



Contents lists available at SciVerse ScienceDirect

Applied Mathematics Lettersjournal homepage: www.elsevier.com/locate/aml**Inequalities for the Lugo and Euler–Mascheroni constants****Chao-Ping Chen**

School of Mathematics and Informatics, Henan Polytechnic University, Jiaozuo City 4540003, Henan Province, People's Republic of China

ARTICLE INFO**Article history:**

Received 9 April 2011

Accepted 28 September 2011

Keywords:

Euler–Mascheroni constant

Lugo's constant

The psi (or digamma) function

Speed of convergence

Approximation

ABSTRACTLugo's constant L given by

$$L = -\frac{1}{2} - \gamma + \ln 2$$

is defined as the limit of the sequence $(L_n)_{n \in \mathbb{N}}$ defined by

$$L_n := \sum_{i=1}^n \sum_{j=1}^n \frac{1}{i+j} - (2 \ln 2)n + \ln n \quad (n \in \mathbb{N} := \{1, 2, 3, \dots\})$$

as $n \rightarrow \infty$, where γ denotes the Euler–Mascheroni constant. In this work, we establish new inequalities for the Lugo and Euler–Mascheroni constants.

© 2011 Elsevier Ltd. All rights reserved.

1. Introduction

The Euler constant (or, more popularly, the Euler–Mascheroni constant) γ was first introduced by Leonhard Euler (1707–1783) in 1734 as follows:

$$\begin{aligned} \gamma &= \lim_{n \rightarrow \infty} D_n \quad \left(D_n := \sum_{k=1}^n \frac{1}{k} - \ln n \right) \\ &\cong 0.577215664901532860606512090082402431042\dots \end{aligned} \tag{1}$$

The constant γ is closely related to the celebrated gamma function $\Gamma(x)$ by means of the familiar Weierstrass formula [1, p. 255, Equation (6.1.3)] (see also [2, Chapter 1, Section 1.1]):

$$\frac{1}{\Gamma(z)} = ze^{\gamma z} \prod_{n=1}^{\infty} \left[\left(1 + \frac{z}{n}\right) e^{-z/n} \right] \quad (|z| < \infty). \tag{2}$$

Lugo [3] considered the sequence $(L_n)_{n \in \mathbb{N}}$, which is essentially an interesting analogue of the sequence $(D_n)_{n \in \mathbb{N}}$ occurring in (1), defined by

$$L_n := \sum_{i=1}^n \sum_{j=1}^n \frac{1}{i+j} - (2 \ln 2)n + \ln n. \tag{3}$$

In fact, Lugo [3] proved the following asymptotic formula:

$$L_n \sim -\frac{1}{2} - \gamma + \ln 2 - \frac{5}{8n} + \frac{7}{48n^2} + O(n^{-3}) \quad (n \rightarrow \infty). \tag{4}$$

E-mail address: chenchaoping@sohu.com.

Clearly, we find from (4) that

$$L := \lim_{n \rightarrow \infty} L_n = -\frac{1}{2} - \gamma + \ln 2, \quad (5)$$

where L is usually called *Lugo's constant*.

Recently, Chen and Srivastava [4] established new analytical representations for the Euler–Mascheroni constant γ :

$$\begin{aligned} \gamma &= -\sum_{i=1}^n \sum_{j=1}^n \frac{1}{i+j} + \ln 2 - 1 + \left(n + \frac{1}{2}\right) \psi\left(n + \frac{1}{2}\right) \\ &\quad - \left(n + \frac{3}{2}\right) \psi(n) + (2 \ln 2)n - \frac{3}{2n} \quad (n \in \mathbb{N}), \end{aligned} \quad (6)$$

in terms of the psi (or digamma) function $\psi(z)$ defined by

$$\psi(z) := \frac{d}{dz} \{\ln \Gamma(z)\} = \frac{\Gamma'(z)}{\Gamma(z)} \quad \text{or} \quad \ln \Gamma(z) = \int_1^z \psi(t) dt.$$

Also in [4], the authors proved inequalities for the Lugo and Euler–Mascheroni constants:

$$\frac{5}{8(n + \frac{7}{30})} - \frac{53}{2880(n + \frac{2081}{4770})^3} < L - L_n < \frac{5}{8(n + \frac{7}{30})} \quad (n \in \mathbb{N}). \quad (7)$$

Choi [5] summarized some known representations for the Euler–Mascheroni constant γ . For a rather impressive collection of various classes of integral representations for the Euler–Mascheroni constant γ , the interested reader may be referred to a recent paper by Choi and Srivastava [6].

