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Singular structure of Toda lattices and cohomology of certain compact Lie groups

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Abstract

We study the singularities (blow-ups) of the Toda lattice associated with a real split semisimple Lie algebra $\mathfrak{g}$. It turns out that the total number of blow-up points along trajectories of the Toda lattice is given by the number of points of a Chevalley group $K(\mathbb{F}_q)$ related to the maximal compact subgroup K of the group $\check{G}$ with $\check{\mathfrak{g}} = \mathrm{Lie}(\check{G})$ over the finite field $\mathbb{F}_q$. Here $\check{\mathfrak{g}}$ is the Langlands dual of $\mathfrak{g}$. The blow-ups of the Toda lattice are given by the zero set of the τ-functions. For example, the blow-ups of the Toda lattice of A-type are determined by the zeros of the Schur polynomials associated with rectangular Young diagrams. Those Schur polynomials are the τ-functions for the nilpotent Toda lattices. Then we conjecture that the number of blow-ups is also given by the number of real roots of those Schur polynomials for a specific variable. We also discuss the case of periodic Toda lattice in connection with the real cohomology of the flag manifold associated to an affine Kac–Moody algebra.

Keywords: Toda lattice; Painlevé divisor; Real flag manifold; Cohomology of compact group; Finite Chevalley groups; Schur polynomials

1. Introduction: Toda lattices and the blow-ups

Let us first give some notations and definitions of the real split semisimple Lie algebra $\mathfrak{g}$ of rank l: We fix a split Cartan subalgebra $\mathfrak{h}$ with root system $\Delta = \Delta(\mathfrak{g}, \mathfrak{h}) = \Delta^+ \cup \Delta^-$, real root vectors e_{α_i} associated with simple roots $\Pi = \{\alpha_i : i = 1, \dots, l\}$. We also denote $\{h_{\alpha_i}, e_{\pm\alpha_i}\}$ the Cartan–Chevalley basis of the algebra $\mathfrak{g}$ which satisfies the relations,

$$[h_{\alpha_i}, h_{\alpha_j}] = 0, \quad [h_{\alpha_i}, e_{\pm\alpha_j}] = \pm C_{j,i} e_{\pm\alpha_j}, \quad [e_{\alpha_i}, e_{-\alpha_j}] = \delta_{i,j} h_{\alpha_j},$$

where $(C_{i,j})$ is the $l \times l$ Cartan matrix of the Lie algebra $\mathfrak{g}$ and $C_{i,j} = \alpha_i(h_{\alpha_j})$ (as used in [11]). For example, the Cartan matrices for B_2, C_2 and G_2 are given by,

$$B_2 : \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}, \quad C_2 : \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}, \quad G_2 : \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}.$$

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The Lie algebra $\mathfrak{g}$ admits the decomposition,

$$\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+ = \mathfrak{n}^- \oplus \mathfrak{b}^+ = \mathfrak{b}^- \oplus \mathfrak{n}^+,$$

where $\mathfrak{n}^\pm$ are nilpotent subalgebras defined by $\mathfrak{n}^\pm = \bigoplus_{\alpha\in\Delta^\pm} \mathbb{R}e_\alpha$ with root vectors e_α, and $\mathfrak{b}^\pm = \mathfrak{n}^\pm \oplus \mathfrak{h}$ are Borel subalgebras of $\mathfrak{g}$. We denote by $\check{\mathfrak{g}}$ the real split Lie algebra with Cartan matrix given by the transpose of the Cartan matrix of $\mathfrak{g}$ ($\check{\mathfrak{g}}$ is called the Langlands dual of $\mathfrak{g}$). Note that $\mathfrak{g} = \check{\mathfrak{g}}$ if $\mathfrak{g}$ is simple and not of type B or C.

We also fix a split Cartan subgroup H with $\mathrm{Lie}(H) = \mathfrak{h}$ and a Borel subgroup B with $\mathrm{Lie}(B) = \mathfrak{b}^+$ with $B = HN$ where N is a Lie group having the Lie algebra $\mathfrak{n}^+$. We also denote Lie groups B^- and N^- with $\mathrm{Lie}(B^-) = \mathfrak{b}^-$ and $\mathrm{Lie}(N^-) = \mathfrak{n}^-$. Integral weights on $\mathfrak{h}$ can be exponentiated to H. For example if α is a root then there is a corresponding character χ_α defined on H.

Most of the results presented in this paper can be found in our recent paper [7], and the main purpose of this paper is to give a brief summary of those, putting emphasis on the singular structure of the Toda lattice. In addition, we will also discuss an extension to the case of periodic Toda lattice whose underlying algebra is given by an affine Kac–Moody algebra.

1.1. Toda lattices: non-periodic case

The non-periodic Toda lattice equation related to real split semisimple Lie algebra $\mathfrak{g}$ of rank l is defined by the Lax equation [3,13],

$$\frac{\mathrm{d}L}{\mathrm{d}t} = [L, A], \tag{1.1}$$

where L is a Jacobi element of $\mathfrak{g}$ and A is the $\mathfrak{n}^-$-projection of L, denoted by $\Pi_{\mathfrak{n}^-}L$,

$$\begin{cases} L(t) = \displaystyle\sum_{i=1}^{l} b_i(t)h_{\alpha_i} + \sum_{i=1}^{l}(a_i(t)e_{-\alpha_i} + e_{\alpha_i}), \\ A(t) = \Pi_{\mathfrak{n}^-}L = \displaystyle\sum_{i=1}^{l} a_i(t)e_{-\alpha_i}. \end{cases} \tag{1.2}$$

The Lax equation (1.1) then gives

$$\begin{cases} \dfrac{\mathrm{d}b_i}{\mathrm{d}t} = a_i, \\ \dfrac{\mathrm{d}a_i}{\mathrm{d}t} = -\left(\displaystyle\sum_{j=1}^{l} C_{i,j}b_j\right)a_i. \end{cases} \tag{1.3}$$

The integrability of the system can be shown by the existence of the Chevalley invariants, $\{I_k(L) : k = 1, \ldots, l\}$, which are given by the homogeneous polynomial of $\{(a_i, b_i) : i = 1, \ldots, l\}$. (Recall that those correspond to the basic invariants in $\mathbb{C}[\mathfrak{h}]$ of the Weyl group W, i.e. $\mathbb{C}[\mathfrak{g}]^G \simeq \mathbb{C}[\mathfrak{h}]^W$ with Ad-action of G.) The invariant polynomials also define the commutative equations of the Toda equation (1.1),

$$\frac{\partial L}{\partial t_k} = [L, \Pi_{\mathfrak{n}^-}\nabla I_k(L)] \quad \text{for } k = 1, \ldots, l, \tag{1.4}$$

where ∇ is the gradient with respect to the Killing form, i.e. for any $x \in \mathfrak{g}$, $\mathrm{d}I_k(L)(x) = K(\nabla I_k(L), x)$. Here $\{t_k : k=1, \ldots, l\}$ represent the flow parameters, and we will also denote t_k by t_{m_k} with the exponent m_k of the basic invariant I_k (recall $m_k = d_k - 1$ where d_k is the degree of I_k, see e.g. [4]). For example, in the case of $\mathfrak{g} = \mathfrak{sl}(l + 1; \mathbb{R})$, the invariants $I_k(L)$ and the gradients $\nabla I_k(L)$ are given by

$$I_k(L) = \frac{1}{k+1}\mathrm{Tr}(L^{k+1}) \quad \text{and} \quad \nabla I_k(L) = L^k.$$

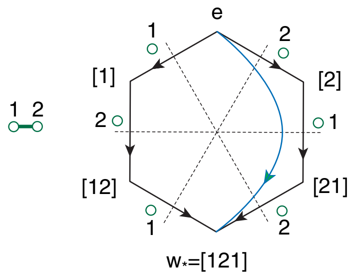

Fig. 1. Isospectral polytope Γ_ε of A_2-Toda lattice. The Dynkin diagram of A_2 is shown on the left, and each edge corresponds to A_1-Toda lattice whose Dynkin diagram is just one circle. For example, the edge from e to $[1]=s_1$ corresponds to the system with $a_2=0$. The element $e\in S_3$ corresponds to the Lax matrix L whose diagonal part is given by $(\lambda_3,\lambda_2,\lambda_1)$. Then the element [1] describes the Lax matrix with $\operatorname{diag}(\lambda_2,\lambda_3,\lambda_1)$, i.e. we have the s_1-action on the diagonal of L. Each element w then corresponds to the Lax matrix with $w^{-1}\cdot(3,2,1)=(w^{-1}(3),w^{-1}(2),w^{-1}(1))$.

In this case, the degree of I_k is $k+1$ and the exponent is $m_k=k$. The set of commutative equations (1.4) is called the Toda lattice hierarchy. Note that Eq. (1.3) is the first member of the hierarchy, i.e. $t=t_1$. Then the *real* isospectral manifold is defined by

$$Z(\gamma)_{\mathbb{R}}=\{(a_1,\ldots,a_l,b_1\ldots,b_l)\in\mathbb{R}^{2l}:I_k(L)=\gamma_k\in\mathbb{R},\ k=1,\ldots,l\}.$$

The manifold $Z(\gamma)_{\mathbb{R}}$ can be compactified by adding the set of points corresponding to the singularities (*blow-ups*) of the solution. Then the compact manifold $\tilde{Z}(\gamma)_{\mathbb{R}}$ is described by a union of convex polytopes Γ_ε with $\varepsilon=(\varepsilon_1,\ldots,\varepsilon_l)$, $\varepsilon_i=\operatorname{sgn}(a_i)$ [5],

$$\tilde{Z}(\gamma)_{\mathbb{R}}=\bigcup_{\varepsilon\in\{\pm\}^l}\Gamma_\varepsilon.$$

Each polytope Γ_ε is expressed as the closure of the orbit of a Cartan subgroup. Thus in an Ad-diagonalizable case with distinct eigenvalues, the compact manifold $\tilde{Z}(\gamma)_{\mathbb{R}}$ is a toric variety, and the vertices of each polytope are labeled by the elements of the Weyl group.

Example 1.1. For the case $\mathfrak{g}=\mathfrak{sl}(l+1;\mathbb{R})$, i.e. A_l-type, we have

$$L=\begin{pmatrix} b_1 & 1 & 0 & \cdots & 0\\ a_1 & b_2-b_1 & 1 & \cdots & 0\\ 0 & a_2 & b_3-b_2 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & -b_l\end{pmatrix}.$$

Notice that a fixed point $(a_1=\cdots=a_l=0)$ is a triangular matrix with the eigenvalues on the diagonal, and the total number of fixed points is given by $(l+1)!=|S_{l+1}|$ (assuming all eigenvalues are distinct). Each fixed point is then labeled by a unique element of the Weyl group S_{l+1}, and it is identified as a vertex of the isospectral polytope Γ_ε. Then the isospectral polytope is just a permutohedron associated with S_{l+1}. For example, A_2-Toda lattice, we have a hexagon as the isospectral polytope whose vertices are the fixed points with the Lax matrices given by

$$L_w:=\begin{pmatrix}\lambda_{w^{-1}(3)} & 1 & 0\\ 0 & \lambda_{w^{-1}(2)} & 1\\ 0 & 0 & \lambda_{w^{-1}(1)}\end{pmatrix}\quad\text{with } \lambda_1>\lambda_2>\lambda_3,$$

where $w^{-1}(i)$ represents a permutation of $(3,2,1)$ associated to $w\in S_3$ (see Fig. 1). In Fig. 2, we show the polytope Γ_ε for A_3-Toda lattice. The vertices of the polytope are marked by the elements of the Weyl group S_4. The faces and

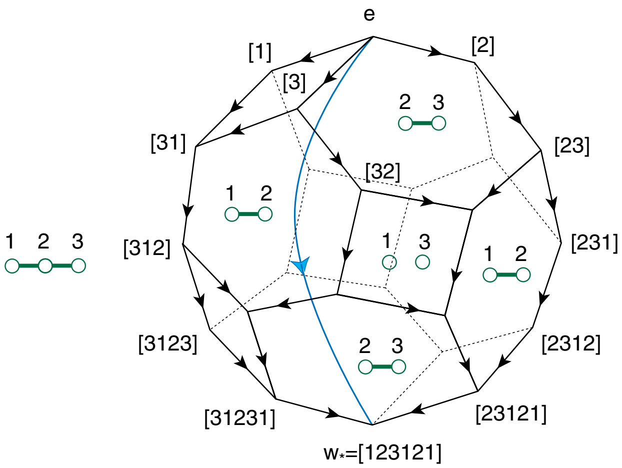

Fig. 2. Isospectral polytope Γ_ε for A_3-Toda lattice. The boundary of the polytope consists of the subsystems of the Toda lattice, in which eight hexagons corresponds to A_2-Toda lattices and six squares corresponds to $A_1 \times A_1$-Toda lattices. Those subsystems are marked by the corresponding sub-Dynkin diagrams.

