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# Cascading gauge theory on $dS_4$ and String Theory landscape

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## Abstract

Placing anti-D3 branes at the tip of the conifold in Klebanov–Strassler geometry provides a generic way of constructing meta-stable de Sitter (dS) vacua in String Theory. A local geometry of such vacua exhibit gravitational solutions with a D3 charge measured at the tip opposite to the asymptotic charge. We discuss a restrictive set of such geometries, where anti-D3 branes are smeared at the tip. Such geometries represent holographic dual of cascading gauge theory in  $dS_4$  with or without chiral symmetry breaking. We find that in the phase with unbroken chiral symmetry the D3 charge at the tip is always positive. Furthermore, this charge is zero in the phase with spontaneously broken chiral symmetry. We show that the effective potential of the chirally symmetric phase is lower than that in the symmetry broken phase, *i.e.*, there is no spontaneous chiral symmetry breaking for cascading gauge theory in  $dS_4$ . The positivity of the D3 brane charge in smooth de-Sitter deformed conifold geometries with fluxes presents difficulties in uplifting AdS vacua to dS ones in String Theory via smeared anti-D3 branes.

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## 1. Introduction and summary

String Theory is expected to have a landscape of (meta-stable) de-Sitter vacua [1]. A generic way to construct such vacua was presented in [2] (KKLT):

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- First, turning on fluxes on Calabi–Yau compactifications of type IIB string theory produces highly warped geometry with stabilized complex structure (but not Kähler) moduli of the compactification [3];
- Next, including non-perturbative effects (which are under control given the unbroken supersymmetry), one obtains anti-de Sitter ( $AdS_4$ ) vacua with *all* moduli fixed;
- Finally, one uses anti-D3 branes of type IIB string theory to uplift  $AdS_4$  to de Sitter ( $dS_4$ ) vacua.

As the last step of the construction completely breaks supersymmetry, it is much less controlled. In fact, in [4–7] it was argued that putting anti-D3 branes at the tip of the Klebanov–Strassler (KS) [8] geometry (as done in KKL construction) leads to a naked singularity. Whether or not the resulting singularity is physical is subject to debates.<sup>1</sup> In [10] it was shown that the singularity cannot be cloaked by a regular event horizon, and thus must be unphysical [11]. This conclusion is reached analyzing local Klebanov–Tseytlin (KT) [12] or KS geometry with regular Schwarzschild horizon. Such geometry is dual to strongly coupled cascading gauge theory plasma with unbroken [13–17] (in KT case) or broken [18] (in KS case) chiral symmetry. It was shown that a D3-brane charge measured at the horizon is always positive, and thus cannot cloak a physical negative-D3-charge singularity.

The *good* versus *bad* gravitational singularity criteria of Gubser [11] is based on a simple principle that singularities in gravitational backgrounds holographically dual to some strongly coupled gauge theories arise in the interior of the bulk space–time geometry, corresponding to the infrared (IR) in the dual gauge theories. Physical infrared singularities in gauge theories can be removed with an infrared cutoff. In the original paper, [11], this cutoff is provided by a temperature. However, the role of the cutoff can be served by a curvature scale of a boundary compactification manifold [19], or by a Hubble scale when the strongly coupled gauge theory is formulated in  $dS_4$  [20]. In this paper we extend analysis of [10] considering<sup>2</sup> de Sitter deformation of the KT/KS geometries (holographically dual to cascading gauge theory in  $dS_4$  with unbroken/broken chiral symmetry). As in [10], we ask the question whether it is possible to construct smooth geometries with a negative D3 charge in the interior of the space.

The analysis presented here closely follow [21]. In Section 2 we review dual five-dimensional effective gravitational actions describing states of cascading gauge theory on  $\mathcal{M}_4$  with (un-)broken chiral symmetry. In Section 3 we construct states of cascading gauge theory in  $dS_4$  with unbroken chiral symmetry. In Section 4 we repeat the exercise for states of the theory with spontaneous broken chiral symmetry. In Section 5 we compare effective potentials of the cascading gauge theory in  $dS_4$  with broken and unbroken chiral symmetry and identify the true ground state of the theory. In Section 6 we compute the D3 charge in the interior of the bulk of de Sitter deformed KT/KS geometries. Using results of [21], we compute the D3 charge in the interior of the bulk of  $S^3$  deformed KT/KS geometries — in this last section we use the radius of the three-sphere  $\ell_3$  as an infrared cutoff to distinguish *good* versus *bad* gravitational singularities.

Our discussion is rather technical; so, for benefits of the readers who are interesting in results only, we collect them here. Recall that cascading gauge theory is a four-dimensional  $\mathcal{N} = 1$  supersymmetric  $SU(K + P) \times SU(K)$  gauge theory with two chiral superfields  $A_1, A_2$  in the  $(K + P, \bar{K})$  representation, and two fields  $B_1, B_2$  in the  $(\bar{K} + P, K)$ . Perturbatively, this gauge

<sup>1</sup> See [9] for arguments in favour of this singularity.

<sup>2</sup> The early discussion of this problem was presented in [20].

theory has two gauge couplings  $g_1, g_2$  associated with two gauge group factors, and a quartic superpotential

$$W \sim \text{Tr}(A_i B_j A_k B_\ell) \epsilon^{ik} \epsilon^{j\ell}. \tag{1.1}$$

The theory has a global  $SU(2) \times SU(2)$  (flavor) symmetry under which  $A_i$  and  $B_k$  (separately) transform as doublets. As this symmetry is always unbroken (both in the field theory and in the gravitational dual) all our conclusions concerning uplifting to de Sitter vacua with anti-D3 branes are strictly applicable when the anti-D3 branes are smeared on the tip of the conifold — it is only in this case that the dual gauge theory flavor symmetry is unbroken. To define a theory, one needs to specify the space–time four-manifold  $\mathcal{M}_4$  in which the theory is formulated. In case when  $\mathcal{M}_4 = R^{3,1}$ , *i.e.*, Minkowski space–time, one finds that the sum of the gauge couplings does not run

$$\frac{d}{d \ln \mu} \left( \frac{\pi}{g_s} \equiv \frac{4\pi}{g_1^2(\mu)} + \frac{4\pi}{g_2^2(\mu)} \right) = 0, \tag{1.2}$$

while the difference between the two couplings is

$$\frac{4\pi}{g_2^2(\mu)} - \frac{4\pi}{g_1^2(\mu)} \sim P[3 + 2(1 - \gamma_{ij})] \ln \frac{\mu}{\Lambda}, \tag{1.3}$$

where  $\Lambda$  is the strong coupling scale of the theory and  $\gamma_{ij}$  are anomalous dimensions<sup>3</sup> of operators  $\text{Tr} A_i B_j$ . For generic  $\mathcal{M}_4$ , the sum of the gauge couplings runs; however, the theory is still determined by 2 parameters: the asymptotic value of the dilaton  $g_0$ ,

$$g_0 \equiv \lim_{\mu \rightarrow \infty} g_s(\mu) = \lim_{\mu \rightarrow \infty} \left( \frac{4}{g_1^2(\mu)} + \frac{4}{g_2^2(\mu)} \right)^{-1}, \tag{1.4}$$

and the strong coupling scale  $\Lambda$  arising in the renormalization group running of the difference of two couplings (1.3). To summarize, cascading gauge theory is characterized by  $\{P, g_0, \Lambda\}$  and the choice of a four-manifold  $\mathcal{M}_4$ . Relevant to the discussion here, when  $\mathcal{M}_4 = dS_4$  or  $R \times S^3$ , the manifold provides one additional scale to the problem: the Hubble scale  $H$  (in case of  $dS_4$ ) or the compactification scale  $\ell_3^{-1}$  (in case of  $S^3$  compactification). Depending on the ratio of the mass scale supplied by  $\mathcal{M}_4$  and the strong coupling scale  $\Lambda$ , the cascading theory might undergo phase transition in the infrared associated with spontaneous breaking of the chiral symmetry<sup>4</sup>  $\mathbb{Z}_{2P} \rightarrow \mathbb{Z}_2$ . Ideally, we would like to explore the phase structure of the theory for arbitrary values of parameters — in practice, we are restricted to regions of parameter space where our numerical code used to generate  $\mathcal{M}_4$  deformed KT/KS throat geometries is stable.

We now present the summary of our results:

- When  $\mathcal{M}_4 = dS_4$  and the chiral symmetry is unbroken, the D3 brane charge at the tip of the conifold is always positive, as long as

$$\ln \frac{H^2}{\Lambda^2 P^2 g_0} \geq -0.4. \tag{1.5}$$

<sup>3</sup> When  $K \gg P$ ,  $\gamma_{ij} \approx -\frac{1}{2}$ , see [8].

<sup>4</sup> When  $\mathcal{M}_4$  is Minkowski, the chiral symmetry is spontaneously broken, see [8].

- When  $\mathcal{M}_4 = dS_4$  and the chiral symmetry is broken, the D3 brane charge at the tip of the conifold is always zero; we managed to construct geometries of this type for

$$\ln \frac{H^2}{\Lambda^2 P^2 g_0} \geq -0.03. \tag{1.6}$$

- Comparing effective potential of the gauge theory in broken  $\mathcal{V}_{eff}^b$  and unbroken  $\mathcal{V}_{eff}^s$  phases we establish that in all cases, when we can construct the phase with spontaneously broken chiral symmetry,

$$\mathcal{V}_{eff}^b > \mathcal{V}_{eff}^s, \quad \text{when } \ln \frac{H^2}{\Lambda^2 P^2 g_0} \geq -0.03, \tag{1.7}$$

*i.e.*, spontaneous symmetry breaking does not happen for given values of the gauge theory parameters. To put these parameters in perspective, note that the (first-order) confinement/deconfinement and chiral symmetry breaking phase transition in cascading gauge theory plasma occurs at temperature  $T$  such that [16]

$$\ln \frac{T_{deconfinement, \chi SB}^2}{\Lambda^2 P^2 g_0} = 0.2571(2), \tag{1.8}$$

and the (first-order) chiral symmetry breaking in cascading gauge theory on  $S^3$  occurs for compactification scale  $\mu_3 \equiv \ell_3^{-1}$  such that [21]

$$\ln \frac{\mu_{3, \chi SB}^2}{\Lambda^2 P^2 g_0} = 0.4309(8). \tag{1.9}$$

- When  $\mathcal{M}_4 = R \times S^3$  and the chiral symmetry is unbroken, the D3 brane charge at the tip of the conifold is **negative** when

$$\ln \frac{\mu_3^2}{\Lambda^2 P^2 g_0} < \ln \frac{\mu_{3, negative}^2}{\Lambda^2 P^2 g_0} = 0.0318(3). \tag{1.10}$$

However, since cascading gauge theory undergoes a first order phase transition with spontaneous breaking of the chiral symmetry at

$$\mu_{3, \chi SB} > \mu_{3, negative}, \tag{1.11}$$

and the D3 brane charge at the tip of the conifold in broken phase is zero, the charge in the ground state is in fact zero whenever

$$\mu_3 \leq \mu_{3, \chi SB}. \tag{1.12}$$

Furthermore, chirally symmetric states of cascading gauge theory on  $S^3$  develop symmetry breaking tachyonic instabilities at  $\mu_{3, tachyon}$  (below the first order chiral symmetry breaking scale  $\mu_{3, \chi SB}$ )

$$\ln \frac{\mu_{3, tachyon}^2}{\Lambda^2 P^2 g_0} = 0.3297(3) \tag{1.13}$$

which is again above  $\mu_{3, negative}$ .

