Isoperimetric inequalities for special classes of curves

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Abstract

In this paper the classical Banchoff–Pohl inequality, an isoperimetric inequality for nonsimple closed curves in the Euclidean plane, involving the square of the winding number, is sharpened for homothetic or Abresch–Langer solutions of curve shortening. For a larger class of curves and for rotationally symmetric curves, further isoperimetric inequalities containing the rotation number and the winding number, are presented.

1. Introduction

In [2], T.F. Banchoff and W.F. Pohl presented an isoperimetric inequality for nonsimple closed curves in the Euclidean plane $E^2$, where the area of the enclosed domain is replaced by the sum of the areas into which the curve divides the plane, each weighted with the square of the winding number:

$$L^2 \geq \iint_{E^2} w^2(x) \, dA,$$

where $L$ is the length of the curve $C$, $w(x)$ the winding number of $x \in E^2 \setminus C$ with respect to $C$ and $dA$ the area element. Their proof uses integral geometric methods.

In [5] the author generalized this inequality to surfaces of nonpositive curvature and proved it by curve shortening. Here we refine the proof of [5] to prove a sharpened inequality for homothetic or Abresch–Langer solutions of curve shortening in the Euclidean plane.

The well known curve shortening problem is described by the partial differential equation

$$X_t = kN$$

for a closed initial curve $C(0)$ of class $C^2$, where $X$ is a parametrization of the evolving family of curves $C(t)$, $N$ the inward unit normal vector, $k$ the curvature, and the subscript $t$ denotes partial derivative by the time $t$. 
Solutions of (1) which do not change their shape during the evolution, are called homothetic, i.e. the deformation is a homothety, or, the evolving family \( C(t) \) is equal to \( C(0) \) under rescaling.

U. Abresch and J. Langer determined these solutions in [1], they are either \( \nu \) times traversed circles \( (\nu \in \mathbb{N}) \) or curves \( C_{\nu,q} \), whose curvature function \( k \) is the solution of the differential equation \( k(k + k^2) = 1 \) (cf. [3, (4.4)]). Here the angle \( \theta \), which the tangent vector encloses with a fixed axis, is chosen as the curve parameter. The first integral of this differential equation is

\[
\frac{k^2}{2} \theta + \frac{k^2}{2} - \log k^2 = a^2, \quad a \in [1, \infty) \text{ constant } [3, (4.6)].
\]

These solutions are periodical with \( q \) periods \( (q \in \mathbb{N} \text{ and } 1 < \frac{\sqrt{2}}{2} \nu < \nu) \), \( \nu \in \mathbb{N} \) is the rotation number of \( C_{\nu,q} \), and \( C_{\nu,q} \) is rotationally symmetric with rotation angles \( 2\pi i/q, i \in \mathbb{Z} \) [1, Theorem A].

There are also curves, which behave asymptotically like Abresch–Langer solutions, i.e. under (1) they converge to a point without the occurrence of any singularities during the evolution, and under rescaling towards an Abresch–Langer curve \( C_{\nu,q} \). They were discovered by C. Epstein and M. Weinstein ([4, Theorem]) or [3, §4]).

A special class of these nonsimple, locally convex curves, which converge to a point, are the by C. Epstein and M. Gage ‘highly symmetric’ called curves because of \( \frac{\nu}{q} < \frac{1}{2} \) (again \( \nu \) is the rotation number and \( q \) the symmetry grade). Their asymptotic shape is a \( \nu \) times traversed circle.

**Theorem 1.** Let \( C \) be a homothetic solution of (1) in \( E^2 \). Let \( \nu \) be the rotation number of \( C \) and \( \gamma \) a positive real number to be determined in the proof.

Then

\[
L^2 \geq 4\pi \int_{E^2} w^2(x) \, dA + 4\gamma \nu \int_{E^2} w(x) \, dA. \tag{2}
\]

Equality holds if and only if \( C \) is, up to homothety and translation, a circle traversed \( \nu \) times in the same direction.