In this sequel, we establish new inequalities for the Lugo and Euler–Mascheroni constants.

2. Sharp inequalities for the Lugo and Euler–Mascheroni constants

In view of the second inequalities in (7) it is natural to ask: What is the smallest number a and what is the largest number b such that the inequality

$$\frac{5}{8(n+a)} \leq L - L_n \leq \frac{5}{8(n+b)} \quad (8)$$

holds for all integers $n \geq 1$? The following theorem answers this question.

Theorem 1. For all integers $n \geq 1$,

$$\frac{5}{8(n+a)} \leq L - L_n < \frac{5}{8(n+b)} \quad (9)$$

with the best possible constants

$$a = \frac{-8\gamma - 13 + 24 \ln 2}{8\gamma + 8 - 24 \ln 2} = 0.244459972 \dots \quad \text{and} \quad b = \frac{7}{30} = 0.233333333 \dots$$

In order to prove Theorem 1, the following results are needed.

It is known from the recent works [7–9] that, for $x > 0$,

$$\ln x + \frac{1}{24x^2} - \frac{7}{960x^4} < \psi\left(x + \frac{1}{2}\right) < \ln x + \frac{1}{24x^2} - \frac{7}{960x^4} + \frac{31}{8064x^6} \quad (10)$$

and

$$\frac{1}{x} - \frac{1}{12x^3} + \frac{7}{240x^5} - \frac{31}{1344x^7} < \psi'\left(x + \frac{1}{2}\right) < \frac{1}{x} - \frac{1}{12x^3} + \frac{7}{240x^5}. \quad (11)$$

It follows from the known results (see [10, Theorem 8 and Theorem 9]) that, for $x > 0$,

$$\ln x - \frac{1}{2x} - \frac{1}{12x^2} + \frac{1}{120x^4} - \frac{1}{252x^6} < \psi(x) < \ln x - \frac{1}{2x} - \frac{1}{12x^2} + \frac{1}{120x^4} \quad (12)$$

and

$$\frac{1}{x} + \frac{1}{2x^2} + \frac{1}{6x^3} - \frac{1}{30x^5} < \psi'(x) < \frac{1}{x} + \frac{1}{2x^2} + \frac{1}{6x^3} - \frac{1}{30x^5} + \frac{1}{42x^7}. \quad (13)$$

Remark 1. The inequalities (10) and (11) were, in fact, derived by means of some results from [11].

We conclude from (10) and (12) that, for $x > 0$,

$$\frac{1}{2x} + \frac{1}{8x^2} - \frac{1}{64x^4} < \psi\left(x + \frac{1}{2}\right) - \psi(x) < \frac{1}{2x} + \frac{1}{8x^2} - \frac{1}{64x^4} + \frac{1}{128x^6}. \quad (14)$$

From (11) and (13), we obtain, for $x > 0$,

$$-\frac{1}{2x^2} - \frac{1}{4x^3} + \frac{1}{16x^5} - \frac{3}{64x^7} < \psi'\left(x + \frac{1}{2}\right) - \psi'(x) < -\frac{1}{2x^2} - \frac{1}{4x^3} + \frac{1}{16x^5}. \quad (15)$$

From (13)–(15), we obtain

$$\begin{aligned} & \psi\left(x + \frac{1}{2}\right) - \psi(x) + \left(x + \frac{1}{2}\right) \left(\psi'\left(x + \frac{1}{2}\right) - \psi'(x)\right) - \psi'(x) + \frac{1}{x} + \frac{3}{2x^2} \\ & < \frac{1}{2x} + \frac{1}{8x^2} - \frac{1}{64x^4} + \frac{1}{128x^6} + \left(x + \frac{1}{2}\right) \left(-\frac{1}{2x^2} - \frac{1}{4x^3} + \frac{1}{16x^5}\right) + \frac{1}{x^2} - \frac{1}{6x^3} + \frac{1}{30x^5} \\ & = \frac{5}{8x^2} - \frac{7}{24x^3} + \frac{3}{64x^4} + \frac{31}{480x^5} + \frac{1}{128x^6} \quad (x > 0). \end{aligned} \quad (16)$$

