Oscillation of solutions of parabolic differential equations of neutral type ♠

Peiguang Wang a,*, K.L. Teo b

a College of Electronic and Information Engineering, Hebei University, Baoding 071002, People’s Republic of China
b Department of Applied Mathematics, The Hong Kong Polytechnic University, Hung Hom, Kowloon, Hong Kong

Received 12 January 2004
Available online 8 April 2005
Submitted by C.V. Pao

Abstract

In this paper, we investigate a class of parabolic differential equations of neutral type, and obtain some sufficient conditions of the oscillation for such equations satisfying two kinds of boundary value conditions.

Keywords: Oscillation; Parabolic equation; Boundary value problem; Distributed deviating arguments

1. Introduction

In this paper, we consider the following parabolic differential equations of neutral type:

$$\frac{\partial}{\partial t} \left[ u - \lambda(t)u(x, t - \tau) \right] = a_0(t) \Delta u + a_1(t) \Delta u(x, t - \rho) - p(x, t)u$$

* This work was supported by the Natural Science Foundation of Hebei Province (A2004000089) and partially supported by the RGC grant (Polyu 5197/03E).
* Corresponding author.
E-mail addresses: pgwang@mail.hbu.edu.cn (P. Wang), mateokl@polyu.edu.hk (K.L. Teo).
\[- \int_{a}^{b} q(x,t,\xi)u[x, g(t,\xi)] \, d\mu(\xi), \quad (x,t) \in \Omega \times R_+ \equiv G, \tag{1}\]

with the boundary value conditions
\[
\begin{align*}
\frac{\partial u}{\partial n} + \nu(x,t)u &= 0, & (x,t) \in \partial\Omega \times \mathbb{R}^+, & (B_1) \\
u(x,t) &= 0, & (x,t) \in \partial\Omega \times \mathbb{R}^+, & (B_2)
\end{align*}
\]

where \(\Omega\) is a bounded domain in \(\mathbb{R}^n\) with a piecewise smooth boundary \(\partial\Omega\), \(\mathbb{R}^+ = [0, \infty)\), \(u = u(x,t)\), \(\Delta\) is the Laplacian in \(\mathbb{R}^n\), \(\tau > 0\) and \(\rho > 0\) are constants, \(n\) is the unit exterior normal vector to \(\partial\Omega\), and \(\nu(x,t)\) is a nonnegative continuous function on \(\partial\Omega \times \mathbb{R}^+\).

The oscillation theory for functional differential equations and partial differential equations has been studied intensively in the last decades. For the study of parabolic partial differential equations of neutral type, we mention here the monograph [1], literatures with deviating arguments [2–7] and with distributed deviating arguments [8–10] and references cited therein. We note that these articles consider only the case where the neutral coefficient number is such that \(-1 \leq \lambda(t) \leq 0\). To the best of our knowledge, very little has been done for other cases (see, for example, [11–13]). The purpose of this paper is to establish some oscillation theorems for the case where the neutral coefficient number is such that \(0 \leq \lambda(t) \leq 1\). The results are different from those reported in [11, 12] and generalize those obtained in [13].

We assume throughout this paper that the following conditions hold:

\((H_1)\) \(\lambda(t) \in C'(R_+, R_+), a_0(t), a_1(t) \in C(R_+, R_+);\)

\((H_2)\) \(p(x,t) \in C(\bar{G}, R_+), q(x,t,\xi) \in C(\bar{G} \times [a,b], R_+);\)

\((H_3)\) \(g(t,\xi) \in C(R_+ \times [a,b], R)\) is nondecreasing with respect to \(t\) and \(\xi\), respectively, and is such that \(g(t,\xi) \leq t\) for \(\xi \in [a,b]\), and \(\lim \inf_{t \to \infty, \xi \in [a,b]}\{g(t,\xi)\} = \infty;\)

\((H_4)\) \(\mu(\xi) \in ([a,b], R)\) is nondecreasing, and the integral of Eq. (1) is to be understood as in the sense of Stieltjes.

