A note on vector-valued rational interpolation

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Received 15 August 2004; received in revised form 23 March 2005

Abstract

Graves-Morris (see [P.R. Graves-Morris, Vector valued interpolants I, Numer. Math. 42 (1983) 331–348 and P.R. Graves-Morris and C.D. Jenkins, Vector valued rational interpolants III, Constr. Approx. 2 (1986) 263–289]) defined the generalized inverse rational interpolants (GIRIs) in the form of \( R(x) = N(x)/D(x) \) with the divisibility condition \( D(x) \mid \|N(x)\|^2 \), and proved the Uniqueness Theorem for GIRIs. However, this condition is found not necessary in some cases. In this paper, we remove this divisibility condition, define the extended generalized inverse rational interpolants (EGIRIs) and establish the Uniqueness Theorem for EGIRIs. One can see that the Uniqueness Theorem for GIRIs is the special case of the one for EGIRIs.

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Keywords: Extended generalized inverse rational interpolants; Reduced vector-valued rational interpolants; Uniqueness

1. Introduction

Given a set of \( n + 1 \) distinct real points \( \{x_i : i = 0, 1, \ldots, n\} \), let \( \{V_i \in C^d : i = 0, 1, \ldots, n\} \) be a set of complex-valued vectors, each \( V_i \) of which is associated with \( x_i, i = 0, 1, \ldots, n \). We show that these vectors can be normally interpolated by the vector-valued rational function

\[
R(x) = N(x)/D(x)
\]
in the sense of
\[ R(x_i) = N(x_i)/D(x_i) = V_i, \quad i = 0, 1, \ldots, n, \] (1.2)
where \( N(x) = \{ \lambda_1(x), \lambda_2(x), \ldots, \lambda_d(x) \} \) is a \( d \)-dimensional polynomial-valued vector, \( \lambda_j(x) \) is a polynomial, \( j = 1, 2, \ldots, d \), and \( D(x) \) is a real polynomial. \( \delta N := \max_{1 \leq j \leq d} \{ \lambda_j \} \). The construction process is based on the use of the Samelson inverse for any non-zero vector \( V = [a_1, a_2, \ldots, a_d] \):
\[ V^{-1} = V^* / \|V\|^2, \quad V^* \text{ is the conjugate vector of } V, \] (1.3)
where \( \|V\|^2 = \sum_{i=1}^{d} |a_i|^2 \) and \( |a_i| \) is the modulus of \( a_i, i = 1, 2, \ldots, d \).

In [1,4], Graves-Morris gave the construction and definition of the generalized inverse rational interpolants (GIRIs) as follows:

**Construction 1 of GIRIs.**

**Initialization:** Define
\[ b_0 := V_0 \] (1.4)
and for \( i = 1, 2, \ldots, n, \)
\[ R_1^i := \frac{x_i - x_0}{V_i - b_0}. \] (1.5)

**Iteration:** For \( k = 1, 2, \ldots, n - 1, \) define
\[ b_k := R_k^k \] (1.6)
and for \( i = k + 1, k + 2, \ldots, n, \)
\[ R_{i+1}^k := \frac{x_i - x_k}{R_i^k - b_k}. \] (1.7)

**Termination:** Define
\[ b_n := R_n^n. \] (1.8)

The resulting construct is
\[ R(x) = b_0 + \frac{x - x_0}{b_1} + \frac{x - x_1}{b_2} + \cdots + \frac{x - x_{n-1}}{b_n}. \] (1.9)

By the tail-to-head rationalization, detailed by construction 2 (see [1]), we observe that \( R(x) \) takes the form (1.1).

**Definition 1.1 (Graves-Morris [1])**. A vector-valued rational fraction \( R(x) = N(x)/D(x) \), as defined in (1.1), is irreducible if \( D(x) \) is real and if there is no non-trivial real polynomial which is a common factor of \( d \) polynomials of \( N(x) \) and \( D(x) \).