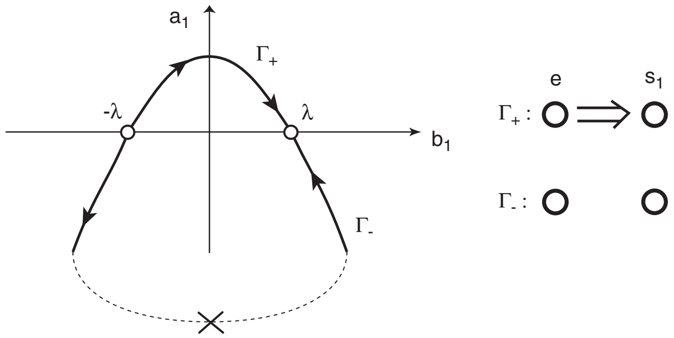

Fig. 3. The isospectral manifold for the A_1-Toda lattice. The left figure shows the invariant curve $I_1 = a_1 + b_1^2 = \lambda^2$, and the mark $\times$ indicates the blow-up point, i.e. the divisor $\mathscr{D}_{\{1\}}$. The right figure shows the graphs of $\Gamma_\pm$-polytopes with the Weyl action on the signs. This shows that Γ_+ is connected, and Γ_- has two connected components with $\mathscr{D}_{\{1\}}$.

edges on the boundary of the polytope correspond to the subsystems given by $a_k = 0$ for some k. In particular, each one dimensional edge of the isospectral polytope corresponds to a A_1-Toda lattice, which is given by the Lax pair,

$$L = \begin{pmatrix} \tilde{b}_k & 1 \\ a_k & \tilde{b}_{k+1} \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 0 \\ a_k & 0 \end{pmatrix}.$$

If $a_k(0) > 0$, the flow is complete, and $a_k(t_1) \to 0$ as $t_1 \to \pm\infty$. The isospectral polytope is a connected line segment, denoted by Γ_+, with the end-points corresponding to the matrices with $\lambda > \mu$,

$$L(t_1 = -\infty) = \begin{pmatrix} \mu & 1 \\ 0 & \lambda \end{pmatrix}, \quad L(t_1 = \infty) = \begin{pmatrix} \lambda & 1 \\ 0 & \mu \end{pmatrix}.$$

If $a_k(0) < 0$, the flow has a singularity (blow-up) in finite time. The isospectral polytope consists of two line segments, denoted by Γ_- (see Fig. 3).

1.2. The τ-functions and Painlevé divisors

The analytical structure of the blow-ups can be obtained by the τ-functions, which are defined by

$$b_k = \frac{\mathrm{d}}{\mathrm{d}t_1} \ln \tau_k, \quad a_k = a_k^0 \prod_{j=1}^{l} (\tau_j)^{-C_{k,j}}, \tag{1.5}$$

where a_k^0 are some constants. The tau-functions are given by [9],

$$\tau_j(t_1, \ldots, t_l) = \langle g(t_1, \ldots, t_l) \cdot v^{\omega_j}, v^{\omega_j} \rangle, \quad g = \exp\left(\sum_{k=1}^{l} t_k \nabla I_k(L^0)\right). \tag{1.6}$$

Here v^{ω_j} is the highest weight vector in the fundamental representation of G, and $\langle \cdot, \cdot \rangle$ is a pairing on the representation space, and L^0 is an initial data of $L(t_1, \ldots, t_l)$. The blow-up points (i.e. the singular points of (a_j, b_j)) are given by the zeros of the τ-functions, $\tau_j(t_1, \ldots, t_l) = 0$ for some $j \in \{1, \ldots, l\}$. We then define the Painlevé divisor $\mathscr{D}_J$ for a subset $J \subset \{1, \ldots, l\}$ as [5]

$$\mathscr{D}_J := \bigcap_{j \in J} \{\tau_j(t_1, \ldots, t_l) = 0\}, \quad \mathrm{codim}_{\mathbb{R}} \mathscr{D}_J = |J|.$$

The $\mathscr{D}_J$ can be also described by the intersection with the Bruhat cell $N^- w_J B/B$ with the longest element w_J of the Weyl subgroup $W_J = \langle s_j : j \in J \rangle$ [9, Theorem 3.3]. In particular, the divisor $\mathscr{D}_{\{1,\ldots,l\}}$ is a unique point, denoted as p_0, in the variety $\tilde{Z}(\gamma)_{\mathbb{R}}$, and it is contained in the Γ_ε-polytope with $\varepsilon = (- \cdots -)$. Then the geometry of the divisor $\mathscr{D}_0 = \bigcup_{j=1}^{l} \{\tau_j = 0\}$, the union of the Painlevé divisors $\mathscr{D}_{\{j\}}$, near the point p_0 can be expressed as the product of τ-functions,

$$F(t_1, \ldots, t_l) := \prod_{j=1}^{l} \tau_j(t_1, \ldots, t_l) = F_d(t_1, \ldots, t_l) + F_{d+1}(t_1, \ldots, t_l) + \cdots, \tag{1.7}$$

where each $F_k(t_1, \ldots, t_l)$ is a homogeneous polynomial of degree k. The algebraic variety $V := \{F_d = 0\}$ defines the tangent cone at the point p_0, and the degree d is the *multiplicity* of the singularity of V at p_0. The number d has several surprising connections with other numbers, such as the number of $\mathbb{F}_q$ points on the maximal compact subgroup of the underlying group of the Toda lattice and the number of real roots of certain symmetric functions (e.g. Schur polynomials). Here $\mathbb{F}_q$ is a finite field with q elements, with q a power of a prime. One of the main purpose of this paper is to explain those connections (the details can be found in our recent paper [7]).

1.3. Action of the Weyl group on the signs of the Toda lattice

Here we give an algebraic description of the blow-ups, so that one can compute the number of blow-ups in the Toda flow. The following action of the Weyl group W describes how the signs of the functions a_j for $j = 1, \ldots, l$ change when a_i blows up.

Definition 1.2 (*[5, Proposition 3.16]*). For any set of signs $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_l) \in \{\pm\}^l$, a simple reflection $s_i := s_{\alpha_i} \in W$ acts on the sign ε_j by

$$s_i : \varepsilon_j \longmapsto \varepsilon_j \varepsilon_i^{-C_{j,i}}.$$

The sign change is defined on the group character χ_{α_i} with $\varepsilon_i = \mathrm{sign}(\chi_{\alpha_i})$ (recall $s_i \cdot \alpha_j = \alpha_j - C_{j,i}\alpha_i$). We also identify the sign ε_i as that of a_i, since the functions a_k are given by (1.5) and the τ-functions are given by the fundamental weights ω_j in (1.6), which relate to the equation $\chi_{\alpha_j} = \prod_{k=1}^{l} (\chi_{\omega_k})^{C_{j,k}}$ (recall $\alpha_j = \sum_{k=1}^{l} C_{j,k}\omega_k$).

We now define the relation $\Rightarrow$ between the vertices of the polytope Γ_ε as follows: if $\varepsilon_i = +$ then we write: $\varepsilon \stackrel{s_i}{\Rightarrow} \varepsilon'$ where $\varepsilon' = s_i \varepsilon$. We also write $w \Rightarrow w s_i$ (see Example 1.4).

Under the action of the Weyl group, not every simple reflection s_i changes the sign ε. The following is an alternative way to measure the size of w which only takes into account simple reflections that change the sign ε, that is, a trajectory of a Toda lattice having a blow-up point. These numbers will later reappear in the context of the computation of certain Frobenius eigenvalues.

Now the following definition gives the number of blow-ups in the Toda orbit from the top vertex e to the vertex labeled by $w \in W$:

Definition 1.3. Choose a reduced expression $w = s_{j_1} \cdots s_{j_r}$. Consider the sequence of signs as the orbit given by w-action:

$$\varepsilon \to s_{j_1}\varepsilon \to s_{j_2}s_{j_1}\varepsilon \to \cdots \to w^{-1}\varepsilon.$$

We then define the function $\eta(w, \varepsilon)$ as the number of $\to$ which are not of the form $\overset{s_i}{\Rightarrow}$ as in Definition 1.2. The number $\eta(w_*, \varepsilon)$ for the longest element w_* gives the total number of blow-ups along the Toda flow in Γ_ε-polytope. Whenever $\varepsilon = (- \cdots -)$ we will just denote $\eta(w, \varepsilon) = \eta(w)$.

Note that each reduced expression of w corresponds to a path following Toda lattice trajectories along one-dimensional subsystems leading to w. Each one-dimensional subsystem is equivalent to A_1-Toda lattice. In [7, Corollary 5.2] it is shown that $\eta(w, \varepsilon)$ is independent of the reduced expression. Hence the number of blow-up points along trajectories in one-dimensional subsystems in the boundary of the Γ_ε-polytope is independent of the trajectory (parametrized by a reduced expression).

Example 1.4. We consider the $\mathfrak{sl}(2;\mathbb{R})$-Toda lattice which is the simplest case, but provides the basic structure of the general case. The Lax pair (L, A) is given by

$$L = \begin{pmatrix} b_1 & 1 \\ a_1 & -b_1 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 0 \\ a_1 & 0 \end{pmatrix}.$$

The Chevalley invariant is given by $I_1 = \frac{1}{2}\operatorname{Tr}(L^2) = a_1 + b_1^2$. Then the isospectral manifold $Z(\gamma)_\mathbb{R}$ for a real split case is given by the curve $I(a_1, b_1) = \gamma_1 > 0$ (see Fig. 3 where $\gamma_1 = \lambda^2$). The compactified manifold $\tilde{Z}(\gamma)_\mathbb{R}$ consists of two polytopes (line segments) Γ_+ and Γ_-,

$$\tilde{Z}(\gamma)_\mathbb{R} = \Gamma_+ \cup \Gamma_- \cong S^1.$$

Here the Γ_- is compactified by adding the blow-up point marked by $\times$ in Fig. 3 i.e. $\tau_1 = 0$. The end-points (vertices) of each segment are marked by the Weyl elements, e and s_1. Fig. 3 also shows the graphs associated with the polytopes $\Gamma_\pm$ where the connection with the arrow in Γ_+ indicate no blow-up in the flow between the vertices (see Definition 3.1 for more details).

The solution (a_1, b_1) can be expressed by the τ-function:

(a) For the case $a_1 > 0$ (i.e. Γ_+-polytope), Eq. (1.6) gives

$$\tau_1(t_1) = \cosh(\lambda t_1),$$

which leads to the solution

$$a_1(t_1) = \lambda^2 \operatorname{sech}^2(\lambda t_1), \quad b_1(t_1) = \lambda \tanh(\lambda t_1).$$

Since there is no blow-up in this case, we have $\eta(e, +) = \eta(s_1, +) = 0$. This implies that there is an edge between the vertices in Γ_+ (see Fig. 3).

(b) For the case $a_1 < 0$ (i.e. Γ_--polytope), we have

$$\tau_1(t_1) = \frac{1}{\lambda}\sinh(\lambda t_1),$$

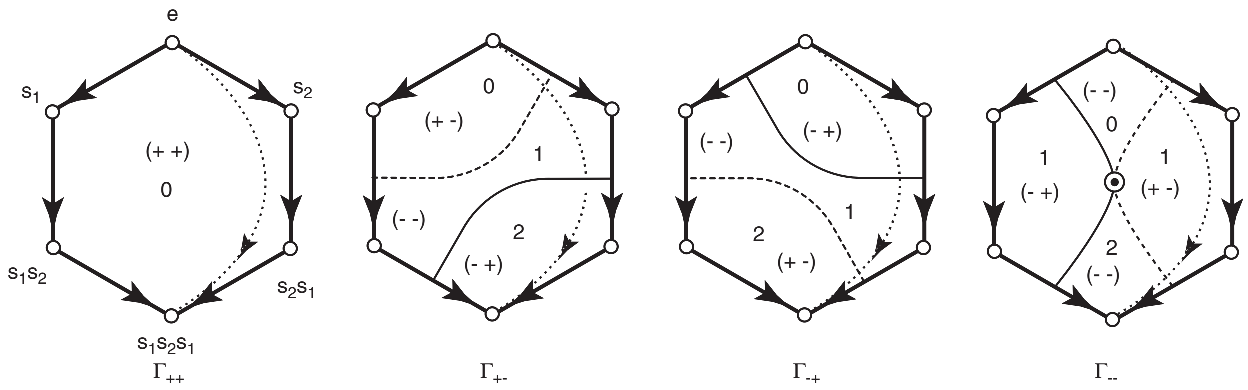

Fig. 4. The four hexagons associated to the signs $\varepsilon = (\varepsilon_1, \varepsilon_2)$. The Painlevé divisors $\mathscr{D}_{\{1\}}$ and $\mathscr{D}_{\{2\}}$ are indicated by the solid curves for the blow-ups of a_1 (i.e. $\tau_1 = 0$) and the dashed curves for the blow-up of a_2 (i.e. $\tau_2 = 0$). The double circle in Γ_{--} indicate the point p_0 corresponding to $\bigcap_{j=1}^{2} \mathscr{D}_{\{j\}}$. The boundaries of the hexagons describe the subsystems given by $a_i = 0$ for $i = 1, 2$. The compactification can be done uniquely by gluing the boundaries according to the sign changes. Each hexagon is divided by the Painlevé divisors into connected components. A Toda flow in t_1-variable is shown as the dotted curve starting from the vertex marked by the identity element e, and ending to the vertex by the longest element $w_* = s_1 s_2 s_1$.

