Topology and its Applications 159 (2012) 2235–2247

Contents lists available at SciVerse ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

Equivariant extension properties of coset spaces of locally compact groups and approximate slices ☆

Sergey A. Antonyan

Departamento de Matemáticas, Facultad de Ciencias, Universidad Nacional Autónoma de México, 04510 México Distrito Federal, Mexico

ARTICLE INFO

MSC:
22D05
22F05
54C55
54H15

Keywords:
Locally compact group
Coset space
Proper G-space
G-ANE
Equivariant embedding
Approximate slice

ABSTRACT

We prove that for a compact subgroup H of a locally compact Hausdorff group G, the following properties are mutually equivalent: (1) G/H is finite-dimensional and locally connected, (2) G/H is a smooth manifold, (3) G/H satisfies the following equivariant extension property: for every paracompact proper G-space X having a paracompact orbit space, every G-map $A \to G/H$ from a closed invariant subset $A \subset X$ extends to a G-map $U \to G/H$ over an invariant neighborhood U of A. A new version of the Approximate Slice Theorem is also proven in the light of these results.

1. Introduction

In the fundamental work of R. Palais [35], it was established that for any compact Lie group G and any compact subgroup H of it, the coset space G/H has the following equivariant extension property: for every normal G-space X and a closed invariant subset $A \subset X$, every G-map $f : A \to G/H$ extends to a G-map $f' : U \to G/H$ defined on an invariant neighborhood U of A. In this case one writes $G/H \in G$-ANE. This property of G is equivalent to the so-called Exact Slice Theorem: every orbit in a completely regular G-space X is a neighborhood G-equivariant retract of X (see [35]).

In general, when G is a compact non-Lie group, the Exact Slice Theorem is no longer true (see [7]). At the same time, among the coset spaces G/H of a compact non-Lie group G, still there are many which posses the property $G/H \in G$-ANE. This observation led the author in [7] to the, so-called, Approximate Slice Theorem, which is valid for every compact group of transformations. It claims that given a point x and its neighborhood O in a G-space, there exists a G-invariant neighborhood U of x that admits a G-map $f : U \to G/H$ to a coset space G/H with a compact subgroup $H \subset G$ such that $G/H \in G$-ANE and $x \in f^{-1}(eH) \subset O$.

On the other hand, in 1961 the Exact Slice Theorem was extended by R. Palais [36] to the case of proper actions of non-compact Lie groups. The validity of the property $G/H \in G$-ANE for every compact subgroup H of a non-compact Lie group G follows from Palais' Exact Slice Theorem; this was proved later in E. Elfving [19, pp. 23–24].

It is one of the purposes of this paper to prove (see Proposition 3.7 and Theorem 5.3) that if G is a locally compact group and H a compact subgroup of G then the following properties are mutually equivalent: (1) G/H is finite-dimensional and locally connected, (2) G/H is a smooth manifold, (3) G/H is a (metrizable) G-ANE for paracompact proper G-spaces having a paracompact orbit space. A compact subgroup $H \subset G$ satisfying these equivalent properties is called a *large* subgroup.

☆ This research was supported in part by grants IN-117511 from PAPIIT (UNAM) and 79536 from CONACYT (Mexico).
E-mail address: antonyan@unam.mx.

0166-8641/$ – see front matter
doi:10.1016/j.topol.2011.10.019

One should clarify that in the case of non-compact group actions the property $G/H \in G$-ANE reads as follows: for every paracompact proper G-space X having a paracompact orbit space, every G-map $f : A \to G/H$ from a closed invariant subset $A \subset X$ extends to a G-map $f' : U \to G/H$ over an invariant neighborhood U of A.

As in the compact case [7], the equivariant extension properties of coset spaces are conjugated with approximate slices. In Section 6 we prove a new version of the Approximate Slice Theorem valid for proper actions of arbitrary locally compact groups. Section 4 plays an auxiliary role. Here we establish a special equivariant embedding of a coset space G/H into a G-AE($\mathcal{P}$)-space (see Proposition 4.6), which is further used in the proof of Theorem 5.3.

In the last section we discuss one more possible characterization of large subgroups; our Conjecture 7.2 states that a compact subgroup H of a locally compact group G is large iff the coset space G/H is locally contractible. A possible proof of this conjecture is based on a result of J. Szenthe [39] to the effect that a locally compact almost connected group which acts effectively and transitively on a locally contractible space is a Lie group. Since there is a gap in Szenthe's argument (see Section 7), our conjecture remains open.

2. Preliminaries

Throughout the paper the letter G will denote a locally compact Hausdorff group unless otherwise stated; by e we shall denote the unity of G.

All topological spaces and topological groups are assumed to be Tychonoff (= completely regular and Hausdorff). The basic ideas and facts of the theory of G-spaces or topological transformation groups can be found in G. Bredon [16] and in R. Palais [35]. Our basic reference on proper group actions is Palais' article [36]. Other good sources are [1,30]. For equivariant theory of retracts the reader can see, for instance, [4,5] and [9].

For the convenience of the reader, we recall however some more special definitions and facts below.

By a G-space we mean a triple (G, X, α), where X is a topological space, and $\alpha : G \times X \to X$ is a continuous action of the group G on X. If Y is another G-space, a continuous map $f : X \to Y$ is called a G-map or an equivariant map if $f(gx) = gf(x)$ for every $x \in X$ and $g \in G$. If G acts trivially on Y then we will use the term "invariant map" instead of "equivariant map".

By a normed linear G-space (resp., a Banach G-space) we shall mean a G-space L, where L is a normed linear space (resp., a Banach space) on which G acts by means of *linear isometries*, i.e., $g(\lambda x + \mu y) = \lambda(gx) + \mu(gy)$ and $\|gx\| = \|x\|$ for all $g \in G$, $x, y \in L$ and λ, $\mu \in \mathbb{R}$.

If X is a G-space then for a subset $S \subset X$, $G(S)$ denotes the saturation of S; i.e., $G(S) = \{gs \mid g \in G, s \in S\}$. In particular, $G(x)$ denotes the G-orbit $\{gx \in X \mid g \in G\}$ of x. If $G(S) = S$ then S is said to be an invariant or G-invariant set. The orbit space is denoted by X/G.

For a closed subgroup $H \subset G$, by G/H we will denote the G-space of cosets $\{gH \mid g \in G\}$ under the action induced by left translations.

If X is a G-space and H a closed normal subgroup of G then the H-orbit space X/H will always be regarded as a G/H-space endowed with the following action of the group G/H: $(gH) * H(x) = H(gx)$, where $gH \in G/H$, $H(x) \in X/H$.

For any $x \in X$, the subgroup $G_x = \{g \in G \mid gx = x\}$ is called the stabilizer (or stationary subgroup) at x.

A compatible metric ρ on a G-space X is called invariant or G-invariant if $\rho(gx, gy) = \rho(x, y)$ for all $g \in G$ and $x, y \in X$.

A locally compact group G is called *almost connected* whenever its quotient space of connected components is compact.

Let X be a G-space. Two subsets U and V in X are called *thin* relative to each other [36, Definition 1.1.1] if the set

$$\langle U, V\rangle = \{g \in G \mid gU \cap V \neq \emptyset\},$$

called *the transporter* from U to V, has a compact closure in G.

A subset U of a G-space X is called *small* if every point in X has a neighborhood thin relative to U. A G-space X is called *proper* (in the sense of Palais) if every point in X has a small neighborhood.

Each orbit in a proper G-space is closed, and each stabilizer is compact [36, Proposition 1.1.4]. Furthermore, if X is a compact proper G-space, then G has to be compact as well.

Important examples of proper G-spaces are the coset spaces G/H with H a compact subgroup of a locally compact group G. Other interesting examples the reader can find in [1,8,13,29] and [30].

In the present paper we are especially interested in the class G-$\mathcal{P}$ of all paracompact proper G-spaces X that have paracompact orbit space X/G. It is an open problem of long standing whether the orbit space of any paracompact proper G-space is paracompact (see [22] and [1]).

A G-space Y is called an equivariant neighborhood extensor for a given G-space X (notation: $Y \in G$-ANE(X)) if for any closed invariant subset $A \subset X$ and any G-map $f : A \to Y$, there exist an invariant neighborhood U of A in X and a G-map $\psi \to U \to Y$ such that $\psi|_A = f$. If in addition one always can take $U = X$, then we say that Y is an equivariant extensor for X (notation: $Y \in G$-AE(X)). The map ψ is called a G-extension of f.

