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On bounds for weighted norms for matrices and integral operators

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Abstract

In this note we consider inequalities of the form $\|\mathbf{Ax}\|_{\omega,q} \leqslant \lambda\|\mathbf{Bx}\|_{v,p}$, where **A** and **B** are matrices or integral operators, **x** decreasing sequence or function and ω and v are weights. Obtained results are generalizations of results of G. Bennett [Linear Algebra Appl. 82 (1986) 81] and P.E. Renaud [Bull. Aust. Math. Soc. 34 (1986) 225]. © 2001 Elsevier Science Inc. All rights reserved.

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1. Introduction

We shall be concerned with the spaces ℓ^p, $0 < p < \infty$, of sequences of real numbers satisfying

$$\|x\|_p = \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{1/p} < \infty.$$

Bennett [1] considered inequality

$$\|\mathbf{Ax}\|_q \geqslant \lambda\|\mathbf{x}\|_p \tag{1}$$

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for $x \in \ell^p$ with $x_1 \geqslant x_2 \geqslant \cdots \geqslant 0$ and $\mathbf{A}$ is a matrix with nonnegative entries, assumed to map ℓ^p into ℓ^q, and λ is a constant not depending on $\mathbf{x}$.

In this paper, we shall consider the inequality of the form

$$\|\mathbf{A}\mathbf{x}\|_{\omega,q} \geqslant \lambda \|\mathbf{x}\|_{v,p} \tag{2}$$

and inequality of the form

$$\|\mathbf{A}\mathbf{x}\|_{\omega,q} \leqslant \lambda \|\mathbf{B}\mathbf{x}\|_{v,p}, \tag{3}$$

where $\|\mathbf{x}\|_{v,p}$ is defined by

$$\|\mathbf{x}\|_{v,p} = \left(\sum_{k=1}^{\infty} v_k x_k^p\right)^{1/p}$$

and ω and $\mathbf{v}$ are nonnegative weights.

In Section 2, we start with generalization of formula for summation by parts in ℓ^p obtained by Bennett [1, Proposition 1]. In Section 3 we consider inequalities (2) and (3), while in Section 4 we show that result obtained by Bennett [1, Theorem 4] and Renaud [7] is also valid for sequences nonincreasing in mean. In Section 5 we consider integral analogues of such inequalities, proved in [2], but our proofs are simpler and in agreement with [1]. We also obtain integral analogues of the summation by parts in ℓ^p.

2. Generalized formula for summation by parts

In this part the following elementary lemma will be needed (see [1]).

Lemma 1. *Let $a, b, c \geqslant 0$ with $a \geqslant b$. If $p > 1$, then*

$$(a+c)^p - a^p > (b+c)^p - b^p, \tag{4}$$

unless $a = b$ or $c = 0$. If $0 < p < 1$, inequality in (4) is reversed.

The following result is the generalization of Proposition 1 in [1].

Proposition 1. *Let $a_1, a_2, \ldots, a_n \geqslant 0$, $v_1, v_2, \ldots, v_n \geqslant 0$ and $x_1 \geqslant x_2 \geqslant \cdots \geqslant x_n \geqslant 0$. If $p \geqslant 1$ and $0 < q \leqslant p$, then*

$$\left(\sum_{k=1}^{n} a_k x_k\right)^q \left[\left(\sum_{i=1}^{n} v_i x_i^p\right)^{1/p}\right]^{p-q} \geqslant \sum_{r=1}^{n-1} \left(\sum_{i=1}^{r} a_i\right)^q \left(\sum_{j=1}^{r} v_j\right)^{1-(q/p)} (x_r^p - x_{r+1}^p)$$

$$+\left(\sum_{i=1}^{n} a_i\right)^q \left(\sum_{j=1}^{n} v_j\right)^{1-(q/p)} x_n^p. \tag{5}$$

If $p \leqslant 1$ *and* $q \geqslant p$, *the inequality in* (5) *is reversed. There is equality in* (*either version of*) (5) *if and only if at least one condition holds from each of the following pairs* (1), (2) *and* (3), (4):

(1) $p = 1$,
(2) $x_u = \cdots = x_v$, *where* u *is the smallest and* v *the largest value of* k *such that* $a_k x_k > 0$.
(3) $q = p$,
(4) $(\sum_{j=1}^{r} v_j)^{-1}(\sum_{i=1}^{r} a_i)^p$ *is constant for those values of* r, $1 \leqslant r \leqslant n$, *satisfying* $x_r > x_{r+1}$.

Proof. The idea of the proof is similar to that of Bennet's. We prove the case $p > 1$; the case $0 < p \leqslant 1$ is similar. It is convenient to set $x_{n+1} = 0$, $s_r = a_1 + \cdots + a_r$, and to consider first the special case, $q = p$. Inequality (5) then reduces to

$$\left(\sum_{k=1}^{n} a_k x_k\right)^p \geqslant \sum_{r=1}^{n} s_r^p \left(x_r^p - x_{r+1}^p\right), \tag{6}$$

what was proved in [1] as a consequence of Lemma 1. Equality is valid in (6) if and only if either (1) or (2) is valid.