From (12) and (14), we obtain

$$\begin{aligned} & \frac{1}{2} - \left(x + \frac{1}{2}\right) \left(\psi\left(x + \frac{1}{2}\right) - \psi(x)\right) + \psi(x) - \ln x + \frac{3}{2x} \\ & > \frac{1}{2} - \left(x + \frac{1}{2}\right) \left(\frac{1}{2x} + \frac{1}{8x^2} - \frac{1}{64x^4} + \frac{1}{128x^6}\right) + \frac{1}{x} - \frac{1}{12x^2} + \frac{1}{120x^4} - \frac{1}{252x^6} \\ & = \frac{5}{8x} - \frac{7}{48x^2} + \frac{1}{64x^3} + \frac{31}{1920x^4} - \frac{1}{128x^5} - \frac{127}{16128x^6} \quad (x > 0). \end{aligned} \quad (17)$$

We are now in a position to prove our **Theorem 1**.

Proof of Theorem 1. Indeed, by using (6), $L - L_n$ can be written as follows:

$$L - L_n = \frac{1}{2} - \left(n + \frac{1}{2}\right) \psi\left(n + \frac{1}{2}\right) + \left(n + \frac{3}{2}\right) \psi(n) - \ln n + \frac{3}{2n} \quad (n \in \mathbb{N}). \quad (18)$$

The inequality (9) can be written as

$$a \geq \frac{1}{\frac{8}{5} \left(\frac{1}{2} - \left(n + \frac{1}{2}\right) (\psi\left(n + \frac{1}{2}\right) - \psi(n)) + \psi(n) - \ln n + \frac{3}{2n}\right)} - n > b \quad (n \in \mathbb{N}).$$

In order to prove (9), we consider the function f defined by

$$f(x) = \frac{1}{\frac{8}{5} \left(\frac{1}{2} - \left(x + \frac{1}{2}\right) (\psi\left(x + \frac{1}{2}\right) - \psi(x)) + \psi(x) - \ln x + \frac{3}{2x}\right)} - x.$$

We conclude from the asymptotic formula of ψ [1, p. 259] that

$$f(x) \sim \frac{7}{30} + \frac{53}{1800x} + O\left(\frac{1}{x^2}\right),$$

which implies

$$\lim_{x \rightarrow \infty} f(x) = \frac{7}{30}.$$

Differentiating $f(x)$ and applying inequalities (16) and (17) yields

$$\begin{aligned} & \frac{8}{5} \left(\frac{1}{2} - \left(x + \frac{1}{2}\right) \left(\psi\left(x + \frac{1}{2}\right) - \psi(x)\right) + \psi(x) - \ln x + \frac{3}{2x}\right)^2 f'(x) \\ & = \psi\left(x + \frac{1}{2}\right) - \psi(x) + \left(x + \frac{1}{2}\right) \left(\psi'\left(x + \frac{1}{2}\right) - \psi'(x)\right) - \psi'(x) + \frac{1}{x} + \frac{3}{2x^2} \end{aligned}$$

$$\begin{aligned}
& -\frac{8}{5} \left(\frac{1}{2} - \left(x + \frac{1}{2} \right) \left(\psi \left(x + \frac{1}{2} \right) - \psi(x) \right) + \psi(x) - \ln x + \frac{3}{2x} \right)^2 \\
& < \frac{5}{8x^2} - \frac{7}{24x^3} + \frac{3}{64x^4} + \frac{31}{480x^5} + \frac{1}{128x^6} - \frac{8}{5} \left(\frac{5}{8x} - \frac{7}{48x^2} + \frac{1}{64x^3} + \frac{31}{1920x^4} - \frac{1}{128x^5} - \frac{127}{16128x^6} \right)^2 \\
& = -\frac{p(x)}{4064256000x^{12}}
\end{aligned}$$

with

$$\begin{aligned}
p(x) &= 38241275209 + 289354688532(x-3) + 543104914296(x-3)^2 \\
&\quad + 490633633728(x-3)^3 + 254598194004(x-3)^4 + 80399052720(x-3)^5 \\
&\quad + 15345282960(x-3)^6 + 1634169600(x-3)^7 + 74793600(x-3)^8 \\
&> 0 \quad (x \geq 3).
\end{aligned}$$

