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On Structure of Some Plane Graphs with Application to Choosability[1]

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A graph $G=(V,E)$ is (x,y)-choosable for integers $x>y\geqslant 1$ if for any given family $\{A(v) \mid v\in V\}$ of sets $A(v)$ of cardinality x, there exists a collection $\{B(v) \mid v\in V\}$ of subsets $B(v)\subset A(v)$ of cardinality y such that $B(u)\cap B(v)=\varnothing$ whenever $uv\in E(G)$. In this paper, structures of some plane graphs, including plane graphs with minimum degree 4, are studied. Using these results, we may show that if G is free of k-cycles for some $k\in\{3,4,5,6\}$, or if any two triangles in G have distance at least 2, then G is $(4m,m)$-choosable for all nonnegative integers m. When $m=1$, $(4m,m)$-choosable is simply 4-choosable. So these conditions are also sufficient for a plane graph to be 4-choosable. © 2001 Academic Press

Key Words: choosable; plane graph; cycle; triangle.

1. INTRODUCTION

In this paper, unless stated otherwise, "graph" means simple plane (finite) graph. Undefined symbols and concepts can be found in [4].

Let $G=(V,E,F)$ be a plane graph, where V, E and F denote the set of vertices, edges and faces of G, respectively. The degree of a vertex v is denoted by $d(v)$. A vertex v is called a k-vertex or a k^+-vertex if $d(v)=k$ or if $d(v)\geqslant k$ respectively. The set of all k-vertices and the set of all k^+-vertices will be written as V_k and V_{k^+} respectively. We denote by $\delta(G)$, or δ

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for simplicity, the minimum degree of G and by $N_G(v)$ the set of vertices in G adjacent to vertex v. A face of a graph is said to be *incident* with all edges and vertices on its boundary. Two faces sharing an edge e are called *adjacent* at e. The *degree* of a face f of graph, denoted by $d(f)$, is the number of edges incident with it, where each cut-edge is counted twice. A k-face or a k^+-face is a face of degree k or of degree at least k respectively. The set of all k-faces of G is denoted by F_k. An i-cycle is a cycle of length i. A graph is called *C_i-free graph* if it contains no i-cycles. We use $d(u, v)$ to denote the distance between vertices u and v in G. The *distance* of two triangles $u_1u_2u_3u_1$ and $v_1v_2v_3v_1$ is $\min\{d(u_i, v_j) \mid i, j=1, 2, 3\}$.

Suppose $G=(V, E)$ is a graph, not necessary planar. Let f, g: $V \to \mathbb{N}$ be two functions, where $\mathbb{N}$ is the set of positive integers. We say G is *(f, g)-choosable* [9] if given arbitrary color-sets $A(v)$ with $|A(v)| = f(v)$ to each of $v \in V$, we can choose subsets $B(v) \subseteq A(v)$ with $|B(v)| = g(v)$ such that $B(u) \cap B(v) = \emptyset$ whenever $uv \in A$. A (f, g)-choosable graph is also (f', g)-choosable if $f'(v) \geqslant f(v)$ for all $v \in V$. Suppose G is (f, g)-choosable, where f and g are constant functions $f(v)=x$ and $g(v)=y$ for all vertices $v \in V$ with $x > y \geqslant 1$, then G is called *(x, y)-choosable*. A $(k, 1)$-choosable is simply *k-choosable*. Graph-choosability is a generalization of graph-colorability. It was first introduced by Vizing [17] and independently by Erdős, Rubin and Taylor [6] nearly two decades ago; and was investigated by many researches in recent years [1–3, 7–11, 13–16].

In Section 2, we study the structure of some plane graphs with minimum degree 4. Using these results, we obtain in Section 3 some sufficient conditions for a plane graph to be $(4m, m)$-choosable for all m. When $m=1$, $(4m, m)$-choosable is simply 4-choosable. So these conditions are also sufficient for a plane graph to be 4-choosable.