The following isoperimetric inequality also contains the rotation number and is valid for a larger class of curves in \( E^2 \):

**Theorem 2.** Let \( C \) be a closed, immersed \( C^2 \) curve in \( E^2 \) with rotation number \( \nu \geq 1 \), which converges to a point under curve shortening without developing singularities.

Then

\[
L^2 \geq 4\pi \nu \int_{E^2} |w(x)| \, dA \tag{3}
\]

with equality if and only if \( C \) is, up to homothety and translation, a circle traversed \( \nu \) times in the same direction.

We have another result for rotationally symmetric curves:

**Theorem 3.** Let \( C \) be a closed, immersed, smooth curve in \( E^2 \) with strictly positive curvature and rotation number \( \nu \geq 1 \). Additionally, let \( C \) be rotationally symmetric with rotation angles \( 2\pi i/q \), with \( i \in \mathbb{Z}, q \in \mathbb{N} \) and \( q \geq \nu \).

Then

\[
L^2 \geq 4\pi \nu \int_{E^2} w(x) \, dA \tag{4}
\]

with equality if and only if \( C \) is, up to homothety and translation, a circle traversed \( \nu \) times in the same direction.

**2. The time derivatives**

We introduce the notions

\[
A := \int_{E^2} w(x) \, dA,
\]

\[
\tilde{A} := \int_{E^2} |w(x)| \, dA,
\]

\[
\hat{A} := \int_{E^2} w^2(x) \, dA.
\]

Let \( C \) be a closed, immersed \( C^2 \) curve in \( E^2 \) with parametrization \( X \) and arclength parameter \( s \). Then we define
\[ \hat{w}(s) := i^+(s) + i^-(s), \]
\[ \hat{w}(s) := \text{sign} \hat{w}(s), \]

where \( i^+(s) \) is the algebraic intersection number of a ray starting in \( X(s) \), in nontangential direction \( \xi \), with \( C \); and \( i^-(s) \) is the algebraic intersection number of the ray starting in \( X(s) \), in direction \(-\xi\), with \( C \). \( \hat{w} \) neither depends on the parametrization of \( C \), nor on the direction \( \xi \).

**Lemma.** The derivatives \( A'(t), \bar{A}'(t), \hat{A}'(t) \) and \( L'(t) \) for a family of curves \( C(t) \) evolving according to (1) are

\[ L'(t) = - \int_0^{L(t)} k^2 \, ds, \]
\[ A'(t) = - \int_0^{L(t)} k \, ds = -2\pi \nu, \]
\[ \bar{A}'(t) = - \int_0^{L(t)} \hat{w} k \, ds, \]
\[ \hat{A}'(t) = - \int_0^{L(t)} \hat{w} k \, ds. \]

(7) can be obtained through a straightforward calculation, see [5, Section 4]. The proofs of (9) and (10) are somewhat more subtle, cf. [5, Section 3]. (8) can easily be derived from (9).

### 3. Proof of Theorem 1

We prove the theorem by curve shortening, i.e. we take the curve \( C \) as initial value at time \( t = 0 \) for the curve shortening problem (1). We show that the isoperimetric deficit

\[ L^2 - 4\pi \int w^2(x) \, dA - 4\pi \int w(x) \, dA \]

is decreasing during the evolution, and nonnegative at final time \( T \). In fact, it converges to 0, since the curve converges to a point. This proves the inequality (2) for the initial curve \( C \).