To prove the general case $q \leqslant p$ we rewrite the right-hand side of (5) as

$$\begin{aligned}
&\left(\sum_{k=1}^{n} a_k x_k\right)^q \left[\left(\sum_{i=1}^{n} v_i x_i^p\right)^{1/p}\right]^{p-q} \\
&\quad = \left[\left(\sum_{k=1}^{n} a_k x_k\right)^p\right]^{q/p} \left(\sum_{i=1}^{n} v_i x_i^p\right)^{(p-q)/p} \\
&\quad \geqslant \left[\sum_{r=1}^{n} s_r^p \left(x_r^p - x_{r+1}^p\right)\right]^{q/p} \left[\sum_{i=1}^{n}\left(\sum_{j=1}^{i} v_j\right)\left(x_i^p - x_{i+1}^p\right)\right]^{(p-q)/p}.
\end{aligned} \tag{7}$$

Applying Hölder's inequality with exponents p/q and $p/(p-q)$, we get

$$\begin{aligned}
&\left(\sum_{k=1}^{n} a_k x_k\right)^q \left[\left(\sum_{i=1}^{n} v_i x_i^p\right)^{1/p}\right]^{p-q} \\
&\quad \geqslant \sum_{r=1}^{n} s_r^q \left(x_r^p - x_{r+1}^p\right)^{q/p} \left[\left(\sum_{j=1}^{r} v_j\right)\left(x_r^p - x_{r+1}^p\right)\right]^{(p-q)/p}
\end{aligned}$$

$$= \sum_{r=1}^{n-1} \left(\sum_{i=1}^{r} a_i\right)^q \left(\sum_{j=1}^{r} v_j\right)^{1-(q/p)} \left(x_r^p - x_{r+1}^p\right)$$
$$+ \left(\sum_{i=1}^{n} a_i\right)^q \left(\sum_{j=1}^{n} v_j\right)^{1-(q/p)} x_n^p. \tag{8}$$

Equality in the last inequality is valid if and only if there is $\mu \in \mathbf{R}$ such that

$$s_r^p \left(x_r^p - x_{r+1}^p\right) = \mu \left(\sum_{j=1}^{r} v_j\right) \left(x_r^p - x_{r+1}^p\right), \quad r = 1, \dots, n$$

and condition (4) follows. □

Remark 1. For $v_1 = v_2 = \cdots = v_n = 1$ we have Proposition 1 from [1].

3. Bounds for matrices

In this part our consideration is finite-dimensional (except Theorem 4). Infinite-dimensional case can be deduced from this in usual way. The following result is a generalization of Theorem 2 from [1].

Theorem 1. *Let $x_1 \geqslant x_2 \geqslant \cdots \geqslant x_n \geqslant 0$, $p \geqslant 1$, $0 < q \leqslant p$ and let $\mathbf{A}$ be an $m \times n$ matrix with nonnegative entries. Then*

$$\|\mathbf{Ax}\|_{\omega,q} \geqslant \lambda \|\mathbf{x}\|_{v,p}, \tag{9}$$

where

$$\lambda^q = \min_{1 \leqslant r \leqslant n} \left(\sum_{i=1}^{r} v_i\right)^{-(q/p)} \sum_{j=1}^{m} \omega_j \left(\sum_{k=1}^{r} a_{jk}\right)^q. \tag{10}$$

There is equality in (9) *if x has the form*

$$x_k = \begin{cases} x_1, & k \leqslant s, \\ 0, & k > s, \end{cases} \tag{11}$$

where s is any value of r at which the minimum in (10) *occurs. If $0 < p \leqslant 1$, $p \leqslant q$ the inequality in* (9) *is reversed where λ is similarly defined with* max *instead of* min.

Proof. The proof is similar to that of Bennett's. We prove the case $p \geqslant 1$, $0 < q \leqslant p$. We may assume, by homogenity, that $\|\mathbf{x}\|_{v,p} = 1$. Applying Proposition 1, we have

$$
\begin{aligned}
\|\mathbf{Ax}\|_{\omega,q}^{q} &= \sum_{i=1}^{m} \omega_i \left(\sum_{j=1}^{n} a_{ij} x_j \right)^{q} \\
&\geqslant \sum_{i=1}^{m} \omega_i \sum_{r=1}^{n} \left(\sum_{k=1}^{r} v_k \right)^{1-(q/p)} \left(\sum_{j=1}^{r} a_{ij} \right)^{q} \left(x_r^p - x_{r+1}^p \right) \\
&= \sum_{r=1}^{n} \left(\sum_{k=1}^{r} v_k \right)^{-(q/p)} \sum_{i=1}^{m} \omega_i \left(\sum_{j=1}^{r} a_{ij} \right)^{q} \left(\sum_{k=1}^{r} v_k \right) \left(x_r^p - x_{r+1}^p \right) \\
&\geqslant \lambda^q \sum_{r=1}^{n} \left(\sum_{k=1}^{r} v_k \right) \left(x_r^p - x_{r+1}^p \right) = \lambda^q \sum_{k=1}^{n} v_k x_k^p \\
&= \lambda^q \|\mathbf{x}\|_{v,p}^{p}.
\end{aligned}
$$

Recalling that $\|\mathbf{x}\|_{v,p} = 1$ we see that (9) follows by taking qth roots.