2. STRUCTURE OF SOME PLANE GRAPHS

THEOREM 2.1. *Let G be a plane graph with $\delta \geqslant 4$. Then either G contains a 4-cycle or G contains a 6-cycle $u_1u_2 \cdots u_6u_1$ with exactly one chord u_1u_3 and each of the six vertices are 4-vertices.*

Proof. If $\delta = 4$, the theorem follows from Lemma 1 in [10]. Suppose $\delta \geqslant 5$ and G contains no 4-cycles. Then $|V| \leqslant \frac{2}{5}|E|$. Since G contains no 4-cycles, so $F_4 = \emptyset$ and no two triangles are adjacent. Therefore $3|F_3| \leqslant |E|$, and consequently

$$2|E| = \sum_{i=3}^{\infty} i|F_i| \geqslant 5|F| - 2|F_3| \geqslant 5|F| - \tfrac{2}{3}|E|,$$

i.e. $|F| \leqslant \frac{8}{15}|E|$. Substituting upper bounds of V and F into the Euler's formula $|V|+|F|-|E|=2$, we have $2 \leqslant [\frac{2}{5}+\frac{8}{15}-1]\,|E| = -\frac{1}{15}|E|$. This contradiction shows that G contains 4-cycles. ▮

THEOREM 2.2. *Let G be a plane graph with $\delta \geqslant 4$. Then G contains a 5-cycle.*

Proof. Suppose G contains no 5-cycles. Define a weight w on elements of $V \cup F$ by letting $w(x) = 2[d(x)-3]$ if $x \in F$ and $w(x) = d(x)-6$ if $x \in V$. Applying Euler's formula for plane graphs, $|V|+|F|-|E|=2$, we have $\sum_{x \in V \cup F} w(x) = -12$. If we obtain a new weight $w^*(x)$ for all $x \in V \cup F$ by transferring weights from one element to another, then we also have $\sum_{x \in V \cup F} w^*(x) = -12$. If these transfers result in $w^*(x) \geqslant 0$ for all $x \in V \cup F$, then we get a contradiction and the Theorem is proved. Weights will be transferred according to the following rule:

(R) From each $f \in F_{4^+}$, transfer $\frac{w(f)}{d(f)}$ to each vertex in $V_4 \cup V_5$ incident with f.

It is clear that $w^*(x) \geqslant 0$ if $x \in F$ or if $x \in V_{6^+}$. Suppose $v \in V_4 \cup V_5$. Because there are no 5-cycles, a triangle cannot be adjacent to a 4-face, or to two other triangles. Therefore v is incident with at least 2 faces in F_{6^+}. Since the weight that each vertex receives from an incident 6^+-face f is $\frac{2d(f)-6}{d(f)} \geqslant 1$, therefore $w^*(v) \geqslant w(v)+2 \geqslant 0$. ▮

A graph G is said to be *k-degenerate* if for every induced subgraph H of G, $\delta(H) \leqslant k$. We have the following Corollary of Theorem 2.2.

COROLLARY 2.3. *A C_5-free plane graph is 3-degenerate.*

THEOREM 2.4. *Let $G=(V,E,F)$ be a graph with $\delta \geqslant 4$, then either G contains a 6-cycle or G contains a 4-cycle incident to four 4-vertices.*

Proof. Suppose $G=(V,E,F)$ is a graph with $\delta \geqslant 4$, without 6-cycles, and without 4-cycles incident to four 4-vertices. For each $x \in V \cup F$, let $w(x) = \frac{d(x)}{8} - \frac{1}{2}$ be a weight assigned to x. Applying Euler's formula for plane graphs, we can show that $\sum_{x \in V \cup F} w(x) = -1$. If we obtain a new weight $w^*(x)$ for all $x \in V \cup F$ by transferring weights from one element to another, then we also have $\sum_{x \in V \cup F} w^*(x) = -1$. Moreover, if $w^*(x) \geqslant 0$ for all $x \in V \cup F$, then the Theorem is proved. Weights will be transferred according to the following rules:

(R_1) From each 5-vertex to each of its incident triangles or 4-faces, transfer $\frac{1}{24}$;

(R_2) From each 6^+-vertex to each of its incident 3- or 4-faces, transfer $\frac{1}{16}$,

(R_3) From each 4-face to an adjacent triangle, transfer $\frac{1}{24}$;

(R_4) From each 7^+-face f to an adjacent triangle f', transfer

$(R_{4.1})$ $\frac{1}{24}$ if f' is not adjacent to any triangles, or is adjacent to exactly one triangle and incident with one 5^+-vertex, or is adjacent to two triangles and incident with two 5^+-vertices;

$(R_{4.2})$ $\frac{1}{16}$ if f' is adjacent to exactly one triangle and is incident with three 4-vertices, or is adjacent to two triangles and incident with two 4-vertices and one 6^+-vertex;

$(R_{4.3})$ $\frac{1}{12}$ if f' is adjacent to two triangles and is incident with two 4-vertices and one 5-vertex;

$(R_{4.4})$ $\frac{1}{8}$ if f' is adjacent to two triangles and is incident with three 4-vertices.