**Definition 1.** A function \(u(x,t) \in C^2(\Omega \times [t-1, \infty), R) \cap C^1(\bar{\Omega} \times [t-1, \infty), R)\) is called a solution of the boundary value problem (1), \((B_1)\) or (1), \((B_2)\), if it satisfies Eq. (1) in the domain \(G\) along with the corresponding boundary condition, where \(t-1 = \min\{-\tau, -\rho, g(0,a)\}\).

**Definition 2.** A solution \(u(x,t)\) of the boundary value problem (1), \((B_1)\) or (1), \((B_2)\) is said to be oscillatory in the domain \(G\) if, for each positive number \(t_\mu\), there exists a point \((x_0, t_0) \in \Omega \times [t_\mu, \infty)\) such that the condition \(u(x_0, t_0) = 0\) holds.

2. Main results

To obtain our main results, we first consider the following neutral differential inequalities:
\[
\frac{d}{dt} \left[ y(t) - \lambda(t) y(t - \tau) \right] + p(t) y(t) + \int_a^b q(t, \xi) y \left[ g(t, \xi) \right] d\mu(\xi) \leq 0, \quad t \geq 0, \tag{2}
\]

\[
\frac{d}{dt} \left[ y(t) - \lambda(t) y(t - \tau) \right] + p(t) y(t) + \int_a^b q(t, \xi) y \left[ g(t, \xi) \right] d\mu(\xi) \geq 0, \quad t \geq 0, \tag{3}
\]

and neutral differential equations

\[
\frac{d}{dt} \left[ y(t) - \lambda(t) y(t - \tau) \right] + p(t) y(t) + \int_a^b q(t, \xi) y \left[ g(t, \xi) \right] d\mu(\xi) = 0, \quad t \geq 0, \tag{4}
\]

where \( \lambda(t) \in C'(R_+, R_+) \), \( p(t) \in C(R_+, R_+) \), \( q(t, \xi) \in C(R_+ \times [a, b], R_+) \).

**Lemma 1** [14]. Assume that the following conditions hold:

(A1) There exists a function \( h(t, \xi) \in C(R_+ \times [a, b], R_+) \), such that \( h(h(t, \xi), \xi) = g(t, \xi) \); \( h(t, \xi) \) is nondecreasing with respect to \( t \) and \( \xi \), and \( t \geq h(t, \xi) \geq g(t, \xi) \);

(A2) \( \liminf_{t \to \infty} \int_{g(t, \xi)}^t q(s, \xi) d\mu(\xi) d\mu(s) > \frac{1}{e} \), and

(A3) \( \liminf_{t \to \infty} \int_{h(t, \xi)}^t q(s, \xi) d\mu(\xi) d\mu(s) > 0 \).

Then, the first order differential inequality

\[
x'(t) + \int_a^b q(t, \xi) x \left[ g(t, \xi) \right] d\mu(\xi) \leq 0 \tag{5}
\]

has no eventually positive solutions.

**Remark.** The existence of such a function \( h(t, \xi) \) has been established in [14].

**Lemma 2.** Assume that \( 0 \leq \lambda(t) \leq 1 \) and (A1) holds, and that there exists a constant \( m > 0 \) such that

\[
q(t, \xi) \geq m, \quad t \geq 0, \quad \xi \in [a, b]. \tag{6}
\]

If

\[
\liminf_{t \to \infty} \int_{g(t, \xi)}^t \int_a^b q(s, \xi) d\mu(\xi) d\mu(s) > \frac{1}{e} \exp \left[ - \liminf_{t \to \infty} \int_{g(t, \xi)}^t p(s) d\mu(s) \right],
\]

for some \( \xi_0 \in [a, b] \),

then

(I) inequality (2) has no eventually positive solutions;

(II) inequality (3) has no eventually negative solutions;