**Definition 1.2 (Graves-Morris [1])**. A vector-valued rational fraction \( R(x) = N(x)/D(x) \) is defined to be a generalized inverse rational fraction (GIRF) if \( D(x) \) is real and \( D(x) \|N(x)\|^2 \). A GIRI is defined to be a GIRF with the interpolating property (1.2). A GIRI or GIRF, \( R(x) = N(x)/D(x) \), is said to be
reduced if $D(x)$ is real and all possible real polynomial common factors of $N(x)$ and $D(x)$ have been removed, consistently with $D(x) | \|N(x)\|^2$.

**Definition 1.3 (Graves-Morris [1]).** A vector-valued rational fraction $R(x) = N(x)/D(x)$, as defined in (1.1), is said to be of type $[L/M]$ if

\[ \delta \lambda_j \leq L, \]

\[ \delta \lambda_j = L \quad \text{for some } j, \quad 1 \leq j \leq d, \]

\[ \delta D = M. \]  

(1.10)

**Characterization Theorem (Graves-Morris [1]).** Consider a GIRI of the form

\[ R(x) = \frac{N(x)}{D(x)} = b_0 + x - x_0 \frac{b_1}{b_2} + x - x_1 \frac{b_2}{b_3} + \cdots + x - x_{n-1} \frac{b_{n-1}}{b_n}, \]

(1.11)

where $N(x)$ and $D(x)$ have been found by construction 2[1]. If $n$ is even, $R(x)$ is normally of type $[n/n]$. If $n$ is odd, $R(x)$ is normally of type $[n/n - 1]$.

Based on the definition of GIRI and the Characterization Theorem, Graves-Morris established the Uniqueness Theorem (see [1]) and made further study on GIRIs (see [2,3]).

**Uniqueness Theorem (Graves-Morris [1]).** Any two GIRIs, $R(x)$ and $r(x)$ which interpolate the same set of (finite-valued) vectors at $n + 1$ distinct points, i.e.

\[ R(x_i) = r(x_i) = V_i, \quad i = 0, 1, \ldots, n, \]

(1.12)

and of the same type (i.e. $[n/n]$ if $n$ is even or $[n/n - 1]$ if $n$ is odd) are equal.

The author claimed, in [1], that the above Uniqueness Theorem shows that the interpolant $R(x) = N(x)/D(x)$ with the divisibility condition $D(x) | \|N(x)\|^2$ produced by construction 1 and construction 2 is unique up to irrelevant common factors, regardless of the actual ordering of the interpolation points used in the construction process. However, not all interpolants produced by construction 1 and construction 2 satisfy the divisibility condition, which could be seen in the following examples.

**Example 1.1.** For $x_i = i - 1$, $i = 0, 1, 2, 3, 4$, we have $V_0 = \{\frac{2}{33}, -\frac{1}{33}\}$, $V_1 = \{0, 0\}$, $V_2 = \{\frac{2}{15}, \frac{1}{15}\}$, $V_3 = \{\frac{2}{17}, \frac{1}{34}\}$, $V_4 = \{\frac{54}{481}, \frac{9}{481}\}$. By construction 1 and construction 2, one gets a vector-valued rational interpolant

\[ R(x) = \frac{N(x)}{D(x)} = \frac{\{2x^3, x^2\}}{(1 + 4x^2)(5x - 2)}. \]

We find that it does not satisfy $D(x) | \|N(x)\|^2$, but there is a non-trivial common factor $\omega(x) = 1 + 4x^2$ of $D(x)$ and $\|N(x)\|^2$.  