Figure 1 depicts some circumstances where the various sub-rules of (R_4) applies. We shall first make the following observations. Because G contains no 6-cycles, therefore,

(i) A k-vertex, where $k \geqslant 5$, is incident with at most $\lfloor \frac{3k}{4} \rfloor$ in total, of 3- or 4-faces.

(ii) A 5-face is not adjacent to any triangle;

(iii) A 4-face is adjacent to at most one triangle;

(iv) A triangle is adjacent to at most two triangles;

(v) If f_1 and f_2 are adjacent triangles, then none of them is adjacent to a 4-face.

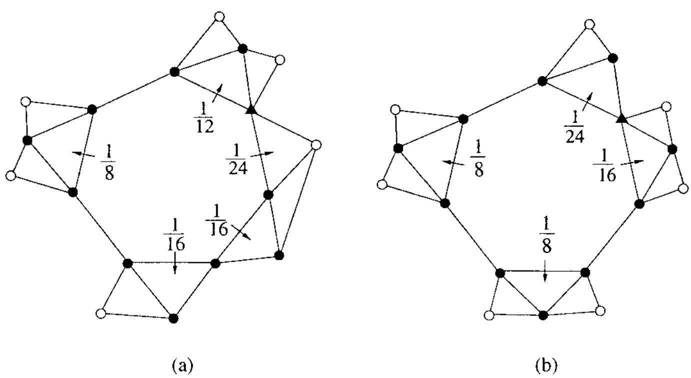

FIG. 1. Weights transferred across edges. Circular and triangular dots are 4- and 5-vertices respectively.

We shall now establish the following claims. Suppose st, tu, uv and vw are four consecutive edges on the boundary of a 7^+-face f.

Claim (A). *If by* ($R_{4.2}$), $\frac{1}{16}$ *is transferred from face* f *across both of* tu *and* uv, *then at most* $\frac{1}{24}$ *is transferred across each of the two edges* st *or* vw.

Claim (B). *If by* ($R_{4.3}$), $\frac{1}{12}$ *is transferred from face* f *across* uv, *then* 0 *is transferred across* one *of* tu *and* vw, *and at most* $\frac{1}{24}$ *is transferred across the other.*

Claim (C). *If by* ($R_{4.4}$), $\frac{1}{8}$ *is transferred from face* f *across* uv *on the boundary of* f, *then* 0 *is transferred across both* tu *and* vw.

Proof of Claim (A). Suppose f is adjacent to triangles $f' = utt'$ and $f'' = vuu'$. Moreover, $\frac{1}{16}$ is transferred from f to f' and f'' across tu and uv respectively.

We shall first consider f'. If f' is adjacent to two triangles $f_1^* = tt't''$ and $f_2^* = t'uu''$ as in Fig. 2(a), then f' is incident with two 4-vertices and one 6^+-vertex. Because $\delta \geqslant 4$, $t'' \neq s$ and $u'' \neq v$. Also $u'' \neq u'$, otherwise $t''t'u'vutt''$ is a 6-cycle. Therefore $d(u) \geqslant 5$. Because f' is incident with two 4-vertices and one 6^+-vertex, u and t must be a 6^+- and a 4-vertex respectively. If f is adjacent to any triangle at st, then that triangle must be adjacent to f_1^*, which is impossible because G is C_6-free. Therefore the weight to be transferred across st is 0. If f' is adjacent to exactly one triangle f^* at $t'u$ as in Fig. 2(b), then f' is incident with three 4-vertices and in particular, u is a 4-vertex. This is possible only if $u'' = u'$ and $tt'u'u$ is a 4-cycle. Because t, u and t' are 4-vertices, so u' is a 5^+-vertex. Moreover, f'' cannot be adjacent to a triangle at $u'v$ because G is C_6-free. Therefore f'' is adjacent to one triangle and is incident with at least one 5^+-vertex. So at most $\frac{1}{24}$ is transferred from f to f'' across uv, which is a contradiction. Similarly, if