(III) every solution of Eq. (4) oscillates.
**Proof.** (I) Suppose that \( y(t) \) is an eventually positive solution of inequality (2), then from (H3), there exists a \( t_1 \geq 0 \) such that \( y(t) > 0, \ y(t - \tau) > 0 \) and \( y[g(t, \xi)] > 0 \) for \( t \geq t_1 \) and \( \xi \in [a, b] \). Let

\[
z(t) = y(t) - \lambda(t)y(t - \tau).
\]

From (2), there exists a \( t_2 \geq t_1 \), such that \( z'(t) < 0 \). Thus, \( z(t) \) is monotone decreasing, and we can assert that \( z(t) \) is bounded below. In fact, assume that it is not true, then \( \lim_{t \to \infty} z(t) = -\infty \). From (8), we have

\[
y(t) = z(t) + \lambda(t)y(t - \tau).
\]

Thus, it is clear that \( y(t) \) is unbounded, and hence, there exists a \( t_3 > t_2, t_3 - \tau \geq t_2 \), such that

\[
z(t_3) < 0, \quad y(t_3) = \max_{t_2 \leq s \leq t_3} y(s).
\]

By directly calculus, we have

\[
z(t_3) = y(t_3) - \lambda(t_3)y(t_3 - \tau) \geq y(t_3)\left(1 - \lambda(t_3)\right) > 0,
\]

which contradicts (10). Let

\[
\lim_{t \to \infty} z(t) = L.
\]

Then, by integrating inequality (2) on \([t_3, \infty)\), and using \( z(t) \leq y(t) \), we obtain

\[
m \int_{t_3}^{t} \int_{a}^{b} z[g(s, \xi)]d\mu(\xi)ds \leq \int_{t_3}^{t} \int_{a}^{b} q(s, \xi)z[g(s, \xi)]d\mu(\xi)ds
\]

\[
\leq \int_{t_3}^{t} \int_{a}^{b} q(s, \xi)y[g(s, \xi)]d\mu(\xi)ds
\]

\[
\leq z(t_3) - z(t) < \infty.
\]

Consequently, it follows from (H3) and (6) that \( m \int_{a}^{b} d\mu(\xi) \int_{t_3}^{t} z(s)ds < \infty \). That is, \( z(t) \in L^1[t_3, \infty) \). Furthermore, we can assert that \( \liminf_{t \to \infty} y(t) = 0 \). In fact, assume the contrary, that \( \liminf_{t \to \infty} y(t) = d > 0 \). Then, there exists a \( t_4 \geq t_3 \) such that \( y(t) > \frac{d}{2} \) for \( t \geq t_4 \). From (2), we have

\[
z'(t) \leq -p(t)y(t) - \int_{a}^{b} q(t, \xi)y[g(t, \xi)]d\mu(\xi)
\]

\[
\leq -p(t)y(t) - \int_{a}^{b} q(t, \xi)z[g(t, \xi)]d\mu(\xi).
\]

Then, by integrating the above inequality from \( t_4 \) to \( t \), we obtain

\[
z(t) - z(t_4) < -\frac{d}{2} \int_{t_4}^{t} p(s)ds - \int_{t_4}^{t} \int_{a}^{b} q(s, \xi)z[g(s, \xi)]d\mu(\xi)ds.
\]
which implies that
\[ z(t_0) \geq z(t) + \frac{d}{2} \int_{t_4}^{t} p(s) \, ds + \int_{t_4}^{b} \int_{a}^{s} q(s, \xi)z[g(s, \xi)] \, d\mu(\xi) \, ds. \]  

(13)

Thus, by (7), we have \( \lim_{t \to \infty} z(t) = -\infty \), which contradicts (11). Since \( \lim\inf_{t \to \infty} y(t) = 0 \), there exists a sequence \( \{t_n\} \), such that \( \lim_{t \to \infty} t_n = \infty \) and \( \lim_{t \to \infty} y(t_n) = 0 \). Thus,
\[
0 = \lim_{n \to \infty} (z(t_n + \tau) - z(t_n)) \\
= \lim_{n \to \infty} [y(t_n + \tau) - (\lambda(t_n + \tau) + 1)y(t_n) + \lambda(t_n)y(t_n - \tau)].
\]