Example 1.2. For \( x_i = i - 1, i = 0, 1, 2, 3 \), we have \( V_0 = \{0, -8, -5\}, V_1 = \{1, 5, 3\}, V_2 = \{\frac{2}{7}, \frac{4}{7}, \frac{5}{7}\}, V_3 = \{\frac{3}{5}, 1, \frac{11}{5} \} \). By construction 1 and construction 2, one gets a vector-valued rational interpolant

\[
N(x) = \frac{1 + x, 5 - 2x + x^2, 3 + 2x^2}{1 + 2x}.
\]

Obviously it does not satisfy \( D(x) | \|N(x)\|^2 \), and there is no non-trivial real common factor of \( D(x) \) and \( \|N(x)\|^2 \).

Since the interpolants obtained in Examples 1.1 and 1.2 do not satisfy the divisibility condition, they are not GIRIs, so, by the above Uniqueness Theorem, we do not determine whether these interpolants are unique or not. Examples 1.1 and 1.2 also show that not all Thiele-type interpolants (1.9) are GIRIs. In this paper, we define the extended generalized inverse rational interpolants (EGIRIs) by removing the divisibility condition \( D(x) | \|N(x)\|^2 \), and prove the Uniqueness Theorem for EGIRIs. One can see that the interpolants obtained in Examples 1.1 and 1.2 as well as all Thiele-type interpolants (1.9) are EGIRIs and unique by the Uniqueness Theorem for EGIRIs, and the Uniqueness Theorem for GIRIs is the special case of the one for EGIRIs. In Section 2 of this paper, we modify the definition of the generalized inverse rational interpolants and prove some properties. In Section 3, we prove the Uniqueness Theorem for EGIRIs.

2. Properties and definition

We first introduce some notations and show some properties for vector-valued rational fractions. Let \( R(x) = N(x)/D(x) \), as defined in (1.1), be a vector-valued rational fraction of type [\( L/M \)]. We denote by \( \langle A(x), B(x) \rangle \) the greatest real common factor of \( A(x) \) and \( B(x) \), where \( A(x) \) and \( B(x) \) are two real polynomials, by \( \lceil a \rceil \) the greatest integer not exceeding \( a \). One has the following results:

Properties. (1) \( x = \bar{x} \) is a zero of \( \|N(x)\|^2 \) if and only if \( \lambda_j(\bar{x}) = 0 \), \( j = 0, 1, \ldots, d \).

(2) If \( (x - \bar{x}) | \|N(x)\|^2 \), then \( (x - \bar{x})^2 | \|N(x)\|^2 \).

(3) If \( R(x) = N(x)/D(x) \) is irreducible and \( \langle D(x), \|N(x)\|^2 \rangle = \omega(x) \) with \( 0 < \bar{\omega} = l \le 2 \left\lceil \frac{M}{2} \right\rceil \), then \( l \) is even, and there is no factor as \( (x - \bar{x}) \) in \( \omega(x) \).

(4) If \( R(x) = N(x)/D(x) \) is irreducible and \( D(x) | \|N(x)\|^2 \), then \( D(x) > 0 \) or \( D(x) < 0 \) for all \( x \).

Proof. The property (1) is obviously and the property (2) is immediately obtained by the property (1). Next we prove properties (3) and (4).

If \( l \) is odd, then there is a factor as \( (x - \bar{x}) \) in \( \omega(x) \) and \( x = \bar{x} \) is a zero of \( \|N(x)\|^2 \). By the property (1), we get \( \lambda_j(\bar{x}) = 0 \), \( j = 0, 1, \ldots, d \). Hence \( N(x) \) and \( D(x) \) have a factor \( (x - \bar{x}) \), and this contradicts the condition of \( R(x) = N(x)/D(x) \) being irreducible. From the same reason, we know that there is no factor as \( (x - \bar{x}) \) in \( \omega(x) \). Hence the property (3) is proved.