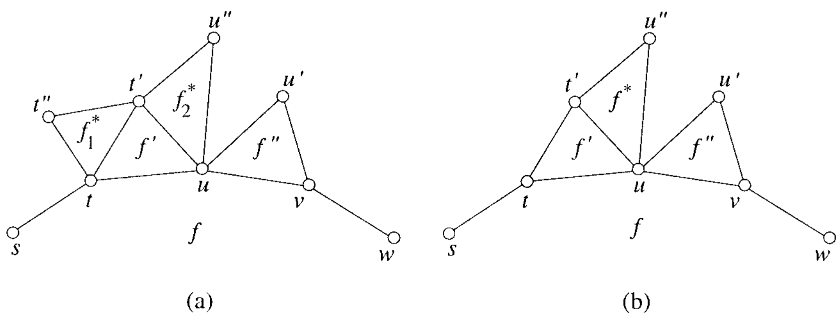

FIG. 2. Weights transferred across consecutive edges.

f' is adjacent to exactly one triangle at tt', then at most $\frac{1}{24}$ is transferred out of f across st.

Considering f'' and repeating the above argument, we can also show that at most $\frac{1}{24}$ is transferred out of f across vw. This completes the proof of Claim (A).

Proofs of Claim (B) and Claim (C) are similar, but, simpler, and are therefore omitted.

We shall now show that $w^*(x) \geqslant 0$ for all $x \in V \cup F$. Suppose v is a k-vertex. Clearly $w^*(x) = w(x) = 0$ if $k = 4$. Because of Observation (i), $w^*(v) \geqslant w(v) - \frac{3}{24} = 0$, $w^*(v) \geqslant w(v) - \frac{4}{16} = 0$ and $w^*(v) \geqslant w(v) - \frac{1}{16} \cdot \frac{3k}{4} \geqslant 0$ if $k = 5$, $k = 6$ and $k \geqslant 7$ respectively.

Suppose f is a triangle. If f is not adjacent to another triangle, then each face adjacent to f is a k-face, where $k = 4$ or $k \geqslant 7$. Thus $w^*(f) \geqslant w(f) + 3 \cdot \frac{1}{24} = 0$. If f is adjacent to exactly one triangle, then f is also adjacent to two 7^+-faces. Moreover, f will receive $\frac{1}{24}$ from each of the two 7^+-faces and at least $\frac{1}{24}$ from an incident 5^+-vertex, or $\frac{1}{16}$'s from each of the two 7^+-faces. Consequently, $w^*(f) \geqslant 0$. Similarly, we can verify that $w^*(f) \geqslant 0$ if f is adjacent to two triangles.

Suppose f is a 4-face. By Observation (iii), f is adjacent to at most one triangle. Since the boundary of f is a 4-cycle, it is incident with at least one 5^+-vertex and we can verify that $w^*(f) \geqslant 0$. It is also clear from Observation (ii) that $w^*(f) > 0$ if f is a 5-face.

Suppose f is a k-face with $k \geqslant 7$, $e_1, e_2, ..., e_k$ are consecutive edges on the boundary of f, and z_i is the weight transferred from f across e_i for $1 \leqslant i \leqslant k$.

Suppose that $z_i \geqslant z_{i+1}$ for some i, where $1 \leqslant i \leqslant k$ and z_{k+1} is identified with z_1. If $z_i \leqslant \frac{1}{16}$, then $z_{i+1} \leqslant \frac{1}{16}$. If $z_i = \frac{1}{12}$, then $z_{i+1} \leqslant \frac{1}{24}$ by Claim B. If $z_i = \frac{1}{8}$, then $z_{i+1} = 0$ by Claim C. In all three cases, $z_i + z_{i+1} \leqslant \frac{1}{8}$. This inequality also holds if $z_i \leqslant z_{i+1}$ by similar argument. It follows that $\sum_{i=1}^{k} z_i = \frac{1}{2} \cdot \sum_{i=1}^{k} (z_i + z_{i+1}) \leqslant \frac{k}{16}$.