Furthermore, we obtain
\[
\lim_{n \to \infty} (y(t_n + \tau) + \lambda(t_n)y(t_n - \tau)) = 0.
\]

(14)

Therefore, \( \lim_{n \to \infty} \lambda(t_n)y(t_n - \tau) = 0 \),
and
\[
L = \lim_{n \to \infty} (y(t_n) - \lambda(t_n)y(t_n - \tau)).
\]

(15)

Since \( z(t) \) is monotone decreasing, there exists a \( t_5 \geq t_4 \) such that \( z(t) > 0, \quad t \geq t_5 \). Noting (8) and the fact that \( z(t) \leq y(t) \), it follows from (2) that
\[
z'(t) + p(t)z(t) + \int_{a}^{b} q(t, \xi)z[g(t, \xi)] \, d\mu(\xi) \leq 0, \quad t \geq t_5.
\]

(16)

Let
\[
w(t) = z(t) \exp\left[ \int_{t_4}^{t} p(s) \, ds \right].
\]

(17)

Then, \( w(t) > 0 \), and
\[
w'(t) + \int_{a}^{b} r(t, \xi)w[g(t, \xi)] \, d\mu(\xi) \leq 0,
\]

(18)
in which \( r(t, \xi) = q(t, \xi) \exp[\int_{g(t, \xi)}^{t} p(s) \, ds] \). Thus, it follows from \( (A_2) \) and (7) that
\[
\liminf_{t \to \infty} \int_{g(t, b)}^{t} \int_{a}^{b} r(s, \xi) \, d\mu(\xi) \, ds \\
\geq \left( \liminf_{t \to \infty} \int_{g(t, b)}^{t} \int_{a}^{b} q(s, \xi) \, d\mu(\xi) \, ds \right) \left( \exp \left[ \liminf_{t \to \infty} \int_{g(t, \xi)}^{t} p(s) \, ds \right] \right) \\
\geq \left( \liminf_{t \to \infty} \int_{g(t, b)}^{t} \int_{a}^{b} q(s, \xi) \, d\mu(\xi) \, ds \right) \left( \exp \left[ \liminf_{t \to \infty} \int_{g(t, b)}^{t} p(s) \, ds \right] \right).
\]
\[ > \frac{1}{e} \exp \left[ - \liminf_{t \to \infty} \int_{g(t,b)}^{t} p(s) \, ds \right] \exp \left[ \liminf_{t \to \infty} \int_{g(t,b)}^{t} p(s) \, ds \right] = \frac{1}{e}, \]

and

\[ \liminf_{t \to \infty} \int_{h(t,b)}^{b} \int_{a}^{t} r(s, \xi) \, d\mu(\xi) \, ds \]
\[ \geq \left( \liminf_{t \to \infty} \int_{g(t,b)}^{t} \int_{a}^{b} q(s, \xi) \, d\mu(\xi) \, ds \right) \left( \exp \left[ \liminf_{t \to \infty} \int_{g(t,\xi)}^{t} p(s) \, ds \right] \right) > 0. \]

Consequently, by Lemma 1, inequality (18) has no eventually positive solutions. This is a contradiction with \( w(t) > 0 \).

(I) If \( y(t) \) is an eventually negative solution of inequality (3), then \(-x(t)\) is an eventually positive solution of inequality (2). The result of (II) follows from an argument similar to that given for the result of (I).

(III) The result of (III) follows from those of (I) and (II). This completes the proof. \( \square \)

We are now in a position to present the main results of the paper. Let

\[ P(t) = \min_{x \in \Omega} \{ p(x, t) \}, \quad Q(t, \xi) = \min_{x \in \Omega} \{ q(x, t, \xi) \}. \tag{19} \]

With each solution \( u(x, t) \) of the boundary value problem (1), \( (B_1) \), we associate with a function \( U(t) \) defined by

\[ U(t) = \int_{\Omega} u(x, t) \, dx, \quad t > 0. \tag{20} \]