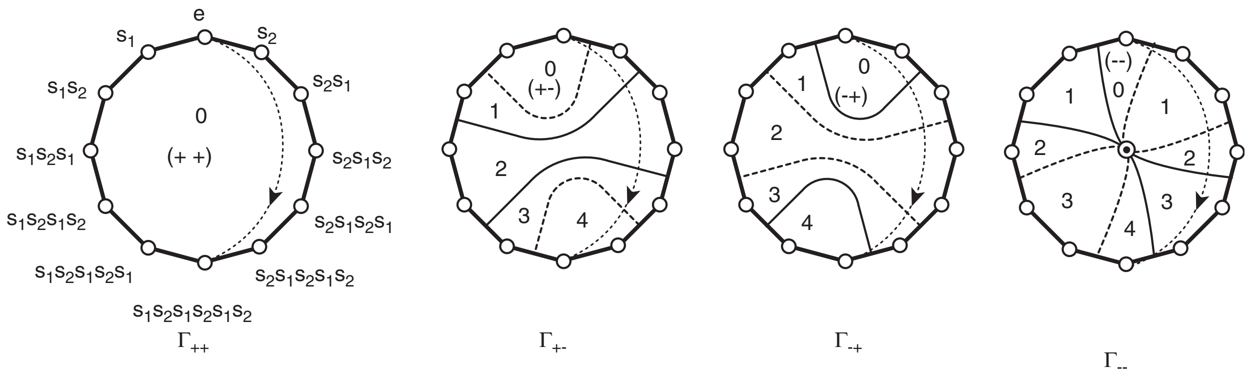

Fig. 5. The 12-gons corresponding to signed Toda lattice for G_2. The Painlevé divisors $\mathscr{D}_{\{1\}}$ and $\mathscr{D}_{\{2\}}$ corresponding to the blow ups of a_1 and a_2 are indicated by the solid and dashed curves inside the 12-gons. A Toda flow in t_1-variable is shown as the dotted curve starting from the vertex marked by the identity element e, and ending to the vertex by the longest element $w_* = s_1 s_2 s_1 s_2 s_1 s_2$.

which gives

$$a_1(t_1) = -\lambda^2 \operatorname{csh}^2(\lambda t_1), \quad b_1(t_1) = \lambda \coth(\lambda t_1).$$

Thus the solution $(a_1(t_1), b_1(t_1))$ blows up at $t_1 = 0$. We have $\eta(e) = 0$ and $\eta(s_1) = 1$, which gives no connecting arrow between the vertices in Γ_- as in Fig. 3.

Note that in both cases the solution approaches the fixed points $(a_1 = 0, b_1 = \pm\lambda)$ as $t \to \pm\infty$, which are the vertices of the polytope.

Example 1.5. The cases of A_2 and G_2 are illustrated in Figs. 4 and 5. In these figures the four hexagons and 12-gons are shown as the ε-polytope Γ_ε with the signs $\varepsilon = (\varepsilon_1 \varepsilon_2)$. These polytopes glue together to form a compact isospectral manifold $\tilde{Z}(\gamma)_{\mathbb{R}}$. Trajectories of the Toda lattice starts in the vertex associated to e and move towards the vertex corresponding to the longest element in the Weyl group.

In the case of A_2, the W-action on the signs $\varepsilon = (\varepsilon_1 \varepsilon_2)$ gives $s_1(--) = (-+)$, $s_2(-+) = (-+)$ and $s_1(-+) = (--)$. From those we obtain $\eta(e) = 0$, $\eta(s_1) = \eta(s_1 s_2) = 1$, $\eta(s_2) = \eta(s_2 s_1) = 1$ and $\eta(s_1 s_2 s_1) = 2$. Those give the numbers of blow-ups in the Toda flow (see Fig. 4).

In the case of G_2, we obtain $\eta(e)=0$, $\eta(s_1)=\eta(s_2)=\eta(s_1s_2)=\eta(s_2s_1)=1$, $\eta(s_1s_2s_1)=\eta(s_2s_1s_2)=2$, $\eta(s_1s_2s_1s_2)=\eta(s_2s_1s_2s_1)=\eta(s_1s_2s_1s_2s_1)=\eta(s_2s_1s_2s_1s_2)=3$ and $\eta(w_*)=4$. The total number of blow-ups is then 4 (see Fig. 5).

1.4. Toda lattice: periodic case

Here we give a brief background of periodic Toda lattice for affine A_l Toda lattice (the details can be found in [12]). The periodic Toda lattice is also give by the Lax equation (1.1) with

$$L_P=\begin{pmatrix} b_1 & 1 & 0 & \cdots & 0 & a_0z^{-1} \\ a_1 & b_2-b_1 & 1 & \cdots & 0 & 0 \\ 0 & a_2 & b_3-b_2 & \cdots & \cdot & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdot & \cdot & \cdots & b_l-b_{l-1} & 1 \\ z & 0 & \cdot & \cdots & a_l & -b_l \end{pmatrix}.$$

The characteristic equation for L_P defines the algebraic curve,

$$\det(L_P-\lambda I)=-\left(z+\frac{\prod_{i=0}^{l}a_i}{z}-P(\lambda)\right)=0, \tag{1.8}$$

which is a one-dimensional affine variety on $(\lambda, z)\in\mathbb{C}^2$. Here $P(\lambda)$ is an $(l+1)$th polynomial of λ given by

$$P(\lambda):=\Delta_l(\lambda)-a_0\Delta_{l-1}(\lambda),$$

where $\Delta_l(\lambda)=\det(L-\lambda I)$ for the Lax matrix L of the non-periodic Toda lattice, and $\Delta_{l-1}(\lambda)$ is the determinant $\Delta_l(\lambda)$ after removing the first row and the last column. Then the polynomial $P(\lambda)$ gives the set of integrals $\{I_k(L_P): k=1,\dots,l\}$ of the Toda flow, i.e.

$$P(\lambda)=-(-1)^l\left(\lambda^{l+1}+\sum_{k=1}^{l}(-1)^kI_k(L_P)\lambda^{l-k}\right).$$

There exists an additional integral obtained from the residue with respect to the spectral parameter z, i.e. $I_0=\prod_{i=0}^{l}a_i$. One should note that the case with $a_0=0$, i.e. $I_0=0$ corresponds to the non-periodic Toda lattice. Then the isospectral set is defined by

$$Z_{\mathbb{R}}^P(\gamma):=\{(a_0,a_1,\dots,a_l,b_1,\dots,b_l)\in Z_{\mathbb{R}}^P: I_k(L_P)=\gamma_k\in\mathbb{R},\ \ k=0,1,\dots,l\}$$

which is the affine part of the compactified manifold $\hat{Z}_{\mathbb{R}}^P(\gamma)$ of $\dim Z_{\mathbb{R}}^P(\gamma)=l$, and with a divisor Θ associated to the blow-ups of a_k, $k=0,1,\dots,l$, we have [1],

$$Z_{\mathbb{R}}^P(\gamma)=\hat{Z}_{\mathbb{R}}^P(\gamma)\backslash\mathscr{D}_{\{0,1,\dots,l\}}\quad \text{with } \mathscr{D}_{\{0,1,\dots,l\}}=\bigcup_{k=0}^{l}\{a_k^{-1}=0\}.$$

It turns out that the flow of Toda lattice can be described as a trajectory on the Riemann surface, $y^2=P(\lambda)^2-4I_0$, and through the Abel–Jacobi map, the compactified isospectral manifold $\hat{Z}_{\mathbb{R}}^P(\gamma)$ can be identified as the real part of the Jacobian $\mathbb{C}^l/\Lambda$ with the lattice Λ defined by the period matrix Ω associated with the Riemann surface of genus l. The divisors $\mathscr{D}_{\{j\}}=\{a_j^{-1}=0\}$ are given by the theta divisor and its translates, that is, the zeros of the Riemann theta function associated with the hyperelliptic Riemann surface, $y^2=P(\lambda)^2-4I_0$. Here we take appropriate values of the integrals $I_k=\gamma_k$ (e.g. $I_0=\prod_k a_k>0$) so that all the roots of the hyperelliptic curve are real and distinct , i.e. $y^2=\prod_{k=1}^{2l+2}(\lambda-\lambda_k)$ with $\lambda_k\in\mathbb{R}$. Then the number of connected components in the compact manifold $\hat{Z}_{\mathbb{R}}^P(\gamma)$ is given by 2^l, that is, the real part of the Jacobian consists of 2^l number of l-dimensional tori (see [12]).

In the case of periodic Toda lattice, the signs of a_k are determined by the action of the *affine* Weyl group associated with the affine Kac–Moody algebra (see [12]). For example, for the case of $A_l^{(1)}$ Toda lattice with $l \geqslant 2$, we have

$$\hat{W} = \left\langle s_0, s_1, \ldots, s_l : \begin{array}{l} s_k^2 = e, \ (s_k s_{k+1})^3 = e \bmod (l+1) \\ (s_k s_j)^2 = e, \quad 1 < |k-j| < l \end{array} \right\rangle$$

and for $A_1^{(1)}$, we have $\hat{W} = \langle s_0, s_1 : s_0^2 = s_1^2 = e \rangle$. The action of $\hat{W}$ is defined by the same way as in Definition 1.2 (see [12, Eq. (4.5) in p. 1710]), i.e. for each $s_i \in \hat{W}$ and $\varepsilon_j = \mathrm{sgn}(a_j)$,

$$s_i : \varepsilon_j \longmapsto \varepsilon_j \varepsilon_i^{-\hat{C}_{j,i}},$$

where $\hat{C}$ is the extended Cartan matrix.

Example 1.6. The affine $\mathfrak{sl}(2)$ Toda lattice: The Lax matrix is given by

$$L_P = \begin{pmatrix} b_1 & 1 + a_0 z^{-1} \\ z + a_1 & -b_1 \end{pmatrix}$$

whose spectral curve defines an elliptic curve,

$$\det(L_P - \lambda I) = -\left(z + \frac{a_0 a_1}{z} - P(\lambda)\right) = 0.$$

The polynomial $P(\lambda)$ is given by

$$P(\lambda) = \lambda^2 - I_1 \quad \text{with } I_1 = b_1^2 + a_1 + \frac{1}{a_1}, \tag{1.9}$$

where we have assumed $I_0 = a_0 a_1 = \gamma_0 = 1$. Then the affine part of the isospectral manifold is given by

$$Z_{\mathbb{R}}^P(\gamma) = \left\{ (a_1, b_1) \in \mathbb{R}^2 : b_1^2 + a_1 + \frac{1}{a_1} = \gamma_1 \in \mathbb{R} \right\} \cong \begin{cases} S^1 \sqcup \mathbb{R} \sqcup \mathbb{R} & \text{if } \gamma_1 > 2, \\ \mathbb{R} \sqcup \mathbb{R} & \text{if } \gamma_1 < 2. \end{cases}$$

Two disconnected pieces of $\mathbb{R}$ are compactified to make a circle S^1, that is, the corresponding solution blows up once in each point p_+ or p_-, which are the infinite points of the Riemann surface, and total twice in one cycle. Thus the compactified manifold $\hat{Z}_{\mathbb{R}}^P(\gamma)$ is just a disjoint union of two circles, and each circle can be marked by the signs of a_k. In Fig. 9, those circles are denoted by S_{++} and S_{--}. The flow on the circle S_{++} is complete, and the flow on S_{--} has blow-ups.

1.5. The group G and the group $\check{G}$

We give here some remarks on the Langlands dual $\check{G}$ of the real connected Lie group G with $\mathrm{Lie}(G) = \mathfrak{g}$ and finite Chevalley groups in connection with our present study: Let $G(\mathbb{C})$ be a connected semisimple Lie group with Lie algebra $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} + \sqrt{-1}\mathfrak{g}$. This can be regarded as an algebraic group defined over a finite extension of the field $\mathbb{Q}$ of rational numbers. It is then possible to consider the group of real points $G(\mathbb{R})$. We denote by G the real connected Lie subgroup of $G(\mathbb{C})$ with Lie algebra $\mathfrak{g}$. Hence $G \subset G(\mathbb{R}) \subset G(\mathbb{C})$. It is also possible to consider this algebraic group over fields of characteristic p, with p a prime.

We also use the notation $\check{G}$ to denote any group associated to $\check{\mathfrak{g}}$ in the same way as G is associated to $\mathfrak{g}$. We will refer to $\check{G}$ loosely by the term *Langlands dual*. In the case of simple Lie algebras not of type B_l or C_l with $l \geqslant 3$ we may just assume $\check{G} = G$ in all the statements. The Langlands dual will be needed to explain the connection between the blow-ups of the Toda lattice associated with $\mathfrak{g}$ and the cohomology of the real flag manifold for $\check{\mathfrak{g}}$.

Example 1.7. We consider $G(\mathbb{C}) = SL(n; \mathbb{C}) = \check{G}(\mathbb{C})$, i.e. the set of all the $n \times n$ complex matrices A satisfying the polynomial equation, $\det(A) = 1$. Then the complex solutions of $\det(A) = 1$ define $G(\mathbb{C})$. The group of real points is, of course, $G(\mathbb{R}) = SL(n;\mathbb{R})$; and, since this group is connected, $G(\mathbb{R}) = G$. There is also an involution θ given by $\theta(A) = A^*$, the inverse of the transpose of A. We then have that a maximal compact Lie subgroup of G is $K = SO(n)$ given as the set of matrices satisfying $\theta(A) = A$.