It is clear, by inspection, that equality holds in (9) whenever (11) is satisfied. □

Remark 2. For $v_1 = \cdots = v_n = 1 = \omega_1 = \cdots = \omega_m$ we have Theorem 2 in [1].

The following theorem should be compared with Theorem 3.2 in [2].

Theorem 2. *Let $x_1 \geqslant x_2 \geqslant \cdots \geqslant x_n \geqslant 0$, let $\mathbf{A}$ and $\mathbf{B}$ be $m \times n$ matrices with non-negative entries and $0 < p \leqslant 1 \leqslant q < \infty$. Then*

$$\|\mathbf{Ax}\|_{\omega,q} \leqslant \lambda \|\mathbf{Bx}\|_{v,p}, \tag{12}$$

where

$$\lambda = \max_{1 \leqslant r \leqslant n} \frac{\left(\sum_{j=1}^{m} \omega_j \left(\sum_{k=1}^{r} a_{j,k} \right)^q \right)^{1/q}}{\left(\sum_{j=1}^{m} v_j \left(\sum_{k=1}^{r} b_{j,k} \right)^p \right)^{1/p}}. \tag{13}$$

There is an equality in (12) if $\mathbf{x}$ has the form

$$x_k = \begin{cases} x_1, & k \leqslant s, \\ 0, & k > s, \end{cases} \tag{14}$$

where s is any value of r at which the maximum in (13) occurs.

Proof. Applying Abel's identity and Minkowski inequality twice, we have

$$\|\mathbf{Ax}\|_{\omega,q} = \left(\sum_{j=1}^{m} \omega_j \left(\sum_{k=1}^{n} a_{j,k} x_k \right)^q \right)^{1/q}$$

$$
\begin{aligned}
&= \left(\sum_{j=1}^{m} \omega_j \left[\sum_{r=1}^{n} \left(\sum_{k=1}^{r} a_{j,k} \right) (x_r - x_{r+1}) \right]^q \right)^{1/q} \\
&\leqslant \sum_{r=1}^{n} \left(\sum_{j=1}^{m} \omega_j \left(\sum_{k=1}^{r} a_{j,k} \right)^q \right)^{1/q} (x_r - x_{r+1}) \\
&\leqslant \lambda \sum_{r=1}^{n} \left(\sum_{j=1}^{m} v_j \left(\sum_{k=1}^{r} b_{j,k} \right)^p \right)^{1/p} (x_r - x_{r+1}) \\
&\leqslant \lambda \left(\sum_{j=1}^{m} v_j \left[\sum_{r=1}^{n} (x_r - x_{r+1}) \left(\sum_{k=1}^{r} b_{j,k} \right) \right]^p \right)^{1/p} \\
&= \lambda \|\mathbf{Bx}\|_{v,p}. \qquad \square
\end{aligned}
$$

In view of the discussion below Theorem 3 from [1, Theorem 2] gives the only possible case for general matrices.

The following theorem can be deduced from Proposition 1 (take w_k's instead of a_k's, x_k^q's instead of x_k's, 1 instead of q and $p/q \leqslant 1$ instead of p), but we choose to give an independent proof (for that reason compare [10, p. 176] and [8, p. 148]).

Theorem 3. *Let* $x_1 \geqslant x_2 \geqslant \cdots \geqslant x_n \geqslant 0$, $0 < p \leqslant q < \infty$. *Then*

$$\|\mathbf{x}\|_{\omega,q} \leqslant \lambda \|\mathbf{x}\|_{v,p}, \tag{15}$$

where

$$\lambda = \max_{1 \leqslant r \leqslant n} \frac{\left(\sum_{j=1}^{r} \omega_j\right)^{1/q}}{\left(\sum_{j=1}^{r} v_j\right)^{1/p}}. \tag{16}$$