If \( x = \bar{x} \) is a real zero of \( D(x) \), then \( x = \bar{x} \) is a zero of \( \|N(x)\|^2 \). By the property (1), we get \( \lambda_j(\bar{x}) = 0 \), \( j = 0, 1, \ldots, d \). Hence \( N(x) \) and \( D(x) \) have a factor \( (x - \bar{x}) \), and this contradicts the condition of \( R(x) = N(x)/D(x) \) being irreducible. So, \( D(x) \) has no real zero, that is, \( D(x) > 0 \) or \( D(x) < 0 \) for all \( x \), and the property (4) is proved. \( \square \)
Let \( \mathcal{R}[L/M, t] := \{ R(x) = N(x)/D(x) \mid R(x) \text{ is a vector-valued rational fraction of type } [L/M], \) as defined in (1.1), with \( D(x) \text{ is real, and } \langle D(x), \|N(x)\|^2 \rangle = \omega(x) \text{ with } \hat{\omega} = t \}. \) (2.1)

By the property (3), we know that \( t \) is even and
\[
0 \leq t \leq 2 \left\lfloor \frac{M}{2} \right\rfloor = \begin{cases} M & \text{if } M \text{ is even,} \\ M - 1 & \text{if } M \text{ is odd.} \end{cases}
\]

**Definition 2.1.** We define a vector-valued rational fraction \( R(x) = N(x)/D(x) \) to be an extended generalized inverse rational fraction (EGIRF) if \( D(x) \text{ is real and } \langle D(x), \|N(x)\|^2 \rangle = \omega(x) \text{ with } \hat{\omega} = t \leq 2 \left\lfloor \frac{\delta D}{2} \right\rfloor. \) An EGIRI is defined to be an EGIRF with the interpolating property (1.2). An EGIRI or EGIRF, \( R(x) = N(x)/D(x) \), is said to be reduced if \( D(x) \text{ is real and all possible real polynomial common factors of } D(x) \text{ and } N(x) \text{ have been removed, consistently with } \langle D(x), \|N(x)\|^2 \rangle = \omega(x) \text{ with } \hat{\omega} = t \leq 2 \left\lfloor \frac{\delta D}{2} \right\rfloor. \)

**Remarks.**
(1) GIRIs as defined in Definition 1.2 are the special case of EGIRIs with \( \hat{\omega}(x) = D(x) \).
(2) Definition 2.1 shows that all Thiele-type interpolants (1.9) are EGIRIs, so the interpolants obtained in Examples 1.1 and 1.2 are EGIRIs.
(3) Definition 2.1 and (2.1) yield that \( \mathcal{R}[L/M, t] \) is a set of EGIRFs of type \( [L/M] \).

### 3. Uniqueness

Just as stated in [1, p. 334], although \( V^{-1} \) is uniquely specified by (1.3), in no sense is it a unique inverse of \( V \). Consequently, the EGIRI \( R(x) \) of (1.9) is uniquely specified but is in no sense a unique rational interpolant. We seek to characterize the EGIRIs so as to establish some uniqueness properties for them. We start by proving a property of the reduced form of the EGIRIs of the form (1.9).

**Characterization Theorem.** Let \( R(x) = N(x)/D(x) \) be an EGIRI as defined in (1.9) and its reduced form be
\[
R(x) = P(x)/Q(x) \in \mathcal{R}[L/M, t]
\]
with
\[
R(x_i) = P(x_i)/Q(x_i) = V_i, \quad i = 0, 1, \ldots, n.
\]

(1) If \( n \) is even, then
\[
L \leq M \quad \text{and} \quad M = \frac{n + t}{2}.
\]

(2) If \( n \) is odd, then
\[
L > M \quad \text{and} \quad L = \frac{n + t + 1}{2}.
\]

**Proof.** The proof will be classified into two parts.
Part 1: In this part, we will, by induction, prove that $L \leq M$ if $n$ is even or $L > M$ if $n$ is odd.

For $n = 1$, we have
\[
R(x) = b_0 + \frac{x - x_0}{b_1} = \frac{b_0 \|b_1\|^2 + (x - x_0)b_1^*}{\|b_1\|^2} = \frac{P(x)}{Q(x)}
\]  
(3.5)

with $\partial P = L = 1$, $\partial Q = M = 0$. Hence $L > M$.