In the case of a Lie algebra of type A_1 we then have two possibilities for G namely $G(\mathbb{R}) = G = SL(2;\mathbb{R})$ or $G(\mathbb{R}) = G = \mathrm{Ad}SL(2;\mathbb{R})$ the adjoint group. This depends on whether we pick $G(\mathbb{C}) = SL(2;\mathbb{C})$ or $G(\mathbb{C}) = \mathrm{Ad}SL(2;\mathbb{C})$. If we now let $G(\mathbb{R}) = G = SL(2;\mathbb{R})$ then both $\check{G} = SL(2;\mathbb{R})$ or $\check{G} = \mathrm{Ad}SL(2;\mathbb{R})$ are possible.

The equation $\det(A) = 1$ has integral coefficients. The integral coefficients make it possible to reduce modulo a prime p. Let q be a power of a prime p and k_q an algebraic closure of the finite field with q elements denoted $\mathbb{F}_q$. We may then consider $G(k_q) = SL(n; k_q)$ i.e. the solutions in k_q of $\det(A) = 1$. By also reducing modulo p the involution $\theta(A) = A^*$ we obtain $SO(n; k_q)$ as the set of fixed points and then the finite group $SO(n;\mathbb{F}_q)$. In the simplest example of $n = 2$, $SO(n; k_q)$ consists of 2×2 matrices $\begin{pmatrix} x & y \\ -y & x \end{pmatrix}$ satisfying $x^2 + y^2 = 1$ with $x, y \in k_q$. The number of points of the finite group $SO(2;\mathbb{F}_q)$ is then the number of solutions of $x^2 + y^2 = 1$ over the field $\mathbb{F}_q$. The answer is given by the polynomial $q - 1$ if we assume that $\sqrt{-1} \in \mathbb{F}_q$.

2. The polynomial $p(q)$ as alternating sum of the blow-ups

In the non-periodic case, we introduce polynomials in terms of the numbers $\eta(w, \varepsilon)$ in Definition 1.3, which play a key role for counting the number of blow-ups. For each Γ_ε-polytope we then define $p_\varepsilon(q)$.

Definition 2.1. We define a polynomial, the alternating sum of the blow-ups,

$$p_\varepsilon(q) = (-1)^{l(w_*)} \sum_{w \in W} (-1)^{l(w)} q^{\eta(w,\varepsilon)}, \tag{2.1}$$

where w_* is the longest element of the Weyl group and $l(w)$ indicates the length of w. When $\varepsilon = (- \cdots -)$, we simply denote this polynomial by $p(q)$.

One can easily show that $p_\varepsilon(q) = 0$ unless $\varepsilon = (- \cdots -)$. Hence the only relevant polynomial here is $p(q)$, the polynomial for the $\Gamma_{-\dots-}$-polytope. This corresponds to the fact that the *rational* cohomology of K and $\mathscr{B} = K/T$ actually agree. Recall that we are dealing only with the case when the Lie algebra is split and the group T is then a finite group. The polynomial $p_\varepsilon(q)$ corresponds to a Lefschetz number for the Frobenius action over a field of positive characteristic for cohomology with local coefficients [7]. When $q = 1$ these polynomials all vanish, including $p(q)$. This reflects the fact that the Euler characteristic of K is zero.

As we will explain below that the numbers $\eta(w, \varepsilon)$ are deeply tied up with the cohomology of the flag manifold. The polynomial $p(q)$ contains all the information regarding the cohomology ring of the compact Lie group K. We discuss below our approach to obtaining this strange relations among the Toda lattice, the flag manifold and K. One of our main results is the computation of the polynomials $p(q)$ in terms of K. However, there is an important technical point here. The polynomial $p(q)$ from the Toda Lattice associated to a Lie algebra $\mathfrak{g}$ is given in terms of the maximal compact subgroup of $\check{G}$. More precisely, one must consider the group $\check{K}$ over $\mathbb{F}_q$ and $p(q)$ is given in terms of the polynomial $|\check{K}(\mathbb{F}_q)|$. If the Lie algebra is simple and not of type B_l or C_l with $l \geqslant 3$ then we can just write $\check{G} = G$ and $\check{K} = K$ in all the statements.

Example 2.2. From Figs. 4 and 5, we note that the numbers $\eta(w, \varepsilon)$ are constant on the connected components in a given Γ_ε-polytope. The polynomials $p(q)$ that are obtained from (2.1) by counting $\eta(w)$ are $p(q) = q^2 - 1$ for type A_2, and $p(q) = (q^2 - 1)^2$ for G_2. Note that the multiplicities d are the degrees of these polynomials. In the case of G_2 each divisor contributes 2 to the multiplicity. Note in Fig. 5 that the divisors shown in the polytopes Γ_ε with $\varepsilon = (+-)$ or $(-+)$ come together in the Γ_{--}-polytope giving a multiplicity $d = 4$ at the center p_0 of the Γ_{--}-polytope.

One can also directly compute the polynomial $p(q)$ from Figs. 6, 7 and 8 for the cases of A_3, B_3 and C_3. Namely we obtain $p(q) = (q^2 - 1)^2$ for A_3, $p(q) = (q - 1)(q^2 - 1)(q^3 - 1)$ for B_3, and $p(q) = (q^2 - 1)^3$ for C_3. The polynomial obtained from the B_3 case can be written as $q^{-3}|\check{K}(\mathbb{F}_q)|$ where $\check{K} = U(3)$ is the maximal compact subgroup of the Langlands dual $Sp(3;\mathbb{R})$, that is, C_3. This serves to illustrate the fact that the polynomial $p(q)$ is related to the Chevalley group that results from a maximal compact subgroup of the Langlands dual $\check{G}$.

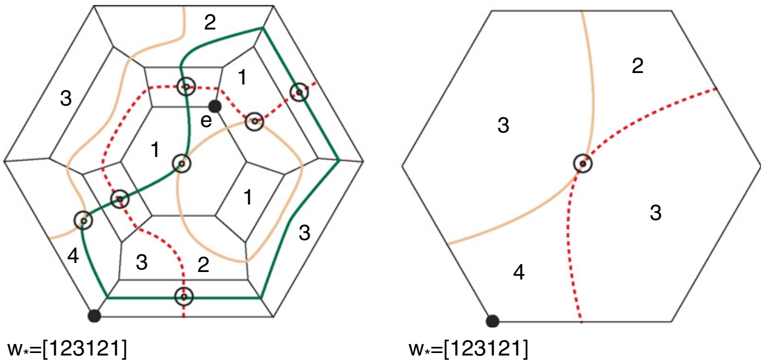

Fig. 6. The Γ_--polytope for type A_3 and the Painlevé divisors (the right figure is the back view of the left one). The Painlevé divisors are shown by the dotted curve for $\mathscr{D}_{\{1\}}$, by the light color one for $\mathscr{D}_{\{2\}}$, and by the dark one for $\mathscr{D}_{\{3\}}$. The double circles indicate the divisor $\mathscr{D}_{\{i,j\}} = \mathscr{D}_{\{i\}} \cap \mathscr{D}_{\{j\}}$, which are all connected at the center of the polytope p_0. The divisor $\mathscr{D}_{\{2\}}$ has the A_1-type singularity at p_0, i.e. a double cone with $t_2^2 - t_1 t_3 = 0$. The numbers indicate $\eta(w)$ which are obtained by using any path from e to w along edges of the polytope, following the direction of the Toda flow, i.e. the Bruhat order.

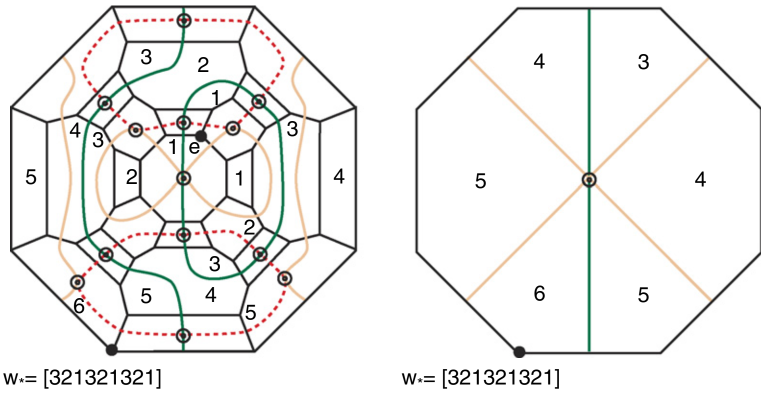

Fig. 7. The polytope Γ_- associated to the B_3-Toda lattice and the Painlevé divisors. The description of the Painlevé divisors are the same as in the case of A_3. The singularities of $\mathscr{D}_{\{1\}}$ and $\mathscr{D}_{\{3\}}$ are both of A_1-type, while the singularity of $\mathscr{D}_{\{2\}}$ is not isolated, a line singularity attached to two double cones of A_1-type (notice two eight-figures of $\mathscr{D}_{\{2\}}$). The numbers indicate $\eta(w)$. The subsystems on the boundary of Γ_- consist of the octagons for B_2, the hexagons for A_2 and the squares for $A_1 \times A_1$.

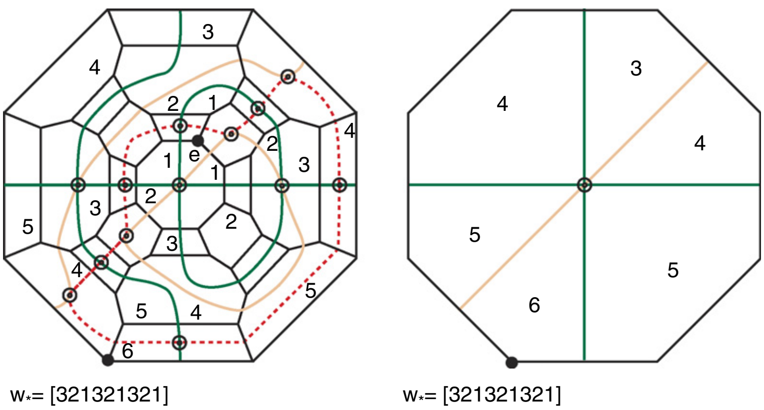

Fig. 8. The polytope Γ_- associated to the C_3-Toda lattice and the Painlevé divisors. The description of the divisors are again the same as in the case of A_3. Notice that both polytopes for B_3 and C_3 are the same, but the geometry of the Painlevé divisors are quite different. The singularity of $\mathscr{D}_{\{2\}}$ is of A_1-type, and that of $\mathscr{D}_{\{3\}}$ is a reducible one with $t_1(t_3^2 - t_1 t_5) = 0$.

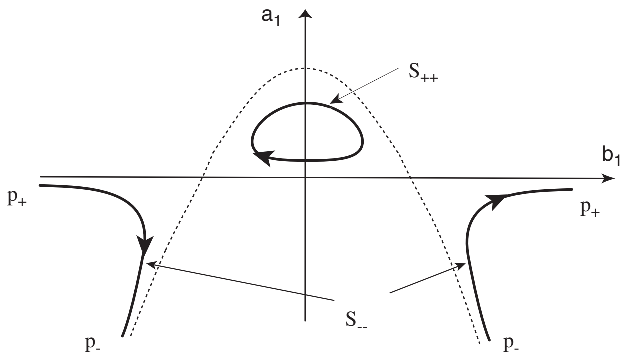

Fig. 9. The isospectral manifold for the affine A_1-Toda lattice. The solid curves show the invariant curve $I_1 = b_1^2 + a_1 + \frac{\gamma_0}{a_1} = \gamma_1$ for $\gamma_1 > 2$ and $\gamma_0 = a_0 a_1 = 1$, and the points $p_\pm$ indicate the blow-up points, i.e. the divisors $\mathscr{D}_{\{0\}}$ and $\mathscr{D}_{\{1\}}$. The compactified manifold $\hat{Z}^P_{\mathbb{R}}(\gamma)$ is given by a disjoint union of two circles, S_{++} and S_{--}, where the signs $(\varepsilon_0, \varepsilon_1)$ correspond to $\varepsilon_k = \mathrm{sgn}(a_k)$. The dashed curve indicates the non-periodic limit of the Toda lattice, i.e. $\gamma_0 \to 0$.

Note also that A_3 case gives the same polynomial $p(q) = (q^2 - 1)^2$ obtained in the G_2 case. The reason is that the maximal compact group K is in both cases essentially $SU(2) \times SU(2)$. This will be explained in Theorem 5.3 which gives the general formulae for the polynomials $p(q)$ for all the cases.