We define a process of “peeling off” the curve: Since \( C \) is a closed \( C^2 \) curve, \( E^2 \setminus C \) consists of a finite number of disjoint domains. Exactly one of these domains is unbounded, called \( M_1 \). We define \( C_1 \) as the boundary of \( E^2 \setminus \text{cl}M_1 \), where \( \text{cl}M_1 \) is the topological closure of \( M_1 \). Then we consider \( E^2 \setminus \text{cl}(C \setminus C_1) \), which consists of a finite number of disjoint domains as well. We set \( C_2 \) as the boundary of \( E^2 \setminus \text{cl}M_2 \), where \( M_2 \) is the only unbounded domain at this step. At step \( i \), \( i \in \{1, \ldots, n\} \), we define \( C_i \) as the boundary of \( E^2 \setminus \text{cl}M_i \), where \( M_i \) is the only unbounded domain among \( E^2 \setminus \text{cl}(C \setminus \bigcup \{C_1 \cup \cdots \cup C_{i-1}\}) \). We proceed until \( C_1 \cup \cdots \cup C_i = C \). The number \( n \) is finite, since \( E^2 \setminus C \) consisted only of a finite number of domains. From every \( C_i \), \( 1 \leq i \leq n \), we obtain a positively oriented curve \( \tilde{C}_i \) by changing the orientation of each single arc of \( C_i \) such that the normal vector along \( \tilde{C}_i \) points inwardly. Each \( \tilde{C}_i \) possesses \( \mu_i \in \mathbb{N} \) connected parts.

Now we prove the monotonicity of \( L^2 - 4\pi \bar{A} \), i.e.

\[ \frac{d}{dt}(L^2 - 4\pi \bar{A})(t) \leq 0. \]

\( t \) is fixed during the following estimates, and for simplicity, we omit it.

We have

\[ \frac{1}{2}(L^2 - 4\pi \bar{A})' = -L \int \bar{A}(x) \, dA + 2\pi \int \bar{A} \, dA \leq - \left( \int \bar{A} \, dA \right)^2 + 2\pi \int |\bar{A}| \, dA \]

with (7), (10) and the Cauchy–Schwarz inequality.
Along $C_i$, $1 \leq i \leq n$, we have $|\dot{w}(s)| \leq |\dot{t}^+(s)| + |\dot{t}^-(s)| \leq 2i - 1$ using (5) (see also [5, p. 210]). This leads to

$$\frac{1}{2}(L^2 - 4\pi \dot{A})' \leq -\left(\sum_{i=1}^n \int_{C_i} |k| \, ds\right)^2 + 2\pi \sum_{i=1}^n (2i - 1) \int_{C_i} |k| \, ds$$

$$= \int_{C_n} |k| \, ds \left(-2 \sum_{i=1}^{n-1} \int_{C_i} |k| \, ds - \int_{C_n} |k| \, ds + 2\pi (2n - 1)\right)$$

$$+ \int_{C_{n-1}} |k| \, ds \left(-2 \sum_{i=1}^{n-2} \int_{C_i} |k| \, ds - \int_{C_{n-1}} |k| \, ds + 2\pi (2n - 3)\right) + \cdots$$

$$+ \int_{C_2} |k| \, ds \left(-2 \int_{C_1} |k| \, ds - |k| \, ds + 6\pi\right) + \int_{C_1} |k| \, ds \left(- \int_{C_1} |k| \, ds + 2\pi\right)$$

$$= \int_{C_n} |k| \, ds \left(- \sum_{i=1}^n \int_{C_i} |k| \, ds - \sum_{i=1}^{n-1} \int_{C_i} |k| \, ds + 2\pi (2n - 1)\right)$$

$$+ \int_{C_{n-1}} |k| \, ds \left(- \sum_{i=1}^{n-1} \int_{C_i} |k| \, ds - \sum_{i=1}^{n-2} \int_{C_i} |k| \, ds + 2\pi (2n - 3)\right) + \cdots$$

$$+ \int_{C_2} |k| \, ds \left(- 2 \sum_{i=1}^2 \int_{C_i} |k| \, ds - \int_{C_2} |k| \, ds + 6\pi\right) + \int_{C_1} |k| \, ds \left(- \int_{C_1} |k| \, ds + 2\pi\right).$$