Suppose $k \geqslant 8$. Then $w^*(f) \geqslant w(f) - \sum_{i=1}^{k} z_i \geqslant w(f) - \frac{k}{16} \geqslant 0$. Suppose $k = 7$ and $z_i \geqslant \frac{1}{12}$ for some i. It follows from Claims B and Claim C that either $z_{i-1} = 0$ or $z_{i+1} = 0$. Without loss of generality, assume that $z_1 = 0$. Then $\sum_{i=1}^{7} z_i = \sum_{i=1}^{3} (z_{2i} + z_{2i+1}) \leqslant \frac{3}{8}$, and $w^*(f) \geqslant 0$. Suppose $z_i \leqslant \frac{1}{16}$ for all i. By Claim A, $z_i \leqslant \frac{1}{24}$ for at least one edge among any three consecutive edges. Therefore $z_i \leqslant \frac{1}{24}$ on at least three edges and consequently $w^*(f) \geqslant w(f) - \frac{4}{16} - \frac{3}{24} = 0$. ∎

In the following Theorem, a 4- or 5-vertex incident with a triangle is called a 4^3- or 5^3-vertex respectively. A 4- or 5-vertex not incident with any triangle is called a 4^{-3}- or 5^{-3}-vertex respectively. Also, a 4-face incident with one 5^{-3}-vertex, one 4^{-3}-vertex and two 4^3-vertices will be called a *light* face. A face is called *normal* if it is not light.

THEOREM 2.5. *Let $G=(V, E, F)$ be a plane graph with $\delta \geqslant 4$ in which any two triangles have distance at least two. Then G contains a 4-face such that each vertex incident with it is a 4-vertex. Moreover, this 4-face has a common vertex with a triangle.*

Proof. Let $w(v)=\frac{3d(v)}{10}-1$ when $v \in V$ and $w(f)=\frac{d(f)}{5}-1$ when $f \in F$ be a weight assigned to the elements of $V \cup F$. Then using Euler's formula again, we can show that $\sum_{x \in V \cup F} w(x)=-2$. Now we construct a new weight w^* by transferring weights between elements of $V \cup F$, keeping the relationship $\sum_{x \in V \cup F} w^*(x)=-2$. Let F^* be the subset of F consisting of 4-faces incident with four 4-vertices, at least one of which is a 4^3-vertex. If we define the transferring rules in such a way so that $w^*(x) \geqslant 0$ for all $x \in (V \cup F) \backslash F^*$, then we can conclude that $F^* \neq \varnothing$ and the Theorem is proved. Weights are transferred according to the following rules:

(R_1) From a 6^+-vertex to each of its incident face, transfer $\frac{2}{15}$;

(R_2) From a 5^3-vertex to an incident face f, transfer

$(R_{2.1})$ $\frac{2}{15}$ if f is a triangle;

$(R_{2.2})$ $\frac{11}{120}$ otherwise;

(R_3) From a 5^{-3}-vertex to an incident face f, transfer

$(R_{3.1})$ $\frac{19}{180}$ if f is a light face;

$(R_{3.2})$ $\frac{7}{90}$ otherwise;

(R_4) From a 4^3-vertex to an incident face f, transfer

$(R_{4.1})$ $\frac{2}{15}$ if f is a triangle;

$(R_{4.2})$ $\frac{1}{45}$ otherwise;

(R_5) From a 4^{-3}-vertex to each of its incident face f, transfer $\frac{1}{20}$.