For $n = 2$, we have
\[
R(x) = b_0 + \frac{x - x_0}{b_1 + \frac{x - x_1}{b_2}} = \frac{b_0 \|b_1\|^2 \|b_2\|^2 + (x - x_0)b_1^* \|b_2\|^2 + 2(x - x_1)\Re{b_1 \cdot b_2}b_0 + (x - x_0)(x - x_1)b_2 + (x - x_1)^2b_0}{\|b_1\|^2 \|b_2\|^2 + 2(x - x_1)\Re{b_1 \cdot b_2} + (x - x_1)^2} = \frac{P(x)}{Q(x)}
\]  
(3.6)

with $\partial P = L \leq 2$, $\partial Q = M = 2$. Hence $L \leq M$.

Assume that the property holds for $n = k$, i.e. $L \leq M$ if $k$ is even or $L > M$ if $k$ is odd. Then, for $n = k + 1$, we have
\[
P(x) = b_0 + \frac{x - x_0}{R_1(x)},
\]
(3.7)

where $R_1(x) = P_1(x)/Q_1(x)$ is a reduced vector-valued rational fraction of type $[L_1/M_1]$ satisfying
\[
R_1(x_i) = R_i^1 = \frac{x_i - x_0}{V_i - b_0}, \quad i = 1, 2, \ldots, k + 1
\]

and
\[
\langle Q_1(x), \|P_1(x)\|^2 \rangle = \omega_1(x) \quad \text{with} \; \partial \omega_1 = t_1 \leq 2 \left[ \frac{M_1}{2} \right].
\]

From (3.7),
\[
P(x) = b_0 + \frac{x - x_0}{R_1(x)} = b_0 + \frac{(x - x_0)Q_1(x)P_1^*(x)}{\|P_1(x)\|^2} = \frac{b_0 Q(x) + (x - x_0)\tilde{Q}_1(x)P_1^*(x)}{Q(x)},
\]

where
\[
\|P_1(x)\|^2 = Q(x)\omega_1(x), \quad Q_1(x) = \tilde{Q}_1(x)\omega_1(x),
\]
\[
\partial Q = M = 2L_1 - t_1, \quad \partial P = L = L_1 + M_1 - t_1 + 1.
\]

If $n = k + 1$ is even, then $k$ is odd and, by the induction assumption, we know
\[
\partial P_1 = L_1 > \partial Q_1 = M_1.
\]
Hence
\[
L = L_1 + M_1 - t_1 + 1 = L_1 + (M_1 + 1) - t_1 \leq L_1 + L_1 - t_1 = 2L_1 - t_1 = M,
\]
namely $L \leq M$. 

If \( n = k + 1 \) is odd, then \( k \) is even and, by the induction assumption, we know
\[
\partial P_1 = L_1 \leq \partial Q_1 = M_1.
\]
Hence
\[
L = L_1 + M_1 - t_1 + 1 = L_1 + (M_1 + 1) - t_1 > L_1 + L_1 - t_1 = 2L_1 - t_1 = M,
\]
namely \( L > M \).

**Part 2:** In this part, we will, by induction, prove that
\[
M = (n + t)/2 \text{ if } n \text{ is even or } L = (n + t + 1)/2 \text{ if } n \text{ is odd.}
\]
For \( n = 1 \), (3.5) yields \( \partial \omega = t = 0 \) and \( L = 1 = (n + t + 1)/2 \).
For \( n = 2 \), (3.6) yields \( \partial \omega = t = 2 \) and \( M = 2 = (n + t)/2 \).
Assume that the property holds for \( n = k \), i.e.
\[
M = (k + t)/2 \text{ if } k \text{ is even or } L = (k + t + 1)/2 \text{ if } k \text{ is odd.}
\]
Then, for \( n = k + 1 \), we have
\[
\frac{P(x)}{Q(x)} = b_0 + \frac{(x - x_0)p(x)}{Q(x)} = b_0 + \frac{x - x_0}{R_1(x)},
\]
where \( p(x)/Q(x) \) is irreducible.
If \( n = k + 1 \) is even, then, by part 1, we know \( \partial P = L \leq \partial Q = M \).
From (3.8), we have \( \partial P = M - 1 \) and \( (Q(x), \|p(x)\|^2) = \omega(x) \) with \( \partial \omega = t \leq 2 \left\lfloor \frac{M}{2} \right\rfloor \).
The next proof can be classified into two cases:

(a) If there is no non-trivial common factor of all components of \( p(x) \), as defined in (3.8), then
\[
R_1(x) = \frac{Q(x)p^*(x)}{\|p(x)\|^2} = \frac{P_1(x)}{Q_1(x)},
\]
where
\[
\|p(x)\|^2 = Q_1(x)\omega(x), \quad Q(x)p^*(x) = P_1(x)\omega(x),
\]
\[
\partial Q_1 = 2M - t - 2, \quad \partial P_1 = 2M - t - 1.
\]
Obviously \( \partial P_1 > \partial Q_1 \) and \( Q_1(x) \|P_1(x)\|^2 \), namely \( (Q_1(x), \|P_1(x)\|^2) = Q_1(x) \).
Since \( R_1(x) \) is a reduced vector-valued rational fraction satisfying
\[
R_1(x_i) = R_1^i = \frac{x_i - x_0}{V_i - b_0}, \quad i = 1, 2, \ldots, k + 1.
\]
By the induction assumption, we have \( \partial P_1 = 2M - t - 1 = (k + (2M - t - 2) + 1)/2 \), hence \( M = ((k + 1) + t)/2 \).
(b) If the common factor of all components of \( p(x) \), as defined in (3.8), is the polynomial \( \omega_1(x) \) with \( 0 < \partial \omega_1 = t_1 \leq M - 1 \), then \( \omega_1(x)^2 \|p(x)\|^2 \) and
\[
R_1(x) = \frac{Q(x)p^*(x)}{\|p(x)\|^2} = \frac{P_1(x)}{Q_1(x)},
\]
(3.10)
where \( P_1(x) \) and \( Q_1(x) \) are defined by
\[
Q(x)p^*(x) = P_1(x)\omega(x)\omega_1(x), \quad \|p^*(x)\|^2 = Q_1(x)\omega(x)\omega_1(x).
\] (3.11)

Hence
\[
\partial P_1 = 2M - t - t_1 - 1, \quad \partial Q_1 = 2M - t - t_1 - 2.
\] (3.12)

Obviously \( \partial P_1 > \partial Q_1 \).

Let \( \langle Q_1(x), \|P_1(x)\|^2 \rangle = \tilde{Q}_1(x) \), then we get, by \( \omega_1(x) \mid \|P(x)\|^2 \) and (3.11), \( Q_1(x) = \tilde{Q}_1(x)\omega_1(x) \) and \( \partial \tilde{Q}_1 = \partial = 2M - t - 2t_1 - 2 \). Since \( R_1(x) \) is a reduced vector-valued rational fraction satisfying
\[
R_1(x) = R^1_j = \frac{x_i - x_0}{V_i - b_0}, \quad i = 1, 2, \ldots, k + 1,
\]
by the induction assumption, we have
\[
\partial P_1 = 2M - t - t_1 - 1 = (k + \tilde{r} + 1)/2 = (k + 2M - t - 2t_1 - 1)/2.
\] (3.13)

(3.12) and (3.13) yield \( (k + 2M - t - 2t_1 - 1)/2 = 2M - t - t_1 - 1 \), and therefore \( M = ((k + 1 + t)/2 \) if \( n = k + 1 \) is even.