With

$$\int_{C_1} |k| \, ds + \cdots + \int_{C_j} |k| \, ds \geq 2\pi (\mu_1 + \cdots + \mu_j) + \gamma_1 + \cdots + \gamma_j - \delta_1 - \cdots - \delta_j$$

for $1 \leq j \leq n$, where $\gamma_j$ is the sum of internal angles, and $\delta_i$ the sum of external angles of $\tilde{C}_i$, each angle nonnegative and taken such that each connected part of $\tilde{C}_i$ is traversed as one, connected curve (see [5, pp. 205, 206]), we get

$$\frac{1}{2}(L^2 - 4\pi \dot{A})' \leq (2\pi (-2\mu_1 - \cdots - 2\mu_{n-1} - \mu_n + 2n - 1)$$

$$+ 2(-\gamma_1 - \cdots - \gamma_{n-1} + \gamma_n + 2\delta_1 + \cdots + \delta_{n-1}) \int_{C_n} |k| \, ds$$

$$+ (2\pi (-2\mu_1 - \cdots - 2\mu_{n-2} - \mu_{n-1} + 2n - 3)$$

$$+ 2(-\gamma_1 - \cdots - \gamma_{n-2} + \gamma_{n-1} + \gamma_1 + \delta_1 + \cdots + \delta_{n-2} - \gamma_1 + \delta_1) \int_{C_{n-1}} |k| \, ds$$

$$+ (2\pi (-2\mu_1 - 2\mu_2 + 3) + 2(-\gamma_1 + \delta_1) - \gamma_1 + \delta_1 \int_{C_2} |k| \, ds$$

$$+ (2\pi (-\mu_1 + 1) - \gamma_1 + \delta_1) \int_{C_1} |k| \, ds.$$
By inserting \( \int |k| \, ds \geq 2\pi + \gamma_i - \delta_i \) for \( i \in \{1, \ldots, n\} \) (see [5, p. 205]) we receive
\[
\frac{1}{2}(L^2 - 4\pi A) \leq -(\gamma_n + \gamma_{n-1})(2\pi + \gamma_n - \delta_n) - (\gamma_{n-1} + \gamma_{n-2})(2\pi + \gamma_{n-1} - \delta_{n-2}) - \cdots \\
- (\gamma_2 + \gamma_1)(2\pi + \gamma_2 - \delta_2) - \gamma_1(2\pi + \gamma_1)
\]
which we obtain again with the help of \( \gamma_n - \delta_i \geq 0 \). So we have with \( \gamma := \gamma_1 + \cdots + \gamma_n \)
\[
\frac{1}{2}(L^2 - 4\pi A) \leq -4\pi (\gamma_1 + \cdots + \gamma_n) = -4\pi \gamma.
\]
Since \( C \) is a homothetic solution, it’s shape does not change during the evolution and so \( \gamma' = 0 \) and \( \nu' = 0 \) hold. This leads to
\[
\frac{d}{dt}(L^2 - 4\pi A) \leq -8\pi \gamma + 4\pi A' = 0
\]
using (8).

To complete the proof, we need to discuss equality in (2): If we have equality in (2), we must have equality in the Cauchy–Schwarz inequality and so \( k \) must be constant. In the other direction we have \( \gamma = 0 \) and so the classical Banchoff–Pohl inequality.

4. Proof of Theorem 2

\( \nu \) is constant during the evolution, so \( \nu' = 0 \) holds, and we get with (7), (9), the Cauchy–Schwarz inequality as well as (6)
\[
\frac{1}{2}(L^2 - 4\pi \nu A)' = -L \int_C k^2 \, ds + 2\pi \nu \int_C w k \, ds \leq - (\int_C |k| \, ds)^2 + 2\pi \nu \int_C |k| \, ds.
\]
Using \( \int_C |k| \, ds \geq \int_C k \, ds = 2\pi \nu \) we obtain the monotonicity of the isoperimetric deficit. The equality discussion is simple.