Our results represented here, together with those reported in [10], point that the singularity of smeared anti-D3 branes at the tip of the conifold is unphysical: had it been otherwise, we should

have been able to implement an infrared cutoff in the geometry with a D3 brane charge measured at the cutoff being negative. The role of the cutoff is played by the temperature (as discussed in [10]), by the compactification scale (when  $\mathcal{M}_4 = R \times S^3$ ), or by the Hubble scale (when  $\mathcal{M}_4 = dS_4$ ). Interesting, we find that the D3 brane charge can become negative when the KT throat geometry is  $S^3$  deformed; however this occurs in the regime where this phase is unstable both via the first order phase transition and the tachyon condensation to  $S^3$  deformed KS throat geometry — the latter geometry has zero D3 brane charge at the tip. All this raises questions about construction of generic de Sitter vacua in String Theory [2].

We stress, however, that our analysis does not definitely exclude local non-singular supergravity description of de Sitter vacua in String Theory. The issue stems from the anti-D3 brane “smearing approximation” used. Early discussion of the relevant smearing approximation appeared in [6,9]. There, the authors carefully analyzed non-supersymmetric deformations of KS geometry, invariant under the  $SU(2) \times SU(2)$  global symmetry of the latter. They further identified a class of perturbations that is being sources by anti-D3 branes, placed at the tip of the conifold, and then computed the leading-order backreaction of those perturbations on KS geometry. Insistence on preserving the  $SU(2) \times SU(2)$  global symmetry is a smearing approximation — from the brane perspective it implies that anti-D3 branes are uniformly distributed (uniformly smeared) over the transverse compact five-dimensional manifold. Our discussion here shares the same smearing approximation as in [6,9], but extends the analysis to the full (rather than leading-order) backreaction. Smearing approximation is a practical tool enabling the analysis of the complicated cascading geometries involved. However, it must be questioned: it is not clear that non-supersymmetric uniform distribution along  $T^{1,1}$  directions of anti-D3 branes is stable against ‘clumping’. While it is highly desirable to lift this approximation, it is very difficult to do this in practice: one is forced to analyze a coupled nonlinear system of partial differential equations, rather than ordinary differential equations. We feel that until fully localized anti-D3 brane analysis in cascading geometries are performed, the singularity question of local supergravity description of de Sitter vacua in String Theory will remain open.

## 2. Dual effective actions of cascading gauge theory

Consider  $SU(2) \times SU(2) \times \mathbb{Z}_2$  invariant states of cascading gauge theory on a 4-dimensional manifold  $\mathcal{M}_4 \equiv \partial\mathcal{M}_5$ . Effective gravitational action on a 5-dimensional manifold  $\mathcal{M}_5$  describing holographic dual of such states was derived in [18]:

$$\begin{aligned}
 S_5[g_{\mu\nu}, \Omega_i, h_i, \Phi] = & \frac{108}{16\pi G_5} \int_{\mathcal{M}_5} \text{vol}_{\mathcal{M}_5} \Omega_1 \Omega_2^2 \Omega_3^2 \left\{ R_{10} - \frac{1}{2} (\nabla\Phi)^2 \right. \\
 & - \frac{1}{2} e^{-\Phi} \left( \frac{(h_1 - h_3)^2}{2\Omega_1^2 \Omega_2^2 \Omega_3^2} + \frac{1}{\Omega_3^4} (\nabla h_1)^2 + \frac{1}{\Omega_2^4} (\nabla h_3)^2 \right) \\
 & - \frac{1}{2} e^{\Phi} \left( \frac{2}{\Omega_2^2 \Omega_3^2} (\nabla h_2)^2 + \frac{1}{\Omega_1^2 \Omega_2^4} \left( h_2 - \frac{P}{9} \right)^2 + \frac{1}{\Omega_1^2 \Omega_3^4} h_2^2 \right) \\
 & \left. - \frac{1}{2\Omega_1^2 \Omega_2^4 \Omega_3^4} \left( 4\Omega_0 + h_2(h_3 - h_1) + \frac{1}{9} P h_1 \right)^2 \right\}, \tag{2.1}
 \end{aligned}$$

where  $\Omega_0$  is a constant,  $R_{10}$  is given by

$$R_{10} = R_5 + \left( \frac{1}{2\Omega_1^2} + \frac{2}{\Omega_2^2} + \frac{2}{\Omega_3^2} - \frac{\Omega_2^2}{4\Omega_1^2\Omega_3^2} - \frac{\Omega_3^2}{4\Omega_1^2\Omega_2^2} - \frac{\Omega_1^2}{\Omega_2^2\Omega_3^2} \right) - 2\Box \ln(\Omega_1\Omega_2^2\Omega_3^2) - \{(\nabla \ln \Omega_1)^2 + 2(\nabla \ln \Omega_2)^2 + 2(\nabla \ln \Omega_3)^2 + (\nabla \ln(\Omega_1\Omega_2^2\Omega_3^2))^2\}, \tag{2.2}$$

and  $R_5$  is the five-dimensional Ricci scalar of the metric

$$ds_5^2 = g_{\mu\nu}(y) dy^\mu dy^\nu, \tag{2.3}$$

that forms part of the ten-dimensional full metric

$$ds_{10}^2 = ds_5^2 + ds_{T^{1,1}}^2, \tag{2.4}$$

$$ds_{T^{1,1}}^2 = \Omega_1^2(y)g_5^2 + \Omega_2^2(y)(g_3^2 + g_4^2) + \Omega_3^2(y)(g_1^2 + g_2^2).$$

One-forms  $\{g_i\}$  (for  $i = 1, \dots, 5$ ) are the usual forms defined in the warp-squashed  $T^{1,1}$  and are given as in [18], for coordinates  $0 \leq \psi \leq 4\pi$ ,  $0 \leq \theta_a \leq \pi$  and  $0 \leq \phi_a \leq 2\pi$  ( $a = 1, 2$ ).

All the covariant derivatives  $\nabla_\lambda$  are with respect to the metric (2.3). Fluxes (and dilaton  $\Phi$ ) are parametrized in such a way that functions  $h_1(y), h_2(y), h_3(y)$  appear as

$$B_2 = h_1(y)g_1 \wedge g_2 + h_3(y)g_3 \wedge g_4, \tag{2.5}$$

$$F_3 = \frac{1}{9}P g_5 \wedge g_3 \wedge g_4 + h_2(y)(g_1 \wedge g_2 - g_3 \wedge g_4) \wedge g_5 + (g_1 \wedge g_3 + g_2 \wedge g_4) \wedge d(h_2(y)),$$

$$\Phi = \Phi(y),$$

where  $P$  corresponds to the number of fractional branes in the conifold.

Finally,  $G_5$  is the five-dimensional effective gravitational constant

$$G_5 \equiv \frac{729}{4\pi^3} G_{10}, \tag{2.6}$$

where  $G_{10}$  is a 10-dimensional gravitational constant of type IIB supergravity.

Chirally symmetric states of the cascading gauge theory are described by the gravitational configurations of (2.1) subject to constraints

$$h_1 = h_3, \quad h_2 = \frac{P}{18}, \quad \Omega_2 = \Omega_3. \tag{2.7}$$

In what follows, we find it convenient to introduce

$$h_1 = \frac{1}{P} \left( \frac{K_1}{12} - 36\Omega_0 \right), \quad h_2 = \frac{P}{18} K_2, \tag{2.8}$$

$$h_3 = \frac{1}{P} \left( \frac{K_3}{12} - 36\Omega_0 \right),$$

$$\Omega_1 = \frac{1}{3} f_c^{1/2} h^{1/4}, \quad \Omega_2 = \frac{1}{\sqrt{6}} f_a^{1/2} h^{1/4},$$

$$\Omega_3 = \frac{1}{\sqrt{6}} f_b^{1/2} h^{1/4}.$$

### 3. Chirally symmetric phase of cascading gauge theory on $dS_4$

We consider here  $SU(2) \times SU(2) \times U(1) \times SO(4)$  (chirally-symmetric) states of the strongly coupled cascading gauge theory. We find it convenient to use a radial coordinate introduced in [23]:

$$\begin{aligned}
 ds_5^2 &= g_{\mu\nu}(y) dy^\mu dy^\nu \\
 &= h^{-1/2} \rho^{-2} \left( -dt^2 + \frac{1}{H^2} \cosh^2(Ht) (dS^3)^2 \right) + h^{1/2} \rho^{-2} (d\rho)^2,
 \end{aligned} \tag{3.1}$$

where  $h = h(\rho)$ . Furthermore, we use parametrization (2.8) and denote<sup>5</sup>

$$f_c = f_2, \quad f_a = f_b = f_3, \quad K_1 = K_3 = K, \quad \Phi = \ln g, \tag{3.2}$$

with  $f_i = f_i(\rho)$ , and  $K = K(\rho)$ ,  $g = g(\rho)$ .

