Enhancing the frequency stability of a NEMS oscillator with electrostatic and mechanical nonlinearities

J. Juillard\textsuperscript{a*}, G. Arndt\textsuperscript{b}, E. Colinet\textsuperscript{b}

\textsuperscript{a}SSE, SUPELEC, Gif-sur-Yvette, France
\textsuperscript{b}CEA-LETI, MINATEC, Grenoble, France

Abstract

We show how to achieve large-amplitude oscillation and good phase noise characteristics in a nonlinear NEMS oscillator by using a softening nonlinearity to locally balance the hardening behaviour of the Duffing nonlinearity for a given motion amplitude. This design approach makes it possible to relax the constraints on the electronic feedback circuitry of NEMS reference oscillators or resonant sensors. The case when the softening behaviour is obtained via electrostatic biasing is theoretically investigated for a fully-capacitive pulse-actuated resonant structure. This is validated by simulated data, in the case of a resonant clamped-clamped beam.

Keywords: NEMS oscillators; resonant sensors; frequency stability; nonlinear oscillators.

1. Introduction

It is a challenge to achieve large-amplitude motion of NEMS oscillators without increasing their phase noise\textsuperscript{1}. On the one hand, achieving large oscillation amplitude leads to better signal-to-noise ratio (SNR) and, thus, relaxes the constraints on the design of the analog front-end electronics and generally simplifies the design of the electronic feedback loop. On the other hand, large-amplitude motion can usually be achieved only by operating the system out of the mechanical linear regime. The resonant structure, typically a clamped-clamped beam, can then be described as a Duffing oscillator, with a hardening characteristic. It is simple to show\textsuperscript{2} that such a system, when operated in closed-loop, has a nonlinear monotonic frequency-amplitude relationship, so that, as the oscillation amplitude increases, the sensitivity of the oscillation frequency to small amplitude perturbations increases too. Thus, for a system with a purely hardening nonlinearity, the frequency stability becomes poorer as the motion amplitude increases.

In this paper, we show how to achieve large-amplitude oscillation together with good frequency stability in a NEMS oscillator: the governing idea is to use a softening nonlinearity to locally balance the hardening behaviour of the Duffing nonlinearity for a given motion amplitude. We study the case when the softening behaviour is obtained...
via electrostatic biasing. The capacitive detection and actuation scheme is depicted in Fig. 1. The system is actuated with short voltage pulses that are delivered when the position of the resonant element (a clamped-clamped beam) crosses zero: such a feedback scheme ensures that the system oscillates and is stable, even beyond the critical Duffing amplitude\(^2,3\). In section 2, the governing equations of this system are established. The conditions for the existence of an optimal oscillation amplitude are derived in section 3. Finally, in section 4, these theoretical results are compared with transient simulations of a nonlinear CC-beam.

2. Governing model

Consider a clamped-clamped beam with length \(L\), width \(w\), thickness \(h\), Young’s modulus \(E\) and density \(\rho\), placed at equal distances between two electrodes (Fig 1), with the same lengths and widths as the beam. Let \(G\) be the electrostatic gap, \(\omega_0\) the natural pulsation of the beam and \(\xi\) its damping coefficient. The feedback electronics ensure that short pulses of voltage of duration \(T_p\) and amplitude \(V_p\) are delivered through the biasing electrodes whenever the detected signal (the charge accumulated on the beam) crosses zero. Upon projection of the fourth-order Euler-Bernoulli beam equation on the first clamped-clamped beam eigenmode, we find, assuming the pulses are short with respect to the period of oscillation, that the modal coordinate \(a\) (normalized so that \(a=\pm 1\) corresponds to full-gap motion) is governed by\(^4\):

\[
\omega_0^2 a \left(1 + \alpha^2 - \delta \left(1 - a^2 \right)^{3/2} \right) + 2 \xi \omega_0 \dot{a} + \ddot{a} = f_p \Delta(a, \dot{a}),
\]

\[
\gamma \approx 0.72 G^2 / h^2, \quad \delta = V_p^2 / V_{pl}^2, \quad V_{pl} = 9.13 \left( E h^3 G^3 / \epsilon_0 L^4 \right)^{1/2}, \quad f_p \approx 1.32 \epsilon_0 V_p T_p / \rho h G^3.
\]

where \(\Delta(a, \dot{a})\) designates a positive Dirac pulse when \(a=0\) and \(\dot{a}>0\) and a negative Dirac pulse when \(a=0\) and \(\dot{a}<0\). Coefficient \(\gamma\) (resp. \(\delta\)) may be interpreted as the intensity of the nonlinear Duffing hardening (resp. electrostatic softening). The electrostatic softening term in (1) is derived from an approximate expression of the projection of the electrostatic force on the first CC-beam eigenmode, valid with less than 4% error across the gap\(^3\). In the absence of actuation, the central position \((a=0)\) is stable if \(\delta<1\). When pulse-actuation is active, energy is injected into the system when \(a\) crosses zero so that the forcing term balances out the damping term on the left-hand side of (1). Provided \(\xi<<1\), it is reasonable to look for a steady-state solution of (1) of the form \(a=A \sin \omega t\). This can be achieved through describing function analysis\(^4\). It is found that there exists a single oscillation state \((A_{osc}, \omega_{osc})\) for this system, given by:

\[
A_{osc} = f_p / \pi \omega_0 \xi \quad \text{and} \quad \omega_{osc} = f(A_{osc}) = \omega_0 \left[1 + 3 / 4. \pi^2 A_{osc}^2 - \delta \left(1 + \kappa A_{osc}^2 \right) \right]^{1/2}, \quad \kappa = 0.18.
\]
As in (1), the electrostatic term in (3) is obtained via an approximation, valid with less than 2% error across the gap. The oscillation regime defined by (3) is stable provided the oscillation amplitude $A_{osc}$ is smaller than $A_{p_{\pi}}$, where electrostatic pull-in occurs. It is simple to show that, for a pulse-actuated CC-beam, $A_{p_{\pi}}$ is the positive root of:

$$1 + p a^2 - \delta (1 - a^2)^{3/2} = 0.$$  \hfill (4)

3. Existence of an optimal oscillation amplitude

When $\gamma=0$ (resp. $\delta=0$), the nonlinear characteristic of the system is purely softening (resp. hardening) and $\omega_{osc}$ is a decreasing (resp. increasing) function of $A_{osc}$. From (3), we find the first derivative of $\omega_{osc}$ with respect to $A_{osc}$ is zero when:

$$A_{osc} = A_{opt} = \left(1 - \left(\frac{\delta}{\gamma}\right)^{1/2}\right)^{1/2}, \quad \lambda = \frac{4}{3}(1 + \kappa).$$  \hfill (5)

Thus, provided

$$\delta < \frac{\gamma}{\lambda},$$  \hfill (6)

there exists an optimal oscillation amplitude for which small amplitude perturbations have no effect on the oscillation frequency. For this phenomenon to be exploitable, one must make sure that $A_{opt}<A_{p_{\pi}}$. Using (5) and (4), it is simple to show that this condition boils down to:

$$\delta < \frac{\gamma}{\lambda} \left( \frac{1 + \frac{1}{\gamma} + \frac{1}{4 \lambda^2} - \frac{1}{2 \lambda}}{4} \right)^4.$$  \hfill (7)

One may verify that (6) implies (7), as long as $\gamma<\lambda$. These results are illustrated in Fig. 2.

4. Simulation and results

Transient simulations of the closed-loop pulse-actuated nonlinear system are performed, with Matlab-Simulink. The structural parameters are $L = 25\mu m$, $b = 500nm$, $h = G = 250nm$, $Q = 600$, $\rho = 2320kg.m^{-3}$, $E = 149GPa$. The bias voltage is set to $V_b = 12V$ and the duration of the impulses to $T_p = 10ns$ (note that in the simulation, the impulses have a finite duration). These parameters yield $\alpha_b \approx 2.07 \times 10^7 rad.s^{-1}$, $\gamma \approx 0.720$, $V_{p_{\pi}} \approx 29.63V$ and $\delta \approx 0.164$. 

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Fig. 2 - Level sets of $A_{opt}$ (a), $A_{p_{\pi}}$ (b) and $A_{opt}/A_{p_{\pi}}$ (c). The bold black line corresponds to (6) and the dotted line to (7).
Three eigenmodes are used to capture the mechanical behaviour of the beam. An additive white noise source is placed at the input of the impulse-generation block. The system is simulated for 100 values of $V_p$ in the range 5mV-270mV. Although steady-state is reached after 0.5ms, the simulations must be run for much longer so that an empirical estimation of the mean value $m$ and the standard deviation $\sigma$ of $\omega_{osc}$ and $A_{osc}$ can be made. From (3), one may infer that:

$$\sigma_{\omega} = \sigma_{\omega} \frac{df}{dA} \quad \text{(8)}$$

From (4) and (5), we should have $A_{opt} \approx 0.63$ and $A_{pi} \approx 0.88$. The simulation results are plotted in Fig. 3. Note that for small values of $V_p$, $m_A$ is close to zero, meaning that there is so much noise that no oscillation can take place in the system. Beyond a threshold value of $V_p$, which depends on the amount of measurement noise, oscillations appear. Fig. 3 shows that the frequency stability is considerably improved at an oscillation amplitude close to the predicted 0.63. Furthermore, the match between the values of $\sigma_{\omega}$ predicted with (8) and the simulated ones is almost perfect, provided an oscillation actually takes place in the system. This proves that the theory developed in the previous sections is valid not only qualitatively but also quantitatively and emphasizes the advantage that can be drawn from large-amplitude motion of resonant structures.

References