Suppose f is a 5^+-face. Then f receives at least $\frac{1}{45}$ from each of its incident vertices, and therefore $w^*(f)>w(f) \geqslant 0$. Suppose f is a 4-face not in F^*. Then f can be incident to at most a total of two 4^3- or 5^3-vertices, otherwise there will be two triangles at distance strictly less than 2 from each other. If f is not incident with any 4^3-vertex, then f receives at least $\frac{1}{20}$ from each of the 4^+-vertices incident with it and $w^*(f) \geqslant w(f)+\frac{4}{20} \geqslant 0$. If f is incident with at least one 4^3-vertex, then f is incident with at least one 5^+-vertex v. If v is a 6^+-vertex, then f receives $\frac{2}{15}$ from v and at least $\frac{1}{45}$ from each of the other three incident vertices. Therefore $w^*(f) \geqslant w(f)+\frac{2}{15}+\frac{3}{45}=0$. If v is a 5^3-vertex, then at worst f is incident with one 4^3-vertex. Therefore $w^*(f) \geqslant w(f)+\frac{11}{120}+\frac{1}{45}+\frac{2}{20}>0$. If v is a 5^{-3}-vertex and f is incident with exactly one 4^3-vertex, then $w^*(f) \geqslant w(f)+\frac{7}{90}+\frac{1}{45}+\frac{2}{20}=0$. If v is a 5^{-3}-vertex and f is incident with exactly two 4^3-vertices, then $w^*(f) \geqslant w(f)+\frac{19}{180}+\frac{2}{45}+\frac{1}{20}=0$. Suppose f is a 3-face, then f receives $\frac{2}{15}$ from each of its incident vertices and $w^*(f)=w(f)+3 \cdot \frac{2}{15}=0$.

Suppose v is a k-vertex. If $k \geqslant 6$, then $w^*(v) \geqslant w(v) - \frac{2k}{15} = \frac{5k-30}{30} \geqslant 0$. If v is a 4^3-vertex, then $w^*(v) \geqslant w(v) - \frac{2}{15} - \frac{3}{45} = 0$. If v is a 4^{-3}-vertex, then $w^*(v) \geqslant w(v) - \frac{4}{20} = 0$. If v is a 5^3-vertex, then $w^*(v) \geqslant w(v) - \frac{2}{15} - 4 \cdot \frac{11}{120} = 0$. If v is a 5^{-3}-vertex, and at least one incident face is not a light face, then $w^*(v) \geqslant w(v) - 4 \cdot \frac{19}{180} - \frac{7}{90} = 0$. Proof of this Theorem will be complete if we can show that every 5^{-3}-vertex is incident with at least one normal face.

If a 5^{-3}-vertex is incident with a 5^+-face f, then f is a normal face. Suppose v, a 5^{-3}-vertex; is incident with five 4-faces—Fig. 3. If any of the vertices u_i and w_i is a 5^+-vertex, then one of $f_1, ..., f_5$ is a normal face. So we assume that all of u_i and w_i, $1 \leqslant i \leqslant 5$, are 4-vertices. Now consider the face f_1. If both w_1 and w_2 are 4^{-3}-vertices, then f_1 is normal. So one of w_1 and w_2 is a 4^3-vertex. If w_1, say, is a 4^3-vertex, then either w_1u_1 or w_1u_5 is incident with a triangle, otherwise w_1 is a 5^+-vertex and f_1 is a normal face. Without loss of generality, we may assume that f_1 is adjacent to a triangle at w_1u_1. Similar consideration for f_2 forces f_2 to be adjacent to a triangle at u_2w_3, otherwise f_2 would be a normal face. Moreover, f_3 would be a normal face unless f_4 is adjacent to a triangle at w_4u_4. Now neither w_5 nor u_5 can be 4^3-vertices, and therefore f_5 is normal. ∎

COROLLARY 2.6. *Let G be a graph with a two-cell embedding on a torus. Suppose $\delta \geqslant 4$ and two triangles have distance at least two. If G contains either a 5^+-face or a 6^+-vertex, then G contains a 4-face such that each vertex incident with it is a 4-vertex. Moreover, this 4-face has a common vertex with a triangle.*

Proof. Following the notations and definitions in the proof of Theorem 2.5, we have $\sum_{x \in V \cup F} w^*(x) = 0$. Also as in Theorem 2.5, we have $w^*(x) \geqslant 0$ for all $x \in (V \cup F) \backslash F^*$.