There is an equality in (15) *if* $\mathbf{x}$ *has the form*

$$x_k = \begin{cases} x_1, & k \leqslant s, \\ 0, & k > s, \end{cases} \tag{17}$$

where s is any value of r at which the maximum in (13) *occurs.*

Proof. Applying Abel's identity, inequality (6) with exponent $q/p \geqslant 1$ we have

$$
\begin{aligned}
\|\mathbf{x}\|_{\omega,q}^q &= \sum_{k=1}^{n} x_k^q \omega_k = \sum_{k=1}^{n} \left(\sum_{r=1}^{k} \omega_r \right) \left(x_k^q - x_{k+1}^q \right) \\
&\leqslant \left(\sum_{k=1}^{n} \left[\left(\sum_{r=1}^{k} \omega_r \right)^{p/q} - \left(\sum_{r=1}^{k-1} \omega_r \right)^{p/q} \right] x_k^p \right)^{q/p}
\end{aligned}
$$

$$
\begin{aligned}
&= \left(\sum_{k=1}^{n} \left(x_k^p - x_{k+1}^p\right) \left(\sum_{r=1}^{k} \omega_r \right)^{p/q} \right)^{q/p} \\
&\leqslant \lambda^q \left(\sum_{k=1}^{n} \left(x_k^p - x_{k+1}^p\right) \left(\sum_{r=1}^{k} v_r \right) \right)^{q/p} = \lambda^q \left(\sum_{k=1}^{n} x_k^p v_k \right)^{q/p} \\
&= \lambda^q \|\mathbf{x}\|_{v,p}^q. \qquad \square
\end{aligned}
$$

If $0 < q < p < \infty$, then by Hölder inequality we have sharp inequality

$$
\left(\sum_{i=1}^{n} x_i^q \omega_i \right)^{1/q} \leqslant \left(\sum_{i=1}^{n} \omega_i^{r/q} v_i^{-(r/p)} \right)^{1/r} \left(\sum_{i=1}^{n} x_i^p v_i \right)^{1/p}, \tag{18}
$$

where (x_i) is nonnegative sequence and $1/r = 1/q - 1/p$. Thus, inequality (16) is reversed Hölder type inequality.

To complete our discussion we give for $0 < q < p < \infty$ inequality also of the type (18) but for decreasing sequences. In this case we found infinite-dimensional case more appropriate. For integral analogue compare [4,8,10]. We follow the idea from [10].

Theorem 4. *Let* $\mathbf{x} = (x_i)$ *be a decreasing nonnegative sequence*, $0 < q < p < \infty$ *and* $1/r = 1/q - 1/p$. *If*

$$
\lambda^r = \sum_{i=1}^{\infty} \left(\sum_{j=1}^{i} \omega_j \right)^{r/q} \left(\sum_{j=1}^{i} v_j \right)^{-(r/q)} v_{i+1} < \infty, \tag{19}
$$

then there is $C \in \mathbf{R}$ (*not depending on* $\mathbf{x}$) *such that*

$$
\|\mathbf{x}\|_{\omega,q} \leqslant C \|\mathbf{x}\|_{v,p}. \tag{20}
$$

If $\lambda < \infty$ *and* C *is the best possible constant such that* (20) *holds*, *then*

$$
\left(\frac{q}{r}\right)^{1/p} \mu^{r/q} \lambda^{-(r/p)} \leqslant C \leqslant \left(\frac{r}{p}\right)^{1/r} \lambda, \tag{21}
$$

where

$$
\mu^r = \sum_{i=1}^{\infty} \left(\sum_{j=1}^{i} \omega_j \right)^{r/p} \left(\sum_{j=1}^{i+1} v_j \right)^{-(r/p)} \omega_i < \infty. \tag{22}
$$

Proof. To prove first implication and second inequality in (21) set

$$
A_i = \sum_{k=i}^{\infty} \left(\sum_{j=1}^{k} \omega_j \right)^{r/p} \left(\sum_{j=1}^{k} v_j \right)^{-(r/q)} v_{k+1},
$$

and note that by Abel's identity

$$\sum_{i=1}^{\infty} \omega_i A_i = \sum_{i=1}^{\infty} (A_i - A_{i+1}) \sum_{k=1}^{i} \omega_k = \lambda^r.$$

Using Hölder's inequality (with exponents $p/q,\ r/q$) we have

$$\begin{aligned}
\|\mathbf{x}\|_{\omega,q}^q &= \sum_{i=1}^{\infty} x_i^q \omega_i^{q/p} A_i^{-(q/r)} \omega_i^{1-(q/p)} A_i^{q/r} \\
&\leqslant \left(\sum_{i=1}^{\infty} x_i^p \omega_i A_i^{-(p/r)} \right)^{q/p} \left(\sum_{i=1}^{\infty} \omega_i A_i \right)^{q/r} \\
&= \lambda^q \left(\sum_{i=1}^{\infty} x_i^p \omega_i A_i^{-(p/r)} \right)^{q/p} \\
&= \lambda^q \left[\sum_{i=1}^{\infty} \left(\sum_{j=1}^{i} \omega_j A_j^{-(p/r)} \right) \left(x_i^p - x_{i+1}^p \right) \right]^{q/p} \\
&\leqslant \lambda^q \left[\sum_{i=1}^{\infty} A_i^{-(p/r)} \left(\sum_{j=1}^{i} \omega_j \right) \left(x_i^p - x_{i+1}^p \right) \right]^{q/p} \\
&\leqslant \lambda^q \left(\frac{r}{p} \right)^{q/r} \left[\sum_{i=1}^{\infty} \left(\sum_{j=1}^{i} v_j \right) \left(x_i^p - x_{i+1}^p \right) \right]^{q/p} \\
&= \lambda^q \left(\frac{r}{p} \right)^{q/r} \left(\sum_{i=1}^{\infty} x_i^p v_i \right)^{q/p} = \lambda^q \left(\frac{r}{p} \right)^{q/r} \|\mathbf{x}\|_{v,p}^q,
\end{aligned}$$