**Theorem 1.** Assume that \( 0 \leq \lambda(t) \leq 1 \) and \((A_1)\) holds. If

\[ \liminf_{t \to \infty} \int_{h(t,b)}^{b} \int_{a}^{t} Q(s, \xi) \, d\mu(\xi) \, ds > 0, \tag{21} \]

\[ \liminf_{t \to \infty} \int_{g(t,\xi_0)}^{t} \int_{a}^{b} q(s, \xi) \, d\mu(\xi) \, ds > \frac{1}{e} \exp \left[ - \liminf_{t \to \infty} \int_{g(t,\xi_0)}^{t} P(s) \, ds \right], \tag{22} \]

for some \( \xi_0 \in [a, b] \).

Then, each solution \( u(x, t) \) of the boundary value problem (1), \( (B_1) \) is oscillatory in the domain \( G \).

**Proof.** Assume that the boundary value problem (1), \( (B_1) \) has a nonoscillatory solution \( u(x, t) \). Without loss of generality, assume that \( u(x, t) > 0, (x, t) \in \Omega \times R_+ \). (\( u(x, t) < 0 \) can be considered by using the same method.) From \((H_3)\), there exists a \( t_1 > 0 \) such that
\( u(x, t - \tau) > 0, u[x, g(t, \xi)] > 0, u(x, t - \rho) > 0 \) for \( t \geq t_1 \) and \( \xi \in [a, b] \). Integrating with respect to \( x \) over the domain \( \Omega \), for \( t \geq t_1 \), we obtain

\[
\frac{d}{dt} \left[ \int_{\Omega} u \, dx + \lambda(t) \int_{\Omega} u(x, t - \tau) \, dx \right] + \int_{\Omega} p(x, t) u \, dx \\
+ \int_{\Omega} \int_{a}^{b} q(x, t, \xi) u \left[ x, g(t, \xi) \right] d\mu(\xi) \, dx \\
= a_0(t) \int_{\Omega} \Delta u \, dx + a_1(t) \int_{\Omega} \Delta u(x, t - \rho) \, dx.
\]

(23)

It is clear that

\[
\int_{\Omega} \int_{a}^{b} q(x, t, \xi) u \left[ x, g(t, \xi) \right] d\mu(\xi) \, dx = \int_{\Omega} \int_{a}^{b} q(x, t, \xi) u \left[ x, g(t, \xi) \right] \, dx \, d\mu(\xi) \\
\geq \int_{a}^{b} Q(t, \xi) \int_{\Omega} u \left[ x, g(t, \xi) \right] \, dx \, d\mu(\xi).
\]

(24)

From Green’s formula and boundary condition \((B_1)\), we have

\[
\int_{\Omega} \Delta u \, dx = \int_{\partial\Omega} \frac{\partial u}{\partial n} \, d\omega = - \int_{\partial\Omega} vu \, d\omega \leq 0
\]

(25)

and

\[
\int_{\Omega} \Delta u(x, t - \rho) \, dx = - \int_{\partial\Omega} v(x, t - \rho) u(x, t - \rho) \, d\omega \leq 0,
\]

(26)

where \( d\omega \) is the surface integral element on \( \partial\Omega \).

Moreover, it follows from (19) that

\[
\int_{\Omega} p(x, t) u \, dx \geq P(t) \int_{\Omega} u \, dx.
\]

(27)

Combining (20), (24)–(27), it follows from (23) that, for \( t \geq t_1 \),

\[
\frac{d}{dt} \left[ U(t) - \lambda(t) U(t - \tau) \right] + P(t) U(t) + \int_{a}^{b} Q(t, \xi) U \left[ g(t, \xi) \right] \, d\mu(\xi) \leq 0,
\]

(28)

\[
t \geq t_1.
\]

By (20), the function \( U(t) \) is an eventually positive solution of inequality (28), which leads to a contradiction with the result of (I) in Lemma 2. This completes the proof. \( \square \)
To discuss the boundary value problem of (1), \((B_2)\), we consider the following Dirichlet problem in the domain \(\Omega\):

\[
\begin{align*}
\Delta u + \alpha u &= 0, \quad (x, t) \in \Omega \times R_+,
\quad u = 0, \quad (x, t) \in \partial\Omega \times R_+,
\end{align*}
\]

where \(\alpha\) is a constant. It is well known \([15]\) that the least eigenvalue \(\alpha_1\) of the problem (29) is positive and the corresponding eigenfunction \(\phi(x)\) is positive on \(x \in \Omega\).