Notice that parametrization (3.1) is not unique — the diffeomorphisms of the type

$$\begin{pmatrix} \rho \\ h \\ f_2 \\ f_3 \\ K \\ g \end{pmatrix} \implies \begin{pmatrix} \hat{\rho} \\ \hat{h} \\ \hat{f}_2 \\ \hat{f}_3 \\ \hat{K} \\ \hat{g} \end{pmatrix} = \begin{pmatrix} \rho/(1+\alpha\rho) \\ (1+\alpha\rho)^4 h \\ (1+\alpha\rho)^{-2} f_2 \\ (1+\alpha\rho)^{-2} f_3 \\ K \\ g \end{pmatrix}, \quad \alpha = \text{const}, \tag{3.3}$$

preserve the general form of the metric. We can completely fix (3.3), i.e., parameter  $\alpha$  in (3.3), requiring that for a geodesically complete  $\mathcal{M}_5$  the radial coordinate  $\rho$  extends as

$$\rho \in [0, +\infty). \tag{3.4}$$

#### 3.1. Equations of motion

For a background ansatz (3.1), (3.2), the equations of motion obtained from (2.1) take form

$$\begin{aligned}
 0 &= f_2'' + \frac{f_2(g')^2}{8g^2} - \frac{3f_2(K')^2}{16hf_3^2gP^2} + \frac{f_2(h')^2}{8h^2} - \frac{3f_2(f_3')^2}{4f_3^2} \\
 &\quad - \frac{f_2'^2}{2f_2} + \frac{f_2h'}{h\rho} + \left( \frac{3f_3'}{2f_3} - \frac{3}{\rho} \right) f_2' \\
 &\quad + \frac{3gP^2}{4hf_3^2\rho^2} - \frac{K^2}{8h^2f_3^4\rho^2} + \frac{f_2(5f_3^2 - 9f_2 + 6f_3)}{f_3^2\rho^2} - 3hf_2H^2,
 \end{aligned} \tag{3.5}$$

$$\begin{aligned}
 0 &= f_3'' + \frac{(K')^2}{16hf_3gP^2} + \frac{f_3(g')^2}{8g^2} + \frac{f_3(h')^2}{8h^2} + \frac{(f_3')^2}{4f_3} \\
 &\quad - \frac{3f_3'}{\rho} + \frac{f_3h'}{h\rho} - \frac{gP^2}{4f_2hf_3\rho^2} \\
 &\quad - \frac{K^2}{8f_2h^2f_3^3\rho^2} + \frac{5f_3^2 - 6f_3 + 3f_2}{f_3\rho^2} - 3hf_3H^2,
 \end{aligned} \tag{3.6}$$

<sup>5</sup> Recall that for the unbroken chiral symmetry we must set  $K_2(\rho) \equiv 1$ .

$$\begin{aligned}
 0 = & h'' + \frac{3(K')^2}{16gf_3^2P^2} - \frac{h(g')^2}{8g^2} - \frac{9(h')^2}{8h} + \frac{3h(f_3')^2}{4f_3^2} + \left(\frac{2f_3'}{f_3} + \frac{f_2'}{2f_2} - \frac{4}{\rho}\right)h' + \frac{hf_2'}{\rho f_2} \\
 & + \frac{9K^2}{8f_2hf_3^4\rho^2} + \left(\frac{4h}{f_3\rho} + \frac{hf_2'}{2f_3f_2}\right)f_3' \\
 & + \frac{5gP^2}{4f_2f_3^2\rho^2} + \frac{h(f_2 - 13f_3^2 - 6f_3)}{f_3^2\rho^2} + 9h^2H^2,
 \end{aligned} \tag{3.7}$$

$$0 = K'' + \left(\frac{f_2'}{2f_2} - \frac{g'}{g} - \frac{h'}{h} - \frac{3}{\rho}\right)K' - \frac{2gKP^2}{hf_2f_3^2\rho^2}, \tag{3.8}$$

$$0 = g'' - \frac{(g')^2}{g} + \left(\frac{2f_3'}{f_3} + \frac{f_2'}{2f_2} - \frac{3}{\rho}\right)g' + \frac{(K')^2}{4hf_3^2P^2} - \frac{g^2P^2}{hf_2f_3^2\rho^2}. \tag{3.9}$$

Additionally we have the first order constraint

$$\begin{aligned}
 0 = & (K')^2 + \frac{2hf_3^2P^2(g')^2}{g} + \frac{2f_3^2P^2g(h')^2}{h} - 12hP^2g(f_3')^2 - \frac{8f_3hgP^2(f_3'\rho - 2f_3)}{f_2\rho}f_2' \\
 & + \frac{16f_3gP^2(4f_3'h + f_3h')}{\rho} + \left(96hf_3 - 48hf_3^2 - 16hf_2 - \frac{4P^2g}{f_2} - \frac{2K^2}{hf_2f_3^2}\right)\frac{gP^2}{\rho^2} \\
 & + 48gP^2h^2f_3^2H^2.
 \end{aligned} \tag{3.10}$$

We explicitly verified that the constraint (3.10) is consistent with (3.5)–(3.9).

### 3.2. UV asymptotics

The general UV (as  $\rho \rightarrow 0$ ) asymptotic solution of (3.5)–(3.10) describing the symmetric phase of cascading gauge theory takes form

$$\begin{aligned}
 f_2 = & 1 - \alpha_{1,0}(H\rho) + \left(-\frac{3}{8}P^2g_0 - \frac{1}{4}K_0 + \frac{1}{4}(\alpha_{1,0})^2 + \frac{1}{2}P^2g_0 \ln \rho\right)(H\rho)^2 \\
 & + \sum_{n=3}^{\infty} \sum_k a_{n,k}(H\rho)^n \ln^k \rho,
 \end{aligned} \tag{3.11}$$

$$\begin{aligned}
 f_3 = & 1 - \alpha_{1,0}(H\rho) + \left(-\frac{1}{2}P^2g_0 - \frac{1}{4}K_0 + \frac{1}{4}(\alpha_{1,0})^2 + \frac{1}{2}P^2g_0 \ln \rho\right)(H\rho)^2 \\
 & + \sum_{n=3}^{\infty} \sum_k b_{n,k}(H\rho)^n \ln^k \rho,
 \end{aligned} \tag{3.12}$$

$$\begin{aligned}
 h = & \frac{1}{8}P^2g_0 + \frac{1}{4}K_0 - \frac{1}{2}P^2g_0 \ln \rho + \alpha_{1,0}\left(\frac{1}{2}K_0 - P^2g_0 \ln \rho\right)(H\rho) \\
 & + \left(\frac{119}{576}P^4g_0^2 + \frac{31}{96}K_0P^2g_0 - \frac{1}{4}P^2g_0\alpha_{1,0}^2 + \frac{1}{8}K_0^2 + \frac{5}{8}\alpha_{1,0}^2K_0\right)
 \end{aligned}$$



$$\begin{aligned}
 & -\frac{1}{96}P^2g_0(62P^2g_0 + 120\alpha_{1,0}^2 + 48K_0)\ln\rho + \frac{1}{2}P^4g_0^2\ln^2\rho \Big)(H\rho)^2 \\
 & + \sum_{n=3}^{\infty} \sum_k h_{n,k}(H\rho)^n \ln^k\rho,
 \end{aligned} \tag{3.13}$$

$$\begin{aligned}
 K &= K_0 - 2P^2g_0\ln\rho - P^2g_0\alpha_{1,0}(H\rho) \\
 & + \left( \frac{1}{16}P^2g_0(2K_0 + 9P^2g_0 - 4\alpha_{1,0}^2) - \frac{1}{4}P^4g_0^2\ln\rho \right) (H\rho)^2 \\
 & + \sum_{n=3}^{\infty} \sum_k K_{n,k}(H\rho)^n \ln^k\rho,
 \end{aligned} \tag{3.14}$$

$$g = g_0 \left( 1 - \frac{1}{2}P^2g_0(H\rho)^2 + \sum_{n=3}^{\infty} \sum_k g_{n,k}(H\rho)^n \ln^k\rho \right). \tag{3.15}$$

It is characterized by 7 parameters:

$$\{K_0, H, g_0, \alpha_{1,0}, a_{4,0}, a_{6,0}, a_{8,0}, g_{4,0}\}. \tag{3.16}$$

In what follows we developed the UV expansion to order  $\mathcal{O}(\rho^{12})$  inclusive.

### 3.3. IR asymptotics

We use a radial coordinate  $\rho$  that extends to infinity, see (3.4). Introducing

$$y \equiv \frac{1}{\rho}, \quad h^h \equiv y^{-2}h, \quad f_{2,3}^h \equiv yf_{2,3}, \tag{3.17}$$

the general IR (as  $y \rightarrow 0$ ) asymptotic solution of (3.5)–(3.10) describing the symmetric phase of cascading gauge theory takes form

$$\begin{aligned}
 f_2^h &= f_{2,0}^h - \frac{9H^2P^2(f_{3,0}^h)^2g_0^h + 6H^4(K_0^h)^2 - 17(f_{2,0}^h)^2(f_{3,0}^h)^2 + 6f_{2,0}^h(f_{3,0}^h)^3}{5(f_{3,0}^h)^4} y \\
 & + \sum_{n=2} f_{2,n}^h y^n,
 \end{aligned} \tag{3.18}$$

$$\begin{aligned}
 f_3^h &= f_{3,0}^h - \frac{H^2P^2(f_{3,0}^h)^2g_0^h + 6H^4(K_0^h)^2 + 7(f_{2,0}^h)^2(f_{3,0}^h)^2 - 18f_{2,0}^h(f_{3,0}^h)^3}{5f_{2,0}^h(f_{3,0}^h)^3} y \\
 & + \sum_{n=2} f_{3,n}^h y^n,
 \end{aligned} \tag{3.19}$$

$$\begin{aligned}
 h^h &= \frac{1}{4H^2} \left( 1 - \frac{2(3H^2P^2(f_{3,0}^h)^2g_0^h + 10H^4(K_0^h)^2 + (f_{2,0}^h)^2(f_{3,0}^h)^2 - 6f_{2,0}^h(f_{3,0}^h)^3)}{(f_{3,0}^h)^4 f_{2,0}^h} y \right. \\
 & \left. + \sum_{n=2} h_n^h y^n \right),
 \end{aligned} \tag{3.20}$$

$$K = K_0^h + \frac{16K_0^h g_0^h P^2 H^2}{5(f_{3,0}^h)^2 f_{2,0}^h} y + \sum_{n=2} K_n^h y^n, \tag{3.21}$$

$$g = g_0^h \left( 1 + \frac{8g_0^h P^2 H^2}{5(f_{3,0}^h)^2 f_{2,0}^h} y + \sum_{n=2} g_n^h y^n \right). \tag{3.22}$$

It is characterized by 4 additional parameters:

$$\{K_0^h, g_0^h, f_{2,0}^h, f_{3,0}^h\}. \tag{3.23}$$

In what follows we developed the IR expansion to order  $\mathcal{O}(y^6)$  inclusive.

