View metadata, citation and similar papers at core.ac.uk
brought to you by CORE
provided by Elsevier - Publisher Connector

Available online at www.sciencedirect.com
ScienceDirect
ELSEVIER
Linear Algebra and its Applications 423 (2007) 81–89
LINEAR ALGEBRA AND ITS APPLICATIONS
www.elsevier.com/locate/laa

Some notes on graphs whose index is close to 2

Francesco Belardo [a], Enzo Maria Li Marzi [a], Slobodan K. Simić [b,∗]

[a] *Department of Mathematics, University of Messina, 98 166 Sant'Agata, Messina, Italy*
[b] *Faculty of Computer Sciences, Knez Mihailova 6, 11 000 Belgrade, Serbia*

Received 10 May 2006; accepted 12 September 2006
Available online 3 November 2006
Submitted by D. Cvetković

Abstract

We consider two classes of graphs: (i) trees of order n and diameter $d = n - 3$ and (ii) unicyclic graphs of order n and girth $g = n - 2$. Assuming that each graph within these classes has two vertices of degree 3 at distance k, we order by the index (i.e. spectral radius) the graphs from (i) for any fixed k $(1 \leqslant k \leqslant d - 2)$, and the graphs from (ii) independently of k.
© 2006 Elsevier Inc. All rights reserved.

AMS classification: 05C50

Keywords: Tree; Diameter; Unicyclic graph; Characteristic polynomial; Index

1. Introduction

For the basic notions and terminology on spectral graph theory the readers are referred to [4,5]. To make the paper more self-contained, we mention here only a few basic facts. The *spectrum* of a (simple) graph G is the spectrum of A_G, the adjacency matrix of G (note all eigenvalues of simple graphs are real). The *index* (or *spectral radius*) is the largest eigenvalue of a graph. If G is connected, then the index, to be denoted by $\rho(G)$, is its simple eigenvalue (i.e. of multiplicity one). It can be also considered as the largest root of $\Phi(G, x)$, the characteristic polynomial G (recall, $\Phi(G, x) = \det(xI - A_G)$).

∗ Corresponding author.
E-mail addresses: fbelardo@gmail.com (F. Belardo), emlimarzi@dipmat.unime.it (E.M. Li Marzi), ssimic@raf.edu.yu (S.K. Simić).

0024-3795/$ - see front matter © 2006 Elsevier Inc. All rights reserved.
doi:10.1016/j.laa.2006.09.009

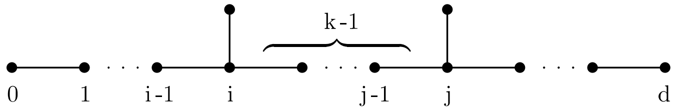

Fig. 1. The graph $M^d_{i,j}$.

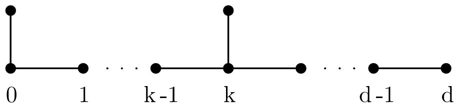

Fig. 2. The graph $M^d_{0,k}$.

Let $\mathscr{T}_{n,d}$ be the class of trees of order n and diameter d, and let $\mathscr{U}_{n,g}$ be the class of unicyclic graphs of order n and girth g. Throughout this paper we will take that $d = n - 3$, while $g = n - 2$ (then, any graph from the observed sets can have at most two vertices of degree 3). By $\mathscr{T}^k_{n,n-3}$ ($\mathscr{U}^k_{n,n-2}$) we denote those graphs from $\mathscr{T}_{n,d}$ (respectively $\mathscr{U}_{n,g}$) having just two vertices of degree 3 at distance k. (Note, any graph from $\mathscr{T}^k_{n,n-3}$ differs only by an edge from a single graph from $\mathscr{U}^k_{n,n-2}$.) We also put $\mathscr{T}^*_{n,n-3} = \bigcup_{k=1}^{n-5} \mathscr{T}^k_{n,n-3}$ ($\mathscr{U}^*_{n,n-2} = \bigcup_{k=1}^{\lfloor n/2\rfloor-1} \mathscr{U}^k_{n,n-2}$).

