The Asymptotic Number of Acyclic Digraphs, II

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Let $A_{n,q}$ be the number of labeled digraphs with $n$ labeled vertices, $q$ edges, and
no directed cycles. Let $C_{n,q}$ be the corresponding number of weakly connected ones,
and let $a_{n,q}$ and $c_{n,q}$ be the corresponding numbers of unlabeled ones. We show that

$$A_{n,q} - C_{n,q} - n! a_{n,q} = n! c_{n,q}$$

for all $q$ when $\varepsilon N < q < (1 - \varepsilon) N$, where $N = \binom{n}{2}$. An asymptotic
formula for $A_{n,q}$ was obtained in an earlier paper.

1. Introduction

An acyclic digraph is a directed graph containing no directed cycles. Let $A_{n,q}$ be the number of acyclic digraphs with $n$ labeled vertices and $q$
unlabeled edges. In [1] it was shown that

**Theorem 1.** Let $\varepsilon > 0$ be given and suppose $q = q(n)$ satisfies $\varepsilon N < q < (1 - \varepsilon) N$ for all large $n$, where $N = \binom{n}{2}$.

$$A_{n,q} \sim n! \left( \frac{N}{q} \right) e^{-\lambda^2/4} f(\lambda, q, r) \frac{1}{\rho^{n+1}},$$

(1.1)

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where
\[ r = \frac{q}{N - q}, \quad \lambda = \frac{N - q}{N}, \quad x = \frac{\lambda \rho f(\lambda^2, r)}{2 f(\lambda \rho, r)}, \quad f(x, y) = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n! (1 + y)^n} \]
and \( \rho = \rho(r) > 0 \) is the smallest solution of the equation \( f(\rho, r) = 0 \).

Let \( C_{n,q} \) be the number of (weakly) connected acyclic digraphs with \( n \) labeled vertices and \( q \) unlabeled edges. Let \( a_{n,q} \) and \( c_{n,q} \) be the corresponding enumerators for acyclic digraphs with unlabeled vertices. Methods for computing the exact values of \( a_{n,q} \) and \( c_{n,q} \) were obtained in [2]. The purpose of this paper is to prove

**Theorem 2.** Let \( \varepsilon > 0 \) be given and suppose \( q = q(n) \) satisfies \( \varepsilon N < q < (1 - \varepsilon) N \) for all large \( n \). Then

\[ A_{n,q} \sim C_{n,q} \sim n! a_{n,q} \sim n! c_{n,q}. \]  \hspace{1cm} (1.2)

**Corollary.** \( \sum_q A_{n,q} \sim \sum_q C_{n,q} \sim \sum_q a_{n,q} \sim \sum_q c_{n,q} \).

These results follow the same pattern as for ordinary graphs. However, the methods of proof are much less direct. The exact results in [2] were not suitable as a point of departure. Instead, properties of the tower construction for acyclic digraphs introduced in [1] are combined with Burnside’s lemma, the asymptotic estimates in Theorem 1, and a method due to Wright to prove \( A_{n,q} \sim n! a_{n,q} \) in the next section. The arguments that are used in Section 3 to prove \( A_{n,q} \sim C_{n,q} \) and \( a_{n,q} \sim c_{n,q} \) are more standard. The corollary is proved in Section 4. Throughout, \( C \) denotes a positive constant, possibly dependent on \( \varepsilon \), and not necessarily the same at each appearance. Also \( N = (\frac{n}{2}) \) throughout.

**2. Unlabeled Acyclic Digraphs**

We assume familiarity with Section 4 of [1]. Let \( \omega \) be a permutation of \( \{1, \ldots, n\} \) that acts on a digraph \( D \) by permuting the vertex labels and let \( F(\omega, T) \) be the number of labeled acyclic \((n, q)\)-digraphs with tower \( T \) that are left fixed by \( \omega \). By Burnside’s lemma [3],

\[ n! a_{n,q} = \sum_{\omega, T} F(\omega, T) = A_{n,q} + \sum_{T} \sum_{\omega \neq 1} F(\omega, T) \]

\[ = A_{n,q} + \sum_1 + \sum_2, \]  \hspace{1cm} (2.1)
where $\sum_1$ is the sum of $F(\omega, T)$ over $T$ and $\omega \neq 1$ with $g(T) \leq n^{1.2}$ and $\sum_2$ is the same sum with $g(T) \geq n^{1.2}$. By the argument leading to [1, (4.2)],

$$
\sum_{g(T) \geq n^{1.2}} F(1, T) \leq A_{n,q} \left( \frac{3N}{q} \right)^n \sum_{f, t \geq n^{1.2}} \left( \frac{N-f}{N} \right)^q \\
\leq A_{n,q} \left( \frac{3N}{q} \right)^n C \exp \left( -n^{1.2} \frac{q}{N} \right).
$$

