R}^n$. For $0 < p < \infty$, $0 < q \leq \infty$ and $\lambda \in \mathbb{R}$, we denote by $\mathcal{L}^{(p, q, \lambda)}$ the Morrey space over C_0. We refer to [8], § 8.A, for the precise definition and properties of these spaces. We only recall that if X is the set of all cubes $C(x, \rho) \subset C_0$ with center is some $x \in C_0$ and edge length equal to ρ, then $\mathcal{L}^{(p, q, \lambda)}$ can be identified with a subspace of $L^{\infty}_{\rho^{(n-\lambda)/p}}(L_{p, q})$ by associating of each $f \in \mathcal{L}^{(p, q, \lambda)}$ the map from X into $L_{p, q}$ defined by

$$
C(x, \rho) \to f_C - \bar{f}_C.
$$

Here f_C is the restriction of f to $C(x, \rho)$ and $\bar{f}_C$ is the mean value of f in $C(x, \rho)$.

Combining (13) and Lemma 8, it is easy to see that if $0 < p_0, p_1, q_0 < \infty$, $0 < q_1 \leq \infty$, $-\infty < \lambda_0, \lambda_1 < \infty$ and $0 < \theta < 1$, then for all $f \in \mathcal{L}^{(p_0, q_0, \lambda_0)} \cap \mathcal{L}^{(p_1, q_1, \lambda_1)}$ we have

$$
\|f\|_{\mathcal{L}^{(p, q, \lambda)}} \leq M \|f\|_{[\mathcal{L}^{(p_0, q_0, \lambda_0)}, \mathcal{L}^{(p_1, q_1, \lambda_1)}]_\theta} \tag{15}
$$

where

$$
\begin{gathered}
1/p = (1-\theta)/p_0 + \theta/p_1, \qquad 1/q = (1-\theta)/q_0 + \theta/q_1, \\
\lambda/p = (1-\theta)\, \lambda_0/p_0 + \theta \lambda_1/p_1.
\end{gathered}
$$

Our last application refers to Schatten-von Neumann classes $S_{p,q}$. Let H and K be Hilbert spaces. Recall that the class $S_{p,q}$ consists of all compact operators $T \in \mathcal{L}(H, K)$ such that its sequence of singular numbers $(s_n(T))$ belongs to the Lorentz sequence space $1_{p,q}$. The quasi-norm of $S_{p,q}$ is given by

$$\|T\|_{S_{p,q}} = (\sum_{n=1}^{\infty} (n^{1/p} s_n(T))^q n^{-1})^{1/q}.$$

The class $S_{p,p}$ is usually denoted by S_p.

We claim that under the same hypothesis on parameters as in (13), we have

$$[S_{p_0,q_0}, S_{p_1,q_1}]_\theta = S_{p,q}. \tag{16}$$

Indeed, take $r_0 < p_0$, $p_1 < r_1$, $0 < \eta_0$, $\eta_1 < 1$ with $1/p_j = (1-\eta_j)/r_0 + \eta_j/r_1$ $(j = 0, 1)$. As it is well-known (see [1] or [14])

$$S_{p_j,q_j} = (S_{r_0}, S_{r_1})_{\eta_j,q_j}. \tag{17}$$

Hence using Theorem 5 it follows from (17) that for all $T \in S_{r_0} = S_{r_0} \cap S_{r_1}$ we have

$$\|T\|_{[S_{p_0,q_0}, S_{p_1,q_1}]_\theta} \leq M_1 \|T\|_{S_{p,q}}. \tag{18}$$

In order to check the reverse inequality, take any $T \in S_{p_0,q_0} \cap S_{p_1,q_1}$. According to the spectral theorem we can find orthonormal systems (x_n) in H and (y_n) in K such that

$$T = \sum_{n=1}^{\infty} s_n(T) \langle \cdot, x_n \rangle y_n.$$

Consider the map $\mathcal{D}$ associating to each operator $R \in \mathcal{L}(H, K)$ the sequence of inner products

$$\mathcal{D}(R) = (\langle Rx_n, y_n \rangle).$$

By [10], Theorem 2.11.18,

$$\mathcal{D} : S_{p_j,q_j} \to \ell_{p_j,q_j}$$

is bounded for $j = 0, 1$. Whence, interpolating by the complex method and using (13), we get that

$$\mathcal{D} : [S_{p_0,q_0}, S_{p_1,q_1}]_\theta \to \ell_{p,q}$$

is bounded as well. Since

$$\mathcal{D}(T) = (s_n(T))$$

we derive that for all $T \in S_{p_0,q_0} \cap S_{p_1,q_1}$ it holds

$$\|T\|_{S_{p,q}} \leq M_2 \|T\|_{[S_{p_0,q_0}, S_{p_1,q_1}]_\theta} \cdot \tag{19}$$

To finish the proof it remains to check that S_{r_0} $(\subset S_{p_0,q_0} \cap S_{p_1,q_1})$ is dense in both spaces of formula (16).

The density in $S_{p,q}$ is clear. On the other hand, (19) shows that $\|\cdot\|_{[S_{p_0,q_0}, S_{p_1,q_1}]_\theta}$ is a quasi-norm in $S_{p_0,q_0} \cap S_{p_1,q_1}$. Therefore, repeating the argument at the beginning of the proof of Corollary 7, we conclude that S_{r_0} is also dense in $[S_{p_0,q_0}, S_{p_1,q_1}]_\theta$.

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