where the last inequality follows from

$$\begin{aligned}
A_i^{-(p/r)} &\leqslant \left(\sum_{j=1}^{i} \omega_j \right)^{-1} \left[\sum_{k=i}^{\infty} \left(\sum_{j=1}^{k} v_j \right)^{-(r/q)} v_{k+1} \right]^{-(p/r)} \\
&\leqslant \left(\frac{p}{r} \right)^{-(p/r)} \left(\sum_{j=1}^{i} \omega_j \right)^{-1} \\
&\quad \times \left[\sum_{k=i}^{\infty} \left(\sum_{j=1}^{k} v_j \right)^{-(r/p)} - \left(\sum_{j=1}^{k+1} v_j \right)^{-(r/p)} \right]^{-(p/r)}
\end{aligned}$$

$$= \left(\frac{r}{p}\right)^{p/r} \left(\sum_{j=1}^{i} \omega_j\right)^{-1} \sum_{j=1}^{i} v_j,$$

where some obvious estimations are used and elementary inequality

$$a^\alpha - b^\alpha \leqslant \alpha a^{\alpha-1}(a-b), \quad a < b,\ \alpha < 0$$

for $\alpha = 1 - r/q,\ a = \sum_{j=1}^{k} v_j,\ b = \sum_{j=1}^{k+1} v_j$.

Suppose now that $\lambda < \infty$ and that (20) holds. First note that

$$\lambda^r = \sum_{i=1}^{\infty} \left(\sum_{j=1}^{i} \omega_j\right)^{r/q} \left(\sum_{j=1}^{i} v_j\right)^{-(r/q)-1} v_{i+1} \left(\sum_{j=1}^{i} v_j\right)$$
$$= \sum_{i=1}^{\infty} v_i \left[\sum_{k=i}^{\infty} \left(\sum_{j=1}^{k} \omega_j\right)^{r/q} \left(\sum_{j=1}^{k} v_j\right)^{-(r/q)-1} v_{k+1}\right]. \tag{23}$$

If x_i^p is defined by expression in square brackets in (23) for $i = 1, 2, \ldots$, then (20) for the sequence $\mathbf{x} = (x_i)$ gives

$$C^q \lambda^{r(q/p)} \geqslant \sum_{i=1}^{\infty} \left[\sum_{k=i}^{\infty} \left(\sum_{j=1}^{k} \omega_j\right)^{r/q} \left(\sum_{j=1}^{k} v_j\right)^{-(r/q)-1} v_{k+1}\right]^{q/p} \omega_i$$
$$\geqslant \sum_{i=1}^{\infty} \left(\sum_{j=1}^{i} \omega_j\right)^{r/p} \left[\sum_{k=i}^{\infty} \left(\sum_{j=1}^{k} v_j\right)^{-(r/q)-1} v_{k+1}\right]^{q/p} \omega_i$$
$$\geqslant \left(\frac{q}{r}\right)^{q/p} \sum_{i=1}^{\infty} \left(\sum_{j=1}^{i} \omega_j\right)^{r/p} \left(\sum_{j=1}^{i} v_j\right)^{-(r/p)} \omega_i = \left(\frac{q}{r}\right)^{q/p} \mu^r,$$

where the last inequality follows using again elementary inequality as above (now for $\alpha = -r/q$). This shows the first inequality in (21) and that $\mu < \infty$. □

Note that from (21) $\mu \leq (r/p)^{q/r^2} (q/r)^{-q/(rp)} \lambda$. We also note that in the same manner we can prove that

$$\mu^r \geqslant \frac{q}{p} \sum_{i=1}^{\infty} \left(\sum_{j=1}^{i} \omega_j\right)^{r/q} \left(\sum_{j=1}^{i+1} v_j\right)^{-(r/q)} v_{i+1} \tag{24}$$

(in the first step take elementary inequality $a^\alpha - b^\alpha \leqslant \alpha a^{\alpha-1}(a-b),\ \alpha > 1,\ a > b$ with $\alpha = r/q$, and in the second step elementary inequality as in the proof of

Theorem 4 with $\alpha = -r/p$). In this way, if $\tilde{\lambda}^r$ denotes the sum in (24), we obtain weaker, but more symmetrical form of (21)

$$\left(\frac{q}{r}\right)^{1/p} \left(\frac{q}{p}\right)^{1/q} \tilde{\lambda}^{r/q} \lambda^{-(r/p)} \leqslant C \leqslant \left(\frac{r}{p}\right)^{1/r} \lambda. \tag{25}$$

Theorem 4 also implies that if $\lambda < \infty$, then identity operator $\iota : \ell_v^p \mapsto \ell_\omega^q$ is bounded on the cone of nonnegative decreasing sequences.