In [3,2] the authors determined all the graphs whose index is in the interval $(2, \sqrt{2+\sqrt{5}})$. This interval is of certain importance in the literature, as it turns out that graphs with index 2 are Smith graphs, while $\sqrt{2+\sqrt{5}}$ is the smallest limit point such that any real greater than this value is a limit point for a class of graphs (see [8]). The complete understanding of graphs from this class (and their ordering by index) is one of a rather classic goal of spectral graph theory. A large subclass of these graphs is in $\mathscr{T}^*_{n,n-3}$ (to be considered in this paper), while the remaining graphs are in $\mathscr{T}_{n,n-3} \setminus \mathscr{T}^*_{n,n-3}$. For some other results (and applications) concerning the ordering of graphs by the index (bounded as above), see [10]. The ordering of graphs belonging to $\mathscr{T}_{n,d}$ ($d < n - 3$) is to a large extent obtained in [9,6].

We now introduce a notation for the trees (i.e. caterpillars) from the set $\mathscr{T}^k_{n,n-3}$. Let $M^d_{i,j}$ (with $d = n - 3$, $j = i + k$) denote a caterpillar of diameter d in which i and j are its vertices of degree 3 (so $i > 0$ and $j < d$) – see Fig. 1 (note, $k - 1$ is the number of the vertices in the interval $[i + 1, j - 1]$). Note also that due to symmetry we can always assume that $i \leqslant \lfloor \frac{d-k}{2} \rfloor$.

Besides, we will also consider the graphs $M^d_{0,k}$ (see Fig. 2) which do not belong to the set $\mathscr{T}^k_{n,n-3}$ (now d is not a diameter of the corresponding graph). Note, such graphs can occur in our class under consideration as the result of some graph perturbations. The ordering of these graphs by index can be easily done by making use of Theorem 6.2.2 [5].

The paper is organized as follows. In Section 2 we give some basic tools to be used later. In Section 3 we give our main results (Theorems 3.1 and 3.4), the complete ordering of trees within the sets $\mathscr{T}^k_{n,n-3}$ ($1 \leqslant k \leqslant n - 5$). In Section 4 we order all unicyclic graphs from the set $\mathscr{U}^*_{n,n-2}$ (Theorem 4.1). Finally, in Section 5, we give some general overview about the whole topic.

2. Basic tools

For any simple graph G, let $\rho(G) = \rho$ be its index, while $\mathbf{x} = (x_1, x_2, \ldots, x_n)^{\mathrm{T}}$ its *Perron eigenvector* (i.e. a positive eigenvector, not necessarily a unit one); x_i is also called the *weight* of the ith vertex (with respect to $\mathbf{x}$). Then we have

$$\rho x_i = \sum_{i \sim j} x_j \quad (i = 1, 2, \dots, n),$$

where $\sim$ denotes the adjacency relation. This equation is called the *eigenvalue equation* for the ith vertex (corresponding to the index).

Lemma 2.1. *Let G be a (simple) graph and v one of its vertices. Denote by $\mathscr{C}(v)$ the set of all cycles in G containing v. Then we have*

$$\Phi(G, x) = x\Phi(G - v, x) - \sum_{w \sim v} \Phi(G - v - w, x) - 2 \sum_{C \in \mathscr{C}(v)} \Phi(G - V(C), x).$$

We assume that $\Phi(G, x) = 1$ if G is the empty graph (i.e. with zero vertices).

The next lemma can be simply verified.

Lemma 2.2. *Given the graphs G' and G'' then:*

(i) *if $\Phi(G', x) - \Phi(G'', x) > 0$, for any $x \geqslant \tau$, where $\tau = \max(\rho(G'), \rho(G''))$, then $\rho(G'') > \rho(G')$;*
(ii) *if $\Phi(G', x) - \Phi(G'', x) > 0$, for any $x \in (\alpha, \beta)$, where $\alpha \leqslant \min(\rho(G'), \rho(G''))$ and $\max(\rho(G'), \rho(G'')) \leqslant \beta$, then $\rho(G'') > \rho(G')$.*

Remark 2.3. It is well known that the index of a path of order n is $2\cos\frac{\pi}{n+1}$ (so less than 2). Next, for a given connected graph G and its proper subgraph G', we have $\rho(G') < \rho(G)$.