With each solution \(u(x, t)\) of the boundary value problem (1), \((B_2)\), we associate with a function \(V(t)\) defined by

\[
V(t) = \int_{\Omega} u(x, t)\phi(x) \, dx, \quad t > 0.
\]

**Theorem 2.** Assume that \(0 \leq \lambda(t) \leq 1\) and \((A_1)\) holds. If

\[
\liminf_{t \to \infty} \int_{h(t,b)}^{b} \int_{a}^{t} Q(s, \xi) \, d\mu(\xi) \, ds > 0,
\]

or (31) holds, and there exists at least a \(j_0 = 0\) or \(1\), such that

\[
\liminf_{t \to \infty} \int_{g(t,\xi_0)}^{b} \int_{a}^{t} Q(s, \xi) \, d\mu(\xi) \, ds > \frac{1}{e} \exp \left[ -\liminf_{t \to \infty} \int_{g(t,\xi_0)}^{t} \{\alpha_1[a_0(s) + a_1(s)] + P(s)\} \, ds \right],
\]

for some \(\xi_0 \in [a, b]\), \(\alpha_1 a_{j_0}(s) ds\), (32)

or (31) holds, and there exists at least a \(j_0 = 0\) or \(1\), such that

\[
\liminf_{t \to \infty} \int_{g(t,\xi_0)}^{b} \int_{a}^{t} Q(s, \xi) \, d\mu(\xi) \, ds > \frac{1}{e} \exp \left[ -\liminf_{t \to \infty} \int_{g(t,\xi_0)}^{t} \alpha_1 a_{j_0}(s) \, ds \right],
\]

for some \(\xi_0 \in [a, b]\).

Then, each solution \(u(x, t)\) of the boundary value problem (1), \((B_2)\) is oscillatory in the domain \(G\).

**Proof.** Assume that the boundary value problem (1), \((B_2)\) has a nonoscillatory solution \(u(x, t)\). Without loss of generality, assume that \(u(x, t) > 0, \quad (x, t) \in \Omega \times R_+\). \((u(x, t) < 0\) can be considered by using the same method.) From \((H_3)\), there exists a \(t_1 \geq 0\) such that \(u(x, t - \tau) > 0, \quad u[x, g(t, \xi)] > 0, \quad u(x, t - \rho) > 0\) for \(t \geq t_1\) and \(\xi \in [a, b]\). Multiplying both sides of Eq. (1) by the eigenfunction \(\phi(x)\) and then performing the integration with respect to \(x\) over the domain \(\Omega\), we obtain, for \(t \geq t_1\),

\[
\frac{d}{dt} \left[ \int_{\Omega} u \phi(x) \, dx + \lambda(t) \int_{\Omega} u(x, t - \tau) \phi(x) \, dx \right] + \int_{\Omega} p(x, t) u \phi(x) \, dx
\]
+ \int_{\Omega}^{b} \int_{a}^{b} q(x, t, \xi) u[x, g(t, \xi)] \phi(x) d\mu(\xi) dx \\
= a_0(t) \int_{\Omega} \Delta u \phi(x) dx + a_1(t) \int_{\Omega} \Delta u(x, t - \rho) \phi(x) dx. \tag{34}

Using Green’s formula and boundary value condition \((B_2)\), we obtain

\[ \int_{\Omega} \Delta u \phi(x) dx = \int_{\partial \Omega} \left( \phi(x) \frac{\partial u}{\partial n} - u \frac{\partial \phi(x)}{\partial n} \right) d\omega + \int_{\Omega} u \Delta \phi(x) dx \]
\[ = -\alpha_1 \int_{\Omega} u \phi(x) dx, \tag{35} \]
\[ \int_{\Omega} \Delta u(x, t - \rho) \phi(x) dx = -\alpha_1 \int_{\Omega} u(x, t - \rho) \phi(x) dx. \tag{36} \]