Since $q/N > \epsilon$ and $F(\omega, T) \leq F(1, T)$, we have

$$
\sum_2 \leq n! \sum_{g(T) \geq n^{1.2}} F(1, T) = o(A_{n,q}). \tag{2.2}
$$

We will use the ideas of Wright [4, Sect. 4] to show that

$$
\sum_1 = o(A_{n,q}). \tag{2.3}
$$

Combining (2.1), (2.2), and (2.3) gives us the desired result that

$$
n! a_{n,q} \sim A_{n,q}. \tag{2.5}
$$

We now turn to (2.3). Let $E = N - h(T)$ and $Q = q - q(T)$. Once $T$ has been specified, there are $E$ potential edges remaining from which $Q$ must be chosen. If $F(\omega, T) \neq 0$, $\omega$ fixes $T$ and so the vertices in a particular set $B_i$ are permuted among themselves. Thus $\omega$ permutes the $E$ potential edges. Let there be $P_i = P_i(\omega, T)$ orbits of size $i$. Following Wright [4], $F(\omega, T)$ is the coefficient of $x^Q$ in $\prod (1 + x^i)^{P_i}$. Thus for any positive real $x$,

$$
F(\omega, T) \leq x^{-Q} \prod (1 + x^i)^{P_i} \\
\leq x^{-Q}(1 + x)^{P_1} \prod_{i \neq 1} (1 + x^i)^{P_i/i} \\
= x^{-Q}(1 + x)^{P_1} (1 + x^2)^{(E - P_1)/2} \\
= x^{-Q}(1 + x)^E \left( \frac{1+x^2}{(1+x)^2} \right)^{(E - P_1)/2}
$$

Set $x = Q/(E - Q)$ and $\beta = (1 + x^2)/(1 + x)^2$. Then

$$
F(\omega, T) \leq \frac{E^E \beta^E P_1^{E - P_1)/2}}{Q^Q (E - Q)^{E - Q}} \leq C \sqrt{N} \left( \frac{E}{Q} \right)^{(E - P_1)/2} \\
\leq C n F(1, T) \beta^{(E - P_1)/2}. \tag{2.4}
$$

By the argument leading to [1, (4.1)] and the fact that $g(T) \leq n^{1.2}$,

$$
b_i \leq C n^{0.6}, \quad q(T) \leq C n^{1.2}, \quad h(T) \leq C n^{1.2}. \tag{2.5}
$$
Since $\varepsilon < q/N < 1 - \varepsilon$, it follows from (2.5) that $\beta$ is bounded above by $\beta(\varepsilon) < 1$ for all $T$ with $g(T) < n^{1.2}$. Let $\omega$ acting on $\{1, ..., n\}$ have $n - a$ fixed points. There are at most $\binom{n}{a} a! < n^a$ such $\omega$. An edge counted by $E$ is fixed if and only if both ends are fixed because the edge is directed. Thus $P_1 \leq \binom{n}{2 - a}$ and so, for $a \geq n/3$ and $n$ large,

$$E - P_1 \geq \binom{n}{2} - h(T) - \binom{n - a}{2} = a(n - a) + \binom{a}{2} - h(T) \geq an/4$$

since $a(n - a) + \binom{a}{2} = a(n - (a + 1)/2) > an/3$ and $h(T) \leq Can^{0.2}$ by (2.5).

Now suppose $a < n/3$. The number of edges counted by $E$ and having exactly one fixed end is at least

$$a(n - a - \max_i (b_{i-1} + b_i + b_{i+1})) \geq an/4$$

by (2.5). Thus

$$E - P_1 \geq an/4$$

(2.6)

for all $a$. Combining (2.4) and (2.6) we obtain

$$\sum_1 \leq \sum_{\omega \neq 1} \sum_T CnF(1, T) \beta^{an/4}$$

$$\leq Cn \sum_T F(1, T) \sum_{a \geq 1} n^a \beta^{an/4}$$

$$= Cn^A_{n,q} \sum_{a \geq 1} (n\beta^{a/4})^a$$

$$\leq Cn^2 A_{n,q} \beta^{an/4} = o(A_{n,q}).$$

3. CONNECTEDACYCLIC DIGRAPHS

We will prove $C_{n,q} \sim A_{n,q}$ and then indicate the minor changes needed to prove $c_{n,q} \sim a_{n,q}$.

To construct an unlabeled acyclic digraph we can take a set of $n$ linearly ordered points and connect them with $q$ edges where each edge is oriented toward the lower point. This leads to redundant constructions so

$$A_{n,q} \leq n! a_{n,q} \leq n! \binom{N}{q}. \quad (3.1)$$

Clearly

$$A_{n,q} - C_{n,q} \leq \sum_{i,j} T_{ij}, \quad (3.2)$$
where
\[ T_{ij} = \binom{n}{i} A_{i,j} A_{n-i,q-j}, \]
so it suffices to show that the right side of (3.2) is \( o(A_{n,q}) \).