3. Main results

In what follows, the graph $M^d_{i,i+k}$ will be abbreviated to G^d_i if k is fixed, and to G_i if both k and d are fixed.

We now recall that a path $v_0, v_1, \dots, v_{k-1}, v_k$ (joining vertices v_0 and v_k) is an *internal path* in a graph if $\deg(v_0), \deg(v_k) \geqslant 3$, while $\deg(v_s) = 2$, $s = 1, \dots, k-1$; if $\deg(v_0) = 1$ or $\deg(v_k) = 1$, then the corresponding path is an *external path*.

Theorem 3.1. *Let $d = 2(k-1)$ and consider $G_i = M^{2(k-1)}_{i,i+k}$ and $G_j = M^{2(k-1)}_{j,j+k}$. Then $\rho(G_i) = \rho(G_j)$ for $0 \leqslant i, j \leqslant \lfloor\frac{d-k}{2}\rfloor$.*

Proof. Without loss of generality let $j = i + 1$ (then $i + 1 \leqslant \lfloor\frac{d-k}{2}\rfloor$). But then $k - i - 3 > 0$. Namely, since $i + 1 \leqslant \lfloor\frac{d-k}{2}\rfloor$ and $d = 2(k-1)$, we get $d - k - i - 1 = k - i - 3 \geqslant i + 1 > 0$, as required.

Assume first that $i \geqslant 1$. We now apply Lemma 2.1 to G_i at the vertex $k - 1$. Then (see also Figs. 3 and 4) we get

$$\begin{aligned}\Phi(G_i, x) &= x\Phi^2(A_{i,k-i-2}, x) - \Phi(A_{i,k-i-3}, x)\Phi(A_{i,k-i-2}, x) \\ &\quad - \Phi(A_{i,k-i-2}, x)\Phi(A_{i-1,k-i-2}, x) \\ &= \Phi(A_{i,k-i-2}, x)[x\Phi(A_{i,k-i-2}, x) - \Phi(A_{i,k-i-3}, x) - \Phi(A_{i-1,k-i-2}, x)].\end{aligned}$$

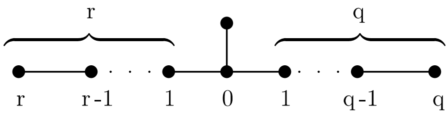

Fig. 3. The graph $A_{r,q}$ $(r > 0)$.

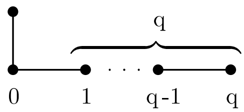

Fig. 4. The graph $A_{0,q}$ $(r = 0)$.

Since $\Phi(A_{i,k-i-1}, x) = x\Phi(A_{i,k-i-2}, x) - \Phi(A_{i,k-i-3}, x)$ we obtain

$$\Phi(G_i, x) = \Phi(A_{i,k-i-2}, x)[\Phi(A_{i,k-i-1}, x) - \Phi(A_{i-1,k-i-2}, x)]. \tag{1}$$

In the same way for G_{i+1} we obtain that

$$\Phi(G_{i+1}, x) = \Phi(A_{i+1,k-i-3}, x)[\Phi(A_{i+1,k-i-2}, x) - \Phi(A_{i,k-i-3}, x)]. \tag{2}$$

We next claim that the second factors in (1) and (2) are equal. Namely, since

$$\Phi(A_{i,k-i-1}, x) = x\Phi(A_{i,k-i-2}, x) - \Phi(A_{i,k-i-3}, x)$$

and

$$\Phi(A_{i+1,k-i-2}, x) = x\Phi(A_{i,k-i-2}, x) - \Phi(A_{i-1,k-i-2}, x),$$

the claim easily follows.

Observe now that the index of G_i, and of G_{i+1}, is the largest root of the second factor in (1) and (2), respectively (since $A_{i,k-i-2}$ and $A_{i+1,k-i-3}$ are induced subgraphs of G_i and G_{i+1}, respectively – see Remark 2.3). But since these factors are the same, we are done.