If \( n = k + 1 \) is odd, we have \( L = (n + t + 1)/2 \) and the proof is similar to the case that \( n = k + 1 \) is even, and the Characterization Theorem is proved. \( \Box \)

Remarks. (1) In Example 1.1, we see that \( n = 4, \partial \omega = t = 2, \partial D = M = (n + t)/2 = 3, \partial N = L = 3 \) and \( L = M \). In Example 1.2, we see that \( n = 3, \partial \omega = t = 0, \partial N = L = (n + t + 1)/2 = 2, \partial D = M = 1 \) and \( L > M \).

(2) If \( n \) is even, by the Characterization Theorem, we have \( L \leq M = (n + t)/2 \) and \( L + M - t \leq M + M - t = (n + t) - t = n \).

If \( n \) is odd, by the Characterization Theorem, we have \( M < L = (n + t + 1)/2 \) and \( L + M - t \leq L + (L - 1) - t = (n + t + 1)/2 + (n + t - 1)/2 - t = n \).

Hence, we may use this property to characterize the EGIRIs so as to establish the Uniqueness Theorem for them.

Uniqueness Theorem. Any two EGIRIs, \( R(x) \in \mathcal{R}[L/M, t_1] \) and \( r(x) \in \mathcal{R}[l/m, t_2] \) which interpolate the same set of (finite-valued) vectors at \( n + 1 \) distinct real points, i.e.
\[
R(x_i) = r(x_i) = V_i, \quad i = 0, 1, \ldots, n,
\] (3.14)
are equal, if
\[
L + M - t_1 \leq n, \quad l + m - t_2 \leq n, \quad n + t_1 \text{ even} \quad \text{or} \quad M < L \leq (n + t_1 + 1)/2 \quad \text{n. odd},
\] (3.15)
\[
L \leq (n + t_1)/2 \quad \text{n. even} \quad \text{or} \quad M < L \leq (n + t_1 + 1)/2 \quad \text{n. odd},
\] (3.16)
\[
l \leq (n + t_2)/2 \quad \text{n. even} \quad \text{or} \quad m < l \leq (n + t_2 + 1)/2 \quad \text{n. odd}.
\] (3.17)
Proof. Let $n$ be even. (The proof of the case of $n$ odd is similarly.) Express $R(x)$ and $r(x)$ in their reduced forms

$$R(x) = P(x)/Q(x) \quad \text{and} \quad r(x) = p(x)/q(x)$$  \hfill (3.18)

with real polynomials $Q(x)$ and $q(x)$. For the sake of simplicity and without loss of generality, we may assume that

$$P(x)/Q(x) \in \mathbb{R}[L/M, t_1] \quad \text{with} \quad L + M - t_1 \leq n \quad \text{and} \quad L \leq M \leq (n + t_1)/2,$$  \hfill (3.19)

$$p(x)/q(x) \in \mathbb{R}[l/m, t_2] \quad \text{with} \quad l + m - t_2 \leq n \quad \text{and} \quad l \leq m \leq (n + t_2)/2.$$  \hfill (3.20)

We will prove $P(x)/Q(x) \equiv p(x)/q(x)$.

Let

$$\langle Q(x), \|P(x)\|^2 \rangle = \omega_1(x) \quad \text{with} \quad \tilde{\omega}_1 = t_1,$$

$$\langle q(x), \|P(x)\|^2 \rangle = \omega_2(x) \quad \text{with} \quad \tilde{\omega}_2 = t_2,$$

$$\langle q(x), Q(x) \rangle = h(x) \quad \text{with} \quad \tilde{\omega}h = s,$$

$$\langle \omega_1(x), \omega_2(x) \rangle = \tilde{\omega}(x) \quad \text{with} \quad \tilde{\omega}\tilde{\omega} = \tilde{t}.$$  \hfill (3.21)

Obviously $\tilde{\omega}(x) \mid h(x)$ and $0 \leq \tilde{t} \leq s$.