A G-space Y is called an equivariant absolute neighborhood extensor for the class G-$\mathcal{P}$ (notation: $Y \in G$-ANE($\mathcal{P}$)) if $Y \in G$-ANE(X) for any G-space $X \in G$-$\mathcal{P}$. Similarly, if $Y \in G$-AE(X) for any $X \in G$–$\mathcal{P}$, then Y is called an equivariant absolute extensor for the class G-$\mathcal{P}$ (notation: $Y \in G$-AE($\mathcal{P}$)).

Theorem 2.1. *Let T be a G-space such that $T \in K$-AE$(\mathcal{P})$ (respectively, $T \in K$-ANE$(\mathcal{P})$) for every compact subgroup $K \subset G$. Then $T \in G$-AE$(\mathcal{P})$ (respectively, $T \in G$-ANE$(\mathcal{P})$).*

Remark 2.2. This theorem is originally proved in [1, Theorem 4.4] only for G-AE$(\mathcal{M})$ while the proof is valid also for G-AE$(\mathcal{P})$, where G-$\mathcal{M}$ stands for the class of all proper G-spaces that are metrizable by a G-invariant metric. The G-ANE$(\mathcal{P})$ case can be reduced to the G-AE$(\mathcal{P})$ case by using the standard cone construction (see [9, Theorem 5 and Remark 3]).

Corollary 2.3. *Any Banach G-space is a G-AE$(\mathcal{P})$.*

Proof. Let B be a Banach G-space. By a result of E. Michael [32], B is an AE$(\mathcal{P})$. Then, it follows from [3, Main Theorem] that B is a K-AE$(\mathcal{P})$ for every compact subgroup K of G. It remains to apply Theorem 2.1. □

Let us recall the well-known definition of a slice [36, p. 305]:

Definition 2.4. Let X be a G-space and H be a closed subgroup of G. An H-invariant subset $S \subset X$ is called an H-slice in X if $G(S)$ is open in X and there exists a G-equivariant map $f : G(S) \to G/H$ such that $S = f^{-1}(eH)$. The saturation $G(S)$ is called a *tubular* set. If $G(S) = X$ then we say that S is *a global* H-slice of X.

The following result of R. Palais [36, Proposition 2.3.1] plays a central role in the theory of topological transformation groups:

Theorem 2.5 *(Exact Slice Theorem). Let G be a Lie group, X a proper G-space and $a \in X$. Then there exists a G_a-slice $S \subset X$ such that $a \in S$.*

In the proof of the next proposition we shall need also the definition of a twisted product $G \times_K S$, where K is a closed subgroup of G, and S a K-space. $G \times_K S$ is the orbit space of the K-space $G \times S$ on which K acts by the rule: $k(g, s) = (gk^{-1}, ks)$. Furthermore, there is a natural action of G on $G \times_K S$ given by $g'[g, s] = [g'g, s]$, where $g' \in G$ and $[g, s]$ denotes the K-orbit of the point (g, s) in $G \times S$. We shall identify S, by means of the K-equivariant embedding $s \mapsto [e, s]$, $s \in S$, with the K-invariant subset $\{[e, s] \mid s \in S\}$ of $G \times_K S$. This K-equivariant embedding $S \hookrightarrow G \times_K S$ induces a homeomorphism of the K-orbit space S/K onto the G-orbit space $(G \times_K S)/G$ (see [16, Chapter II, Proposition 3.3]). It is also useful to note that the twisted product $G \times_K S$ is a proper G-space whenever K is a compact subgroup of G. This follows from the following three well-known easy facts: (1) the map $[g, s] \mapsto gK : G \times_K S \to G/K$ is a G-map, (2) G/K is a proper G-space whenever K is compact, (3) being a proper G-space is inverse invariant under G-maps.

Proposition 2.6. *Let G' be any closed subgroup of G, and $X \in G$-ANE$(\mathcal{P})$ (respectively, a $X \in G$-AE$(\mathcal{P})$). Then $X \in G'$-ANE$(\mathcal{P})$ (respectively, a $X \in G'$-AE$(\mathcal{P})$).*

Proof. Due to Theorem 2.1, it suffices to show that $X \in K$-ANE$(\mathcal{P})$ (respectively, a $X \in K$-AE$(\mathcal{P})$) for every compact subgroup K of G'. Henceforth, assume that K is a compact subgroup of G'. Let Y be a paracompact K-space, B a closed K-invariant subset of Y and $f : B \to X$ a K-equivariant map. Then f induces a G-map $F : G \times_K B \to X$ by the rule: $F([g, b]) = gf(b)$, where $[g, b] \in G \times_K B$ (see [17, Chapter I, Proposition 4.3]).

We claim that the twisted product $G \times_K Y$ belongs to the class G-$\mathcal{P}$. Indeed, first we apply a result of Morita [34] to the effect that paracompactness is stable under multiplication by a locally compact paracompact space. Since every locally compact group is paracompact (even, strongly paracompact [15, Theorem 3.1.1]), we then infer that $G \times Y$ is paracompact. Next, since the K-orbit map $G \times Y \to G \times_K Y$ is closed, it follows from Michael's classical theorem (see, e.g., [20, Theorem 5.1.33]) that the twisted product $G \times_K Y$ is paracompact. Further, $G \times_K Y$ is a proper G-space being an inverse image of the proper G-space G/K under the natural G-map $[g, y] \mapsto gK : G \times_K Y \to G/K$. Besides, $(G \times_K Y)/G = Y/K$ (see e.g., [16, Chapter II, Proposition 3.3]). Since Y is paracompact and K is compact, we infer that Y/K is paracompact. Hence, the G-orbit space $(G \times_K Y)/G$ is paracompact, and thus, $G \times_K Y \in G$-$\mathcal{P}$.

Next, we observe that $G \times_K B$ is a closed G-invariant subset of $G \times_K Y$. Since $X \in G$-ANE$(\mathcal{P})$, there exist a G-neighborhood U of $G \times_K B$ in $G \times_K Y$ and a G-extension $F_1 : U \to X$ of F. Evidently, $V = U \cap Y$ is a K-invariant neighborhood of B in Y, and the restriction $f_1 = F_1|_V$ is the required K-extension of f.

If, in addition, X is a G-AE$(\mathcal{P})$, then one can choose $U = G \times_K Y$, which yields that $V = Y$. This completes the proof. □

3. Large subgroups

Recall that the letter G always denotes a locally compact Hausdorff group unless otherwise stated.

Definition 3.1. A compact subgroup H of G is called *large* if the quotient space G/H is locally connected and finite-dimensional.

The notion of a large subgroup of a compact group first was singled out in 1991 by the author [6] in form of two other its characteristic properties, namely: "G/H is a manifold" and "G/H is a G-ANR". More systematically this notion was studied later in [7] (for compact groups) and in [10] (for almost connected groups). In this section we shall investigate the remaining general case of an arbitrary locally compact group. Large subgroups play a central role also in Section 6.

Lemma 3.2. *Let H be a compact subgroup of G and G_0 the connected component of G. If G/H is locally connected then the subgroup $G_0H \subset G$ is open and almost connected.*

Proof. Since the natural map

$$G/H \to G/G_0H, \qquad gH \mapsto gG_0H$$

is open and the local connectedness is invariant under open maps, we infer that G/G_0H is locally connected. On the other hand

$$G/G_0H \cong \frac{G/G_0}{(G_0H)/G_0}.$$

Consequently, G/G_0H, being the quotient space of the totally disconnected group G/G_0 is itself totally disconnected. Hence, G/G_0H should be discrete, implying that G_0H is an open subgroup of G.

To prove that G_0H is almost connected it suffices to observe that the quotient group G_0H/G_0 is just the image of the compact group H under the natural homomorphism $G \to G/G_0$, and hence, is compact. □

The following result is immediate from Lemma 3.2:

Corollary 3.3. *Let H be a large subgroup of G and G_0 the connected component of G. Then the subgroup $G_0H \subset G$ is open and almost connected.*

Proposition 3.4. *Let H and K be compact subgroups of G such that $H \subset K$. If H is a large subgroup then so is K.*

Proof. Being H a large subgroup of G, the quotient G/H is finite-dimensional and locally connected. Since the map $G/H \to G/K$, $gH \mapsto gK$, is continuous and open, we infer that G/K is locally connected. Its finite-dimensionality follows from the one of G/H and the following equality (see [38, Theorem 10]):

$$\dim G/H = \dim G/K + \dim K/H. \qquad \square \tag{3.1}$$

Proposition 3.5. *Let H and K be two compact subgroups of G such that K is a large subgroup of G while H is a large subgroup of K. Then H is a large subgroup of G.*

Proof. Being K a large subgroup of G, the quotient G/K is finite-dimensional and locally connected. Then the natural map $G/H \to G/K$ is a locally trivial fibration with the fibers homeomorphic to K/H (see [38, Theorem 13′]). But K/H is also locally connected (and finite-dimensional) since H is a large subgroup of K. This yields that G/H is locally connected. Finite-dimensionality of G/H follows from the one of G/K and K/H, and the above equality (3.1).