Assume now that $i = 0$. Applying Lemma 2.1 at vertex $k - 1$ in G_0 we get

$$\begin{aligned}\Phi(G_0, x) &= x\Phi^2(P_k, x) - \Phi(P_{k-1}, x)\Phi(P_k, x) - x\Phi(P_{k-2}, x)\Phi(P_k, x)\\ &= \Phi(P_k, x)[x\Phi(P_k, x) - \Phi(P_{k-1}, x) - x\Phi(P_{k-2}, x)].\end{aligned}$$

In the same way, for G_1 we get

$$\Phi(G_1, x) = \Phi(A_{1,k-3}, x)[\Phi(A_{1,k-2}, x) - \Phi(A_{0,k-3}, x)].$$

Note first that $A_{0,k-3} = P_{k-1}$. Applying Lemma 2.1 to $A_{1,k-2}$ at the vertex 1 we get $\Phi(A_{1,k-2}, x) = x\Phi(P_k, x) - x\Phi(P_{k-2}, x)$, and therefrom we easily get that

$$\Phi(G_1, x) = \Phi(A_{1,k-3}, x)[x\Phi(P_k, x) - \Phi(P_{k-1}, x) - x\Phi(P_{k-2}, x)].$$

Therefore, the second factors in the corresponding characteristic polynomials are again the same and we are again done (similarly to above).

This completes the proof. □

We now define a function $g_k(x)$. Inspecting the above proof we can write

$$\begin{aligned}g_k(x) &= \Phi(A_{i,k-i-1}, x) - \Phi(A_{i-1,k-i-2}, x) = \Phi(A_{1,k-2}, x) - \Phi(A_{0,k-3}, x)\\ &= \Phi(A_{1,k-2}, x) - \Phi(P_{k-1}, x) = \Phi(P_{k+1}, x) - x\Phi(P_{k-2}, x).\end{aligned}$$

Note, the largest root of $g_k(x)$ represents the index of all graphs $M_{i,i+k}^{2(k-1)}$ for $0 \leqslant i \leqslant \lfloor \frac{d-k}{2} \rfloor$. Therefore, for a (unique) unicyclic graph from $\mathscr{U}_{n,n-2}^{k}$ all trees obtained by deleting an edge of the cycle from the internal path of length $k-1$ have the same index.

Remark 3.2. Let ρ, ρ' be the indices of the graphs P_{k+1} and $P_{k-2} \cup K_1$, respectively. Then $\rho' < \rho < 2$. Let τ_k be the largest root of $g_k(x) = 0$, and suppose that $\tau_k > 2$. Then for any $x \in (2, \tau_k)$ we have that $g_k(x) < 0$.

Lemma 3.3. *If* $G_i^d = M_{i,i+k}^d$ *and* $0 \leqslant i < \lfloor \frac{d-k}{2} \rfloor$, *then*

$$\Phi(G_i^d, x) - \Phi(G_{i+1}^d, x) = \Phi(M_{0,k}^{d-2i}, x) - \Phi(M_{1,1+k}^{d-2i}, x)$$

holds for any d.

Proof. For $i = 0$ the proof is trivial. For $i = 1$, we observe that $d - k - 2 \geqslant 2$. Applying Lemma 2.1 for G_1^d at the vertex of degree 1 positioned on the right side we get

$$\Phi(G_1^d, x) = x\Phi(M_{1,k+1}^{d-1}, x) - \Phi(M_{1,k+1}^{d-2}, x).$$

Applying the same lemma for G_2^d at the vertex of degree 1 positioned on the left side we get

$$\Phi(G_2^d, x) = x\Phi(M_{1,k+1}^{d-1}, x) - \Phi(M_{0,k}^{d-2}, x).$$

Therefrom

$$\Phi(G_1^d, x) - \Phi(G_2^d, x) = \Phi(M_{0,k}^{d-2}, x) - \Phi(M_{1,k+1}^{d-2}, x).$$