### 3.4. Symmetries

The background geometry (3.1), (3.2) enjoys 4 distinct scaling symmetries. We now discuss these symmetries and exhibit their action on the asymptotic parameters (3.16).

- First, we have:

$$\begin{aligned} P &\rightarrow \lambda P, & g &\rightarrow \frac{1}{\lambda} g, \\ \{\rho, f_i, h, K\} &\rightarrow \{\rho, f_i, h, K\}, & \{y, f_i^h, h^h\} &\rightarrow \{y, f_i, h^h\}, \end{aligned} \tag{3.24}$$

which acts on the asymptotic parameters as

$$\begin{aligned} g_0 &\rightarrow \frac{1}{\lambda} g_0, \\ \{K_0, H, \alpha_{1,0}, a_{4,0}, a_{6,0}, a_{8,0}, g_{4,0}\} &\rightarrow \{K_0, H, \alpha_{1,0}, a_{4,0}, a_{6,0}, a_{8,0}, g_{4,0}\}, \end{aligned} \tag{3.25}$$

and

$$\{K_0^h, g_0^h, f_{2,0}^h, f_{3,0}^h\} \rightarrow \{K_0^h, \lambda^{-1} g_0^h, f_{2,0}^h, f_{3,0}^h\}. \tag{3.26}$$

We can use the exact symmetry (3.24) to set

$$g_0 = 1. \tag{3.27}$$

- Second, we have:

$$\begin{aligned} P &\rightarrow \lambda P, & \rho &\rightarrow \frac{1}{\lambda} \rho, & h &\rightarrow \lambda^2 h, & K &\rightarrow \lambda^2 K, \\ \{H, f_i, g\} &\rightarrow \{H, f_i, g\}, \\ \{y, f_2^h, f_3^h, h^h\} &\rightarrow \{\lambda y, \lambda f_2^h, \lambda f_3^h, h^h\}, \end{aligned} \tag{3.28}$$

which acts on the asymptotic parameters as

$$g_0 \rightarrow g_0, \tag{3.29}$$

$$\alpha_{1,0} \rightarrow \lambda \alpha_{1,0}, \tag{3.30}$$

$$K_0 \rightarrow \lambda^2 (K_0 - 2P^2 g_0 \ln \lambda), \tag{3.31}$$

$$a_{4,0} \rightarrow \lambda^4 \left( a_{4,0} + \frac{1}{48} P^2 g_0 (3K_0 - P^2 g_0) \ln \lambda - \frac{1}{16} P^4 g_0^2 \ln^2 \lambda \right), \quad (3.32)$$

$$g_{4,0} \rightarrow \lambda^4 \left( g_{4,0} + \left( -\frac{3}{16} P^2 \alpha_{1,0}^2 g_0 - \frac{5}{64} K_0 P^2 g_0 + \frac{37}{96} P^4 g_0^2 + 3a_{4,0} \right) \ln \lambda + \frac{3}{64} P^2 g_0 (P^2 g_0 + 2K_0) \ln^2 \lambda - \frac{1}{16} P^4 g_0^2 \ln^3 \lambda \right), \quad (3.33)$$

$$a_{6,0} \rightarrow \lambda^6 \left( a_{6,0} + \left( \frac{89}{40} P^2 a_{4,0} g_0 - \frac{1}{5} P^2 g_0 g_{4,0} + \frac{1}{5} K_0 a_{4,0} + \frac{1491}{32000} K_0 P^4 g_0^2 + \frac{689743}{3840000} P^6 g_0^3 + \frac{11}{320} K_0 P^2 \alpha_{1,0}^2 g_0 - \frac{197}{640} P^4 \alpha_{1,0}^2 g_0^2 + \frac{419}{38400} K_0^2 P^2 g_0 \right) \ln \lambda + \left( -\frac{1}{64} P^4 \alpha_{1,0}^2 g_0^2 + \frac{1}{160} K_0^2 P^2 g_0 + \frac{171}{3200} K_0 P^4 g_0^2 - \frac{1}{2} P^2 a_{4,0} g_0 - \frac{1733}{16000} P^6 g_0^3 \right) \ln^2 \lambda + \left( -\frac{463}{14400} P^6 g_0^3 - \frac{3}{160} K_0 P^4 g_0^2 \right) \ln^3 \lambda + \frac{3}{320} P^6 g_0^3 \ln^4 \lambda \right), \quad (3.34)$$

$$a_{8,0} \rightarrow \lambda^8 \left( a_{8,0} + \frac{1}{P^2 g_0 (70K_0 - 141P^2 g_0)} \left( -140P^4 a_{8,0} g_0^2 - \frac{11289869889229}{7468070400000} P^{12} g_0^6 + 18K_0^2 a_{4,0}^2 + \frac{79241}{280} K_0 P^4 \alpha_{1,0}^2 a_{4,0} g_0^2 - \frac{67}{2} K_0 P^4 \alpha_{1,0}^2 g_0^2 g_{4,0} + \frac{131}{4} K_0^2 P^2 \alpha_{1,0}^2 a_{4,0} g_0 - 24K_0 P^2 a_{4,0} g_0 g_{4,0} - \frac{17122502251}{790272000} K_0 P^8 \alpha_{1,0}^2 g_0^4 - \frac{1264903}{26880} K_0 P^6 \alpha_{1,0}^4 g_0^3 + \frac{3642629}{537600} K_0^2 P^6 \alpha_{1,0}^2 g_0^3 - \frac{3}{4} K_0^2 P^4 \alpha_{1,0}^4 g_0^2 - \frac{308363}{560} P^6 \alpha_{1,0}^2 a_{4,0} g_0^3 + \frac{135}{4} P^6 \alpha_{1,0}^2 g_0^3 g_{4,0} + \frac{16067}{6720} K_0^3 P^4 \alpha_{1,0}^2 g_0^2 - \frac{53709659}{3087000} K_0 P^6 a_{4,0} g_0^3 - \frac{15332}{1225} K_0 P^6 g_0^3 g_{4,0} - \frac{875}{4} P^4 \alpha_{1,0}^4 a_{4,0} g_0^2 + \frac{1923781}{33600} K_0^2 P^4 a_{4,0} g_0^2 - \frac{2001}{560} K_0^2 P^4 g_0^2 g_{4,0} + 350P^4 \alpha_{1,0}^2 a_{6,0} g_0^2 - 12P^4 a_{4,0} g_0^2 g_{4,0} + \frac{9013}{1120} K_0^3 P^2 a_{4,0} g_0 - \frac{5706}{35} K_0 P^2 a_{4,0}^2 g_0 + \frac{17699297459}{592704000} P^{10} \alpha_{1,0}^2 g_0^5 + \frac{1365178374361}{553190400000} K_0 P^{10} g_0^5 + \frac{4598761}{80640} P^8 \alpha_{1,0}^4 g_0^4 + \frac{2135}{192} P^6 \alpha_{1,0}^6 g_0^3 \right) \right)$$

$$\begin{aligned}
 & + \frac{48\,152\,049\,931}{189\,665\,280\,000} K_0^2 P^8 g_0^4 - \frac{33\,703\,011\,407}{148\,176\,000} P^8 a_{4,0} g_0^4 + \frac{14\,708\,381}{529\,200} P^8 g_0^4 g_{4,0} \\
 & + \frac{402\,129\,463}{210\,739\,200} K_0^3 P^6 g_0^3 + \frac{3\,965\,783}{15\,052\,800} K_0^4 P^4 g_0^2 + \frac{1315}{6} P^6 a_{6,0} g_0^3 \\
 & + \frac{49\,853}{70} P^4 a_{4,0}^2 g_0^2 - 8 P^4 g_0^2 g_{4,0}^2 \Big) \ln \lambda \\
 & + \left( -\frac{5\,436\,207\,853}{30\,732\,800\,000} P^8 g_0^4 - \frac{35\,277}{171\,500} P^6 \alpha_{1,0}^2 g_0^3 - \frac{1\,469\,772\,959}{31\,610\,880\,000} K_0 P^6 g_0^3 \right. \\
 & + \frac{489}{8960} P^4 \alpha_{1,0}^4 g_0^2 + \frac{8889}{89\,600} K_0 P^4 \alpha_{1,0}^2 g_0^2 + \frac{1\,953\,403}{105\,369\,600} K_0^2 P^4 g_0^2 \\
 & + \frac{131}{8960} K_0^2 P^2 \alpha_{1,0}^2 g_0 - \frac{2\,780\,609}{1\,372\,000} P^4 a_{4,0} g_0^2 + \frac{859}{9800} P^4 g_0^2 g_{4,0} \\
 & + \frac{9013}{2\,508\,800} K_0^3 P^2 g_0 - \frac{157}{140} P^2 \alpha_{1,0}^2 a_{4,0} g_0 \\
 & \left. - \frac{2949}{5600} K_0 P^2 a_{4,0} g_0 - \frac{3}{280} K_0 P^2 g_0 g_{4,0} + \frac{9}{560} K_0^2 a_{4,0} - \frac{36}{35} a_{4,0}^2 \right) \ln^2 \lambda \\
 & + \left( \frac{2\,671\,073\,519}{47\,416\,320\,000} P^8 g_0^4 - \frac{180\,151}{2\,822\,400} P^6 \alpha_{1,0}^2 g_0^3 - \frac{3\,778\,787}{56\,448\,000} K_0 P^6 g_0^3 \right. \\
 & - \frac{27}{640} K_0 P^4 \alpha_{1,0}^2 g_0^2 - \frac{4513}{250\,880} K_0^2 P^4 g_0^2 + \frac{8879}{19\,600} P^4 a_{4,0} g_0^2 \\
 & \left. + \frac{1}{140} P^4 g_0^2 g_{4,0} + \frac{3}{8960} K_0^3 P^2 g_0 - \frac{3}{40} K_0 P^2 a_{4,0} g_0 \right) \ln^3 \lambda \\
 & + \left( \frac{3\,590\,117}{112\,896\,000} P^8 g_0^4 + \frac{93}{4480} P^6 \alpha_{1,0}^2 g_0^3 + \frac{4537}{179\,200} K_0 P^6 g_0^3 \right. \\
 & \left. - \frac{3}{1792} K_0^2 P^4 g_0^2 + \frac{3}{70} P^4 a_{4,0} g_0^2 \right) \ln^4 \lambda \\
 & + \left( -\frac{4617}{448\,000} P^8 g_0^4 + \frac{3}{1600} K_0 P^6 g_0^3 \right) \ln^5 \lambda - \frac{1}{1600} P^8 g_0^4 \ln^6 \lambda \Big), \tag{3.35}
 \end{aligned}$$

and

$$\{K_0^h, g_0^h, f_{2,0}^h, f_{3,0}^h\} \rightarrow \{\lambda^2 K_0^h, g_0^h, \lambda f_{2,0}^h, \lambda f_{3,0}^h\}. \tag{3.36}$$

We can use the exact symmetry (3.28) to relate different sets of  $\{K_0, P\}$ . For the study of perturbative in  $P^2/K_0$  expansion we find it convenient to set  $K_0 = 1$  and vary  $P^2$ . To access the infrared properties of the theory we set  $P = 1$  and vary  $K_0$ . Notice that the two approaches connect at  $\{K_0 = 1, P = 1\}$ .