By (3.1) we have
\[
\sum_j T_{ij} \leq \sum_j \binom{n}{i} i! \left( \binom{i}{2} \right) \binom{n-i}{j} (n-i)! \left( \frac{n-i}{q-j} \right)
\]
\[ = n! \left( \frac{i}{2} + \binom{n-i}{2} \right) \]
\[ \leq n! \left( \frac{N}{q} \right)^q \left( \frac{i}{2} \right)^q \]
\[ = n! \left( \frac{N}{q} \right)^q \left( 1 - \frac{i(n-i)}{N} \right)^q \]
\[ \leq n! \left( \frac{N}{q} \right)^q e^{-i(n-i)} \cdot \]

Hence, with \( I = [C \log n, n/2] \),
\[
\sum_{i \in I} \sum_j T_{ij} \leq n! \left( \frac{N}{q} \right)^q \sum_{i \in I} e^{-i(n-i)} \]
\[ \leq n! \left( \frac{N}{q} \right)^q O(e^{-Cn \log n}). \] (3.3)

Since \( \varepsilon < \lambda < 1 - \varepsilon \) in Theorem 1 and \( r \) is bounded away from 0 and \( \infty \), \( x \), \( \rho \), and \( f(\lambda \rho, r) \) are bounded away from 0 and \( \infty \). By (1.1) and (3.3),

\[
\sum_{i \in I} \sum_j T_{ij} = o(A_{n,q}). \] (3.4)

Now suppose \( i \leq C \log n \). Let \( r^* \) and \( \rho^* \) be the values in Theorem 1 when \( n \) and \( q \) are replaced by \( n-i \) and \( q-j \). We have
\[
\frac{1}{A_{n,q}} \sum_j T_{ij} = \sum_j \binom{n}{i} A_{i,j} \frac{A_{n-i,q-j}}{A_{n,q}}.
\]
Use (3.1) to bound \( A_{i,j} \). Since \( j \leq \binom{i}{2} < i^2 \), \( r^* = r + O(i/n) \) and so \( \rho^* = \)
\( \rho + O(i/n) \) by the smoothness of \( \rho \) as in the proof of [1, Lemma 1]. Thus \((\rho/\rho^*)^n = e^{O(i)} \). Hence

\[
\left( \binom{n}{i} A_{i,j} \frac{A_{n-i,q-j}}{A_{n,q}} \right) \leq \binom{n}{i} i! \binom{2}{j} (n-i)! \binom{2}{q-j} / n! \binom{N}{q} e^{O(i)} .
\]

\[
= \binom{n-i}{2} / \binom{N}{q} e^{O(i^2)} .
\]

\[
\leq \binom{q}{j} \binom{n-i}{2} / \binom{N}{q} e^{O(i^2)} .
\]

\[
\leq \left( \frac{1}{2e} \right)^j \binom{n-i}{2} / \binom{N}{q} e^{O(i^2)} .
\]

Thus, for sufficiently large \( n \),

\[
\sum_j T_{ij} = \sum_{j \leq i^2} T_{ij} \leq \left( 1 - \frac{i(n-i)}{N} \right)^q e^{O(i^2)} A_{n,q} \\
\leq \exp \left( -\frac{i(n-i)q}{N} + O(i^2) \right) A_{n,q} \\
\leq e^{-Cn} A_{n,q} .
\]

Thus

\[
\sum_{1 \leq i \leq C \log n} \sum_j T_{ij} \leq A_{n,q} \sum_{i \geq 1} (e^{-Cn})^i = o(A_{n,q}) . 
\]

Combining (3.4) and (3.5), we obtain

\[
A_{n,q} - C_{n,q} = o(A_{n,q}) .
\]

Now consider the unlabeled case. All equations following (3.1) remain valid if \( A_{*,*} \) is replaced by \( a_{*,*} \) and all references to \( (\cdot)!, n!, i!, \) and \((n-i)!\) are dropped. The references to Theorem 1 still apply because \( a_{n,q} \sim A_{n,q}/n! \) by Section 2.
4. PROOF OF COROLLARY

By Lemma 3 in [1] and its proof,

\[ \sum_{q \leq cN} A_{n,q} \leq CNA_{n,cN}. \]

Thus, with \( p \) and \( p^* \) as in Theorem 1 for \( q = N/2 \) and \( q = cN \), respectively,

\[ \sum_{q \leq cN} A_{n,q} \leq CNA_{n,N/2} \left( \frac{p}{p^*} \right)^n \frac{(\frac{N}{cN})}{(N/2)} \]
\[ = A_{n,N/2} e^{O(n)} 2^{-N} \frac{N^{cN}}{(cN/e)^{cN}} \]
\[ = A_{n,N/2} e^{O(n)} \left( \frac{(e/e)^c}{2} \right)^N \]
\[ = o(A_{n,N/2}/n!), \]

for any small enough \( \varepsilon \). A similar argument shows that

\[ \sum_{q \gg (1 - \varepsilon)N} A_{n,q} = o(A_{n,N/2}/n!). \]

The corollary follows easily.

REFERENCES