We suppose now that $i > 1$, and that

$$\Phi(G_{i-1}^d, x) - \Phi(G_i^d, x) = \Phi(M_{0,k}^{d-2(i-1)}, x) - \Phi(M_{1,k+1}^{d-2(i-1)}, x)$$

holds for any d. Consider next the graphs G_i^d and G_{i+1}^d. It follows that $d - k - 2 \geqslant 2i$. By similar calculations we obtain

$$\begin{aligned}\Phi(G_i^d, x) - \Phi(G_{i+1}^d, x) &= \Phi(M_{i-1,k+i-1}^{d-2}, x) - \Phi(M_{i,k+i}^{d-2}, x)\\ &= \Phi(G_{i-1}^{d-2}, x) - \Phi(G_i^{d-2}, x).\end{aligned}$$

Therefore, by induction, we get

$$\Phi(G_i^d, x) - \Phi(G_{i+1}^d, x) = \Phi(M_{0,k}^{d-2i}, x) - \Phi(M_{1,k+1}^{d-2i}, x).$$

This completes the proof. $\square$

Theorem 3.4. *Let* $G_i^d = M_{i,k+i}^d$. *Then*

(i) *if* $d > 2(k-1)$ *then* $\rho(G_i) < \rho(G_j)$ *for* $0 \leqslant i < j \leqslant \lfloor \frac{d-k}{2} \rfloor$;
(ii) *if* $d < 2(k-1)$ *then* $\rho(G_i) > \rho(G_j)$ *for* $0 \leqslant i < j \leqslant \lfloor \frac{d-k}{2} \rfloor$.

Proof. Without loss of generality let $j = i + 1$. Then $i + 1 \leqslant \lfloor \frac{d-k}{2} \rfloor$. Let $s = d - 2i - k - 2$. Clearly, $s \geqslant 0$.

By Lemma 3.3 we have

$$\Delta = \Phi(G_i, x) - \Phi(G_{i+1}, x) = \Phi(M_{0,k}^{k+2+s}, x) - \Phi(M_{1,k+1}^{k+2+s}, x).$$

We first assume that $s = 0$. Then we have

$$\Phi(M_{0,k}^{k+2}, x) - \Phi(M_{1,k+1}^{k+2}, x) = \Phi(P_0, x) f_k(x).$$

(Note, we assume here that $\Phi(P_0, x) = 1$)).

We next assume that $s > 0$ (then $k + 3 \leqslant d$). Applying Lemma 2.1 at the vertex $k + 3$ of both graphs $M_{0,k}^{k+2+s}$ and $M_{1,k+1}^{k+2+s}$ we obtain

$$\begin{aligned}\Phi(M_{0,k}^{k+2+s}, x) &= x\Phi(M_{0,k}^{k+2}, x)\Phi(P_{s-1}, x) - \Phi(A_{1,k+1}, x)\Phi(P_{s-1}, x)\\ &\quad - \Phi(M_{0,k}^{k+2}, x)\Phi(P_{s-2}, x),\end{aligned}$$

and

$$\begin{aligned}\Phi(M_{1,k+1}^{k+2+s}, x) &= x\Phi(M_{1,k+1}^{k+2}, x)\Phi(P_{s-1}, x) - \Phi(A_{1,k+1}, x)\Phi(P_{s-1}, x)\\ &\quad - \Phi(M_{1,k+1}^{k+2}, x)\Phi(P_{s-2}, x).\end{aligned}$$

(Note, we assume here that $\Phi(P_{-1}, x) = 0$.)

Next we have

$$\begin{aligned}\Delta &= \Phi(M_{0,k}^{k+2+s}, x) - \Phi(M_{1,k+1}^{k+2+s}, x)\\ &= (x\Phi(P_{s-1}, x) - \Phi(P_{s-2}, x))(\Phi(M_{0,k}^{k+2}, x) - \Phi(M_{1,k+1}^{k+2}, x)) = \Phi(P_s, x) f_k(x).\end{aligned}$$

At this point we observe that P_s is a proper subgraph of the graphs G_i and G_{i+1}, and that for $x \geqslant \min(\rho(G_i), \rho(G_{i+1}))$, $f_k(x)$ and Δ have the same sign, since $\Phi(P_s, x) > 0$.