- Third, we have:

$$\begin{aligned}
 \rho & \rightarrow \lambda \rho, & H & \rightarrow \frac{1}{\lambda} H, \\
 \{P, f_2, f_3, h, K, g\} & \rightarrow \{P, f_2, f_3, h, K, g\}, \\
 \{y, f_2^h, f_3^h, h^h\} & \rightarrow \{\lambda^{-1} y, \lambda^{-1} f_2^h, \lambda^{-1} f_3^h, \lambda^2 h^h\}.
 \end{aligned} \tag{3.37}$$

This scaling symmetry acts on the asymptotic parameters as

$$\{g_0, \alpha_{1,0}\} \rightarrow \{g_0, \alpha_{1,0}\}, \tag{3.38}$$

$$K_0 \rightarrow K_0 + 2P^2 g_0 \ln \lambda, \tag{3.39}$$

$$a_{4,0} \rightarrow a_{4,0} + \left( \frac{1}{48} P^4 g_0^2 - \frac{1}{16} K_0 P^2 g_0 \right) \ln \lambda - \frac{1}{16} P^4 g_0^2 \ln^2 \lambda, \tag{3.40}$$

$$g_{4,0} \rightarrow g_{4,0} + \left( \frac{3}{16} P^2 \alpha_{1,0}^2 g_0 + \frac{5}{64} K_0 P^2 g_0 - \frac{37}{96} P^4 g_0^2 - 3a_{4,0} \right) \ln \lambda \\ + \left( \frac{3}{64} P^4 g_0^2 + \frac{3}{32} K_0 P^2 g_0 \right) \ln^2 \lambda + \frac{1}{16} P^4 g_0^2 \ln^3 \lambda, \tag{3.41}$$

$$a_{6,0} \rightarrow a_{6,0} + \left( -\frac{89}{40} P^2 a_{4,0} g_0 + \frac{1}{5} P^2 g_0 g_{4,0} - \frac{1}{5} K_0 a_{4,0} - \frac{1491}{32000} K_0 P^4 g_0^2 \right. \\ \left. - \frac{689743}{3840000} P^6 g_0^3 - \frac{11}{320} K_0 P^2 \alpha_{1,0}^2 g_0 + \frac{197}{640} P^4 \alpha_{1,0}^2 g_0^2 - \frac{419}{38400} K_0^2 P^2 g_0 \right) \ln \lambda \\ + \left( -\frac{1}{64} P^4 \alpha_{1,0}^2 g_0^2 + \frac{1}{160} K_0^2 P^2 g_0 + \frac{171}{3200} K_0 P^4 g_0^2 \right. \\ \left. - \frac{1}{2} P^2 a_{4,0} g_0 - \frac{1733}{16000} P^6 g_0^3 \right) \ln^2 \lambda \\ + \left( \frac{463}{14400} P^6 g_0^3 + \frac{3}{160} K_0 P^4 g_0^2 \right) \ln^3 \lambda + \frac{3}{320} P^6 g_0^3 \ln^4 \lambda, \tag{3.42}$$

$$a_{8,0} \rightarrow a_{8,0} + \frac{1}{P^2 g_0 (70K_0 - 141P^2 g_0)} \left( \frac{11289869889229}{7468070400000} P^{12} g_0^6 \right. \\ + \left( -\frac{17699297459}{592704000} \alpha_{1,0}^2 - \frac{1365178374361}{553190400000} K_0 \right) P^{10} g_0^5 \\ + \left( \frac{17122502251}{790272000} K_0 \alpha_{1,0}^2 + \frac{33703011407}{148176000} a_{4,0} \right. \\ \left. - \frac{14708381}{529200} g_{4,0} - \frac{48152049931}{189665280000} K_0^2 - \frac{4598761}{80640} \alpha_{1,0}^4 \right) P^8 g_0^4 \\ + \left( \frac{1264903}{26880} K_0 \alpha_{1,0}^4 + \frac{308363}{560} \alpha_{1,0}^2 a_{4,0} - \frac{135}{4} \alpha_{1,0}^2 g_{4,0} \right. \\ + \frac{53709659}{3087000} K_0 a_{4,0} + \frac{15332}{1225} K_0 g_{4,0} - \frac{402129463}{210739200} K_0^3 \\ \left. - \frac{2135}{192} \alpha_{1,0}^6 - \frac{1315}{6} a_{6,0} - \frac{3642629}{537600} K_0^2 \alpha_{1,0}^2 \right) P^6 g_0^3 \\ + \left( \frac{3}{4} K_0^2 \alpha_{1,0}^4 + 12a_{4,0} g_{4,0} + \frac{2001}{560} K_0^2 g_{4,0} + \frac{67}{2} K_0 \alpha_{1,0}^2 g_{4,0} + \frac{875}{4} \alpha_{1,0}^4 a_{4,0} \right. \\ \left. - \frac{1923781}{33600} K_0^2 a_{4,0} + 140a_{8,0} - 350\alpha_{1,0}^2 a_{6,0} - \frac{79241}{280} K_0 \alpha_{1,0}^2 a_{4,0} \right. \\ \left. - \frac{16067}{6720} K_0^3 \alpha_{1,0}^2 + 8g_{4,0}^2 - \frac{49853}{70} a_{4,0}^2 - \frac{3965783}{15052800} K_0^4 \right) P^4 g_0^2$$

$$\begin{aligned}
 & + \left( -\frac{9013}{1120} K_0^3 a_{4,0} + \frac{5706}{35} K_0 a_{4,0}^2 - \frac{131}{4} K_0^2 \alpha_{1,0}^2 a_{4,0} + 24 K_0 a_{4,0} g_{4,0} \right) P^2 g_0 \\
 & - 18 K_0^2 a_{4,0}^2 \Big) \ln \lambda + \left( -\frac{5436207853}{30732800000} P^8 g_0^4 \right. \\
 & + \left. \left( -\frac{1469772959}{31610880000} K_0 - \frac{35277}{171500} \alpha_{1,0}^2 \right) P^6 g_0^3 \right. \\
 & + \left. \left( \frac{489}{8960} \alpha_{1,0}^4 + \frac{8889}{89600} K_0 \alpha_{1,0}^2 + \frac{1953403}{105369600} K_0^2 \right. \right. \\
 & - \left. \left. \frac{2780609}{1372000} a_{4,0} + \frac{859}{9800} g_{4,0} \right) P^4 g_0^2 \right. \\
 & + \left. \left( \frac{131}{8960} K_0^2 \alpha_{1,0}^2 - \frac{2949}{5600} K_0 a_{4,0} - \frac{3}{280} K_0 g_{4,0} \right. \right. \\
 & + \left. \left. \frac{9013}{2508800} K_0^3 - \frac{157}{140} \alpha_{1,0}^2 a_{4,0} \right) P^2 g_0 \right. \\
 & + \left. \frac{9}{560} K_0^2 a_{4,0} - \frac{36}{35} a_{4,0}^2 \right) \ln^2 \lambda \\
 & + \left( -\frac{2671073519}{47416320000} P^8 g_0^4 + \left( \frac{180151}{2822400} \alpha_{1,0}^2 + \frac{3778787}{56448000} K_0 \right) P^6 g_0^3 \right. \\
 & + \left. \left( -\frac{8879}{19600} a_{4,0} - \frac{1}{140} g_{4,0} + \frac{4513}{250880} K_0^2 + \frac{27}{640} K_0 \alpha_{1,0}^2 \right) P^4 g_0^2 \right. \\
 & + \left. \left( \frac{3}{40} K_0 a_{4,0} - \frac{3}{8960} K_0^3 \right) P^2 g_0 \right) \ln^3 \lambda \\
 & + \left( \frac{3590117}{112896000} P^8 g_0^4 + \left( \frac{93}{4480} \alpha_{1,0}^2 + \frac{4537}{179200} K_0 \right) P^6 g_0^3 \right. \\
 & + \left. \left( \frac{3}{70} a_{4,0} - \frac{3}{1792} K_0^2 \right) P^4 g_0^2 \right) \ln^4 \lambda \\
 & + \left. \left( \frac{4617}{448000} P^8 g_0^4 - \frac{3}{1600} K_0 P^6 g_0^3 \right) \ln^5 \lambda - \frac{1}{1600} P^8 g_0^4 \ln^6 \lambda, \tag{3.43}
 \end{aligned}$$

and

$$\{K_0^h, g_0^h, f_{2,0}^h, f_{3,0}^h\} \rightarrow \{K_0^h, g_0^h, \lambda^{-1} f_{2,0}^h, \lambda^{-1} f_{3,0}^h\}. \tag{3.44}$$