4. The Cesaro matrix

An interesting application of (1) for Cesaro matrix $\mathbf{C}$ was obtained in [1]. The same result has been obtained in [7].

Theorem 5. *Let p be fixed*, $1 < p < \infty$. *Then*

$$\|\mathbf{Cx}\|_p \geqslant \zeta(p)^{1/p} \|\mathbf{x}\|_p \tag{26}$$

for every $x \in \ell^p$ *satisfying* $x_1 \geqslant x_2 \geqslant \cdots \geqslant 0$, *where* $\zeta(p)$ *is Riemann's zeta function. There is equality in* (26) *if and only if* $x_2 = x_3 = \cdots = 0$.

Here we note that inequality (26) holds under weaker assumption that (x_n) is decreasing in mean, that is

$$x_1 \geqslant \frac{x_1 + x_2}{2} \geqslant \cdots \geqslant \frac{1}{n} \sum_{i=1}^{n} x_i \geqslant \cdots \tag{27}$$

The proof of Renaud was based on the following two lemmas:

Lemma 2. *Let* $p > 1$ *and let* $\mathbf{x} = (x_n), n = 1, 2, \ldots$ *be a nonincreasing sequence of nonnegative real numbers. Then*

$$\begin{aligned}&(x_1 + \cdots + x_n)^p - (x_1^p + \cdots + x_n^p)\\ &\quad \geqslant \sum_{k=2}^{n} [k^p - (k-1)^p - 1] x_k^p, \quad n = 2, 3, \ldots\end{aligned}$$

Lemma 3. *For* $n \geqslant 2$, *we have*

$$[n^p - (n-1)^p - 1]\, T_{n-1} \geqslant S_{n-1},$$

where

$$S_n = \sum_{k=1}^{n} \frac{1}{k^p} \quad \textit{and} \quad T_n = \zeta(p) - S_n = \sum_{k=n+1}^{\infty} \frac{1}{k^p}.$$

It was proved in [6] that Lemma 2 is valid for nonnegative sequences which are nonincreasing in mean. So, using idea of proof from [7] we can prove:

Theorem 6. *If* $p > 1$ *and* $\mathbf{x} = (x_n) \in \ell^p$ *is a nonincreasing in mean sequence of nonnegative real numbers, then* (26) *is valid. The constant* $\zeta(p)$ *in* (26) *is best possible.*

5. Integral analogues

In this section we consider an integral version of Proposition 1 and a generalization of Theorem 7 from [1]. We give sharp lower and upper bounds on weighted Lebesgue spaces (to be more precise, on the cone of nonnegative decreasing functions in these spaces) for transformations, $f = Kg$, of the form

$$f(x) = \int_0^a K(x, y)g(y)\,\mathrm{d}y, \quad 0 < x < a,$$

where $K(x, y) \geqslant 0$ is measurable, and, in lower bound cases, for given p and q we assume that K maps L_v^p into L_q^ω. As usual, for given $p > 0$ and nonnegative weight v on $(0, a)$, for $g \in L_v^p(0, a)$ we set

$$\|g\|_{v,p} = \left(\int_0^a g^p(x)v(x)\,\mathrm{d}x\right)^{1/p}.$$

Also, for given weight v we set $V(x) = \int_0^x v(t)\,\mathrm{d}t,\ x \in (0, a)$.

We will need the following theorem (Theorem 2.1 in [2]):

Theorem 7. *Let* $-\infty < a < b \leqslant \infty$ *and* $f \geqslant 0$ *on* (a, b) *and* g *be continuous on* (a, b). *Suppose* $f \uparrow$ *on* (a, b) *and* $g \downarrow$ *on* (a, b) *with* $\lim_{x\to b-} g(x) = 0$. *Then for any* $\gamma \in (0, 1]$

$$\int_a^b f(x)\,\mathrm{d}[-g(x)] \leqslant \left(\int_a^b f^\gamma(x)\,\mathrm{d}\left[-g^\gamma(x)\right]\right)^{1/\gamma}. \tag{28}$$

If $1 \leqslant \gamma < \infty$, *the inequality in* (28) *is reversed.*

Proposition 2. *Suppose that* $p \geqslant 1$, $q \leqslant p$, *f is nonnegative on* $(0, a)$ *and g is absolutely continuous nonincreasing on* $(0, a)$ *such that* $g(a-0) = 0$. *Then*

$$\left(\int_0^a f(x)g(x)\,\mathrm{d}x\right)^q \|g\|_{v,p}^{p-q} \geqslant \int_0^a \left(\int_0^x f(t)\,\mathrm{d}t\right)^q V^{1-q/p}(x)\,\mathrm{d}\left[-g^p(x)\right]. \tag{29}$$