Thus, G/H is locally connected and finite-dimensional, and hence, H is a large subgroup of G, as required. □

The following characterization of large subgroups is well known. For compact groups it is proved in Pontryagin's book [37, Chapter 8, Section 48], for almost connected groups which have a countable base it follows from Montgomery and Zippin [33, Section 6.3], and for arbitrary almost connected groups it is proved in Skljarenko [38, Theorem 3].

Proposition 3.6. *Let H be a compact subgroup of an almost connected group G. Then the following assertions are equivalent*:

(1) *H is a large subgroup,*
(2) *There exists a compact normal subgroup N of G such that $N \subset H$ and G/N is a Lie group. In particular, G/H is a coset space of a Lie group.*

This proposition yields the following characterization of large subgroups of arbitrary locally compact groups (cf. [38, Corollary]):

Proposition 3.7. *Let H be a compact subgroup of G. Then the following properties are equivalent*:

(1) *H is a large subgroup,*

(2) *G/H is a smooth manifold*; *in this case it is the disjoint union of open submani-folds which all are homeomorphic to the same coset space of a Lie group.*

Proof. (1) ⇒ (2) Since, by Corollary 3.3, G_0H is open in G we see that G is the disjoint union of the open cosets xG_0H, $x \in G$. Since the quotient map $p : G \to G/H$ is continuous, open and closed we infer that G_0H/H is open and closed in G/H, and G/H is the disjoint union of its open subsets xG_0H/H, $x \in G$. Observe that each xG_0H/H is homeomorphic to the coset space G_0H/H.

It follows from Corollary 3.3 that G_0H is an open almost connected subgroup of G.

Hence, by virtue of Proposition 3.6, it suffices to show that H is a large subgroup of the almost connected group G_0H (see Corollary 3.3).

But this is easy. Indeed, since G_0H/H is an open subset of the locally connected space G/H, we infer that G_0H/H is locally connected too. Further, since G_0H/H is closed in G/H we infer that $\dim G_0H/H \leqslant \dim G/H$, and hence, G_0H/H is finite-dimensional because G/H is so. Thus, H is a large subgroup of G_0H, as required.

(2) ⇒ (1) is evident. □

Immediate application of the results of K.H. Hofmann and S.A. Morris [25] provides a rather explicit description of the local topological structure of the coset space G/H for large subgroups H. At the same time, on this way, one obtains one more proof of Proposition 3.7 which does not rely on Proposition 3.6. On the other hand, the reduction procedure described below is universal in a certain sense and it may be useful in some other situations as well (see, for instance, Section 7).

Let H be a compact subgroup of a locally compact group G. It is a well-known result of H. Yamabe that every locally compact group G admits an open almost connected subgroup G_1. Then $G_1 \cap H$, being an open subgroup of H, has finite index in H. Due to compactness of H we have the homeomorphism $G_1/(G_1 \cap H) \cong G_1H/H$. Since G_1H is open in G and G_1H/H is open in G/H it suffices, when studying the local structure of G/H, to consider the pair $(G_1, G_1 \cap H)$ instead of the pair (G, H). Assume henceforth that G is almost connected.

Due to a well-known structure theorem, in an almost connected locally compact group G every compact subgroup H is contained in a maximal compact subgroup K; all of the maximal compact subgroups are conjugate, and there is a closed subspace E homeomorphic to $\mathbb{R}^n$ for some nonnegative integer n such that $kEk^{-1} = E$ for every $k \in K$, and the map $(x, k) \mapsto xk : E \times K \to G$ is a homeomorphism (see [28]; for G a Lie group see [23, Chapter XV, Theorem 3.1]). This homeomorphism is equivariant for the action of K on $E \times K$ given by $k * (x, y) = (x, yk^{-1})$ and on G by $k * g = gk^{-1}$. Therefore, there is a natural homeomorphism between the H-orbit spaces:

$$E \times (K/H) \cong (E \times K)/H \cong G/H. \tag{3.2}$$

Since $E \cong \mathbb{R}^n$, we have $\dim G/H = n + \dim K/H$, and G/H is locally connected iff K/H is locally connected. Therefore, in order to establish the equivalence of (1) and (2) in Proposition 3.7, it is no loss of generality to assume that G is compact.

Further, the compact case reduces to the compact connected case. Indeed, due to [25, Corollary 3.8], there is a homeomorphism (even G_0-equivariant):

$$\frac{G_0}{G_0 \cap H} \times \frac{G}{G_0H} \cong \frac{G}{H}.$$

Now, if H is a large subgroup of G then by Corollary 3.3, G_0H is open in G, yielding the finiteness of G/G_0H. Hence, G/H is locally homeomorphic to $G_0/(G_0 \cap H)$. Consequently, it suffices to consider the pair $(G_0, G_0 \cap H)$ instead of the pair (G, H). Assume henceforth that G is compact and connected.

For a compact group G, we denote by $\mathcal{L}(G)$ the Lie algebra of G (see [26, Definition 9.44, Proposition 9.45 and Theorem 9.55]. According to [25, Theorem 3.9], $\dim G/H = \dim \mathcal{L}(G)/\mathcal{L}(H)$, and hence, $\mathcal{L}(G)/\mathcal{L}(H)$ is a finite-dimensional real vector space whenever $\dim G/H < \infty$. Denote by N the kernel of the G-action on G/H, i.e., $N = \bigcap_{g \in G} gHg^{-1}$. Clearly, N is the largest normal subgroup of G contained in H. Then G/N acts effectively on G/H, and

$$G/H \cong \frac{G/N}{H/N} \quad \text{and} \quad \mathcal{L}(G)/\mathcal{L}(H) \cong \frac{\mathcal{L}(G/N)}{\mathcal{L}(H/N)} \quad \text{is finite-dimensional.}$$

Hence, the hypotheses of [25, Proposition 3.3(i)] are fulfilled for the pair $(G/N, H/N)$, according to which there exists a totally disconnected central subgroup D of G/N such that the base point $eH \in G/H$ has a neighborhood homeomorphic to $D \times \mathcal{L}(G)/\mathcal{L}(H)$. Since H is a large subgroup of G, the coset space G/H is locally connected, and it then follows that D is finite. Consequently, G/H is locally homeomorphic to the finite-dimensional real vector space $\mathcal{L}(G)/\mathcal{L}(H)$, and hence, it is a manifold.

4. Equivariant embeddings into a G-AE($\mathcal{P}$)-space

Recall that the letter G always denotes a locally compact Hausdorff group.

The main result of this section is Proposition 4.6 which provides a special equivariant embedding of a coset space G/H into a G-AE($\mathcal{P}$)-space; it is used in the proof of Theorem 5.3 below.

We begin with the following lemma proved in [2, Lemma 2.3]:

Lemma 4.1. *Let H a compact subgroup of G and X a metrizable proper G-space admitting a global H-slice S. Then there is a compatible G-invariant metric d on X such that each open unit ball $O_d(x,1)$ is a small set.*

Lemma 4.2. *Let H be a compact subgroup of G. Then a subset $S \subset G/H$ is small if and only if the closure $\overline{S}$ is compact.*

Proof. Assume that S is a small subset of G/H. Then there is a neighborhood U of the point $eH \in G/H$ such that the transporter

$$\langle S, U\rangle = \{g \in G \mid gS \cap U \neq \emptyset\}$$

has a compact closure. Due to local compactness of G/H one can assume that the closure $\overline{U}$ is compact.

Next, for every $sH \in S$ we have $s^{-1}sH = eH \in U$, i.e., $s^{-1} \in \langle S, U\rangle$, or equivalently, $s \in \langle U, S\rangle$. Hence $sH \in \langle U, S\rangle(U)$ showing that $S \subset \langle U, S\rangle(U) \subset \overline{\langle U, S\rangle}(\overline{U})$. But, due to compactness of the closures $\overline{\langle U, S\rangle}$ and $\overline{U}$, the set $\overline{\langle U, S\rangle}(\overline{U})$ is compact. This yields that the closure $\overline{S}$ is compact, as required.

The converse is immediate from the fact that G/H is a proper G-space and every compact subset of a proper G-space is a small set (see [36, Section 1.2]). □

Corollary 4.3. *Let H be a compact subgroup of G such that the quotient G/H is metrizable. Then there is a compatible G-invariant metric ρ on G/H such that each closed unit ball $B_\rho(x,1)$, $x \in G/H$, is compact.*

Proof. By Lemmas 4.1 and 4.2, there exists a compatible G-invariant metric d in G/H such that each open unit ball $O_d(x,1)$ has a compact closure. Then the metric ρ defined by $\rho(x,y) = 2d(x,y)$, $x, y \in G/H$, has the desired property because $B_\rho(x,1) \subset \overline{O_d(x,1)}$. □

We recall that a continuous function $f : X \to \mathbb{R}$ defined on a G-space X is called G-uniform if for each $\epsilon > 0$, there is an identity neighborhood U in G such that $|f(gx) - f(x)| < \epsilon$ for all $x \in X$ and $g \in U$.