We can use the exact symmetry (3.37) to set

$$H = 1. \tag{3.45}$$

- Forth, we have residual diffeomorphisms (3.3) of the metric parametrization (3.1). The latter transformations act on asymptotic parameters as

$$\{g_0, H, K_0\} \rightarrow \{g_0, H, K_0\}, \tag{3.46}$$

$$\alpha_{1,0} \rightarrow \alpha_{1,0} + 2 \frac{\alpha}{H}, \tag{3.47}$$

$$a_{4,0} \rightarrow a_{4,0} + \frac{1}{4} P^2 \alpha_{1,0} g_0 \frac{\alpha}{H} + \frac{1}{4} P^2 g_0 \frac{\alpha^2}{H^2}, \tag{3.48}$$

$$g_{4,0} \rightarrow g_{4,0} - \frac{3}{2} P^2 \alpha_{1,0} g_0 \frac{\alpha}{H} - \frac{3}{2} P^2 g_0 \frac{\alpha^2}{H^2}, \quad (3.49)$$

$$\begin{aligned} a_{6,0} \rightarrow & a_{6,0} + \left( -\frac{11}{96} P^4 g_0^2 \alpha_{1,0} - \frac{1}{8} P^2 g_0 \alpha_{1,0}^3 + \frac{5}{32} P^2 g_0 K_0 \alpha_{1,0} + 3\alpha_{1,0} a_{4,0} \right) \frac{\alpha}{H} \\ & + \left( -\frac{11}{96} P^4 g_0^2 + \frac{5}{32} K_0 P^2 g_0 + 3a_{4,0} \right) \frac{\alpha^2}{H^2} \\ & + \frac{1}{4} P^2 \alpha_{1,0} g_0 \frac{\alpha^3}{H^3} + \frac{1}{8} P^2 g_0 \frac{\alpha^4}{H^4}, \end{aligned} \quad (3.50)$$

$$\begin{aligned} a_{8,0} \rightarrow & a_{8,0} + \left( \frac{1791949}{2560000} P^6 \alpha_{1,0} g_0^3 + \left( \frac{16839}{64000} K_0 \alpha_{1,0} - \frac{10337}{11520} \alpha_{1,0}^3 \right) P^4 g_0^2 \right. \\ & + \left( -\frac{9}{10} \alpha_{1,0} g_{4,0} - \frac{473}{1920} K_0 \alpha_{1,0}^3 + \frac{1417}{25600} K_0^2 \alpha_{1,0} \right. \\ & + \left. \left. \frac{1}{4} \alpha_{1,0}^5 + \frac{761}{80} \alpha_{1,0} a_{4,0} \right) P^2 g_0 \right. \\ & - \left. \left. 5\alpha_{1,0}^3 a_{4,0} + \frac{9}{10} K_0 \alpha_{1,0} a_{4,0} + 10\alpha_{1,0} a_{6,0} \right) \frac{\alpha}{H} \right. \\ & + \left( \frac{1791949}{2560000} P^6 g_0^3 + \left( -\frac{1793}{1280} \alpha_{1,0}^2 + \frac{16839}{64000} K_0 \right) P^4 g_0^2 \right. \\ & + \left( \frac{761}{80} a_{4,0} - \frac{9}{10} g_{4,0} + \frac{1417}{25600} K_0^2 + \frac{99}{640} K_0 \alpha_{1,0}^2 \right) P^2 g_0 \\ & + \left. \left. 10a_{6,0} + \frac{9}{10} K_0 a_{4,0} \right) \frac{\alpha^2}{H^2} \right. \\ & + \left( -\frac{145}{144} P^4 g_0^2 \alpha_{1,0} + \left( -\frac{5}{12} \alpha_{1,0}^3 + \frac{77}{96} K_0 \alpha_{1,0} \right) P^2 g_0 + 10\alpha_{1,0} a_{4,0} \right) \frac{\alpha^3}{H^3} \\ & + \left( -\frac{145}{288} P^4 g_0^2 + \frac{77}{192} K_0 P^2 g_0 + 5a_{4,0} \right) \frac{\alpha^4}{H^4} \\ & + \frac{1}{4} g_0 P^2 \alpha_{1,0} \frac{\alpha^5}{H^5} + \frac{1}{12} P^2 g_0 \frac{\alpha^6}{H^6}, \end{aligned} \quad (3.51)$$

and

$$\{K_0^h, g_0^h, f_{2,0}^h, f_{3,0}^h\} \rightarrow \{K_0^h, g_0^h, f_{2,0}^h, f_{3,0}^h\}. \quad (3.52)$$

As mentioned earlier, the diffeomorphisms (3.3) can be completely fixed requiring that

$$\lim_{\rho \rightarrow +\infty} h^{-1/2} \rho^{-2} = 0, \quad (3.53)$$

*i.e.*, in the holographic dual to the symmetric phase of cascading gauge theory the manifold  $M_5$  geodesically completes in the interior with smooth shrinking of  $dS_4$  (see (3.1)) as  $\rho \rightarrow +\infty$ .

### 3.5. Keeping the physical parameters fixed

Holographic duality between a gauge theory and a supergravity necessitates the dictionary relating the parameters of the two. Specifically, the non-zero non-normalizable components of the gravitational modes are mapped to parameters of the gauge theory. From (3.11)–(3.15) these are:  $H$  (characterizing the curvature of the boundary metric  $\partial\mathcal{M}_5$  in (3.1)), the asymptotic string coupling  $g_0$ , the number of fractional D3 branes  $P$ , and the asymptotic five-form flux parameter  $K_0$ . It is straightforward to map the former 3 parameters:  $H$  is simply the Hubble constant of the background geometry on which we formulate the cascading gauge theory; the value of  $g_0$  is related to the sum of the gauge couplings of the cascading gauge theory in the far UV (see (1.4)), and the parameter  $P$  is the rank difference of the cascading gauge theory group factors inducing the renormalization group flow. It is a bit more tricky to identify the last gravitational parameter —  $K_0$ . The difficulty arises from the fact that  $K_0$  cannot be identified in the far UV, *i.e.*, as  $\rho \rightarrow \infty$  in (3.14), and thus it is sensitive to the rescaling of the radial coordinate  $\rho$ . To address this question, the authors of [6,9] proposed matching the D3-brane Maxwell charge of two cascading geometries (supposedly dual to the same gauge theory) on a fixed<sup>6</sup> UV holographic screen. An alternative (and equivalent) method, first proposed in [23], is to notice that  $K_0$  must be related to the strong coupling scale  $\Lambda$  of the cascading gauge theory, see (1.3). It becomes clear then why rescaling of the radial coordinate  $\rho$  requires modification of  $K_0$ : holographic radial coordinate serves as an ‘energy scale ruler’, and its rescaling necessitates corresponding rescaling of the dimensionful gauge theory parameters ( $H$  and  $\Lambda$  in our case). It is also clear that the combination of gravitational parameters dual to the ratio of  $\frac{H}{\Lambda}$  must be left invariant under the rescaling. Specifically, in our case the corresponding combination must be invariant under the gravitational symmetry transformations rescaling the asymptotic radial coordinate  $\rho$ , *i.e.*, the symmetries (3.28) and (3.37). Turns out that this is sufficient to unambiguously relate  $K_0$  to the strong coupling scale of the cascading gauge theory. We point out that this approach was used in [23] and [16], and passed a highly nontrivial consistency check of validity of the cascading gauge theory plasma first law of thermodynamics in a dual holographic setting. It was also used in [21].

Recall that a symmetry transformation (3.37) rescales  $H$ , and a symmetry transformation (3.28) rescales  $P$  and affects  $K_0$ , while leaving the combination

$$\frac{K_0}{P^2 g_0} + 2 \ln H + \ln P^2 g_0 = \text{invariant} \equiv -2 \ln \Lambda + 2 \ln H = \ln \frac{H^2}{\Lambda^2} \quad (3.54)$$

invariant. The latter invariant defines the strong coupling scale  $\Lambda$  of cascading gauge theory. In particular, using the symmetry choices (3.27) and (3.45) we identify

$$\frac{K_0}{P^2} = \ln \frac{1}{\Lambda^2 P^2} \equiv \frac{1}{\delta}. \quad (3.55)$$

Notice that (3.55) is not invariant under the symmetry transformation (3.28). This is because such transformation modifies  $P^2 g_0$ , and thus changes the theory; (3.55) is invariant under the residual diffeomorphisms (3.3).

As defined in (3.55), a new dimensionless parameter  $\delta$  is small when the IR cutoff set by the  $dS_4$  is much higher than the strong coupling scale  $\Lambda$  (and thus cascading gauge theory is close to be conformal). In Section 3.7 we develop perturbative expansion in  $\delta$ .

<sup>6</sup> Fixing a UV screen requires a careful matching of the radial coordinates.



### 3.6. Numerical procedure

Although we would like to have an analytic control over the gravitational solution dual to a symmetric phase of cascading gauge theory, the relevant equations for  $\{f_2, f_3, h, K, g\}$  (3.5)–(3.10) are rather complicated. Thus, we have to resort to numerical analysis. Recall that various scaling symmetries of the background equations of motion allowed us to set (see (3.27) and (3.45))

$$\lim_{\rho \rightarrow 0} g \equiv g_0 = 1, \quad H = 1. \tag{3.56}$$

While the metric parametrization (3.1) has residual diffeomorphisms (3.3), the latter are fixed once we insist on the IR asymptotics at  $y \equiv \frac{1}{\rho} \rightarrow 0$  (see (3.53)). Finally, a scaling symmetry (3.28) relates different pairs  $\{K_0, P\}$  so that only the ratio  $\frac{K_0}{P^2} \equiv \frac{1}{\delta}$  is physically meaningful (see (3.55)). In the end, for a fixed  $\delta$ , the gravitational solution is characterized by 5 parameters in the UV and 4 parameters in the IR:

$$\begin{aligned} \text{UV: } & \{\alpha_{1,0}, a_{4,0}, a_{6,0}, a_{8,0}, g_{4,0}\}, \\ \text{IR: } & \{K_0^h, g_0^h, f_{2,0}^h, f_{3,0}^h\}. \end{aligned} \tag{3.57}$$

Notice that  $5 + 4 = 9$  is precisely the number of integration constants needed to specify a solution to (3.5)–(3.10) — we have 5 second order differential equations and a single first order differential constraint:  $2 \times 5 - 1 = 9$ .

In practice, we replace the second-order differential equation for  $f_2$  (3.5) with the constraint equation (3.10), which we use to algebraically eliminate  $f_2'$  from (3.6)–(3.9). The solution is found using the “shooting” method as detailed in [16].