For $0 < p \leqslant 1,\ p \leqslant q$ *the inequality in* (29) *is reversed.*

Proof. Applying integration by parts, Theorem 7 and Hölder inequality we have

$$\begin{aligned}
&\left(\int_0^a f(x)g(x)\,\mathrm{d}x\right)^q \left[\left(\int_0^a g^p(x)v(x)\,\mathrm{d}x\right)^{1/p}\right]^{p-q} \\
&= \left[\left(\int_0^a \left(\int_0^x f(t)\,\mathrm{d}t\right) \mathrm{d}[-g(x)]\right)^p\right]^{q/p} \cdot \left(\int_0^a g^p(x)v(x)\,\mathrm{d}x\right)^{1-(q/p)} \\
&\geqslant \left(\int_0^a \left(\int_0^x f(t)\,\mathrm{d}t\right)^p \mathrm{d}\left[-g^p(x)\right]\right)^{q/p} \cdot \left(\int_0^a V(x)\,\mathrm{d}\left[-g^p(x)\right]\right)^{1-(q/p)} \\
&\geqslant \int_0^a \left(\int_0^x f(t)\,\mathrm{d}t\right)^q V^{1-(q/p)}(x)\,\mathrm{d}\left[-g^p(x)\right].
\end{aligned}$$

The proof for the case $0 < p \leqslant 1$ is similar. □

The following theorem is integral analogue of the Theorem 1 (compare Theorem 3.2 in [2], Theorem 2.1 in [5] and [9]).

Theorem 8. *Suppose that $g \in L_v^p(0,a)$ is nonnegative and nonincreasing, $p \geqslant 1$, $q \leqslant p$. Then*

$$\|Kg\|_{\omega,q} \geqslant \lambda \|g\|_{v,p}, \tag{30}$$

where

$$\lambda = \inf_{0<y<a} \frac{1}{V^{1/p}(y)} \left(\int_0^a \left(\int_0^y K(x,u)\,\mathrm{d}u\right)^q \omega(x)\,\mathrm{d}x\right)^{1/q}. \tag{31}$$

The constant λ is the least possible. If $0 < p \leqslant 1$, $p \leqslant q$, then the inequality in (30) *is reversed with λ defined similarly with* sup *instead of* inf.

Proof. Without loss of generality we can assume that g is absolutely continuous and that a is finite. Suppose also that $g(a) = 0$. Applying Proposition 2 with $\|g\|_{v,p} = 1$ and Fubini theorem we have

$$\begin{aligned}
\|Kg\|_{\omega,q}^q &= \int_0^a \left(\int_0^a K(x,y)g(y)\,\mathrm{d}y\right)^q \omega(x)\,\mathrm{d}x \\
&\geqslant \int_0^a \left[\int_0^a \left(\int_0^y K(x,u)\,\mathrm{d}u\right)^q V^{1-(q/p)}(y)\,\mathrm{d}\left[-g^p(y)\right]\right] \omega(x)\,\mathrm{d}x \\
&= \int_0^a \left[\int_0^a \left(\int_0^y K(x,u)\,\mathrm{d}u\right)^q \omega(x)\,\mathrm{d}x\right] V^{1-(q/p)}(y)\,\mathrm{d}\left[-g^p(y)\right] \\
&\geqslant \lambda^q \int_0^a V(y)\,\mathrm{d}\left[-g^p(y)\right] = \lambda^q \int_0^a g^p(y)v(y)\,\mathrm{d}y,
\end{aligned}$$

which gives (30) for the case $g(a) = 0$. For the absolutely continuous nonincreasing function g such that $g(a) > 0$ we define increasing sequence (g_n) of absolutely continuous nonincreasing functions by

$$g_n(x) = \begin{cases} g(x), & 0 < x < a - 1/n, \\ ng(a - 1/n)(a - x), & a - 1/n \leqslant x \leqslant a. \end{cases}$$

Applying proven case and using limiting procedure we obtain general assertion.

Sharpness of the constant can be obtained in standard way using characteristic function. Remaining case is similar. □

Using techniques from Theorem 8 and discrete case, one can easily proves the integral analogues of Theorems 2–4.

Although we can give applications of Theorem 8 for various kernels (fractional Riemann–Liouville $K(x, y) = (x - y)^{\alpha-1}$, $\alpha > 0$, $y < x$, extended Hilbert $K(x, y) = (x + y)^{-\lambda}$, $x, y \in \mathbf{R}$, $\lambda > 0$; for Hardy kernel see [5]), in our opinion the most interesting one is transformation with Hilbert's kernel $K(x, y) = 1/(x + y)$ on weighted Lebesgue spaces, especially in view of the fact that the only bounded linear operator from L^p to L^q for $0 < p < 1$, $p < q \leqslant \infty$ is trivial one [3, p. 150] and interesting upper bound which appears in this case.

For given $\beta \in \mathbf{R}$ we denote by L^p_β Lebesgue space with weight $v(t) = t^\beta$.