If G contains a 5^+-face f, then it follows from the proof of Theorem 2.5 that $w^*(f) > 0$. If G does not contains a 5^+-face f, but contains a 6^+-vertex v, then v is incident with a 4-face f. Therefore f receives $\frac{2}{15}$ from v, at least

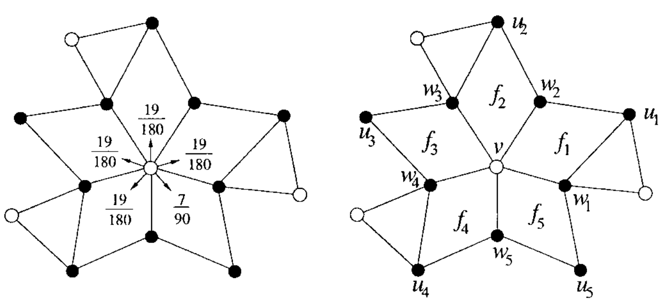

FIG. 3. Weights transferred from a 5-vertex. Black vertices are 4-vertices.

$\frac{1}{45}$ from two other vertices and at least $\frac{1}{20}$ from the fourth vertex. Consequently, $w^*(f) \geqslant w(f) + \frac{2}{15} + \frac{2}{45} + \frac{1}{20} > 0$. In either case, we can conclude that $F^* \neq \emptyset$. ∎

3. APPLICATION TO CHOOSABILITY

LEMMA 3.1 [15]. *For any $m \in \mathbb{N}$, every 2-choosable graph is $(2m, m)$-choosable.*

It follows from Lemma 3.1 that every even cycle is $(2m, m)$-choosable.

LEMMA 3.2. *Let $G = (V, E)$ be a cycle $v_1 v_2 \cdots v_n v_1$ with exactly one chord $v_1 v_k$ $(3 \leqslant k \leqslant n-1)$. If f: $V \to \mathbb{N}$ such that $f(v_1) = f(v_k) = 3$ and $f(v_i) = 2$ when $i \neq 1$, k, then G is (mf, m)-choosable for all $m \in \mathbb{N}$.*

Proof. Let $A(v)$ be color-lists given to v such that $|A(v)| = mf(v)$ for all $v \in V$. We first choose for v_1 color-set $B(v_1)$ with $|B(v_1)| = m$ and $B(v_1) \subset A(v_1) \backslash A(v_n)$, then choose for $v_2, v_3, ..., v_n$ color-sets $B(v_i)$, of order m, successively from $A(v_i) \backslash B(v_{i-1})$ when $i \neq k$, and from $A(v_i) \backslash \{B(v_{k-1}) \cup B(v_1)\}$ when $i = k$. The choice of $B(v_i)$ is always possible because the choice is always made on sets of order not less than m. ∎

LEMMA 3.3. *Let $G = (V, E)$ be a graph, not necessarily planar, and f, g be two functions $V \to \mathbb{N}$. Let $H = G[V']$ for some $V' \subset V$, and $f'(v) = f(v) - \sum_{u \in N_G(v) \backslash V'} g(u)$ for all $v \in V'$. If $f'(v) \geqslant g(v)$ for all $v \in V'$, $G - H$ is (f, g)-choosable and H is (f', g)-choosable, then G is (f, g)-choosable.*

Proof. Let $\{A(v) \mid v \in V\}$ be a family of sets such that $|A(v)| = f(v)$ for all $v \in V$. Because $G - H$ is (f, g)-choosable, then for each $v \in V(G - H)$ there is a subset $B(v) \subset A(v)$ of cardinality $g(v)$, such that $B(u) \cap B(v) = \emptyset$ whenever $uv \in E(G - H)$.

For each $v \in V'$, let $A'(v) = A(v) \backslash \bigcup_{u \in N_G(v) \backslash V'} B(u)$. Then $|A'(v)| \geqslant f'(v)$. Since H is (f', g)-choosable, there exists $B(v) \subset A'(v)$ such that $|B(v)| = g(v)$ and $B(u) \cap B(v) = \emptyset$ whenever $uv \in E(H)$. The (f, g)-choosability of G immediately follows. ∎

THEOREM 3.4. *Let G be a plane graph. If G is C_k-free for some $k \in \{3, 4, 5, 6\}$, then it is $(4m, m)$-choosable for all $m \in \mathbb{N}$.*

Proof. By mathematical induction on the order of $G = (V, E)$. It is trivial if $|V| = 1$. Assume that the theorem holds for $|V| < n$ where $n \geqslant 2$.