Finding a “shooting” solution in 9-dimensional parameter space (3.57) is quite challenging. Thus, we start with (leading) analytic results for  $\delta \ll 1$  (see Section 3.7) and construct numerical solution for  $(K_0 = 1, P^2)$  slowly incrementing  $P^2$  from zero to one. Starting with the solution at  $K_0 = P^2 = 1$  we slowly decrease  $K_0$  while keeping  $P^2 = 1$ .

### 3.7. Symmetric phase of cascading gauge theory at $\frac{H}{\Lambda} \gg 1$

In this section we describe perturbative solution in  $\delta \ll 1$  (3.55) (3.5)–(3.10). Such gravitational backgrounds describe cascading gauge theory on  $dS_4$ , which Hubble scale  $H$  is well above the strong coupling scale  $\Lambda$  of cascading gauge theory.

In the limit  $\delta \rightarrow 0$  (or equivalently  $P \rightarrow 0$ ) the gravitational background is simply that of the Klebanov–Witten model [22] on  $dS_4$  [20]:

$$\begin{aligned} \delta = 0: \quad f_2^{(0)} = f_3^{(0)} = 1 + \sqrt{\hat{K}_0} \rho, \quad h^{(0)} = \frac{\hat{K}_0}{4(1 + \sqrt{\hat{K}_0} \rho)^2}, \\ K^{(0)} = \hat{K}_0, \quad g^{(0)} = 1, \end{aligned} \tag{3.58}$$

where  $\hat{K}_0$  is a constant. Perturbatively, we find

$$f_i(\rho) = f_i^{(0)} \times \sum_{j=0}^{\infty} \left(\frac{P^2}{\hat{K}_0}\right)^j f_{i,j}(\rho^2 \hat{K}_0), \quad h(\rho) = h^{(0)} \times \sum_{j=0}^{\infty} \left(\frac{P^2}{\hat{K}_0}\right)^j h_j(\rho^2 \hat{K}_0),$$

$$K(\rho) = \hat{K}^{(0)} \times \sum_{j=0}^{\infty} \left(\frac{P^2}{\hat{K}_0}\right)^j K_j(\rho^2 \hat{K}_0), \quad g(\rho) = g^{(0)} \times \sum_{j=0}^{\infty} \left(\frac{P^2}{\hat{K}_0}\right)^j g_j(\rho^2 \hat{K}_0). \quad (3.59)$$

Apart from technical complexity, there is no obstacle of developing perturbative solution to any order in  $\frac{P^2}{\hat{K}_0}$ . For our purposes it is sufficient to do so to order  $\mathcal{O}(\frac{P^4}{\hat{K}_0^2})$ . Notice that explicit  $\rho$  dependence enters only in combination  $\rho\sqrt{\hat{K}_0}$ , thus, we can set  $\hat{K}_0 = 1$  and reinstall explicit  $\hat{K}_0$  dependence when necessary.

Substituting (3.59) in (3.5)–(3.10) we find to order  $\mathcal{O}(\delta)$  the following equations

$$0 = f''_{2,1} - \frac{\rho + 6}{2\rho(\rho + 1)} f'_{2,1} + \frac{\rho + 2}{2\rho(\rho + 1)} h'_1 - \frac{3}{4} (K'_1)^2 - \frac{3\rho^2 - 16\rho - 16}{4\rho^2(\rho + 1)^2} h_1 - \frac{4K_1 + 7f_{2,1} - 20f_{3,1} - 3}{(\rho + 1)\rho^2}, \quad (3.60)$$

$$0 = f''_{3,1} - \frac{\rho + 6}{2\rho(\rho + 1)} f'_{3,1} + \frac{1}{4} (K'_1)^2 + \frac{\rho + 2}{2\rho(\rho + 1)} h'_1 - \frac{3\rho^2 - 16\rho - 16}{4\rho^2(\rho + 1)^2} h_1 + \frac{5f_{2,1} + 8f_{3,1} - 4K_1 - 1}{(\rho + 1)\rho^2}, \quad (3.61)$$

$$0 = h''_1 - \frac{\rho + 4}{\rho(\rho + 1)} h'_1 + \frac{3}{4} (K'_1)^2 + \frac{(\rho + 2)(f'_{2,1} + 4f'_{3,1})}{2\rho(\rho + 1)} + \frac{9(\rho^2 - 16\rho - 16)}{4\rho^2(\rho + 1)^2} h_1 - \frac{17f_{2,1} + 68f_{3,1} - 36K_1 - 5}{(\rho + 1)\rho^2}, \quad (3.62)$$

$$0 = K''_1 - \frac{\rho + 6}{2\rho(\rho + 1)} K'_1 - \frac{8}{(\rho + 1)\rho^2}, \quad (3.63)$$

$$0 = g''_1 - \frac{\rho + 6}{2\rho(\rho + 1)} g'_1 + (K'_1)^2 - \frac{4}{(\rho + 1)\rho^2}, \quad (3.64)$$

along with the first order constraint

$$0 = f'_{2,1} + 4f'_{3,1} + h'_1 + \frac{(\rho + 1)\rho}{2(\rho + 2)} (K'_1)^2 + \frac{(\rho + 4)(3\rho + 4)}{2\rho(\rho + 2)(\rho + 1)} h_1 + \frac{2(4f_{3,1} + f_{2,1} - 4K_1 - 1)}{(\rho + 2)\rho}. \quad (3.65)$$

Above equations should be solved with  $\mathcal{O}(\delta)$  UV and the IR boundary conditions prescribed in Sections 3.2 and 3.3. We solve all the equations numerically. Parameterizing the asymptotics as follows:

- UV, i.e.,  $\rho \rightarrow 0$  (the independent coefficients being  $\{\alpha_{1,1,0}, k_{1,4,0}, a_{1,6,0}, a_{1,8,0}, g_{1,4,0}\}$ ):

$$f_{2,1} = \alpha_{1,1,0}\rho + \left(-\frac{3}{8} - \frac{1}{2}\alpha_{1,1,0} + \frac{1}{2}\ln\rho\right)\rho^2 + \left(\frac{1}{8} + \frac{1}{2}\alpha_{1,1,0} - \frac{1}{2}\ln\rho\right)\rho^3 + \left(-\frac{5}{24} - \frac{1}{2}\alpha_{1,1,0} + \frac{4}{3}k_{1,4,0} + \frac{9}{16}\ln\rho\right)\rho^4$$

$$\begin{aligned}
 & + \left( \frac{29}{96} + \frac{1}{2} \alpha_{1,1,0} - \frac{8}{3} k_{1,4,0} - \frac{5}{8} \ln \rho \right) \rho^5 \\
 & + \left( \frac{1}{160} \ln^2 \rho + \frac{4}{15} \ln \rho k_{1,4,0} + \frac{25\,099}{38\,400} \ln \rho + a_{1,6,0} \right) \rho^6 \\
 & + \left( -\frac{3}{160} \ln^2 \rho - \frac{4}{5} \ln \rho k_{1,4,0} - \frac{8379}{12\,800} \ln \rho - \alpha_{1,1,0} - 3a_{1,6,0} \right. \\
 & \left. + \frac{98}{15} k_{1,4,0} - \frac{17\,513}{25\,600} \right) \rho^7 \\
 & + \left( \frac{3}{8960} \ln^3 \rho + \frac{3}{140} \ln^2 \rho k_{1,4,0} + \frac{16}{35} \ln \rho k_{1,4,0}^2 + \frac{89\,373}{2\,508\,800} \ln^2 \rho \right. \\
 & \left. + \frac{29\,791}{19\,600} \ln \rho k_{1,4,0} + \frac{223\,043\,661}{351\,232\,000} \ln \rho + a_{1,8,0} \right) \rho^8 \\
 & + \mathcal{O}(\rho^9), \tag{3.66}
 \end{aligned}$$

$$\begin{aligned}
 f_{3,1} = & \alpha_{1,1,0} \rho + \left( -\frac{1}{2} - \frac{1}{2} \alpha_{1,1,0} + \frac{1}{2} \ln \rho \right) \rho^2 + \left( \frac{1}{4} + \frac{1}{2} \alpha_{1,1,0} - \frac{1}{2} \ln \rho \right) \rho^3 \\
 & + \left( -\frac{41}{192} - \frac{1}{2} \alpha_{1,1,0} + \frac{1}{2} \ln \rho \right) \rho^4 \\
 & + \left( \frac{7}{32} + \frac{1}{2} \alpha_{1,1,0} - \frac{1}{2} \ln \rho \right) \rho^5 \\
 & + \left( -\frac{1}{640} \ln^2 \rho - \frac{1}{15} \ln \rho k_{1,4,0} \right. \\
 & \left. + \frac{19\,049}{38\,400} \ln \rho - \frac{5}{8} \alpha_{1,1,0} - \frac{1}{4} a_{1,6,0} + \frac{353}{480} k_{1,4,0} - \frac{381\,067}{1\,228\,800} \right) \rho^6 \\
 & + \left( \frac{3}{640} \ln^2 \rho + \frac{1}{5} \ln \rho k_{1,4,0} - \frac{6\,229}{12\,800} \ln \rho + \frac{7}{8} \alpha_{1,1,0} \right. \\
 & \left. + \frac{3}{4} a_{1,6,0} - \frac{1043}{480} k_{1,4,0} + \frac{591\,377}{1\,228\,800} \right) \rho^7 \\
 & + \left( \frac{3}{8960} \ln^3 \rho + \frac{3}{140} \ln^2 \rho k_{1,4,0} + \frac{16}{35} \ln \rho k_{1,4,0}^2 \right. \\
 & - \frac{28\,227}{2\,508\,800} \ln^2 \rho - \frac{9409}{19\,600} \ln \rho k_{1,4,0} - \frac{4}{5} k_{1,4,0}^2 + \frac{167\,306\,161}{351\,232\,000} \ln \rho \\
 & \left. - \frac{231}{64} \alpha_{1,1,0} - \frac{231}{32} a_{1,6,0} + a_{1,8,0} + \frac{129\,741}{6400} k_{1,4,0} - \frac{116\,879\,077}{49\,152\,000} \right) \rho^8 \\
 & + \mathcal{O}(\rho^9), \tag{3.67}
 \end{aligned}$$

$$\begin{aligned}
 h_1 = & \frac{1}{2} - 2 \ln \rho + (1 - 2\alpha_{1,1,0}) \rho + \left( \frac{19}{24} + \alpha_{1,1,0} - \ln \rho \right) \rho^2 \\
 & + \left( -\frac{11}{24} - \alpha_{1,1,0} + \ln \rho \right) \rho^3 + \left( \frac{431}{1024} + \alpha_{1,1,0} + \frac{1}{6} k_{1,4,0} - \frac{127}{128} \ln \rho \right) \rho^4
 \end{aligned}$$