Now we prove that $f_k(x) = g_k(x)$. In fact, applying Lemma 2.1 repeatedly we obtain

$$\begin{aligned}f_k(x) &= x\Phi(A_{1,k-1}, x) - \Phi(P_{k+3}, x) = x^2\Phi(P_{k+1}, x) - x^2\Phi(P_{k-1}, x) - \Phi(P_{k+3}, x)\\ &= x\Phi(P_{k+2}, x) + x\Phi(P_k, x) - x\Phi(P_k, x) - x\Phi(P_{k-2}, x) - \Phi(P_{k+3}, x)\\ &= \Phi(P_{k+3}, x) + \Phi(P_{k+1}, x) - x\Phi(P_{k-2}, x) - \Phi(P_{k+3}, x)\\ &= \Phi(P_{k+1}, x) - x\Phi(P_{k-2}, x) = g_k(x).\end{aligned}$$

Therefore τ_k is the largest root of $f_k(x)$. Since $\lim_{x\to\infty} f_k(x) = +\infty$, for $x > \tau_k$ we have $f_k(x) > 0$.

Now we can finally prove the theorem.

(i) Let G be any graph in $\mathscr{T}_{n,d}^k$ ($d = n - 3$) such that $d > 2(k - 1)$. We eliminate some vertices from the external paths of G in order to obtain a graph G' of the type with $d' = 2(k - 1)$. Then the index of G' is exactly τ_k, and it is the same for any choice that we can do for the elimination of the vertices in the external paths. So $\tau_k < \rho(G)$ for any G in question. Consequently $\Phi(G_i, x) - \Phi(G_{i+1}, x) > 0$ for any $x > \tau_k$, and by Lemma 2.2(i) $\rho(G_i) < \rho(G_{i+1})$.

(ii) Let G be any graph in $\mathscr{T}_{n,d}^k$ ($d = n - 3$) such that $d < 2(k - 1)$. We add some vertices to the external paths of G in order to obtain a graph G' of the type with $d' = 2(k - 1)$. Then the index of G' is exactly τ_k, and it is the same for any choice that we can do for the adding of the vertices in the external paths. So $\tau_k > \rho(G)$ for any G in question. We observe that the index of any graph G is greater than 2 unless in a few cases (related to Smith graphs; see [4, p. 79]). But for these cases we can easily show that the theorem holds (by calculating the spectra). Therefore, for all other cases, there exist α and β such that $2 < \alpha < \min(\rho(G_i), \rho(G_{i+1}))$, and $\max(\rho(G_i), \rho(G_{i+1})) < \beta < \tau_k$. Using Remark 3.2, we have that $f_k(x) < 0$ for any $x \in (\alpha, \beta)$. Finally, using Lemma 2.2(ii), we get that $\rho(G_i) > \rho(G_{i+1})$.

This completes the proof. □

Theorem 3.5. *Let τ_k be the largest root of $f_k(x)$. Then $\{\tau_k\}_{k\in\mathbb{N}}$ is an increasing sequence and*

$$\lim_{k\to\infty} \tau_k = \sqrt{2+\sqrt{5}}.$$

Proof. By Theorem 3.1 we can consider τ_k as the index of the graph $M_{0,k}^{2k-2} = A_{k+1,k-2}$, and τ_{k+1} as the index of the graph $M_{0,k+1}^{2k} = A_{k+2,k-1}$. Since $A_{k+1,k-2}$ is subgraph of $A_{k+2,k-1}$, it follows that $\tau_k < \tau_{k+1}$, i.e. τ_k is an increasing sequence (also note that $\tau_k < 3$, so $\lim_{k\to\infty\tau_k}$ exists).