Note that

\[ \int_{\Omega} p(x, t) u \phi(x) dx \geq P(t) \int_{\Omega} u \phi(x) dx, \tag{37} \]
\[ \int_{\Omega}^{b} \int_{a}^{b} q(x, t, \xi) u[x, g(t, \xi)] \phi(x) d\mu(\xi) dx \\
= \int_{\Omega}^{b} \int_{a}^{b} q(x, t, \xi) u[x, g(t, \xi)] \phi(x) dx d\mu(\xi) \\
\geq \int_{\Omega}^{b} Q(t, \xi) \int_{a}^{b} u[x, g(t, \xi)] \phi(x) dx d\mu(\xi), \tag{38} \]

and combining (35)–(38), it follows from (34) that, for \( t \geq t_1 \),

\[ \frac{d}{dt} \left[ \int_{\Omega} u \phi(x) dx + \lambda(t) \int_{\Omega} u(x, t - \tau) \phi(x) dx \right] + P(t) \int_{\Omega} u \phi(x) dx \\
+ \int_{\Omega}^{b} Q(t, \xi) \int_{a}^{b} u[x, g(t, \xi)] \phi(x) dx d\mu(\xi) \\
\leq -\alpha_1 a_0(t) \int_{\Omega} u \phi(x) dx - \alpha_1 a_1(t) \int_{\Omega} u(x, t - \rho) \phi(x) dx. \tag{39} \]

Thus, by (30), we obtain
\[
\frac{d}{dt}\left[V(t) - \lambda(t)V(t - \tau)\right] + \left\{\alpha_1[a_0(t) + a_1(t)] + P(t)\right\}V(t) \\
+ \int_a^b Q(t, \xi)V[g(t, \xi)]d\mu(\xi) \leq 0, \quad t \geq t_1.
\] (40)

Consequently, it follows from (30) that the function \( V(t) \) is an eventually positive solution of inequality (40), which leads to a contradiction with the result of (I) in Lemma 2. This completes the proof. \( \square \)

To assert our results, we give an example.

**Example.** Consider the parabolic equation

\[
\frac{\partial}{\partial t}\left[u - \frac{1}{2}u(x, t - \pi)\right] \\
= (2e^{t+1} - 1)\Delta u + \frac{3}{2}\Delta u\left(x, t - \frac{\pi}{2}\right) - u - 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{t+1}u(x, t + \xi)d\xi,
\]

\((x, t) \in (0, \pi) \times (0, +\infty)\), (41)

and

\[ u(0, t) = 0, \quad u(\pi, t) = 0, \quad t > 0, \] (42)

where \( \lambda(t) = \frac{1}{2} \), \( a = -\frac{3}{2}\pi \), \( b = -\pi ; \ a_0(t) = 2e^{t+1} - 1, \ a_1(t) = \frac{3}{2}, \ p(x, t) = 1, \)

\( q(x, t, \xi) = 2e^{t+1}, \ g(t, \xi) = t + \xi \).

Choosing \( h(t, \xi) = t + \frac{1}{2}\xi, \ Q(t, \xi) = 2e \) and \( P(t) = 1 \), then \( h(h(t, \xi), \xi) = g(t, \xi) \), it is easy to see that the (A1) holds and all the conditions of Theorem 2 are satisfied. then every solution of the problem (41), (42) is oscillatory in \((x, t) \in (0, \pi) \times (0, +\infty)\). In fact, the function \( u(x, t) = \sin x \cos t \) is an oscillatory solution of the problem (41), (42).

**Acknowledgments**

The authors thank the reviewers and the editor for their valuable suggestions and comments.

**References**


[7] X.L. Fu, X.Z. Liu, Oscillation criteria for a class of nonlinear neutral parabolic partial differential equations,
111–120.
661–670.