In the affine (periodic) cases we just consider $p_\varepsilon(q) = \sum_{w \in \hat{W}} (-1)^{l(w)} q^{\eta(w,\varepsilon)}$, which is now given by a power series of q. The numbers $\eta(w, \varepsilon)$ are defined similarly but the elements $w \in \hat{W}$ do not represents points in the compactified isospectral manifold (Fig. 9). In the concrete examples given here the universal cover ($\mathbb{R}^l$) is subdivided into regions by the divisors, which are further subdivided so that they are labeled by Weyl group elements (see Fig. 12). The numbers $\eta(w, \varepsilon)$ are assigned uniquely to the different regions by counting blow-up points along the Toda trajectories ignoring the direction of the flow. If a path of the Toda lattice goes from a region labeled w to a region labeled w', with $w \leqslant w'$ in the Bruhat order, and the path crosses k blow-up points, then $\eta(w', \varepsilon) = \eta(w, \varepsilon) + k$. Setting $\eta(w, \varepsilon) = 0$ determines all the other numbers uniquely. Note that there may not be a concrete path going from the region labeled e to the region labeled w but, still, the number $\eta(w, \varepsilon)$ is still being determined by counting blow-ups along trajectories of the Toda lattice.

3. A graph associated to the blow-ups of the Toda lattice

The following graph $\mathscr{G}_\varepsilon$ was originally motivated by the problem of computing the number of connected components in the Γ_ε-polytope. This problem is analogous to the problem of computing the intersection of two opposite top dimensional Bruhat cells in the case of a real flag manifold (e.g. see [16,15,17]). We then observed that in all examples this was the graph of incidence numbers for a real flag manifold (see [8]). In the affine cases we may consider $\hat{G} = G(\mathbb{R}[t, t^{-1}])$ instead of G and $\hat{B} = \{f \in G(\mathbb{R}[t, t^{-1}]) : f(0) \in B\}$ instead of B. Thus $\hat{\mathscr{B}} = \hat{G}/\hat{B}$.

Definition 3.1. For a fixed $\varepsilon = (\varepsilon_1 \ldots \varepsilon_l)$, we associate a graph $\mathscr{G}_\varepsilon$ to the blow-ups of the Toda lattice. The graph consists of vertices labeled by the elements of the Weyl group W, i.e. the vertices of the Γ_ε-polytope, and oriented edges $\Rightarrow$. The edges are defined as follows: For any $w_1, w_2 \in W$, there exists an edge between w_1 and w_2,

$$w_1 \Rightarrow w_2 \quad \text{iff} \quad \begin{cases} \text{(a) } w_1 \leqslant w_2 \text{ (Bruhat order)}, \\ \text{(b) } l(w_2) = l(w_1) + 1, \\ \text{(c) } \eta(w_1, \varepsilon) = \eta(w_2, \varepsilon), \\ \text{(d) } w_1^{-1}\varepsilon = w_2^{-1}\varepsilon. \end{cases}$$

When $\varepsilon = (- \cdots -)$, we simply denote $\mathscr{G} = \mathscr{G}_\varepsilon$. This graph also makes sense in the periodic case.

Note that in non-periodic cases $w_1 \Rightarrow w_2$ implies that in the polytope there is a path from w_1-vertex to w_2-vertex without crossing a Painlevé divisor (no blow-up). Namely, in this case, $w_1 \Rightarrow w_2$ means that the vertices associated to w_1 and w_2 belong to the *same* connected component of the hexagon (when blow-ups are removed).

In many cases the graph $\mathscr{G}_\varepsilon$ accomplishes the job of joining together in its connected components exactly those vertices w of the polytope Γ_ε belonging to the same connected components. This will be the case in the example considered below.

Example 3.2. In the case of A_2, we have $s_1(--) = (-+)$, $s_2(-+) = (-+)$ which implies $(--) \to (-+) \overset{s_2}{\Rightarrow} (-+)$ and $\eta(s_1) = 1$, $\eta(s_1 s_2) = 1$. Therefore, the graph $\mathscr{G}$ which encodes blow-up information in Γ_{--} of Fig. 2 is (s_i is replaced with i):

$$\begin{array}{ccccc} & e & & : & q^0 \\ 1 & & 2 & : & q^1 \\ \Downarrow & & \Downarrow & & \\ 12 & & 21 & : & q^1 \\ & 121 & & : & q^2. \end{array}$$

Here we have also listed the monomials $q^{\eta(w)}$ (in the variable q) associated to representatives of the integral cohomology ($w \to \eta(w) \to q^{\eta(w)}$). As already noted, the vertices of the hexagon Γ_{--} belonging to a connected component in Fig. 4 form a connected component of this graph (see also Fig. 3). This graph classifying connected components in the hexagon minus the blow-ups, agrees with the graph in [8, p. 465] which is defined very differently in terms of incidence numbers. The graph of incidence numbers gives rise to a chain complex by replacing the edges $\Rightarrow$ with multiplication by 2,

$$\mathbb{Q}\langle e\rangle \xrightarrow{\delta_o} \mathbb{Q}\langle s_1\rangle \oplus \mathbb{Q}\langle s_2\rangle \xrightarrow{\delta_1} \mathbb{Q}\langle s_1 s_2\rangle \oplus \mathbb{Q}\langle s_2 s_1\rangle \xrightarrow{\delta_2} \mathbb{Q}\langle s_1 s_2 s_1\rangle.$$

Here $\langle w\rangle$ is the Bruhat cell associated to the element $w \in W$. The only non-zero map is δ_1 given by a diagonal matrix with 2's in the diagonal corresponding to the $\Rightarrow$. The rational cohomology that results is

$$\begin{cases} H^0(G/B, \mathbb{Q}) = \mathbb{Q} : q^0, \\ H^1(G/B, \mathbb{Q}) = 0 : q^1, \\ H^2(G/B, \mathbb{Q}) = 0 : q^1, \\ H^3(G/B, \mathbb{Q}) = \mathbb{Q} : q^2. \end{cases}$$

Over the rationals this just gives the cohomology of $K = SO(3)$. The alternating sum of the $q^{\eta(w)}$ produces the polynomial $p(q) = q^2 - 1$. Now $p(q)$ multiplied by q^r with $r = \dim(K) - \deg(p(q)) = 1$ gives the number of points of $SO(3)$ over a field with q elements (q is a power of an odd prime p). The explanation for this is that the $q^{\eta(w)}$ listed can be shown to be Frobenius eigenvalues in etale cohomology of the appropriate varieties reduced to a field of positive characteristic. When this is taken into account we see that, in fact, more than the cohomology of K/T or K, we are obtaining etale $\bar{\mathbb{Q}}_l$ cohomology over a field of positive characteristic, including Frobenius eigenvalues. All derived from the structure of the Toda lattice and its blow-up points.

If we start with $\varepsilon = (-+)$ then we obtain the edges $e \Rightarrow s_2$, $s_1 \Rightarrow s_2 s_1$, $s_1 s_2 \Rightarrow s_1 s_2 s_1$. This time the vertices of the hexagon Γ_{-+} belonging to a connected component in Fig. 4 form a connected component of this graph. The graph for Γ_{-+} now corresponds to the graph of incidence numbers computing cohomology with local coefficients. The local system $\mathscr{L}$ can be described by the signs $(-+)$. The $-$ sign indicates that along a circle in G/B that corresponds to s_1 the local system is constant, and the second $+$ that along a circle corresponding to s_2 it is non-trivial. With these local coefficients $H^*(G/B; \mathscr{L}) = 0$, which implies $p_{-+}(q) = 0$ (see Section 2). Similarly for Γ_{+-} and Γ_{++}.

In the case of a Lie algebra of type A_3 we obtain the graph $\mathscr{G}$ in Fig. 10. This graph corresponds to the polytope in Fig. 6 as separated into connected components by the divisors shown. To determine the number $\eta(w)$ for any given w it is enough to go from e to w along any path along the boundary corresponding to a reduced expression $w = s_{n_1} \cdots s_{n_r}$ and count the number of intersections with the divisors. In Fig. 6, we show the path following the expression $w_* = [123121]$, i.e. $e \to s_1 \to s_1 s_2 \to s_1 s_2 s_3 \to \cdots \to w_*$, with the arrows on the edges of the polytope.

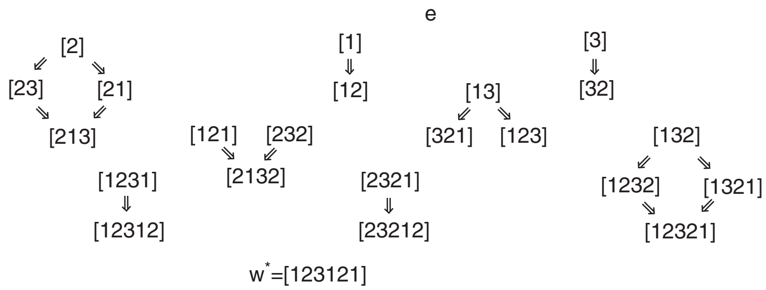

Fig. 10. The graph $\mathcal{G}$ of the real flag manifold for type A_3. The Bruhat cells NwB/B are denoted by $[ij\dots k]$ for $w=s_is_j\dots s_k$. The incidence numbers associated with the edges $\Rightarrow$ are ± 2 (see also [8, Example (8.1)]). There are 10 connected components in this graph corresponding to the 10 connected components in Γ_- after blow-up points are removed. This graph can be also obtained from Fig. 6.

We then state the following theorem showing the equivalence between the connected components in the polytopes Γ_ε and the graphs $\mathcal{G}_\varepsilon$ of the incidence numbers defined in [8]. A proof of the theorem can be found in [7]. We state the case of $\mathcal{G}$ but the remark below explains the general version of the theorem.

Theorem 3.3. *The graph $\mathcal{G}$ is the graph of incidence numbers for the cohomology of the real flag manifold $\check{\mathcal{B}}$ in terms of the Bruhat cells.*

Remark 3.4. Note that while the compactified isospectral manifold of the Toda lattice lives inside $\mathcal{B}$ through the companion embedding [5], this theorem mysteriously involves the topology of the flag manifold associated to the Langlands dual, namely $\check{\mathcal{B}}$. In Section 6 we give an explicit evidence of this duality in the computation of the multiplicity of the τ-function at p_0.

Remark 3.5. In general each $\check{K}$-equivariant local system $\mathcal{L}$ on $\check{\mathcal{B}}$ corresponds to an ε and $\mathcal{G}_\varepsilon$ is the graph of incidence numbers for cohomology of $\check{\mathcal{B}}$ with twisted coefficients in $\mathcal{L}$. There may be some signs ε such that no $\check{K}$-equivariant local system $\mathcal{L}$ corresponds to it. For instance if $G=SL(4;\mathbb{R})$ this is the case. Moreover, Theorem 3.3 can be proved for integral coefficients.

We now discuss the periodic cases through a couple of examples. In the periodic cases, the Toda flow can be described as a flow on a Riemann surface, and the real solutions are determined by the real part of the surface (see Section 1.4). The compactified isospectral manifold of the periodic Toda lattice consists of union of tori, the real part of the Jacobian (see [2,12]). The theta divisors are the blow-ups of the Toda lattice. For each torus we consider its universal cover and we need to pull-back the Toda flow to this universal cover before counting blow-up points.

Example 3.6. In the periodic Toda lattice associated to affine Kac–Moody algebra $A_1^{(1)}$, we have a compactified isospectral manifold consisting of two disjoint circles, i.e. S_{++} and S_{--} in Fig. 9. Since the only possible boundaries in the chain complex that computes integral cohomology involve w and ws_i it is easy to compute the graph of incidence numbers.

In the case of constant coefficients, the graph of incidence numbers consists of the affine Weyl group and there are no edges. The graph $\mathcal{G}_{--}$ can easily be computed by counting the number of blow-ups along the Toda trajectories (see Fig. 11). We have $\eta(w)=l(w)$ for any w in the affine Weyl group. Therefore, there are no edges $\Rightarrow$. We obtain that the graph of incidence numbers of the infinite dimensional real flag manifold $\hat{G}/\hat{B}$ that corresponds agrees with the graph $\mathcal{G}_{--}$. The case of $\varepsilon=(++)$ corresponds to a graph of incidence numbers for integral cohomology with twisted coefficients. Then we have $p(q)=(1-q)/(1+q)$ and $p_{++}(q)=0$. We expect that the rational function $p(q)$ contains some information of the (rational) cohomology of the circle S_{--}, a real part of the Jacobian, e.g. $H^0=\mathbb{Q}$ and $H^1=\mathbb{Q}$ by looking at the degrees of polynomials appearing in $p(q)=p_1(q)/p_0(q)$ (in analogy of the Weil conjecture).

The case of $A_2^{(1)}$ has the complication that at least one sign ε_i must be $+$ (recall $I_0=a_0a_1a_2>0$). Fig. 12 corresponds to the universal cover of a one of these tori (compare Fig. 12 with the Fig. 2 in p. 1516 of [2]). One then necessarily

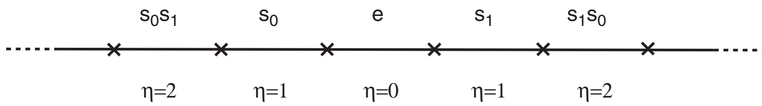

Fig. 11. The compactified isospectral manifold for the periodic Toda Lattice in the case of $A_1^{(1)}$ is a union of two circles S_{++} and S_{--} in Fig. 9. This is the universal cover of S_{--}. The $\times$ are the divisors and we include the number of blow-ups $\eta(w)$. Each line segment separated by the blow-ups is marked by a unique element of the affine Weyl group, $\hat{W} = \langle s_0, s_1 \rangle$. The graph $\mathcal{G}_{--}$ then consists of disconnected vertices marked by the elements of $\hat{W}$, and the function $p(q)$ is given by $p(q) = 1 - 2q + 2q^2 - \cdots = (1-q)/(1+q)$.