Suppose $|V| = n$. If $\delta \leqslant 3$, then let $v \in V$ such that $d(v) = \delta$. Since any subgraph of G is also C_k-free if G is, by induction $G - v$ is $(4m, m)$-choosable.

By an argument similar to that used in the proof of Lemma 3.3, we can show that G is $(4m, m)$-choosable.

If $\delta \geqslant 4$, then G must contain 3-cycles, otherwise $w(x) \geqslant 0$ for all $x \in V \cup F$, where w is defined in the proof of Theorem 2.4. By Theorem 2.2, G contains 5-cycles. If G contains no 4-cycles, then by Theorem 2.1 G contains a 6-cycle $u_1 u_2 \cdots u_6 u_1$ with exactly one chord $u_1 u_3$ such that $d(u_i) = 4$ for $i = 1, ..., 6$. Let $f(u_1) = f(u_3) = 3$, $f(u_i) = 2$ for $2 \leqslant i \leqslant 6$ and $i \neq 3$. Then $G[C]$, where $C = \{u_1, u_2, ..., u_6\}$, is (mf, m)-choosable by Lemma 3.2. The $(4m, m)$-choosability of G follows from Theorem 3.3 and the induction assumption that $G - G[C]$ is $(4m, m)$-choosable.

If G contains no 6-cycles, then by Theorem 2.4, G contains a 4-cycle incident with four 4-vertices. Similar to the last paragraph, G must be $(4m, m)$-choosable because every even cycle is $(2m, m)$-choosable. ∎

The proof of the following Theorem is almost identical to that of Theorem 3.4 and is therefore omitted.

THEOREM 3.5. *Each plane graph in which any two triangles have distance at least two is* $(4m, m)$*-choosable.*

In addition, it is easy to see that the conclusion of the last two theorems also holds for graphs embedded on a surface of characteristic one.

4. REMARKS

Theorems 2.1, 2.2 and 2.4 offer more than a proof that any plane graph without 4-cycles, or without 5-cycles, or without 6-cycles is 4-choosable. They in fact provide a simple method of coloring the graph.

For example, suppose a C_4-free plane graph G, and lists of four colors for each vertex are given. Then either the minimum degree is $\leqslant 3$, or there exists a 6-cycle with exactly one chord and each of the six vertices are of degree 4. Then we delete a vertex of minimum degree 3 (or less) if exists, or the 6-cycle as described above. Since any subgraph of G is also C_4-free, this process can be continued until G has been reduced to a graph of very small order so that coloring the vertices is simple. Then we reverse the reduction process, extending the coloring on the way until the entire graph is restored.

These processes are also available for $(4m, m)$-choosability.

5. OPEN QUESTIONS

In [18], Voigt gave a 3-colorable plane graph which is not 4-choosable. This means that there exists a non-4-choosable K_4-free plane graph. In this paper we proved that each C_4-free plane graph and each plane graph in which any two triangles have distance at least 2 is $(4m, m)$-choosable. It is known that there exists non-3-colorable plane graph in which any two triangles have distance at least 3. Probably each plane graph without adjacent triangles is $(4m, m)$-choosable.

In 1976, Steinberg [12] conjectured that every plane graph without 4- and 5-cycles is 3-colorable. Erdős in 1990 relaxed this Conjecture by asking if there is an integer $k \geqslant 5$ such that every plane graph without i-cycles for $4 \leqslant i \leqslant k$ is 3-colorable. Borodin, [5] proved that $k = 9$ is suitable. In this paper, we proved that for each $i \in \{3, 4, 5, 6\}$, every plane graph without i-cycles is $(4m, m)$-choosable. It is interesting to find the largest integer k such that for $i \in \{3, 4, ..., k\}$, every plane graph without i-cycles is $(4m, m)$-choosable. Voigt's examples [18] indicates that $k \leqslant 75$. Finally, we post a conjecture and a problem:

Conjecture. Every plane graph without adjacent triangles is $(4m, m)$-choosable for $m \in \mathbb{N}$.

Problem. Find the largest integer k such that for all $i \in \{3, 4, ..., k\}$, every plane graph without i-cycles is $(4m, m)$-choosable for $m \in \mathbb{N}$. It follows from Theorem 2.4, [11] and the example in Theorem 1.7 of [18] that $6 \leqslant k \leqslant 75$.

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