5. Proof of Theorem 3

We take \( C \) as initial curve \( C(0) \) of the flow with the evolution equation
\[
X_t = \left( h - \frac{1}{k} \right) N, \tag{12}
\]
where \( h := -\langle X, N \rangle \) \((\ldots) \) the scalar product in \( E^2 \) is the support function of \( C \). For curves with strictly positive curvature we can use the angle \( \theta \in [0, 2\pi \nu) \), which the tangent vector \( T \) encloses with a fixed direction, as curve parameter. A straightforward calculation shows that with this parameter the following evolution equations are valid for (12):
\[
\left( \frac{1}{k} \right)_t = \left( \frac{1}{k} \right)_{\theta \theta} \quad \text{and} \quad h_t = h_{\theta \theta}, \tag{13}
\]
i.e. \( 1/k \) and \( h \) satisfy the linear heat equation. With another straightforward calculation one can show that this flow preserves the length of the curves during the evolution and that the curves \( C(t) \) converge to \( \nu \) times traversed circles for \( t \to \infty \). We choose the rotation center as the origin of the coordinate system, then \( h \) is additionally periodical with period \( 2\pi \nu / q \). From the above mentioned facts we have \( \nu' = 0, \nu_t = 0 \) and \( \lim_{t \to \infty} (L^2 - 4\pi \nu A)(t) = 0 \). With
\[
A = \frac{1}{2} \int_0^{2\pi \nu} h \, d\theta = \frac{1}{2} \int_0^{2\pi \nu} h(h_{\theta \theta} + h) \, d\theta = \frac{1}{2} \int_0^{2\pi \nu} (h^2 - h_{\theta \theta}^2) \, d\theta
\]
we obtain
\[
A' = \frac{1}{2} \int_0^{2\pi \nu} (h h_{\theta\theta} - h_\theta h_{\theta\theta\theta}) \, d\theta = \frac{1}{2} \int_0^{2\pi \nu} (h_{\theta\theta}^2 - h_\theta^2) \, d\theta.
\]

Since \( h \) and so also \( h_\theta \) possess the period \( 2\pi \nu / q \leq 2\pi \), the Wirtinger inequality
\[
\frac{1}{2} \int_0^{2\pi \nu / q} (h_{\theta\theta}^2 - h_\theta^2) \, d\theta \geq 0
\]
holds for \( h_\theta \). Equality holds if and only if \( h_\theta \) vanishes identically (due to the rotational symmetry). With the help of the periodicity of \( h \) we get now
\[
(L^2 - 4\pi \nu A)' = -4\pi \nu A' = -2\pi \nu q \int_0^{2\pi \nu / q} (h_{\theta\theta}^2 - h_\theta^2) \, d\theta \leq 0
\]
with equality if and only if \( h \) is constant for each \( t \), in this case \( C(t) \) is a \( \nu \) times traversed circle for each \( t \).

6. Remarks

The preliminaries of Theorem 1 are fulfilled by e.g. the Epstein–Weinstein solutions, i.e. the curves with strictly positive curvature, which converge to a point under curve shortening and asymptotically to an Abresch–Langer solution.

The homothetic (for those \( 1/2 < \nu / q < 1/\sqrt{2} \) is valid) as well as the highly symmetric (\( \nu / q < 1/2 \)) solutions additionally fulfill the preliminaries of Theorem 3.

For highly symmetric curves C. Epstein and M. Gage proved a Bonnesen type inequality \[3, (5.11)\] and so a sharpened version of (4):
\[
L^2 - 4\pi \nu A \geq \pi^2 \nu^2 (r_{\text{out}} - r_{\text{in}})^2,
\]
where the outer radius \( r_{\text{out}} \) and the inner radius \( r_{\text{in}} \) are defined as for simple curves, with equality exactly for \( \nu \) times traversed circles.

References