Theorem 9. *Let* $0 < p \leqslant 1$, $p \leqslant q < \infty$, $-1 < \beta$, $0 < 1 + \alpha < q$, $q/p = (1 + \alpha)/(1 + \beta)$ *and let g be nonnegative decreasing function on* $(0, \infty)$ *such that* $g \in L^p_\beta(0, \infty)$. *Then*

$$\|Kg\|_{\alpha,q} \leqslant (1 + \beta)^{1/p} \Gamma^{1/q}(1 + q) \zeta^{1/q}(q; \alpha) \|g\|_{\beta,p}, \tag{32}$$

where

$$\zeta(q;\ \alpha) = \sum_{n=0}^{\infty} \binom{\alpha + n + 1}{n} \frac{1}{(\alpha + n + 1)^{q+1}}. \tag{33}$$

For $p \geqslant 1$, $q \leqslant p$, *inequality* (32) *is reversed. There is equality in* (32) *if and only if* $g = A\chi_{[0,b]}$, *where A and b are nonnegative constants and* χ *characteristic function.*

Proof. Using Theorem 8 and simple transformations it is easy to see that

$$\begin{aligned} \lambda^q &= (\beta + 1)^{q/p} \int_0^\infty \frac{\log^q (1 + t)}{t^{\alpha+2}} \, \mathrm{d}t \\ &= (\beta + 1)^{q/p} \int_0^\infty u^q \mathrm{e}^{-(\alpha+1)u} \left(1 - \mathrm{e}^{-u}\right)^{-\alpha-2} \, \mathrm{d}u. \end{aligned}$$

Using binomial expansion of $(1 - \mathrm{e}^{-u})^{-\alpha-2}$ in power series in e^{-u} and integral representation of Γ function, inequality (32) follows. □

It is obvious that $\zeta(q;\ 0) = \zeta(q),\ q > 1$, where $\zeta(q)$ is classical Riemann's zeta function [11], and we see that Theorem 9 reduces for $p = q > 1,\ \alpha = \beta = 0$ to the Hilbert part of Corollary on page 97 in [1].

Appearing in Theorem 9 function $\zeta(q;\ \alpha)$ as a generalization (in some sense very natural one) of classical Riemann's zeta function, needs some attention. First recall that the Bernoulli numbers $B_k^{(n)}$ of order $n,\ n = 0, 1, 2, \ldots$ and degree $k,\ k = 0, 1, 2, \ldots$ are defined by $B_k^{(n)} = B_k^{(n)}(0)$, where $B_k^{(n)}(x)$ are the Bernoulli polynomials defined by expansion

$$\frac{t^n \mathrm{e}^{xt}}{(\mathrm{e}^t - 1)^n} = \sum_{k=0}^{\infty} B_k^{(n)}(x) \frac{t^k}{k!}. \tag{34}$$

In this context we find identity

$$B_k^{(n+1)}(x) = \frac{k!}{n!} \frac{\mathrm{d}^{n-k}}{\mathrm{d}x^{n-k}} [(x-1)(x-2)\cdots(x-n)], \quad 0 \leqslant k \leqslant n \tag{35}$$

more suitable. We also need recursion formula

$$(n+1) B_{n+1-k}^{(n+2)} = k B_{n+1-k}^{(n+1)} - (n+1)(n+1-k) B_{n-k}^{(n+1)}, \quad k \leqslant n. \tag{36}$$

Some properties of function $\zeta(q;\ \alpha)$ are given in the following.

Proposition 3. *If* $0 < \alpha + 1 < q$, *then the function* $\zeta(q;\ \alpha)$ *is well defined by* (33). *The following identities hold:*

$$\zeta(q;\ \alpha) = \frac{1}{\Gamma(q+1)} \int_0^{\infty} \frac{\log^q (1+t)}{t^{\alpha+2}}\, \mathrm{d}t \tag{37}$$

$$(\alpha+2)\zeta(q;\ \alpha+1) = \zeta(q-1;\ \alpha) - (\alpha+1)\zeta(q;\ \alpha),\ \alpha + 2 < q \tag{38}$$

$$\zeta(q;\ n) = \frac{1}{(n+1)!} \sum_{k=0}^{n} \binom{n}{k} B_{n-k}^{(n+1)} \zeta(q-k),$$

$$n + 1 < q,\ n = 0, 1, 2, \ldots [6pt] \tag{39}$$

Proof. Since the calculations and argumentations are elementary, we give just a sketch of the proof. That the function $\zeta(q;\ \alpha)$ is well defined for $0 < \alpha + 1 < q$ can be easily seen from the proof of Theorem 9 where also the first identity (integral representation) is contained. The second identity follows from the first using integration by parts twice (and trivial decomposition $t^{-\alpha-2}(1+t)^{-1} = t^{-\alpha-2} - t^{-\alpha-1}(1+t)^{-1}$). The third identity follows from the second one using induction and recursion formula (36). □

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