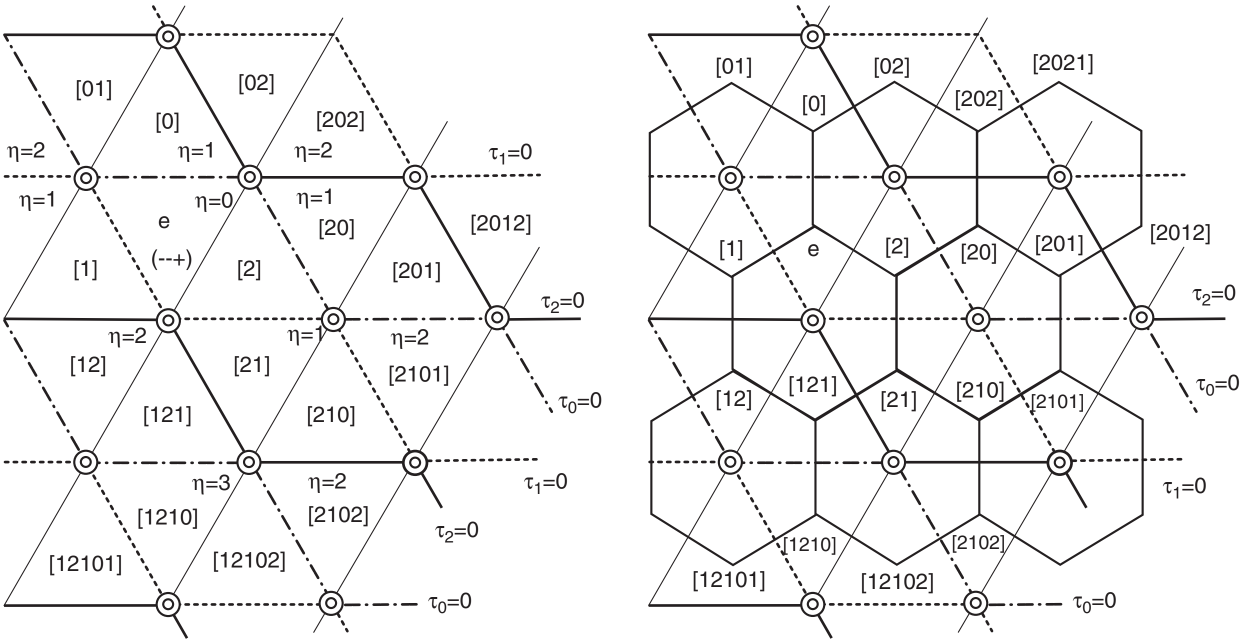

Fig. 12. The universal cover of one of the tori of the real part of the Jacobian for $A_2^{(1)}$ Toda lattice. Each triangular region is marked by the element w of the affine Weyl group, e.g. $[02] = s_0 s_2$. The right figure shows that each element of the affine Weyl group is identified as a unique vertex of the honeycomb of the dual graph of the left figure. Each hexagon represents the Weyl group for A_2. Then some of the chamber walls correspond to the divisors (compare with the hexagons in Fig. 4), and they are indicated here with a solid curve ($\tau_2 = 0$) or a dotted curve ($\tau_1 = 0$) or a dot-dashed curve ($\tau_0 = 0$). On a neighborhood of each double circle ($\tau_k = \tau_{k+1} = 0$, k (mod 3)), the τ_k-functions can be expressed as the Schur polynomials of A_2-nilpotent Toda lattice. We also include the number of blow-ups $\eta(w)$ starting from $\eta(e) = 0$.

obtains cohomology with twisted coefficients and $p_\varepsilon(q) = 0$. In the A_3 case, $\varepsilon = (----)$ should give rise to integral cohomology.

We have the following conjecture which clarifies the relation between the integral cohomology of the real flag manifold and the blow-up structure of the Toda lattice:

Conjecture 3.7. The graph $\mathcal{G}$ is the graph of incidence numbers for the integral cohomology of the real flag manifold $\mathcal{B}$ ($\hat{\mathcal{B}}$) in terms of the Bruhat cells. In general, each ε corresponds to a local system $\mathcal{L}$ in the real flag manifold and the graph $\mathcal{G}_\varepsilon$ is the graph of incidence numbers in the computation of integral cohomology with coefficients in $\mathcal{L}$.

Remark 3.8. This is an extension of Theorem 3.3 for the finite dimensional case (non-periodic Toda case) which has been completed as Theorem 3.5 in [7]. The infinite dimensional case (periodic Toda case) has several steps which were omitted above. We consider a real split semisimple Lie algebra $\mathfrak{g}$ and the corresponding affine Lie algebra $\hat{\mathfrak{g}}$. We also have the Weyl group W and the affine Weyl group $\hat{W}$.

First one considers the universal cover of the compactified isospectral manifold. Since it consists of several tori of dimension l, we end up with several copies of l dimensional vector space which we could identify with the Cartan subalgebra $\mathfrak{h}$. The idea here will be that the structure of these universal covers (of each tori), and the pull-back of the Painlevé divisors, resemble $\mathfrak{h}$ together with certain walls of an action of $\hat{W}$ on $\mathfrak{h}$. Step 1 is a division of each of the universal covers into regions parametrized by $\hat{W}$. This is a refinement of the division into connected components by the Painlevé divisors pulled back to the universal cover. Moreover, the region assigned to the identity has a sign ε and the region corresponding to w will have a sign obtained by applying w to the sign ε. Step 2 is that $\eta(w)$, the number of blow-ups associated to each region is well defined. This now allows us to define the graph $\mathscr{G}_\varepsilon$.

4. The real flag manifold K/T and the variety $\mathscr{O}_o$

For each real split simple Lie algebra $\mathfrak{g}$ we can consider a connected Lie group G and a maximal compact Lie subgroup K which we assume is the set of fixed point sets of a Cartan involution θ. Moreover in the appropriate context of algebraic groups, all these objects can also be considered over a field k_q, an algebraic closure of a finite field $\mathbb{F}_q$ with q elements. We just give a simple example and recall that the real flag manifold can be replaced with a complex manifold $\mathscr{O}_o$ which has the same homotopy type. This is just the unique open dense $K(\mathbb{C})$ orbit in the complex flag manifold $\mathscr{B}_\mathbb{C}$. The advantage of this is that we will be able to consider an analogue of the real flag manifold that makes sense over fields of positive characteristic. Note that the complex flag manifold $G(\mathbb{C})/B(\mathbb{C})$ cannot be chosen to play this role since the topologies of $G(\mathbb{C})/B(\mathbb{C})$ and G/B are very different. For instance, in the A_1 case G/B is a circle, i.e. $\mathbb{P}^1$, and $G(\mathbb{C})/B(\mathbb{C})$ is a two-dimensional sphere, i.e. $\mathbb{CP}^1$.

Example 4.1. Consider the group $=SL(2;k)$. We consider the Cartan involution θ given as follows: $\theta(g) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} g \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Therefore, $K(k)$ consists of the diagonal matrices $\{\mathrm{diag}(z, z^{-1}) : z \in k\}$. The choice of K corresponds to considering the subgroup $SU(1,1)$ of $SL(2;\mathbb{C})$ rather than $SL(2;\mathbb{R})$. The flag manifold is $\mathbb{P}^1 = k \cup \{\infty\}$and the action of $K(k)$ is $g(z)\cdot y = z^2 y$ for $g(z) \in K(k)$ with $z \in k^*$. The $K(k)$ orbits are k^*, $\{0\}$ and $\{\infty\}$. Hence for $k=\mathbb{C}$, we have $\mathscr{O}_o = \mathbb{C}^*$. This orbit has the homotopy type of the real flag manifold of $SU(1,1)$, namely a circle S^1.

5. Computation of $p(q)$ in terms of finite Chevalley groups

The Chevalley groups that will be related to the Toda lattice are the finite groups $\check{K}(\mathbb{F}_q)$. Since Langlands duality $\mathfrak{g} \to \check{\mathfrak{g}}$ only affects type B and C we will, for the sake of notational simplicity, ignore this technical detail, and assume that our Lie algebra does not contain factors of type B or C.

Note that so far we have explained a relation of the Toda lattice with the cohomology of G/B and not with the cohomology of K. However, it turns out that $H^*(G/B;\mathbb{Q}) = H^*(K;\mathbb{Q})$ [7, Proposition 6.3]. This is equivalent to showing that for K-equivariant twisted coefficients $\mathscr{L}$, $H^*(G/B;\mathscr{L})=0$. This, it turns out, is the reason that $p_\varepsilon(q)=0$ whenever ε contains at least one + sign.

The main idea now is to rely on a Lefschetz fixed point theorem. The polynomial $p(q)$ is defined as an alternating sum of powers $q^{\eta(w)}$ the polynomial that computes the order of the finite group $K(\mathbb{F}_q)$ is also given as a polynomial in q. This second statement can be seen by direct computation, but it can also be seen as a consequence of a Lefschetz fixed point theorem. This second way of looking at it is much more complicated than the direct computation but it will be more useful for us in this case. This requires that we first consider the group $K(k_q)$ with k_q an algebraic closure of the finite field $\mathbb{F}_q$. Then we need to consider its cohomology, but, since this algebraic group is not over $\mathbb{C}$ or $\mathbb{R}$, ordinary cohomology does not make sense. We then use *etale cohomology*. In etale cohomology there is also an action induced by the Frobenius map. The eigenvalues resulting from this action are, in this case, powers of q. Using a version of the Lefschetz fixed point theorem applied to the Frobenius map Fr one obtains that $|K(\mathbb{F}_q)|$ is an alternating sum of powers of q. Since the Lefschetz fixed point theorem involves not cohomology, but cohomology with compact supports, we will not get exactly $|K(\mathbb{F}_q)|$ but rather $q^{-r}|K(\mathbb{F}_q)|$ for some r. The main point will then be that these Frobenius eigenvalues are exactly the numbers $q^{\eta(w)}$ obtained from the Toda lattice.

Example 5.1. In the case of $K = SO(2)$ i.e. of a circle, we can give a characteristic zero analogue of this explanation involving the Frobenius map: Let us consider the map, $\Phi_q : S^1 \to S^1$, $z \mapsto z^q$, a map of degree q. Then we have

$$\begin{cases} H_0(S^1; \mathbb{Q}) = \mathbb{Q} : q^0, \\ H_1(S^1; \mathbb{Q}) = \mathbb{Q} : q^1 \end{cases}$$

and now the number of fixed points is $q - 1$, i.e. the number of non-zero roots of $z^q = z$, or as in the Lefschetz fixed point theorem,

$$L(\Phi_q) = \mathrm{Tr}((\Phi_q)_*|_{H_1(S^1;\mathbb{Q})}) - \mathrm{Tr}((\Phi_q)_*|_{H_0(S^1;\mathbb{Q})}) = q - 1.$$

If we replace $SO(2)$ with its complexification, we obtain $\mathbb{C}^* = \mathbb{C}\backslash\{0\}$. Reduction to positive characteristic means we consider $k_q^* = k_q\backslash\{0\}$. Now the Frobenius map is $Fr(z) = z^q$ and the fixed points of this map are the $\mathbb{F}_q$ points i.e., the elements in $\mathbb{F}_q\backslash\{0\}$. We have then $q - 1 = |K(\mathbb{F}_q)|$. The polynomial $q - 1$ is just $p(q)$ obtained in the case of A_1 from the Toda lattice. The number of blow-ups in Fig. 3 for Γ_- is 1 and q^1 is the Frobenius eigenvalue appearing in etale cohomology with proper supports of k_q^*, corresponding to the degree of the map Φ_q.

We now expand a little on this explanation and the full details are in [7]: Recall that there is filtration by Bruhat cells,

$$\emptyset \subset \mathscr{B}_0 \subset \mathscr{B}_1 \subset \cdots \subset \mathscr{B}_{l(w_*)} = G/B,$$

where $\mathscr{B}_j := \bigcup_{l(w)\leqslant j} NwB/B$. There is a similar filtration of $\mathscr{Y}_o$: $\cdots \subset \mathscr{Y}_j \subset \mathscr{Y}_{j+1} \subset \cdots$ given by intersection of $\mathcal{O}_o$ with $N(\mathbb{C})$ cells inside $G(\mathbb{C})/B(\mathbb{C})$. We obtain coboundary maps: $H^j(\mathscr{Y}_j, \mathscr{Y}_{j-1}; \mathbb{C}) \to H^{j+1}(\mathscr{Y}_{j+1}, \mathscr{Y}_j; \mathbb{C})$ which give rise to a chain complex computing the cohomology of $\mathcal{O}_o$. For example in the case of $SU(1, 1)$ above, $\mathscr{Y}_0 = \{\infty\}$ and $\mathscr{Y}_1 = \mathbb{C}\backslash\{0\}$. Each w corresponds to a dual of a Bruhat cell and contributes to $H^{l(w)}(\mathscr{Y}_{l(w)}, \mathscr{Y}_{l(w)-1}; \mathbb{C})$ giving rise to a cohomology class $[w]_{\mathbb{C}}$. This can be done with etale cohomology with coefficients in $\bar{\mathbb{Q}}_m$ and a field of positive characteristic where m is relatively prime to p and $p \neq 2$ and $x^2 + 1$ factors over $\mathbb{F}_q$. In this case a Frobenius action arises in cohomology.