For a proper G-space X we denote by $\mathcal{P}(X)$ the linear space of all bounded G-uniform functions $f : X \to \mathbb{R}$ whose support $\operatorname{supp} f = \{x \in X \mid f(x) \neq 0\}$ is a small subset of X. We endow $\mathcal{P}(X)$ with the sup-norm and the following G-action:

$$(g, f) \mapsto gf, \qquad (gf)(x) = f\left(g^{-1}x\right), \quad x \in X.$$

It is easy to see that $\mathcal{P}(X)$ is a normed linear G-space. It was proved in [2, Proposition 3.1] that the complement $\mathcal{P}_0(X) = \mathcal{P}(X) \setminus \{0\}$ is a proper G-space. Moreover, it follows immediately from [12, Propositions 3.4 and 3.5] that the complement $\widetilde{\mathcal{P}}_0(X) = \widetilde{\mathcal{P}}(X) \setminus \{0\}$ is also a proper G-space, where $\widetilde{\mathcal{P}}(X)$ denotes the completion of $\mathcal{P}(X)$.

The following result follows immediately from [2, Lemma 3.3] and [2, proof of Proposition 3.4]:

Proposition 4.4. *Let (X, ρ) be a metric proper G-space with an invariant metric ρ such that each closed unit ball in X is a small set. Then X admits a G-embedding $i : X \hookrightarrow \mathcal{P}_0(X)$ such that*:

(1) $\|i(x) - i(y)\| \leqslant \rho(x,y)$ *for all* $x, y \in X$,
(2) $\rho(x,y) = \|i(x) - i(y)\|$ *whenever* $\rho(x,y) \leqslant 1$,
(3) $\|i(x) - i(y)\| \geqslant 1$ *whenever* $\rho(x,y) > 1$,
(4) *the image $i(X)$ is closed in its convex hull.*

We aim at applying this result to the case $X = G/H$, where H is a compact subgroup of G such that G/H is metrizable.

As it follows from Lemma 4.2, in this specific case, $\mathcal{P}(G/H)$ is just the space of all continuous functions $G/H \to \mathbb{R}$ having a precompact support. Respectively, $\widetilde{\mathcal{P}}(G/H)$ is the Banach space of all continuous functions $G/H \to \mathbb{R}$ vanishing at infinity.

Proposition 4.5. *Let H be a compact subgroup of G such that the quotient space G/H is metrizable. Choose, by Corollary 4.3, a compatible G-invariant metric ρ on G/H such that each closed unit ball $B_\rho(x,1)$, $x \in G/H$, is compact. Then G/H admits a G-embedding $i : G/H \hookrightarrow \widetilde{\mathcal{P}}_0(G/H)$ such that*:

(a) $\|i(x) - i(y)\| \leqslant \rho(x,y)$ *for all* $x, y \in G/H$,
(b) $\rho(x,y) = \|i(x) - i(y)\|$ *whenever* $\rho(x,y) \leqslant 1$,
(c) $\|i(x) - i(y)\| \geqslant 1$ *whenever* $\rho(x,y) > 1$,
(d) *the image $i(G/H)$ is closed in $\widetilde{\mathcal{P}}(G/H)$.*

Proof. Since $(G/H, \rho)$ satisfies the hypothesis of Proposition 4.4, there exists a topological G-embedding $j : G/H \hookrightarrow \mathcal{P}_0(G/H)$ satisfying all the four properties in Proposition 4.4. Composing this G-embedding with the isometric G-embedding $\mathcal{P}_0(G/H) \hookrightarrow \widetilde{\mathcal{P}}_0(G/H)$ we get a G-embedding $i : G/H \hookrightarrow \widetilde{\mathcal{P}}_0(G/H)$ that also satisfies the four properties in Proposition 4.4. Hence, the above properties (a), (b) and (c) are fulfilled. So, only the last property (d) needs to be verified.

Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in G/H such that $(i(x_n))_{n\in\mathbb{N}}$ converges to a point $f \in \widetilde{\mathcal{P}}(G/H)$. One should to check that $f \in i(X)$. Since $(i(x_n))_{n\in\mathbb{N}}$ is a Cauchy sequence, it follows from the above property (b) that $(x_n)_{n\in\mathbb{N}}$ is also Cauchy. Since each closed unit ball $B_\rho(x,1)$ is a compact subset of G/H, it then follows from [18, Chapter XIV, Theorem 2.3] that ρ is a complete metric. Then $(x_n)_{n\in\mathbb{N}}$ converges to a limit, say $y \in G/H$. By continuity of i, this implies that $i(x_n) \rightsquigarrow i(y)$, and hence, $f = i(y) \in i(G/H)$, as required. □

Proposition 4.6. *Let H be a compact subgroup of G such that G/H is metrizable and locally connected. Then there exists a closed G-embedding $i : G/H \hookrightarrow \widetilde{\mathcal{P}}_0(G/H)$ such that*

$$\begin{cases} \|i(x) - i(y)\| > 1/2 \quad \text{whenever } x \text{ and } y \text{ belong} \\ \quad \text{to different connected components of } G/H. \end{cases} \tag{4.1}$$

Proof. Since G/H is metrizable, by virtue of Corollary 4.3, there exists a compatible G-invariant metric d on G/H such that each closed unit ball $B_d(x,1)$, $x \in G/H$, is compact.

To shorten our notation, set $S = G_0H/H$, where G_0 stands for the identity component of G. Since S is the image of G_0 under the quotient map $G \to G/H$ we infer that S is connected. On the other hand, it follows from Lemma 3.2 that S is an open and closed subset of G/H, so S should be a connected component of G/H. Further, it is easy to check that S is a global G_0H-slice for G/H. Thus,

$$G/H = G(S) = \bigsqcup_{g\in G} gS,$$

the disjoint union of closed and open connected components gS, one g out of every coset in G/G_0H.

Next we define a new metric ρ on G/H as follows:

$$\begin{cases} \text{if two points } x \text{ and } y \text{ of } G/H \text{ belong to the same connected component,} \\ \quad \text{then we put } \rho(x,y) = d(x,y); \text{ otherwise we set } \rho(x,y) = d(x,y) + 1/2. \end{cases}$$

Clearly, ρ is a compatible metric for G/H. Since $B_\rho(x,1) \subset B_d(x,1)$ for every $x \in X$, we see that each closed unit ball $B_\rho(x,1)$ is compact.

To see the G-invariance of ρ assume that $x, y \in X$ and $h \in G$. If the points x and y are in the same connected component gS then the points hx and hy belong to the same connected component hgS. But then $\rho(hx,hy) = d(hx,hy) = d(x,y) = \rho(x,y)$, as required.

By the same argument, if x and y are in two different connected components, then hx and hy also belong to different connected components. In this case $\rho(hx,hy) = d(hx,hy) + 1/2 = d(x,y) + 1/2 = \rho(x,y)$, as required. Thus ρ is G-invariant.

Now, let $i : G/H \hookrightarrow \widetilde{\mathcal{P}}_0(G/H)$ be the closed G-embedding from Proposition 4.5. It then follows from the properties (a) and (b) of Proposition 4.5 that (4.1) is satisfied, as required. □

5. Equivariant extension properties of coset spaces

The following result is proved in Elfving [19, pp. 23–24] in a different way:

Proposition 5.1. *Let G be a Lie group and H a compact subgroup of G. Then G/H is a G-ANE($\mathcal{P}$).*

Proof. By virtue of Corollary 4.3, there exists a compatible G-invariant metric ρ on G/H such that each closed unit ball $B_\rho(x,1)$, $x \in G/H$, is compact.

Next, by Proposition 4.5, one can assume that G/H is an invariant closed subset of the proper G-space $\widetilde{\mathcal{P}}_0(G/H)$. Now, due to Exact Slice Theorem 2.5 (see also [36, Corollary 1]), G/H is a G-retract of some invariant neighborhood U in $\widetilde{\mathcal{P}}_0(G/H)$. Since by Corollary 2.3, $\widetilde{\mathcal{P}}(G/H) \in G$-AE($\mathcal{P}$), we conclude that $G/H \in G$-ANE($\mathcal{P}$), as required. □

Proposition 5.2. *Let G be an almost connected group and H a large subgroup of G. Then G/H is a metrizable G-ANE($\mathcal{P}$).*

Proof. According to Proposition 3.6, there exists a compact normal subgroup N of G such that $N \subset H$ and G/N is a Lie group.