Straightforward calculation produces

$$\begin{aligned}
f(1) &= \frac{-8\gamma - 13 + 24\ln 2}{8\gamma + 8 - 24\ln 2} = 0.244459972\dots, \\
f(2) &= \frac{-107 - 48\gamma + 192\ln 2}{46 + 24\gamma - 96\ln 2} = 0.2425032898\dots, \\
f(3) &= \frac{-1149 - 360\gamma - 360\ln 3 + 2520\ln 2}{358 + 120\gamma - 840\ln 2 + 120\ln 3} = 0.240541882\dots,
\end{aligned}$$

and thus, the sequence

$$f(n) = \frac{1}{\frac{8}{5} \left(\frac{1}{2} - \left(n + \frac{1}{2} \right) \left(\psi \left(n + \frac{1}{2} \right) - \psi(n) \right) + \psi(n) - \ln n + \frac{3}{2n} \right)} - n \quad (n \in \mathbb{N})$$

is strictly decreasing. This leads to

$$\frac{7}{30} = \lim_{n \rightarrow \infty} f(n) < f(n) \leq f(1) = \frac{-8\gamma - 13 + 24\ln 2}{8\gamma + 8 - 24\ln 2}.$$

The proof of **Theorem 1** is complete. \square

3. Continued fraction approximations for $L - L_n$

We define the sequence $(v_n)_{n \in \mathbb{N}}$ by

$$v_n = L - L_n - \frac{\frac{5}{8}}{n + a + \frac{b}{n + c + \frac{p}{n + q}}}. \quad (19)$$

We are interested in finding the values of the parameters a, b, c, p and q such that $(v_n)_{n \in \mathbb{N}}$ is the fastest sequence which would approximate zero. Our study is based on the following **Lemma 1**, which provides a method for measuring the speed of convergence.

Lemma 1 ([12,13]). *If the sequence $(\lambda_n)_{n \in \mathbb{N}}$ converges to zero and if there exists the following limit:*

$$\lim_{n \rightarrow \infty} n^k(\lambda_n - \lambda_{n+1}) = l \in \mathbb{R} \quad (k > 1),$$

then

$$\lim_{n \rightarrow \infty} n^{k-1}\lambda_n = \frac{l}{k-1} \quad (k > 1).$$

Theorem 2. *Let the sequence $(v_n)_{n \in \mathbb{N}}$ be defined by (19). Suppose also that*

$$a = \frac{7}{30}, \quad b = \frac{53}{1800}, \quad c = \frac{1339}{1590}, \quad p = \frac{15975}{22472} \quad \text{and} \quad q = -\frac{6528287}{59267250}. \quad (20)$$

Then

$$\lim_{n \rightarrow \infty} n^8(v_n - v_{n+1}) = -\frac{6164042747}{257644800000} \quad \text{and} \quad \lim_{n \rightarrow \infty} n^7(v_n - L) = -\frac{6164042747}{1803513600000}. \quad (21)$$

The speed of convergence of the sequence $(v_n)_{n \in \mathbb{N}}$ is given by the order estimate $O(n^{-7})$ as $n \rightarrow \infty$.

Proof. First of all, we write the difference $v_n - v_{n+1}$ as the following power series in n^{-1} :