Define polynomials $Q_1(x)$ and $q_1(x)$ by

$$Q(x) = Q_1(x)h(x) \quad \text{and} \quad q(x) = q_1(x)h(x).$$  \hfill (3.22)

Define polynomials $\omega_1(x)$ and $\omega_2(x)$ by

$$\omega_1(x) = \tilde{\omega}_1(x)\tilde{\omega}(x) \quad \text{and} \quad \omega_2(x) = \tilde{\omega}_2(x)\tilde{\omega}(x).$$  \hfill (3.23)

Define $T(x)$ by

$$T(x) = P(x)q_1(x) - p(x)Q_1(x).$$  \hfill (3.24)

From (3.19)–(3.22), we obtain

$$\tilde{\omega}T \leq \max\{L + m - s, l + M - s\} \leq M + m - s \leq n + (t_1 + t_2)/2 - s$$  \hfill (3.25)

and

$$T(x_i) = 0, \quad i = 0, 1, \ldots, n.$$  \hfill (3.26)

Define $Z(x)$ by

$$Z(x) = (x - x_0)(x - x_1) \cdots (x - x_n)$$  \hfill (3.27)

and a polynomial-valued vector $V(x)$ by

$$T(x) = Z(x)V(x).$$  \hfill (3.28)

From (3.23), we obtain

$$\|T(x)\|^2 = \|P(x)\|^2[q_1(x)]^2 + \|p(x)\|^2[Q_1(x)]^2 - 2q_1(x)Q_1(x)\text{Re}[p(x) \cdot P^*(x)].$$  \hfill (3.29)
From (3.28), we obtain
\[ \bar{\omega}_1(x) \parallel T(x) \parallel^2 \text{ and } \bar{\omega}_2(x) \parallel T(x) \parallel^2. \] (3.29)

Because the vectors \( \{V_i\} \) are finite, we deduce from (3.27) and (3.29) that
\[ \bar{\omega}_1(x) \parallel V(x) \parallel^2 \text{ and } \bar{\omega}_2(x) \parallel V(x) \parallel^2. \] (3.30)

If \( V(x) \neq O \), then
\[ \partial \parallel V \parallel^2 = 2\partial V \geq \delta \bar{\omega}_1 + \delta \bar{\omega}_2 = (t_1 - \bar{t}) + (t_2 - \bar{t}) = (t_1 + t_2) - 2\bar{t}. \] (3.31)

From (3.24),
\[ \partial \parallel T \parallel^2 = 2\partial T \leq 2n + (t_1 + t_2) - 2s. \] (3.32)

From (3.27) and (3.31),
\[ \partial \parallel T \parallel^2 = 2\partial Z + \partial \parallel V \parallel^2 \geq 2(n + 1) + (t_1 + t_2) - 2\bar{t} \geq 2(n + 1) + (t_1 + t_2) - 2s. \] (3.33)

(3.32) and (3.33) show a contradiction, and so \( V(x) = O \). Hence \( P(x)/Q(x) \equiv p(x)/q(x) \) and the Uniqueness Theorem is proved. \( \square \)

**Remarks.** (1) The Uniqueness Theorem for GIRIs is the special case of the one for EGIRIs with \( L = M = l = m = n, t_1 = t_2 = n \) (even) or \( L - 1 = M = l - 1 = m = n - 1, t_1 = t_2 = n - 1 \) (odd) which obviously satisfy (3.15)–(3.17).

(2) The Uniqueness Theorem for EGIRIs shows that the interpolant \( R(x) = N(x)/D(x) \) produced by construction 1 and construction 2 (see [1]) is unique up to irrelevant common factors, regardless of the actual ordering of the interpolation points used in the construction process. So, we can determine that the interpolants obtained in Examples 1.1 and 1.2 are unique by the Uniqueness Theorem for EGIRIs.

**References**