Note next that $\tau_k < \rho(A_{k+1,k+1}) < \tau_{k+3}$ for any $k \in \mathbb{N}$ (since $A_{k+1,k-2}$ is a proper subgraph of $A_{k+1,k+1}$, while the latter graph is a proper subgraph of $A_{k+4,k+1}$). Therefore

$$\lim_{k\to\infty} \tau_k = \lim_{r\to\infty} \rho(A_{r,r}).$$

On the other hand (see [7])

$$\lim_{r\to\infty} \rho(A_{r,r}) = \sqrt{2+\sqrt{5}}.$$

This completes the proof. □

Corollary 3.6. *The value $\sqrt{2+\sqrt{5}}$ is the best possible upper bound for the index of the graphs $M_{i,k+i}^{d}$, where $d = 2(k-1)$.*

Remark 3.7. It can be shown that in the above corollary we can put that $d \leqslant 2(k-1)$, but to prove this some results from [1] are necessary.

4. A result on unicyclic graphs

Let C_g^k be a unique graph from $\mathcal{U}_{n,g}^k$ $(g = n-2)$. So, it is a graph consisting of a cycle of length g having two pendant edges at distance $k \neq 0$ (then $1 \leqslant k \leqslant \lfloor \frac{g}{2} \rfloor$). We denote by C_g^+ the graph consisting of a cycle with one hanging edge added. In the following theorem we prove that for any fixed g if k increases then the index decreases.

Theorem 4.1. *Let $2 \leqslant k \leqslant \lfloor \frac{d+1}{2} \rfloor = \lfloor \frac{g}{2} \rfloor$, then $\rho(C_g^{k-1}) > \rho(C_g^k)$.*

Proof. Let $\Delta = \Phi(C_g^{k-1}, x) - \Phi(C_g^k, x)$. Applying Lemma 2.1 at a pendant vertex for both graphs C_g^{k-1} and C_g^k we get

$$\Phi(C_g^{k-1}) = x\Phi(C_g^+, x) - \Phi(A_{k-2,d-k+1}, x),$$

and

$$\Phi(C_g^k, x) = x\Phi(C_g^+, x) - \Phi(A_{k-1,d-k}, x).$$

So

$$\Delta = \Phi(A_{k-1,d-k}, x) - \Phi(A_{k-2,d-k+1}, x).$$

Applying Lemma 2.1 at the vertex of degree 1 positioned on the left side of $A_{k-1,d-k}$ and the same lemma at the vertex of degree 1 positioned on the right side of $A_{k-2,d-k+1}$ we get

$$\Delta = \Phi(A_{k-2,d-k-1}, x) - \Phi(A_{k-3,d-k}, x).$$

Next, applying repeatedly (in the same way) the same lemma we obtain

$$\begin{aligned}\Delta &= \Phi(A_{2,d-2k+3}, x) - \Phi(A_{1,d-2k+4}, x)\\ &= \Phi(A_{1,d-2k+2}, x) - \Phi(A_{0,d-2k+3}, x)\\ &= \Phi(P_{d-2k+3}, x) - x\Phi(P_{d-2k+2}, x) = -\Phi(P_{d-2k+1}, x).\end{aligned}$$

Next, since $\rho(P_{d-2k+1}) < \min(\rho(C^g_{k-1}), \rho(C^g_k))$ we have $\Delta < 0$, for any $x > \rho(P_{d-2k+1})$. Finally, by Lemma 2.2(i), the proof follows. □

5. Concluding remarks

In [3,2], all graphs with index in the interval $(2, \sqrt{2+\sqrt{5}})$ were determined (see also [10]). These graphs are the trees of two types (star-like trees with three legs or the trees as we considered in this paper). In this section we will give some comments about the trees from $\mathscr{T}^*_{n,n-3}$ whose indices fall in the above interval. Finally, we list all the graphs with index in $(2, \sqrt{2+\sqrt{5}})$ in order to show the reader to what extent this class of graphs is covered by the graphs we considered in this paper. As we will see below, it turns out that only the graphs from {d} are not covered.