Observe that the following G/N-equivariant homeomorphism holds:

$$G/H \cong \frac{G/N}{H/N}. \tag{5.1}$$

Since G/N is a Lie group, it then follows from Proposition 5.1 and from the homeomorphism (5.1) that G/H is a G/N-ANE($\mathcal{P}$). Further, G/H is metrizable since it is homeomorphic to the coset space of the metrizable (in fact Lie) group G/N.

Now, since N acts trivially on G/H, it then follows from [4, Proposition 3] that G/H is a G-ANE($\mathcal{P}$). □

We have developed all the tools necessary to prove our main result:

Theorem 5.3. *Let H be a compact subgroup of a locally compact group G. Then the following properties are equivalent*:

(1) *H is a large subgroup,*
(2) *G/H is a (metrizable) G-ANE($\mathcal{P}$).*

Proof. (1) ⇒ (2) By Proposition 3.7, G/H is metrizable. Then, by Proposition 4.6, one can assume that G/H is a G-invariant closed subset of $\widetilde{\mathcal{P}}_0(G/H)$ and

$$\begin{cases} \|x-y\| > 1/2 \quad \text{whenever } x \text{ and } y \text{ belong} \\ \quad \text{to different connected components of } G/H. \end{cases} \tag{5.2}$$

Set $S = G_0H/H$ and denote by W the 1/4-neighborhood of S in $\widetilde{\mathcal{P}}_0(G/H)$, i.e.,

$$W = \{z \in \widetilde{\mathcal{P}}_0(G/H) \mid \operatorname{dist}(z, S) < 1/4\}.$$

Claim. *W is a G_0H-slice in $\widetilde{\mathcal{P}}_0(G/H)$.*

Indeed, W is G_0H-invariant since S is so and the norm of $\widetilde{\mathcal{P}}(G/H)$ is G-invariant. Further, $G(W)$ is open in $\widetilde{\mathcal{P}}_0(G/H)$ since W is so.

Check that W and gW are disjoint whenever $g \in G \setminus G_0H$. In fact, if $gw \in W$ for some $w \in W$ then $\|w - s\| < 1/4$ and $\|gw - s_1\| < 1/4$ for some $s, s_1 \in S$. By the invariance of the norm we have $\|gw - gs\| = \|w - s\| < 1/4$. Hence,

$$\|gs - s_1\| \leqslant \|gs - gw\| + \|gw - s_1\| < 1/4 + 1/4 = 1/2.$$

Consequently, by (5.2), s_1 and gs must belong to the same connected component of G/H. Since $s_1 \in S$ we infer that $gs \in S$. Thus, $S \cap gS \neq \emptyset$. But, since S is a global G_0H-slice of G/H (see the proof of Proposition 4.6), it then follows that $g \in G_0H$, as required.

Thus, $G(W)$ is the disjoint union of its open subsets gW. In particular, each gW is also closed in $G(W)$.

So, we have verified that W is a G_0H-slice in $\widetilde{\mathcal{P}}_0(G/H)$.

Further, since G_0H is an almost connected group (see Corollary 3.3), it follows from Proposition 5.2 that S is a G_0H-ANE($\mathcal{P}$). Now, since $W \in G_0H$-$\mathcal{P}$, it then follows that there exists a G_0H-equivariant retraction $r : V \to S$ for some G_0H-invariant neighborhood V of S in W.

Then r induces a G-map $R : G(V) \to G/H$ by the rule: $R(gv) = gr(v)$, where $g \in G$ and $v \in V$ (see [17, Chapter I, Proposition 4.3]). Clearly, R is a G-retraction, and hence, G/H being a G-neighborhood retract of the G-AE($\mathcal{P}$)-space $\widetilde{\mathcal{P}}(G/H)$ (see Corollary 2.3), is itself a G-ANE($\mathcal{P}$)-space, as required.

(2) ⇒ (1) Let H be a compact subgroup of a locally compact group G. It is a well-known result of H. Yamabe that every locally compact group G admits an open (and hence, closed) almost connected subgroup G_1. It is easy to verify that the following natural G_1-equivariant homeomorphism holds:

$$G_1/(G_1 \cap H) \cong G_1H/H. \tag{5.3}$$

Since, in general, G_1H need not be a group, it is in order to clarify the meaning of G_1H/H on the right side of this homeomorphism. It is just the H-orbit space of G_1H under the H-action given by $h * (g_1h_1) = g_1h_1h^{-1}$, where $g_1 \in G_1$ and $h, h_1 \in H$. Then G_1 acts on G_1H/H by left translations.

Next, it follows from Proposition 2.6 that G/H is a G_1-ANE($\mathcal{P}$).

Since G_1H is an open G_1-invariant subset of G it then follows that G_1H/H is an open G_1-invariant subset of G/H, and hence, it is a G_1-ANE($\mathcal{P}$). This, together with (5.3) yields that $G_1/(G_1 \cap H)$ is a G_1-ANE($\mathcal{P}$). Since, by (5.3), $G_1/(G_1 \cap H)$ homeomorphic to an open subset of G/H, it is sufficient to prove that $G_1/(G_1 \cap H)$ is a manifold. Hence, it is no loss of generality to consider the pair $(G_1, G_1 \cap H)$ instead of the pair (G, H).

Assume henceforth that G is almost connected. Consider the homeomorphism $(x, k) \mapsto xk : E \times K \to G$ given by the structure theorem, where E is a subset of G such that $kEk^{-1} = E$ for every $k \in K$ (see the paragraph before the formula (3.2)). In this case, there is a natural homeomorphism $E \times (K/H) \cong G/H$, see (3.2). Hence, it suffices to prove that K/H is a manifold.

We claim that K/H is a K-equivariant retract of G/H. Indeed, each element $g \in G$ has a unique decomposition $g = x_gk_g$ with $x_g \in E$ and $k_g \in K$. We claim that the retraction $r : G \to K$ given by $r(g) = k_g$ is $K \times K$-equivariant, where the action of the group $K \times K$ on G is given by the rule: $(t, s) * g = tgs^{-1}$. In fact, if $g = x_gk_g$ is the decomposition of $g \in G$ then for each $(t, s) \in K \times K$ one has:

$$tgs^{-1} = tx_gt^{-1} \cdot tk_gs^{-1}. \tag{5.4}$$

Since $tx_gt^{-1} \in E$ and $tk_gs^{-1} \in K$ we infer that (5.4) is the (unique) decomposition of the element tgs^{-1}, and hence, $r(tgs^{-1}) = tk_gs^{-1}$. This shows that r is a $K \times K$-equivariant retraction. Consequently, the map $\tilde{r} : G/H \to K/H$ induced by r is a K-equivariant retraction, and hence, K/H is a K-equivariant retract of G/H, as claimed.

Next, since G/H is a G-ANE($\mathcal{P}$) we infer that it is a K-ANE($\mathcal{P}$) (see Proposition 2.6). Hence K/H, being a K-equivariant retract of G/H, is itself a K-ANE($\mathcal{P}$). Therefore, by considering the pair (K,H) instead of (G,H), we see that it is no loss of generality to assume that G is compact.

Assume henceforth that G is compact. It is a well-known classical result (see, e.g., [26, Chapter 2, Corollary 2.43]) that every compact group G can be represented as the limit of a projective (or an inverse) system $\{f_{ij}: G_j \to G_i \mid (i,j) \in J \times J, i \leqslant j\}$ indexed by a directed set J, where all G_i, $i \in J$, are compact Lie groups, and all bonding maps f_{ij}, as well as all limit maps $f_i: G \to G_i$ are continuous epimorphisms.

We shall briefly consider the equivariant analog of Mardešić's construction [31] described in [5]. Let G^* be the disjoint union $G \sqcup \bigsqcup_{i\in J} G_i$, and for every $i \in J$ we set $G_i^* = G \sqcup \bigsqcup_{j \geqslant i} G_j$. Define the map $r_i: G_i^* \to G_i$ such that it coincides with f_i on G and with f_{ij} on G_j for $j \geqslant i$. Consider a topology on G^* whose open base consists of all possible sets $U_i \subset G_i$ which are open in G_i, and the sets of the form $U_i^* = r_i^{-1}(U_i)$, $i \in J$. It is known [31] that G^* becomes then a paracompact space. The natural action of the group G on G^* is continuous [5, Lemma 8]. Thus, G^* is a paracompact G-space.