$$\begin{aligned}
v_n - v_{n+1} = & \frac{30a - 7}{24n^3} - \frac{-31 - 120b + 120a + 120a^2}{64n^4} + \frac{200a - 300b - 200bc + 300a^2 - 400ab + 200a^3 - 49}{80n^5} \\
& - (1200a - 2400b - 2400abc - 2400bc + 1200bp - 1200bc^2 \\
& + 2400a^2 - 4800ab + 2400a^3 + 1200b^2 - 3600ba^2 + 1200a^4 - 263) \frac{1}{384n^6} \\
& + (1680a - 4200b + 1680a^5 - 1680bc^3 - 8400abc - 5600bc + 4200bp \\
& - 4200bc^2 + 3360bc + 1680bpq + 3360b^2c + 4200a^2 - 11200ab \\
& - 6720ba^3 + 5040ab^2 + 5600a^3 + 4200b^2 - 12600ba^2 + 4200a^4 \\
& - 321 + 3360abp - 3360abc^2 - 5040bca^2) \frac{1}{448n^7} \\
& - (-1503 + 8960a - 26880b + 26880a^5 - 26880bc^3 + 8960bpq^2 - 89600abc - 44800bc \\
& + 44800bp - 44800bc^2 - 8960bc^4 - 8960bp^2 + 53760bc + 26880bpq + 53760b^2c \\
& + 26880a^2 - 89600ab + 26880pbc^2 - 107520ba^3 + 80640ab^2 \\
& + 17920bc + 8960b^3 + 44800a^3 + 44800b^2 - 134400ba^2 + 17920abpq \\
& + 35840abcp - 17920abc^3 - 26880a^2bc^2 + 26880a^2bp + 53760b^2ca \\
& + 26880b^2c^2 - 17920b^2p + 53760b^2a^2 + 44800a^4 - 35840bca^3 \\
& + 8960a^6 - 44800ba^4 + 53760abp - 53760abc^2 - 80640bca^2) \frac{1}{2048n^8} \\
& + O\left(\frac{1}{n^9}\right) \quad (n \rightarrow \infty). \tag{22}
\end{aligned}$$

The fastest sequence $(v_n)_{n \in \mathbb{N}}$ is obtained when the first five coefficients of this power series vanish. In this case $a = \frac{7}{30}$, $b = \frac{53}{1800}$, $c = \frac{1339}{1590}$, $p = \frac{15975}{22472}$ and $q = -\frac{6528287}{59267250}$, we have

$$v_n - v_{n+1} = -\frac{6164042747}{257644800000n^8} + O\left(\frac{1}{n^9}\right) \quad (n \rightarrow \infty). \tag{23}$$

Finally, by using Lemma 1, we obtain assertion (21) of Theorem 2. \square

Motivated by Theorem 2, we establish the following:

Theorem 3. For all integers $n \geq 2$, then

$$\frac{\frac{5}{8}}{n + \frac{7}{30} + \frac{\frac{53}{1800}}{n + \frac{1339}{1590}}} < L - L_n < \frac{\frac{5}{8}}{n + \frac{7}{30} + \frac{\frac{53}{1800}}{n + \frac{1339}{1590} + \frac{\frac{15975}{22472}}{n - \frac{6528287}{59267250}}}}. \tag{24}$$

Proof. We only prove the left-hand inequality in (24). The proof of the right-hand inequality in (24) is similar.

The lower bound in (24) is obtained by considering the function $g(x)$ which is defined by

$$g(x) = \frac{1}{2} - \left(x + \frac{1}{2}\right) \left(\psi\left(x + \frac{1}{2}\right) - \psi(x)\right) + \psi(x) - \ln x + \frac{3}{2x} - \frac{\frac{5}{8}}{x + \frac{7}{30} + \frac{\frac{53}{1800}}{x + \frac{1339}{1590}}}.$$

We conclude from a well-known asymptotic formula for ψ (see, for example, [1, p. 259, Equation (6.3.18)]) that

$$\lim_{x \rightarrow \infty} g(x) = 0.$$

Differentiating $g(x)$ and applying inequalities (13)–(15), we find that

$$g'(x) = -\psi\left(x + \frac{1}{2}\right) + \psi(x) - \left(x + \frac{1}{2}\right) \left(\psi'\left(x + \frac{1}{2}\right) - \psi'(x)\right) + \psi'(x) - \frac{1}{x} - \frac{3}{2x^2}$$