We have the following proposition for the Frobenius eigenvalue of the cohomology class $[w]_{k_q}$ in etale cohomology over a field k_q algebraic closure of $\mathbb{F}_q$ of characteristic p. This assumes $\bar{\mathbb{Q}}_m$ coefficients.

Proposition 5.1. *The cohomology class* $[w]_{k_q}$ in $H^{l(w)}(\mathscr{Y}_{l(w)}, \mathscr{Y}_{l(w)-1}; \bar{\mathbb{Q}}_m)$ *corresponding to* $w \in W$ *has Frobenius eigenvalue given by* $q^{\eta(w)}$.

If one really tries to see where this comes from, this really is Corollary 5.1 in [7]. The computation of $\eta(w)$ agrees with the computation of certain coefficients that arise from a module over the Hecke algebra introduced in [14] . These coefficients are already Frobenius eigenvalues in that paper and are seen to come from the cohomology of the flag manifold and thus of $K(k_q)$.

Example 5.2. In the case of $SU(1, 1)$ which was done earlier, we obtain Frobenius eigenvalues 1 and q, respectively, in the degree 0 and 1.

This proposition is obtained by expressing [8, Proposition 9.5] in terms of new notation motivated by the Toda lattice. The upshot of this is that the number of blow-up associated to a vertex w in the Toda lattice and the Frobenius eigenvalue of $[w]_k$ in $H^{l(w)}(\mathscr{Y}(k)_{l(w)}, \mathscr{Y}(k)_{l(w)-1}; \bar{\mathbb{Q}}_m)$ are given by the same formula in terms of $\eta(w)$. Then applying the Lefschetz fixed point theorem to the Frobenius map to count the number of $\mathbb{F}_q$ points. We obtain the following Theorem [7, Theorem 6.5]:

Theorem 5.3. *The polynomial* $p(q)$ *satisfies* $p(q) = q^{-r}|\check{K}(\mathbb{F}_q)|$ *with* $r = \dim(\check{K}) - \deg(p(q))$. *Moreover* $p(q)$ *factors as* $p(q) = \prod_{i=1}^{g}(q^{d_i} - 1)$ *where* g *is the rank of* $\check{K}$. *The polynomial* $p(q)$ *is given by the following* *explicit formulas*:

A_l: $\check{K} = SO(l+1)$,

$$l \textit{ even}:\ \ p(q) = (q^2-1)(q^4-1)\cdots(q^{l-2}-1)(q^l-1), \quad g = l/2,$$

$$l \textit{ odd}:\ \ p(q) = (q^2-1)(q^4-1)\cdots(q^{l-3}-1)(q^{l-1}-1)(q^g-1), \quad g = (l+1)/2,$$

B_l: $\check{K} = U(l)$,

$$p(q) = (q-1)(q^2-1)(q^3-1)\cdots(q^l-1), \quad g = l,$$

C_l: $\check{K} = SO(l) \times SO(l+1)$,

$$l \textit{ even}:\ \ p(q) = (q^2-1)^2(q^4-1)^2\cdots(q^{l-2}-1)^2(q^l-1)(q^{l/2}-1), \quad g = l,$$

$$l \textit{ odd}:\ \ p(q) = (q^2-1)^2(q^4-1)^2\cdots(q^{l-1}-1)^2(q^{(l+1)/2}-1), \quad g = l,$$

D_l: $\check{K} = SO(l) \times SO(l)$,

$$l \textit{ even}:\ \ p(q) = (q^2-1)^2(q^4-1)^2(q^6-1)^2\ldots(q^{l-2}-1)^2(q^{l/2}-1)^2, \quad g = l,$$

$$l \textit{ odd}:\ \ p(q) = (q^2-1)^2(q^4-1)^2(q^6-1)^2\ldots(q^{l-1}-1)^2, \quad g = l-1,$$

E_6: $\mathrm{Lie}(\check{K}) = \mathfrak{sp}(4)$,

$$p(q) = (q^2-1)(q^4-1)(q^6-1)(q^8-1), \quad g = 4,$$

E_7: $\mathrm{Lie}(\check{K}) = \mathfrak{su}(8)$,

$$p(q) = (q^2-1)(q^3-1)(q^4-1)(q^5-1)(q^6-1)(q^7-1)(q^8-1), \quad g = 7,$$

E_8: $\mathrm{Lie}(\check{K}) = \mathfrak{so}(16)$,

$$p(q) = (q^2-1)(q^4-1)(q^6-1)(q^8-1)(q^{10}-1)(q^{12}-1)(q^{14}-1)(q^8-1), \quad g = 8,$$

F_4: $\mathrm{Lie}(\check{K}) = \mathfrak{sp}(1) \times \mathfrak{sp}(3)$,

$$p(q) = (q^2-1)^2(q^4-1)(q^6-1), \quad g = 4,$$

G_2: $\mathrm{Lie}(\check{K}) = \mathfrak{su}(2) \times \mathfrak{su}(2)$,

$$p(q) = (q^2-1)^2, \quad g = 2.$$

In the factorization $p(q) = \prod_{i=1}^{g}(q^{d_i}-1)$, the d_i are the degrees of the basic Weyl group invariant polynomials for the compact Lie group K (see [4]). The numbers $2d_i - 1$ are the degrees of the generators of the cohomology ring of K, i.e. $H^*(K;\mathbb{Q}) \cong \Lambda(x_1,\ldots,x_g)$ with $\deg(x_i) = 2d_i - 1$. In this sense the cohomology ring of K over the rationals can be derived from the structure of the blow-up points of the Toda lattice. This can be made more explicit and also allows us to write cocycles representing the generators of K in terms of duals of the Bruhat cells.

6. Zeros of Schur polynomials and $p(q)$

We now show that the singular structure of the Painlevé divisor $\mathscr{D}_0 = \bigcup_{j=1}^{l}\mathscr{D}_j$ at the point p_0 is related to the zeros of certain Schur polynomials. We have the following conjecture for the non-periodic case:

Conjecture 6.1. The degree $\eta(w_*)$ of the polynomial $p(q)$ is related to the multiplicity d of the singularity of the divisor given by $\mathscr{D}_0 = \bigcup_{j=1}^{l}\mathscr{D}_j$ at the point p_0 where all the divisors $\mathscr{D}_j = \{\tau_j = 0\}$ intersect. Namely the number

d is given by the minimal degree of the product of the τ-functions, that is, the degree of the tangent cone given by (1.7). Furthermore, the degree $\eta(w_*)$ also gives the number of *real* t_1 roots of the product of the Schur polynomials $S_{\lambda_k}(t_1, \ldots, t_l)$ for generic values of $t_2, \ldots, t_l$. Here λ_k, $k = 1, \ldots, l$, indicate the Young diagrams λ_k, and those Schur polynomials are associated to the τ_k-function of the nilpotent Toda lattices.

The first part of the conjecture can be verified directly for the Toda lattice associated with any Lie algebra of the classical type or type G_2 by counting the minimal degrees of Schur polynomials, which are the leading terms of the τ-functions near the point p_0. The second part is verified for the cases with lower ranks (see below).

6.1. Schur polynomials appearing in the nilpotent Toda lattices

Let us first show that the τ-functions in the semisimple case near the point p_0 can be approximated by those in the nilpotent case (see also [6]). We here explain the case of A_l, and other cases of B, C, D and G can be discussed in the similar way. In the case of A_l, the τ_1 function defined by $\tau_1 = \langle ge_1, e_1\rangle$ with $g = \exp(\sum_{j=1}^{l}(L^0)^j t_j)$, i.e. (1.6) can be expressed by

$$\tau_1(t_1, \ldots, t_l) = \sum_{k=0}^{l} \rho_k \exp\left(\sum_{j=1}^{l} \lambda_k t_j\right)$$
$$= \sum_{n=0}^{\infty} h_n(t_1, \ldots, t_l)\left(\sum_{k=0}^{l} \lambda_k^n \rho_k\right),$$

where λ_k are the eigenvalues of the $(l+1) \times (l+1)$ Lax matrix L^0, and ρ_k are constants determined by L^0. The functions $h_k(t_1, \ldots, t_l)$ are the complete homogeneous symmetric functions given by

$$h_k(t_1, \ldots, t_l) = \sum_{i_1+2i_2+\cdots+li_l=k} \frac{t_1^{i_1} t_2^{i_2} \cdots t_l^{i_l}}{i_1! i_2! \cdots i_l!}. \tag{6.1}$$

We set $t = (0, \ldots, 0)$ to be the point p_0, that is,

$$\tau_1(0, \ldots, 0) = \tau_2(0, \ldots, 0) = \cdots = \tau_l(0, \ldots, 0) = 0.$$

This implies $(\partial^k \tau_1/\partial t_1^k)(0, \ldots, 0) = 0$ for $k = 0, 1, \ldots, l-1$, and $(\partial^l \tau_1/\partial t_1^l)(0, \ldots, 0) = 1$, from which we obtain

$$\tau_1(t_1, \ldots, t_l) = \sum_{n=l}^{\infty} h_n(t_1, \ldots, t_l)\left(\sum_{k=0}^{l} \lambda_k^n \rho_k\right),$$

where ρ_k are determined by

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ \lambda_0 & \lambda_1 & \cdots & \lambda_l \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_0^l & \lambda_1^l & \cdots & \lambda_l^l \end{pmatrix} \begin{pmatrix} \rho_0 \\ \rho_1 \\ \vdots \\ \rho_l \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}.$$

Notice that in the nilpotent case (all $\lambda_k = 0$), we have $\tau_1 = h_l = S_{(l)}$ (recall that $\tau_1 = \langle ge_{l+1}, e_1\rangle$ with $g \in G^{C_0}$). Now using the equation of the τ-functions which is derived from (1.3) and (1.5), i.e.

$$\tau_k \frac{\partial^2 \tau_k}{\partial t_1^2} - \left(\frac{\partial \tau_k}{\partial t_1}\right)^2 = a_k^0 \prod_{j \neq k} (\tau_j)^{-C_{k,j}}, \tag{6.2}$$

we obtain

$$\tau_k = (-1)^{(k(k-1)/2)} S_{(l-k+1,\ldots,l)}(t_1, \ldots, t_l) + (\text{higher degree terms}), \tag{6.3}$$

where $S_{(i_1,\ldots,i_k)}$ with $1 \leqslant i_1 < \cdots < i_k \leqslant l$ is the Schur polynomial defined as the Wronskian determinant with respect to the t_1-variable (see [6]),

$$S_{(i_1,\ldots,i_k)} = \mathrm{Wr}(h_{i_1}, \ldots, h_{i_k}) = |(h_{i_\alpha - \beta + 1})_{1 \leqslant \alpha,\beta, \leqslant k}|.$$

This implies that the τ-functions in the semisimple case near the point p_0 can be approximated by the Schur polynomials which are the τ-functions in the nilpotent case. In particular, the multiplicity of the zero of τ_k is given by $k(l-k+1)$ which is the degree of the Schur polynomial in (6.3). Also notice that the τ functions are weighted homogeneous polynomials where the weight is defined by k for h_k.

The multiplicities of the τ functions for any Lie algebra $\mathfrak{g}$ can be obtained from (6.2) (see also [9]): Let v_k be the multiplicity of the zero for $\tau_k(t_1)$, that is, $\tau_k(t_1) \sim t_1^{v_k}$. Then from (6.2), we have $2(v_k - 1) = -\sum_{j \neq k} C_{k,j} m_j$ which leads to

$$v_k = 2 \sum_{j=1}^{l} (C^{-1})_{k,j},$$

(see also [9, Proposition 2.4]). This number is also related to the total number of the root α_k in the positive root system Δ^+ for the *Langlands dual* algebra $\check{\mathfrak{g}}$: To show this, we calculate 2ρ, the sum of all the positive roots in $\mathfrak{g}$,

$$2\rho = \sum_{\alpha \in \Delta_+} \alpha = \sum_{k=1}^{l} n_k \alpha_k.$$

Note that ρ is also given by the sum of the fundamental weights, $\rho = \sum_{k=1}^{l} \omega_k$. Then using the relation $\alpha_i = \sum_{j=1}^{l} C_{i,j} \omega_j$, the number n_k is given by

$$n_k = 2 \sum_{j=1}^{l} (C^{-1})_{j,k}.$$

Thus the multiplicity v_k for the τ_k function associated to $\mathfrak{g}$ is related to the number n_k for the Langlands dual algebra $\check{\mathfrak{g}}$ whose Cartan matrix is given by the transpose of that for $\mathfrak{g}$. This duality leads to the appearance of the Langlands dual in Theorems 3.3 and 5.3 (see [7] for the details). The multiplicity of the product of the τ functions at the point p_0 is then given by (see also [1,9])

$$|2\rho| = \sum_{k=1}^{l} v_k = \sum_{k=1}^{l} n_k = 2 \sum_{i,j=1}^{l} (C^{-1})_{i,j}.$$

In the general case, the τ-functions for the nilpotent Toda lattices are obtained from (1.6). Then one finds explicit forms of the highest weight vectors and the companion matrix (the regular nilpotent element) for each algebra. The following is the result based on the τ-functions of the nilpotent Toda lattices (see [6] for the nilpotent Toda lattices in general):

A_l-Toda lattices: The τ-functions are given by the Schur polynomials in (6.3), i.e.

$$\tau_k(t_1, \ldots, t_l) = (-1)^{(k(k-1))/2} S_{(l-k+1,\ldots,l)}(t_1, \ldots, t_l), \quad k = 1, \ldots, l.$$

The minimal degrees of those Schur polynomials are given as follows:

- For l even, the minimal degrees of τ_j, $j = 1, \ldots, l$, are given by

  $$1,\ 2, \ldots, \frac{l}{2}, \frac{l}{2},\ , \ldots, 2,\ 1,$$

  (e.g. $\tau_1 \sim t_l$, $\tau_2 \sim t_{l-1}^2$ and $\tau_l \sim t_l$). The sum of those degrees then gives $d = l(l+2)/4$.