Let $f: G \to G/H$ be the natural projection given by the formula $f(g) = gH$, $g \in G$. Since f is a G-map and $G/H \in G$-ANE($\mathcal{P}$), there exists an equivariant extension $F: V \to G/H$ of f over an invariant neighborhood V of G in G^*. In accordance with [31], there exists an index $i \in J$ such that $G_i \subset V$. Then the restriction $F|_{G_i}: G_i \to G/H$ is a G-map, and hence, for every point $z \in G_i$, the stabilizer G_z is contained in the stabilizer $G_{F(z)}$ of its image $F(z) \in G/H$. Due to equivariance, $F|_{G_i}$ is an onto map, so there exists a point $z \in G_i$ such that $F(z) = eH$, and hence, $G_{F(z)} = H$. Further, since the limit projection $f_i: G \to G_i$ is an epimorphism, we infer that $G_i = G/N_i$ with N_i the kernel of f_i. Therefore, $N_i \subset G_z$. In sum, we get that $N_i \subset H$ and the quotient group G/N_i is a Lie group. This, by Proposition 3.6, yields that H is a large subgroup, as required. □

We conclude this section with the following two corollaries, which were used in the proof of [11, Theorem 1.1]:

Corollary 5.4. *([11], Lemma 2.5) Let H be a large subgroup of G. Assume that A is a closed invariant subset of a proper G-space $X \in G$-$\mathcal{P}$, and S is a global H-slice of A. Then there exists an H-slice $\widetilde{S}$ in X such that $\widetilde{S} \cap A = S$.*

Proof. Let $f: A \to G/H$ be a G-map with $f^{-1}(eH) = S$. By Theorem 5.3, $G/H \in G$-ANE($\mathcal{P}$), and hence, there exists a G-extension $F: U \to G/H$ over an invariant neighborhood U of A in X. It is easy to see that the preimage $\widetilde{S} = F^{-1}(eH)$ is the desired H-slice. □

Corollary 5.5. *([11], Lemma 3.2) Let H be a closed normal subgroup of G, and K a large subgroup of G. Then $(KH)/H$ is a large subgroup of G/H.*

Proof. Since $(KH)/H$ is the image of the compact subgroup K under the continuous homomorphism $G \to G/H$ we see that it is a compact subgroup of G/H. Further, observe that the following homeomorphism (even G-equivariant) holds:

$$\frac{G/H}{(KH)/H} \cong G/KH. \tag{5.5}$$

Since K is a large subgroup and $K \subset KH$, it follows from Proposition 3.4 that KH is so. Hence, the coset space G/KH is finite-dimensional and locally connected. It remains to apply (5.5). □

6. Approximate slices for proper actions of non-Lie groups

In [1] and [10] approximate versions of the Exact Slice Theorem 2.5 were established which are applicable also to proper actions of non-Lie groups.

In this section we shall prove the following new version of the Approximate Slice Theorem for proper actions of *arbitrary locally compact groups* which improves the one in [10, Theorem 3.6]:

Theorem 6.1 *(Approximate Slice Theorem). Assume that X is a proper G-space, $x \in X$ and O a neighborhood of x. Denote by $\mathcal{N}(x, O)$ the set of all large subgroups H of G such that $G_x \subset H$ and $H(x) \subset O$. Then:*

(1) *$\mathcal{N}(x, O)$ is not empty.*
(2) *For every $K \in \mathcal{N}(x, O)$, there exists a K-slice S with $x \in S \subset O$.*

In the proof of this theorem we shall need the following lemma:

Lemma 6.2. *Let X be a proper G-space, H a compact subgroup of G, and S a global H-slice of X. Then the restriction $f: G \times S \to X$ of the action is an open map.*

Proof. Let O be an open subset of G and U be an open subset of S. It suffices to show that the set $OU=\{gu \mid g\in O,\ u\in U\}$ is open in X.

Define $W=\bigcup_{h\in H}(Oh^{-1})\times(hU)$. We claim that

$$X\setminus OU=f\big((G\times S)\setminus W\big). \tag{6.1}$$

Indeed, since $OU=f(W)$ and $X=f(G\times S)$, the inclusion $X\setminus OU\subset f((G\times S)\setminus W)$ follows.

Let us establish the converse inclusion $f((G\times S)\setminus W)\subset X\setminus OU$.

Assume the contrary, that there exists a point $gs\in f(G\times S)\setminus W)$ with $(g,s)\in(G\times S)\setminus W$ such that $gs\in OU$. Then $gs=tu$ for some $(t,u)\in O\times U$. Denote $h=g^{-1}t$. Then one has:

$$s=g^{-1}tu=hu \quad\text{and}\quad (g,s)=\big(tt^{-1}g,g^{-1}tu\big)=\big(th^{-1},hu\big)\in\big(Oh^{-1}\big)\times(hU).$$

Since both s and u belong to S, and $s=hu$, we conclude that $h\in H$. Consequently, $(Oh^{-1})\times(hU)\subset W$, yielding that $(g,s)\in W$, a contradiction. Thus, the equality (6.1) is proved.

Being a global H-slice, the set S is a closed small subset of X. Consequently, by virtue of [1, Proposition 1.4], the restriction of the action map $G\times S\to X$ is closed. Then, since $(G\times S)\setminus W$ is a closed subset of $G\times S$, the image $f((G\times S)\setminus W)$ is closed in X. Finally, together with the equality (6.1), this implies that OU is open in X, as required. □

Proof of Theorem 6.1. We first consider two special cases, namely G totally disconnected and G almost connected, and combine the two to get the general result.

Consider the set $V=\{g\in G\mid gx\in O\}$ which is an open neighborhood of the compact subgroup G_x in G.

Case 1. Let G be totally disconnected. Then there exists a compact open subgroup H of G such that $G_x\subset H\subset V$ (see [33, Chapter II, Section 2.3]). Therefore G/H is discrete, and hence, H is a large subgroup of G. Thus, $H\in\mathcal{N}(x,O)$.

Now assume that $K\in\mathcal{N}(x,O)$. Then K is a compact open subgroup of G (see, e.g., Corollary 3.3). Since $K(x)\subset O$, there exists a neighborhood Q of x such that $KQ\subset O$. Since K is open, by [36, Proposition 1.1.6], there exists a neighborhood W of the point x in X such that $\langle W,W\rangle\subset K$. Then the set $S=K(Q\cap W)$ is a K-invariant neighborhood of x with $S\subset O$ and $\langle S,S\rangle=K^{-1}\langle Q\cap W,Q\cap W\rangle K=K$. Now, the saturation $U=G(S)$ is the disjoint union of open subsets gS, one g out of every coset in G/K. So, the map $f:U\to G/K$ with $f(u)=gK$ if $u\in gS$, is a well-defined G-map and $f^{-1}(eK)=S$. Since $x\in S\subset O$, we are done.

Case 2. Let G be almost connected. By compactness of G_x, there exists a unity neighborhood V_1 in G such that $V_1\cdot G_x\subset V$. By a result of Yamabe (see [33, Chapter IV, Section 46] or [21, Theorem 8]), V_1 contains a compact normal subgroup N of G such that G/N is a Lie group; in particular, N is a large subgroup of G. Setting $H=N\cdot G_x$ we get a compact subgroup H of G such that $G_x\subset H\subset V$. Since $N\subset H$ and N is a large subgroup, it follows from Proposition 3.4 that H is also a large subgroup. Thus, $H\in\mathcal{N}(x,O)$.

Now assume that $K\in\mathcal{N}(x,O)$. Since G/K is finite-dimensional and locally connected, by Proposition 3.6, there exists a compact normal subgroup M of G such that $M\subset K$ and G/M is a Lie group.

Since K is compact and $K(x)\subset O$, there exists a K-invariant neighborhood Q of x such that $Q\subset O$. Let $p:X\to X/M$ be the M-orbit map. Then X/M is a proper G/M-space [36, Proposition 1.3.2], and it is easy to see that the G/M-stabilizer of the point $p(x)\in X/M$ is just the group K/M. Now, by the Exact Slice Theorem 2.5, there exists an invariant neighborhood $\widetilde{U}$ of $p(x)$ in X/M and a G/M-equivariant map

$$\tilde{f}:\widetilde{U}\to\frac{G/M}{K/M}$$

such that $\tilde{f}(p(x))=K/M$.

Next we shall consider X/M (and its invariant subsets) as a G-space endowed with the action of G defined by the natural homomorphism $G\to G/M$. In particular, $\widetilde{U}$ is a G-space.

Since the two G-spaces $\frac{G/M}{K/M}$ and G/K are naturally G-homeomorphic, we can consider $\tilde{f}$ as a G-equivariant map from $\widetilde{U}$ to G/K with $\tilde{f}(p(x))=eK$.

Let $S=\tilde{f}^{-1}(eK)$ and $S_1=S\cap p(Q)$. Then S is a global K-slice for $\widetilde{U}$ and S_1 is an open K-invariant subset of S.