We now list the graphs (trees) that we considered in the previous sections of this paper:

- {1} The graphs $M^d_{0,k}$, where $2k \leqslant d-1$ (note, d is not a diameter; also note that they were sometimes indicated as $A_{k+1,d-k}$). We exclude from this set of graphs those which are Smith graphs, or their proper subgraphs.
- {2} The graphs $M^d_{i,i+k}$ ($1 \leqslant i \leqslant \lfloor\frac{d-k}{2}\rfloor$, d is the diameter) with $d \leqslant 2(k-1)$.
- {3} The graphs $M^d_{i,i+k}$ ($1 \leqslant i \leqslant \lfloor\frac{d-k}{2}\rfloor$, d is the diameter) with $d > 2(k-1)$.

We next list all the graphs with index in the interval $(2, \sqrt{2+\sqrt{5}})$. We will specify for each of them, to which of the above sets it belongs.

- {a} The graphs $A_{2,m}$ ($m > 5$) and $A_{l,m}$ ($l > 2, m > 3$). Clearly, this set of graphs is the set {1}.
- {b} The graphs $M^d_{i,i+k}$, where $1 \leqslant i \leqslant \lfloor\frac{d-k}{2}\rfloor$ and $d \leqslant 2(k-1)$. This set of grphs is the set {2}.
- {c} The graphs $M^4_{1,2}$, $M^8_{2,6}$, $M^{10}_{2,7}$, $M^{13}_{3,10}$ and $M^{15}_{3,11}$; the graphs $M^d_{1,k+1}$, where $2k-1 \leqslant d \leqslant 2k+1$; and the graphs $M^d_{2,k+2}$, where $d = 2k-1$. This set is a subset of {3}.
- {d} The graphs depicted in Fig. 5 do not belong to any of the set {1}, {2} or {3}. This last set of graphs is a subset of $\mathscr{T}_{n,n-3}\backslash\mathscr{T}^*_{n,n-3}$.

An interesting problem is to completely order all graphs of $\mathscr{T}^*_{n,n-3}$ with respect to the index. We gave some further contributions in [1].

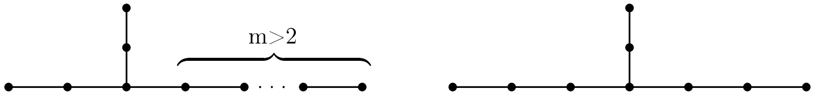

Fig. 5. Graphs with index in $(2, \sqrt{2+\sqrt{5}})$ which are not in $\mathscr{T}^*_{n,n-3}$.

Acknowledgments

Research of the first and second author was supported by MIUR and INDAM. Research of the third author was partly supported by INDAM and Project 144015G of Serbian Ministry of Sciences.

References

[1] F. Belardo, E.M. Li Marzi, S.K. Simić, Path-like graphs ordered by the index, Int. J. Algebra, in press.

[2] A.E. Brouwer, A. Neumaier, The graphs with spectral radius between 2 and $\sqrt{2+\sqrt{5}}$, Linear Algebra Appl. 114–115 (1989) 273–276.

[3] D. Cvektović, M. Doob, I. Gutman, On graphs whose eigenvalues do not exceed $\sqrt{2+\sqrt{5}}$, Ars Combin. 14 (1982) 225–239.

[4] D. Cvetković, M. Doob, H. Sachs, Spectra of Graphs, third ed., Johann Ambrosius Barth, Heidelberg, 1995.

[5] D. Cvetković, P. Rowlinson, S. Simić, Eigenspaces of Graphs, Cambridge Univ. Press, Cambridge, 1997.

[6] J.-M. Guo, J.-Y. Shao, On the spectral radius of trees with fixed diameter, Linear Algebra Appl. 413 (2006) 131–147.

[7] A.J. Hoffman, On limit points of spectral radii of non-negative symmetric integral matrices, Graph Theory and applications, in: Lecture Notes, vol. 303, Springer-Verlag, 1972, pp. 165–172.

[8] J.B. Shearer, On the distribution of the maximum eigenvalue of graphs, Linear Algebra Appl. 114/115 (1989) 17–20.

[9] S.K. Simić, B. Zhou, Indices of trees with a prescribed diameter, submitted for publication.

[10] F. Zhang, Z. Chen, Ordering graphs with small index and its application, Discrete Appl. Math. 121 (2002) 295–306.