- For l odd, the minimal degrees are

  $$1,\ 2, \ldots, \frac{l-1}{2}, \frac{l+1}{2}, \frac{l-1}{2}, \ldots, 2, 1,$$

  from which we have $d = ((l+1)/2)^2$.

For example, in the case of $l=2$, we have $\tau_1 = S_{(2)} = t_2 + t_1^2/2,\ \tau_2 = -S_{(1,2)} = t_2 - t_1^2/2$. Thus we have the degree four polynomial for $F(t_1, t_2) = \tau_1\tau_2(t_1, t_2)$, and the minimal degree is two which is the number of blow-ups $\eta(w_*)$. Note here that $|2\rho| = 4$ is the number of complex roots. The second part of Conjecture 6.1 then states that $F(t_1, t_2) = -S_{(1)}S_{(1,2)}(t_1, t_2)$ has two *real* roots in t_1 for a generic value of $t_2 =$ constant (i.e. $t_2 \neq 0$). One should also note that the sum of the minimal degrees of the pair τ_k and τ_{l-k+1} is equal to the degree d_k in Theorem 5.3.

B_l-Toda lattice: The τ-functions are given by

$$\tau_k(t_1, t_3, \ldots, t_{2l-1}) = \mathrm{Wr}(h_{2l}, \ldots, h_{2l-k+1}), \quad k = 1, \ldots, l-1$$

and

$$\tau_l(t_1, t_3, \ldots, t_{2l-1}) = \sqrt{|\mathrm{Wr}(h_{2l}, \ldots, h_{l+1})|},$$

where h_k are given in (6.1) with $t_{2k} = 0$ for all even parameters (see [6]). Note the indices of the flow parameters, $2k-1, k=1, \ldots, l$, are given by the exponents m_k of the root system Δ. Note that the τ_l is given by the pfaffian related to the Schur Q-function. More precisely, we have $\tau_l(t_1, t_3, \ldots, t_{2l-1}) \sim S_{(1,3,\ldots,2l-1)}(t_1/2, t_3/2, \ldots, t_{2l-1}/2)$ which is the Schur Q-polynomial, $Q_{(l,l-1,\ldots,1)}(t_1, t_3, \ldots, t_{2l-1})$ [10]. Then we have:

- For l even, the minimal degrees are given by

  $$2,\ 2,\ 4,\ 4, \ldots, l-2, l-2, l, \frac{l}{2}$$

  (e.g. $\tau_1 \sim t_1 t_{2l-1},\ \tau_2 \sim t_{2l-1}^2$ and $\tau_l \sim (t_{l+1})^{l/2}$). The degree d is then given by $d = l(l+1)/2$.
- For l odd, the minimal degrees are

  $$2,\ 2,\ 4,\ 4, \ldots, l-1, l-1, \frac{l+1}{2},$$

  from which we have $d = l(l+1)/2$.

For the case B_2, we have $\tau_1 = S_{(4)} = t_1(t_3 + t_1^3/24)$ and $\tau_2 = \sqrt{S_{(3,4)}} = t_3 - t_1^3/12$. The degree of $F = \tau_1\tau_2$ in t_1 is seven which is the height $|2\rho|$, and the minimal degree is three which is the number of blow-ups $\eta(w_*)$ and also the number of real roots of $F(t_1, t_3)$ in t_1 for $t_3 \neq 0$. One should note that the minimal degree of τ_k also gives the degree d_k in Theorem 5.3 for the Langlands dual algebra C_l.

C_l-Toda lattice: The τ-functions are

$$\tau_k(t_1, t_3, \ldots, t_{2l-1}) = \mathrm{Wr}(h_{2l-1}, \ldots, h_{2l-k}), \quad k = 1, \ldots, l.$$

Again one takes $t_{2k} = 0$ for h_n. Then the minimal degrees are given by

$$1,\ 2,\ 3, \ldots, l-1, l.$$

This gives $d = l(l+1)/2$ (which is the same as B_l-case). For the case C_2, we have $\tau_1 = S_{(3)} = t_3 + t_1^3/6$ and $\tau_2 = -S_{(2,3)} = t_1(t_3 - t_1^3/12)$. The product $F = \tau_1\tau_2$ has the degree seven which is the height $|2\rho|$, and the minimal degree is three which is $\eta(w_*)$ and also the number of real roots of $F(t_1, t_3)$ in t_1 for $t_3 \neq 0$. The minimal degree of τ_k then gives d_k for the Langlands dual algebra B_l in Theorem 5.3.

D_l-Toda lattice: The τ-functions are given as follows:

- For l even, they are given by, for $k = 1, \ldots, l-2$,

  $$\tau_k(t_1, t_3, \ldots, t_{2l-3}, s) = \mathrm{Wr}(sh_{l-1} + 2h_{2l-2}, sh_{l-2} + 2h_{2l-3}, \ldots, sh_{l-k} + 2h_{2l-1-k}).$$

The τ_{l-1} and τ_l are given by

$$[\tau_{l-1}\cdot\tau_l](t_1,t_3,\dots,t_{2l-3},s)=\mathrm{Wr}(sh_{l-1}+2h_{2l-2},\dots,sh_2+2h_l),$$

and

$$(\tau_l(t_1,t_3,\dots,t_{2l-3},s))^2=\pm\begin{vmatrix} sh_{l-1}+2h_{2l-2} & sh_{l-2}+2h_{2l-3} & \cdots & sh_1+2h_l & s+h_{l-1}\\ sh_{l-2}+2h_{2l-3} & sh_{l-3}+2h_{2l-4} & \cdots & s+2h_{l-1} & h_{l-2}\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ sh_1+2h_l & s+2h_{l-1} & \cdots & 2h_2 & h_1\\ s+h_{l-1} & h_{l-2} & \cdots & h_1 & 0\end{vmatrix}.$$

Here the even parameters are all zero $t_{2k}=0$, and s is a flow parameter associated with the Chevalley invariant with the degree l, i.e. the exponent $m_l=l-1$. The exponents m_k of the root system of D-type are given by $(1,3,\dots,2l-3,l-1)$ (see e.g. [4]). Counting the minimal degrees of the τ-functions, we have

$$2,\ 2,\ 4,\ 4,\dots,l-2,\ l-2,\ \frac{l}{2},\frac{l}{2}$$

(e.g. $\tau_1\sim st_{l-1},\ \tau_2\sim t_{2l-3}^2$ and $\tau_l\sim s^{l/2}$). Then we have $d=l^2/2$. The minimal degree of τ_k then gives the degree d_k in Theorem 5.3.

- For l odd, the τ-functions are given by, for $k=1,\dots,l-2$,

$$\tau_k(t_1,t_3,\dots,t_{2l-3},s)=\mathrm{Wr}(s^2+2h_{2l-2},2h_{2l-3},\dots,2h_{2l-1-k}).$$

The last two τ-functions are

$$[\tau_{l-1}\cdot\tau_l](t_1,t_3,\dots,t_{2l-3},s)=\mathrm{Wr}(s^2+2h_{2l-2},2h_{2l-3},\dots,2h_l),$$

and

$$(\tau_l(t_1,t_3,\dots,t_{2l-3},s))^2=\pm\begin{vmatrix} s^2+2h_{2l-2} & 2h_{2l-3} & \cdots & 2h_l & s+h_{l-1}\\ 2h_{2l-3} & 2h_{2l-4} & \cdots & 2h_{l-1} & h_{l-2}\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 2h_l & 2h_{l-1} & \cdots & 2h_2 & h_1\\ s+h_{l-1} & h_{l-2} & \cdots & h_1 & 1\end{vmatrix}.$$

Now the minimal degrees of the τ-functions are

$$2,\ 2,\dots,l-3,l-3,l-1,\frac{l-1}{2},\frac{l-1}{2}.$$

Then we have $d=(l^2-1)/2$.

G_2-Toda lattice: We have

$$\tau_1=S_{(6)}=t_1\left(t_5+\frac{t_1^5}{720}\right),\qquad \tau_2=-S_{(5,6)}=t_5^2-\frac{t_5t_1^5}{40}+\frac{t_1^{10}}{86400}.$$

Here the flow parameters are only t_1 and t_5 and all others take zero, i.e. $t_2=t_3=t_4=0$. Those indices 1, 5 in the non-zero parameters are the exponents of the Weyl group of G-type. The degree of $F=\tau_1\tau_2$ is 16 $(=|2\rho|)$, and the minimal degree is four which is $\eta(w_*)$, and this also gives the number of real roots of $F(t_1,t_5)$ in t_1 for $t_5\neq 0$. Each minimal degree of τ_k is also d_k in Theorem 5.3.

6.2. The degree of the tangent cone at p_0 and $p(q)$

We can now state the following Proposition from those computations:

Proposition 6.1. *Let $\mathfrak{g}$ be a split semisimple Lie algebra not containing factors of type E or F. The degree d of the tangent cone at p_0 of the Painlevé divisor $\mathscr{D}_0 = \bigcup_{j=1}^{l} \mathscr{D}_j$, which is given as the minimal degrees of Schur polynomials in* (6.3) *(see* (1.7)*), is the dimension of any Borel subalgebra of* $\mathrm{Lie}(\check{K}(\mathbb{C}))$. *Moreover, d is also the degree of the polynomial $\tilde{p}(q) = q^{-r}|\check{K}(\mathbb{F}_q)|$.*

Then from Theorem 5.3, we obtain the following Proposition (the proof can be found in [7]):

Proposition 6.2. *The number $\eta(w_*) = \deg(p(q))$ satisfies $\eta(w_*) = d_1 + \cdots + d_l$. Moreover we have $\eta(w_*) = d$ for any semisimple Lie algebra not containing factors of type E or F. We have the following formulas for $\eta(w_*)$. The number $\eta(w_*)$ is, in each case, the complex dimension of any Borel subalgebra of* $\mathrm{Lie}(\check{K}(\mathbb{C}))$,

$$A_l : \eta(w_*) = \frac{l(l+2)}{4} \text{ if } l \text{ is even}; \quad \eta(w_*) = \frac{(l+1)^2}{4} \text{ if } l \text{ is odd},$$

$$B_l \text{ or } C_l : \eta(w_*) = \frac{l(l+1)}{2},$$

$$D_l : \eta(w_*) = \frac{l^2}{2} \text{ if } l \text{ is even}; \quad \eta(w_*) = \frac{l^2-1}{2} \text{ if } l \text{ is odd},$$

$$E_l : \eta(w_*) = 20 \text{ if } l = 6; \quad \eta(w_*) = 35 \text{ if } l = 7; \quad \eta(w_*) = 64 \text{ if } l = 8,$$

$$F_4 : \eta(w_*) = 14,$$

$$G_2 : \eta(w_*) = 4.$$

We especially note that d gives the multiplicity of a singularity at p_0, and $\eta(w_*)$ is defined differently as the maximal number of blow-ups encountered along the Toda flow, counted along one-dimensional subsystems. The polynomials $p(q)$, the alternating sum of the blow-ups, are shown to agree with the polynomials $\tilde{p}(q)$. As was noticed, the minimal degree for each τ-functions is related to each degree d_i of the basic W-invariant polynomial of the Chevalley group $\check{K}$. For example, the degree d_i and the minimal degree of τ_i-function are the same for the cases having the same ranks, $l = \mathrm{rank}(\mathfrak{g}) = \mathrm{rank}(K)$, i.e. the cases of B, C and D_l with l even. We did not compute the exact relation between those degrees, but we expect that each degree d_i is related to the number of real intersection points on the tangent cone $V = \{F_d = 0\}$ defined in (1.7) with a linear line corresponding to the t_1-flow of the Toda lattice. This may be stated as

$$d = \deg(p(q)) = \max_{c \in U} |\{\mathscr{D}_0 \cap L_c| \text{ transversal intersection}\}|,$$

where $U \subset \mathbb{R}^{l-1}$ is a neighborhood of $\mathrm{t} = 0$, and $L_c := \{(t_1, c_2, \ldots, c_l) : c = (c_2, \ldots, c_l) \in U\}$ is the linear line of t_1-flow. This statement is equivalent to the second part of Conjecture 6.1, that is, the number of real t_1-roots of the product $F = \prod_{j=1}^{l} \tau_j$ in the nilpotent limit.

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