$$\begin{aligned}
& + \frac{25(337\,080x^2 + 567\,736x + 229\,131)}{3(2120x^2 + 2280x + 479)^2} \\
& < -\frac{1}{2x} - \frac{1}{8x^2} + \frac{1}{64x^4} - \left(x + \frac{1}{2}\right) \left(-\frac{1}{2x^2} - \frac{1}{4x^3} + \frac{1}{16x^5} - \frac{3}{64x^7}\right) \\
& \quad - \frac{1}{x^2} + \frac{1}{6x^3} - \frac{1}{30x^5} + \frac{1}{42x^7} + \frac{25(337\,080x^2 + 567\,736x + 229\,131)}{3(2120x^2 + 2280x + 479)^2} \\
& = -\frac{q(x)}{13\,440x^7(2120x^2 + 2280x + 479)^2}
\end{aligned}$$

with

$$\begin{aligned}
q(x) &= 377\,588\,384\,377 + 902\,538\,003\,828(x - 3) + 737\,355\,609\,978(x - 3)^2 \\
&\quad + 278\,168\,322\,870(x - 3)^3 + 49\,996\,036\,960(x - 3)^4 + 3\,479\,238\,000(x - 3)^5 \\
&> 0 \quad (x \geq 3).
\end{aligned}$$

Hence, $g'(x) < 0$ for $x \geq 3$.

Clearly, for $x = 1, x = 2$ and $x = 3$, direct computation would yield

$$\begin{aligned}
g(1) &= -\frac{43\,919}{29\,274} - \gamma + 3\ln 2 = 0.00195259\dots, \\
g(2) &= -\frac{118\,709}{54\,076} - \gamma + 4\ln 2 = 0.00014781\dots, \\
g(3) &= -\frac{1\,676\,957}{527\,980} - \gamma + 7\ln 2 - \ln 3 = 0.00002707\dots.
\end{aligned}$$

Consequently, the sequence $(g(n))_{n \in \mathbb{N}}$ is strictly decreasing. This leads us to

$$g(n) > \lim_{n \rightarrow \infty} g(n) = 0,$$

which means that the lower bound in the assertion (24) of Theorem 3 holds true for all $n \in \mathbb{N}$. The proof of Theorem 3 is thus completed. \square

Remark 2. The first inequality in (24) is valid for all integers $n \geq 1$, while the second inequality in (24) is valid for $n \geq 2$.

Acknowledgment

The numerical calculations presented in this work were performed by using the *Maple* software for symbolic computations.

References

- [1] M. Abramowitz, I.A. Stegun (Eds.), Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, Ninth printing, in: Applied Mathematics Series, vol. 55, National Bureau of Standards, Washington, DC, 1972.
- [2] H.M. Srivastava, J. Choi, Series Associated with the Zeta and Related Functions, Kluwer Academic Publishers, Dordrecht, Boston and London, 2001.
- [3] I. Lugo, An Euler–Maclaurin summation. Available at: <http://www.math.upenn.edu/~mlugo/114-07C/euler-maclaurin.pdf>.
- [4] C.-P. Chen, H.M. Srivastava, New representations for the Lugo and Euler–Mascheroni constants, Appl. Math. Lett. 24 (7) (2011) 1239–1244.
- [5] J. Choi, Some mathematical constants, Appl. Math. Comput. 187 (2007) 122–140.
- [6] J. Choi, H.M. Srivastava, Integral representations for the Euler–Mascheroni constant γ , Integral Transforms Spec. Funct. 21 (2010) 675–690.
- [7] C.-P. Chen, Inequalities and monotonicity properties for some special functions, J. Math. Inequal. 3 (2009) 79–91.
- [8] C.-P. Chen, Inequalities for the Euler–Mascheroni constant, Appl. Math. Lett. 23 (2010) 161–164.
- [9] C.-P. Chen, Monotonicity properties of functions related to the psi function, Appl. Math. Comput. 217 (2010) 2905–2911.
- [10] H. Alzer, On some inequalities for the gamma and psi functions, Math. Comp. 66 (1997) 373–389.
- [11] G. Allasia, C. Giordano, J. Pečarić, Inequalities for the gamma function relating to asymptotic expansions, Math. Inequal. Appl. 5 (2002) 543–555.
- [12] C. Mortici, New approximations of the gamma function in terms of the digamma function, Appl. Math. Lett. 23 (2010) 97–100.
- [13] C. Mortici, Product approximations via asymptotic integration, Amer. Math. Monthly 117 (2010) 434–441.