We claim that the G-saturation $U_1=G(S_1)$ is a tubular set with S_1 as a K-slice. Indeed, the openness of U_1 in $\widetilde{U}$, and hence in X/M, follows from Lemma 6.2.

To prove that S_1 is a global K-slice of U_1 it suffices to show that ${f_1}^{-1}(eK)=S_1$, where $f_1:U_1\to G/K$ is the restriction $\tilde{f}|_{U_1}$.

To this end, choose $x\in {f_1}^{-1}(eK)$ arbitrary. Since ${f_1}^{-1}(eK)=S\cap U_1$ then $x=gs_1$ for some $s_1\in S_1$ and $g\in G$. Hence, $gs_1\in S\cap gS$, which implies that $g\in K$. Since S_1 is K-invariant we infer that $x=gs_1\in S_1$. Thus, ${f_1}^{-1}(eK)\subset S_1$. The converse inclusion $S_1\subset {f_1}^{-1}(eK)$ is evident, so we get the desired equality ${f_1}^{-1}(eK)=S_1$.

Thus, S_1 is a K-slice lying in $p(Q)$ and containing the point $p(x) \in X/M$.

Now we set $U = p^{-1}(U_1)$, $S = p^{-1}(S_1)$, and let $f : U \to G/K$ be the composition $f_1 p$. Since $S = f^{-1}(eK) \subset p^{-1}(p(Q)) = Q \subset O$ and $x \in S$, we conclude that S is the desired K-slice.

Case 3. Let G be arbitrary. First we show that $\mathcal{N}(x, O) \neq \emptyset$.

Denote by G_0 the identity component of G and $\widetilde{G} = G/G_0$. Set $\widetilde{X} = X/G_0$ and let $p : X \to \widetilde{X}$ be the G_0-orbit map. Then $\widetilde{X}$ is a proper $\widetilde{G}$-space [36, Proposition 1.3.2], and the stabilizer $\widetilde{G}_{p(x)}$ of the point $p(x) \in X/M$ in $\widetilde{G}$ is just the group $(G_0 \cdot G_x)/G_0$.

Since $\widetilde{G}$ is totally disconnected, there exists a compact open subgroup M of $\widetilde{G}$ such that $\widetilde{G}_{p(x)} \subset M$ (see [33, Chapter II, Section 2.3]).

Denote by $\pi : G \to \widetilde{G}$ the natural homomorphism and let $L = \pi^{-1}(M)$. Then L is a closed-open subgroup of G. Since, clearly, the quotient group G/G_0 is topologically isomorphic to the compact group M we infer that L is almost connected. Hence, we can apply the first part of Case 2 to the almost connected group L, the proper L-space X and the neighborhood $O \subset X$ of the point $x \in X$. Accordingly, there exists a large subgroup N of L such that $L_x \subset N$ and $N(x) \subset O$.

We claim that $N \in \mathcal{N}(x, O)$. Indeed, since $(G_0 \cdot G_x)/G_0 = \widetilde{G}_{p(x)} \subset M$ we infer that $G_0 \cdot G_x \subset L$. In particular, this yields that $G_x = L_x$, and hence, $G_x \subset N$. It remains to check that N is a large subgroup of G. In fact, since N is a large subgroup of L, due to Theorem 3.7, the quotient L/N is locally contractible. But G/N is the disjoint union of its open subsets of the form xL/N, $x \in G$, each of which is homeomorphic to L/N. Consequently, G/N is itself locally contractible, and again by Theorem 3.7, this yields that N is a large subgroup of G. Thus, we have proved that $N \in \mathcal{N}(x, O)$, as required.

Next, we assume that $K \in \mathcal{N}(x, O)$. Since K is a large subgroup of G, by Corollary 3.3, $H = G_0 K$ is an open almost connected subgroup of G. Hence $\widetilde{H} = G_0 K/G_0$ is a compact open subgroup of $\widetilde{G}$. The inclusion $G_x \subset K$ easily implies that $\widetilde{G}_{p(x)} \subset \widetilde{H}$. Respectively, the inclusion $K(x) \subset O$ yields that $\widetilde{H} \subset p(O)$. Then, according to Case 1, there exists a $\widetilde{G}$-map $f_1 : U_1 \to \widetilde{G}/\widetilde{H}$ of an open $\widetilde{G}$-invariant neighborhood U_1 of $p(x)$ in $\widetilde{X}$ to the discrete $\widetilde{G}$-space $\widetilde{G}/\widetilde{H}$ with $p(x) \in f_1^{-1}(\tilde{e}\widetilde{H}) \subset p(O)$.

The inverse image $W_1 = f_1^{-1}(\tilde{e}\widetilde{H})$ is an open $\widetilde{H}$-invariant subset of $\widetilde{X}$; so the set $W = p^{-1}(W_1)$ is an open H-invariant subset of X with $x \in W$, $G_x \subset K \subset H$ and $K(x) \subset W \cap O$. Since H/K is open in G/K we infer that K is a large subgroup of H (for instance, by Theorem 3.7).

Hence, we can and do apply Case 2 of this proof to the almost connected group H, the proper H-space W, the neighborhood $O \cap W$ of the point $x \in W$ and the large subgroup K of H. Then there exist an H-neighborhood U of x in W and an H-map $f_0 : U \to H/K$ with $x \in f_0^{-1}(eK) \subset O \cap W$. Next we want to extend f_0 to a G-map $f : G(U) \to G/K$. Since $H/K \subset G/K$, we simply define $f(gu) = gf_0(u)$ for $g \in G$, $u \in U$.

It is easy to check that f is a well-defined G-map. Thus, the K-slice $S = f^{-1}(eK)$ is the desired one. □

If G is a Lie group then, clearly, each compact subgroup of G is large. So, in this case, Theorem 6.1 has the following simpler form:

Corollary 6.3. *Assume that G is a Lie group, X a proper G-space, $x \in X$ and O a neighborhood of x. Then for each compact subgroups K of G such that $G_x \subset K$ and $K(x) \subset O$, there exists a K-slice S such that $x \in S \subset O$.*

We derive from Theorem 6.1 yet another corollary applicable to the, so-called, *rich* G-spaces.

Recall that a G-space X is called *rich*, if for any point $x \in X$ and for any its neighborhood $U \subset X$, there exists a point $y \in U$ such that the stabilizer G_y is a large subgroup of G and $G_x \subset G_y$ (see [7,9]).

Corollary 6.4. *Assume that X is a rich proper G-space, $x \in X$ and O a neighborhood of x. Then there exist a point $y \in O$ with a large stabilizer G_y containing G_x, and a G_y-slice S such that $x \in S \subset O$.*

Proof. Choose a neighborhood O' of x such that $G_y(x)$ is contained in O for all $y \in O'$ (see [10, Lemma 3.9]). Further, since X is a rich proper G-space, we can choose a point $y \in O'$ such that G_y is a large subgroup of G and $G_x \subset G_y$. Thus, $G_y \in \mathcal{N}(x, O)$, the set defined in Theorem 6.1.

If $G(y) = G(x)$ then the stabilizer G_x, being conjugate to G_y, is also a large subgroup, and clearly, $G_x \in \mathcal{N}(x, O)$. Next, we apply item (2) of Theorem 6.1 to $K = G_x$; the resulting K-slice S is the desired one.

If $G(y) \neq G(x)$ then we first choose (due to [36, Proposition 1.2.8]) disjoint invariant neighborhoods A_x and A_y of $G(x)$ and $G(y)$, respectively. Next, we apply twice the first assertion of the statement (2) of Theorem 6.1: first, to $x \in O \cap A_x$ and $K = G_y$, and then to $y \in O \cap A_y$ and $K = G_y$. As a result we get two G_y-slices $S_x \subset O \cap A_x$ and $S_y \subset O \cap A_y$ which contain the points x and y, respectively. Since $A_x \cap A_y = \emptyset$ the union $S = S_x \cup S_y$ is the desired G_y-slice. □

In conclusion we show that there are sufficiently many rich G-spaces. Indeed, it was proved in [7] that if G is a compact group then every metrizable G-ANE($\mathcal{M}$) is a rich G-space, where G-$\mathcal{M}$ stands for the class of all proper G-spaces that are metrizable by a G-invariant metric. The same was proved in [10, Proposition 3.10] for proper actions of almost connected groups. Below we show that it is true also for proper actions of arbitrary locally compact groups.

Proposition 6.5. *Every metrizable proper G-ANE($\mathcal{M}$) is a rich G-space.*

Proof. Let X be a metrizable proper G-ANE($\mathcal{M}$), $x \in X$ and O a neighborhood of x. Then by Theorem 6.1, there is a compact large subgroup $K \subset G$, and a K-slice S such that $x \in S \subset O$. The tube $G(S)$, being an open subset of X, is a G-ANE($\mathcal{M}$) as well. This yields that $G(S)$ is a K-ANE($\mathcal{M}$) (see [10, Proposition 3.4]). Hence, according to [7, Proposition (2)], $G(S)$ is a rich K-space, so there exists a point $y \in G(S) \cap O$ such that $K_x \subset K_y$, and K_y is a large subgroup of K. It remains to show that G_y is a large subgroup of G.

First, it follows from Proposition 3.5 that K_y is a large subgroup of G. Next, since the point y belongs to the K-slice S we infer that $K_y = G_y$, and therefore, G_y is a large subgroup of G, as required. □

7. Concluding remarks

The author passionately believes in the following characterization of large subgroups:

Conjecture 7.1. *A compact subgroup H of G is large iff the coset space G/H is locally contractible.*

In fact, this conjecture was an *assertion* in the initial version of the paper (see also [14, Proposition 4.11]). Our proof in [14] relies on J. Szenthe's Theorem 4 from [39] which claims that if an almost connected group acts effectively and transitively on a locally contractible space then it is a Lie group. After this paper was submitted the author realized that there is a gap in Szenthe's proof. Namely, [39, Lemma 6] is an essential ingredient of Szenthe's argument, however it is false. Indeed, this lemma claims that if H is a closed subgroup of a compact group G, then every closed normal subgroup A of G whose image under the natural projection $G \to G/H$ is contractible in G/H is necessarily contained in H. Here is an easy counterexample to this assertion: take $G = \mathbb{S}^1$ the circle group, $H = \{1\}$ the trivial subgroup, and A any nontrivial finite subgroup of $\mathbb{S}^1$.

Thus, [39, Theorem 4] is not proved correctly and it now becomes an important conjecture:

Conjecture 7.2 *(J. Szenthe). Let H be a closed subgroup of a compact connected group G such that the coset space G/H is locally contractible and G acts effectively on it. Then G is a Lie group.*

In this connection it is in order to remember a result by K.H. Hofmann [24] (in fact a corollary of [24, Main Lemma]) which states that each locally contractible locally compact group is necessarily a Lie group. The same is true also for (not necessarily locally compact) pro-Lie groups [27, Theorem 1.1].

In fact, Conjecture 7.1 follows from the special case of Conjecture 7.2 when the subgroup H is compact. In turn, by virtue of the reduction procedure described at the end of Section 3, this special case of Conjecture 7.2 follows from the following its particular case:

Conjecture 7.3. *Let H be a closed subgroup of a compact connected group G such that the coset space G/H is locally contractible and G acts effectively on it. Then G is a Lie group.*

Acknowledgement

The author is obliged to the referee for many valuable comments and stimulating suggestions which lead to a substantial improvement of the paper.

References

[1] H. Abels, A universal proper G-space, Math. Z. 159 (1978) 143–158.
[2] N. Antonyan, S. Antonyan, L. Rodríguez-Medina, Linearization of proper group actions, Topology Appl. 156 (2009) 1946–1956.
[3] S.A. Antonyan, Retracts in categories of G-spaces, Izvestiya Akad. Nauk Arm. SSR Ser. Matem. 15 (1980) 365–378, English transl.: Soviet J. Contemp. Math. Anal. 15 (1980) 30–43.
[4] S.A. Antonyan, Equivariant embeddings into G-AR's, Glasnik Matematički 22 (42) (1987) 503–533.
[5] S.A. Antonyan, Retraction properties of the orbit space, Matem. Sbornik 137 (3) (1988) 300–318 (in Russian), English transl.: Math. USSR Sbornik 65 (2) (1990) 305–321.
[6] S.A. Antonyan, Equivariant embeddings and free G-spaces, in: Il VI^{sto} Simposio Tiraspoliano di Topologia Generale e sue Applicazioni, Institutul Pedagogio din Tiraspol, Chişinău, 1991, pp. 9–10.
[7] S.A. Antonyan, Existence of a slice for arbitrary compact transformation groups, Matematicheskie Zametki 56 (5) (1994) 3–9 (in Russian), English transl.: Math. Notes 56 (5–6) (1994) 1101–1104.
[8] S.A. Antonyan, The Banach–Mazur compacta are absolute retracts, Bull. Acad. Polon. Sci. Ser. Math. 46 (1998) 113–119.
[9] S.A. Antonyan, Extensorial properties of orbit spaces of proper group actions, Topology Appl. 98 (1999) 35–46.
[10] S.A. Antonyan, Orbit spaces and unions of equivariant absolute extensors, Topology Appl. 146–147 (2005) 289–315.
[11] S.A. Antonyan, Orbit spaces of proper equivariant absolute extensors, Topology Appl. 153 (2005) 698–709.
[12] S.A. Antonyan, Proper actions of locally compact groups on equivariant absolute extensors, Fund. Math. 205 (2009) 117–145.
[13] S.A. Antonyan, Proper actions on topological groups: Applications to quotient spaces, Proc. Amer. Math. Soc. 138 (10) (2010) 3707–3716.

[14] S.A. Antonyan, Equivariant extension properties of coset spaces of locally compact groups and approximate slices, arXiv:1103.0804v1 [math.GN], 2011.
[15] A. Arhangel'skii, M. Tkachenko, Topological Groups and Related Structures, Atlantis Press/World Scientific, Amsterdam/Paris, 2008.
[16] G. Bredon, Introduction to Compact Transformation Groups, Academic Press, New York/London, 1972.
[17] T. tom Dieck, Transformation Groups, Walter de Gruyter, Berlin/New York, 1987.
[18] J. Dugundji, Topology, Allyn and Bacon Inc., Boston, 1966, MR 33:1824.
[19] E. Elfving, The G-homotopy type of proper locally linear G-manifolds, Ann. Acad. Sci. Fenn. Math. Dissertationes 108 (1996).
[20] R. Engelking, General Topology, PWN-Pol. Sci. Publ., Warsaw, 1977.
[21] V.M. Glushkov, The structure of locally compact groups and Hilbert's fifth problem, Amer. Math. Soc. Transl. 15 (2) (1960) 59–93.
[22] O. Hájek, Parallelizability revisited, Proc. Amer. Math. Soc. 27 (1) (1971) 77–84.
[23] G. Hochschild, The Structure of Lie Groups, Holden-Day Inc., San Francisco, 1965.
[24] K.H. Hofmann, Homogeneous locally compact groups with compact boundary, Trans. Amer. Math. Soc. 106 (1963) 52–63.
[25] K.H. Hofmann, S.A. Morris, Transitive actions of compact groups and topological dimension, J. Algebra 234 (2000) 454–479.
[26] K.H. Hofmann, S.A. Morris, The Structure of Compact Groups, 2nd revised and augmented edition, de Gruyter, Berlin/New York, 2006.
[27] K.H. Hofmann, K.-H. Neeb, Pro-Lie groups which are infinite-dimensional Lie groups, Math. Proc. Camb. Soc. 146 (2009) 351–378.
[28] K.H. Hofmann, C. Terp, Compact subgroups of Lie groups and locally compact groups, Proc. Amer. Math. Soc. 120 (2) (1994).
[29] S. Illman, Every proper smooth action of a Lie group is equivalent to a real analytic action: a contribution to Hilbert's fifth problem, in: Frank Quinn (Ed.), Prospects in Topology, Proceedings of a Conference in Honor of William Browder, in: Ann. of Math. Stud., vol. 138, 1995, pp. 189–220.
[30] J.L. Koszul, Lectures on Groups of Transformations, Tata Institute of Fundamental Research, Bombay, 1965.
[31] S. Mardešić, On inverse limits of compact spaces, Glasnik Matemat. Fiz. Astr. 13 (4) (1958) 249–255.
[32] E. Michael, Some extension theorems for continuous functions, Pacific J. Math. 3 (4) (1953) 789–806.
[33] D. Montgomery, L. Zippin, Topological Transformation Groups, Interscience Publ., Inc., New York, 1955.
[34] K. Morita, On the dimension of product spaces, Amer. J. Math. 75 (2) (1953) 205–223.
[35] R. Palais, The classification of G-spaces, Mem. AMS 36 (1960).
[36] R. Palais, On the existence of slices for actions of non-compact Lie groups, Ann. of Math. 73 (1961) 295–323.
[37] L.S. Pontryagin, Selected Works, vol. 2. Topological Groups, Classics of Soviet Math., Gordon & Breach Sci. Publ., New York–London–Paris–Montreux–Tokyo, 1986.
[38] E.G. Skljarenko, On the topological structure of locally bicompact groups and their quotient spaces, Amer. Math. Soc. Transl. Ser. 2 39 (1964) 57–82.
[39] J. Szenthe, On the topological characterization of transitive Lie group actions, Acta Sci. Math. Szeged. 36 (3–4) (1974) 323–344.