$$\begin{aligned}
& + \left( -\frac{3359}{7680} - \alpha_{1,1,0} - \frac{1}{3}k_{1,4,0} + \frac{63}{64} \ln \rho \right) \rho^5 \\
& + \left( \frac{656813}{1536000} + \alpha_{1,1,0} + \frac{189}{200}k_{1,4,0} - \frac{12233}{12800} \ln \rho \right) \rho^6 \\
& + \left( -\frac{4213513}{10752000} - \alpha_{1,1,0} - \frac{1201}{600}k_{1,4,0} + \frac{11599}{12800} \ln \rho \right) \rho^7 \\
& + \left( -\frac{9}{7168} \ln^3 \rho - \frac{9}{112} \ln^2 \rho k_{1,4,0} - \frac{12}{7} \ln \rho k_{1,4,0}^2 + \frac{12441}{2007040} \ln^2 \rho \right. \\
& + \frac{4147}{15680} \ln \rho k_{1,4,0} + \frac{212}{105} k_{1,4,0}^2 - \frac{238628771}{280985600} \ln \rho + \frac{637}{64} \alpha_{1,1,0} \\
& \left. + \frac{693}{32} a_{1,6,0} - \frac{15}{4} a_{1,8,0} - \frac{17860741}{313600} k_{1,4,0} + \frac{110837461177}{16859136000} \right) \rho^8 \\
& + \mathcal{O}(\rho^9); \tag{3.68}
\end{aligned}$$

$$\begin{aligned}
K_1 = & -2 \ln \rho + \rho - \frac{1}{8} \rho^2 - \frac{1}{24} \rho^3 + \left( k_{1,4,0} + \frac{3}{64} \ln \rho \right) \rho^4 \\
& + \left( \frac{33}{640} - 2k_{1,4,0} - \frac{3}{32} \ln \rho \right) \rho^5 \\
& + \left( -\frac{307}{3072} + \frac{35}{12} k_{1,4,0} + \frac{35}{256} \ln \rho \right) \rho^6 + \left( \frac{1031}{7168} - \frac{15}{4} k_{1,4,0} - \frac{45}{256} \ln \rho \right) \rho^7 \\
& + \left( -\frac{24077}{131072} + \frac{1155}{256} k_{1,4,0} + \frac{3465}{16384} \ln \rho \right) \rho^8 + \mathcal{O}(\rho^9); \tag{3.69}
\end{aligned}$$

$$\begin{aligned}
g_1 = & -\frac{1}{2} \rho^2 + \frac{1}{2} \rho^3 + \left( g_{1,4,0} + \left( -\frac{33}{64} + 4k_{1,4,0} \right) \ln \rho + \frac{3}{32} \ln^2 \rho \right) \rho^4 \\
& + \left( -\frac{31}{128} - 2g_{1,4,0} - 2k_{1,4,0} + \left( \frac{15}{16} - 8k_{1,4,0} \right) \ln \rho - \frac{3}{16} \ln^2 \rho \right) \rho^5 \\
& + \left( \frac{3671}{9216} + \frac{35}{12} g_{1,4,0} + \frac{161}{36} k_{1,4,0} \right. \\
& \left. + \left( -\frac{497}{384} + \frac{35}{3} k_{1,4,0} \right) \ln \rho + \frac{35}{128} \ln^2 \rho \right) \rho^6 + \left( -\frac{533}{1024} - \frac{15}{4} g_{1,4,0} - \frac{83}{12} k_{1,4,0} \right. \\
& \left. + \left( \frac{103}{64} - 15k_{1,4,0} \right) \ln \rho - \frac{45}{128} \ln^2 \rho \right) \rho^7 + \left( \frac{81683}{131072} - \frac{1}{2} k_{1,4,0}^2 + \frac{1155}{256} g_{1,4,0} \right. \\
& \left. + \frac{7117}{768} k_{1,4,0} + \left( -\frac{15499}{8192} + 18k_{1,4,0} \right) \ln \rho + \frac{27}{64} \ln^2 \rho \right) \rho^8 + \mathcal{O}(\rho^9); \tag{3.70}
\end{aligned}$$

- IR, *i.e.*,  $y = \frac{1}{\rho} \rightarrow 0$ , (the independent coefficients being  $\{a_{1,0}^h, b_{1,0}^h, g_{1,0}^h, k_{1,0}^h\}$ :

$$\begin{aligned}
f_{2,1} = & a_{1,0}^h + \mathcal{O}(y), & f_{3,1} = & b_{1,0}^h + \mathcal{O}(y), & g_1 = & g_{1,0}^h + \mathcal{O}(y), \\
K_1 = & k_{1,0}^h + \mathcal{O}(y), & h_1 = & \left( -\frac{6}{5} + \frac{18}{5} a_{1,0}^h + \frac{72}{5} b_{1,0}^h - 8k_{1,0}^h \right) y + \mathcal{O}(y^2), \tag{3.71}
\end{aligned}$$

we find

$$\begin{aligned}
\alpha_{1,1,0} &= 0.43427(8), & k_{1,4,0} &= 0.04829(9), & a_{1,6,0} &= -0.40703(7), \\
a_{1,8,0} &= -0.42707(1), & g_{1,4,0} &= -0.26443(7), & a_{1,0}^h &= -0.15661(4), \\
b_{1,0}^h &= -0.37883(6), & g_{1,0}^h &= -0.72222(2), & k_{1,0}^h &= -1.10592(2).
\end{aligned} \tag{3.72}$$

In an analogous way, it is possible to go to second order in  $\delta$  by taking Eqs. (3.5)–(3.10) and evaluate them with the expansion (3.59) to second order in  $\delta$ . Then, we will get equations for functions  $f_{2,2}$ ,  $f_{3,2}$ ,  $h_2$ ,  $K_2$ ,  $g_2$ . As with the first order equations, one uses the UV and IR boundary conditions prescribed in Sections 3.2 and 3.3. Setting  $H = 1$ , we get that the independent coefficients in the UV are  $\{\alpha_{2,1,0}, k_{2,4,0}, a_{2,6,0}, a_{2,8,0}, g_{2,4,0}\}$ , while those in the IR are  $\{a_{2,0}^h, b_{2,0}^h, g_{2,0}^h, k_{2,0}^h\}$ . Solving numerically, we find the values of these constants to be

$$\begin{aligned}
\alpha_{2,1,0} &= 0.35729(1), & k_{2,4,0} &= 0.18423(1), & a_{2,6,0} &= -0.48877(2), \\
a_{2,8,0} &= -0.60853(7), & g_{2,4,0} &= -0.64457(3), & a_{2,0}^h &= 0.54009(5), \\
b_{2,0}^h &= 0.63805(4), & g_{2,0}^h &= 0.31165(0), & k_{2,0}^h &= 1.65246(0).
\end{aligned} \tag{3.73}$$

We can now identify the leading  $\mathcal{O}(\delta^2)$  values of general UV and IR parameters (see (3.57)):

$$\begin{aligned}
\alpha_{1,0} &= -1 - \alpha_{1,1,0} \delta - \alpha_{2,1,0} \delta^2, \\
a_{4,0} &= \left( -\frac{1}{12} + \frac{4}{3}k_{1,4,0} \right) \delta + \left( -\frac{139}{1152} + \frac{a_{1,1,0}}{24} + \frac{2g_{1,4,0}}{3} - \frac{22k_{1,4,0}}{9} + \frac{4k_{2,4,0}}{3} \right) \delta^2, \\
g_{4,0} &= g_{1,4,0} \delta + g_{2,4,0} \delta^2, \\
a_{6,0} &= \left( a_{1,6,0} + \frac{29}{96} - \frac{8}{3}k_{1,4,0} + \frac{1}{2}\alpha_{1,1,0} \right) \delta \\
&\quad + \left( \frac{145}{576} - \frac{5a_{1,1,0}}{32} - \frac{a_{1,1,0}^2}{4} + \frac{a_{2,1,0}}{2} - \frac{4g_{1,4,0}}{3} + \frac{44k_{1,4,0}}{9} \right. \\
&\quad \left. - \frac{4a_{1,1,0}k_{1,4,0}}{3} - \frac{8k_{2,4,0}}{3} + a_{2,6,0} \right) \delta^2, \\
a_{8,0} &= \left( a_{1,8,0} - 3a_{1,6,0} - \frac{17513}{25600} - \alpha_{1,1,0} + \frac{98}{15}k_{1,4,0} \right) \delta \\
&\quad + \left( -\frac{87973}{192000} - \frac{15353a_{1,1,0}}{25600} - \frac{a_{1,1,0}^2}{2} \right. \\
&\quad \left. - 2a_{1,1,0}a_{1,6,0} - a_{2,1,0} - 3a_{2,6,0} + a_{2,8,0} + \right. \\
&\quad \left. + \frac{101g_{1,4,0}}{30} - \frac{2423k_{1,4,0}}{180} + \frac{178a_{1,1,0}k_{1,4,0}}{15} + \frac{98k_{2,4,0}}{15} \right) \delta^2,
\end{aligned} \tag{3.74}$$

$$\begin{aligned}
K_0^h &= 1 + k_{1,0}^h \delta + k_{2,0}^h \delta^2, & g_0^h &= 1 + g_{1,0}^h \delta + g_{2,0}^h \delta^2, \\
f_{2,0}^h &= 1 + a_{1,0}^h \delta + a_{2,0}^h \delta^2, & f_{3,0}^h &= 1 + b_{1,0}^h \delta + b_{2,0}^h \delta^2,
\end{aligned} \tag{3.75}$$

where we set  $K_0 = 1$ .

Fig. 1 compares the values of general UV and IR parameters  $\alpha_{1,0}$ ,  $a_{4,0}$ ,  $a_{6,0}$ ,  $a_{8,0}$ ,  $g_{4,0}$ ,  $K_0^h$ ,  $g_0^h$ ,  $f_{2,0}^h$ ,  $f_{3,0}^h$  (see (3.57)), with their perturbative predictions at linear and quadratic order. The results for first and second order will help to correctly initialize the fully non-linear calculation and at the same time provide a verification of the results, at least for small enough  $\delta$ .

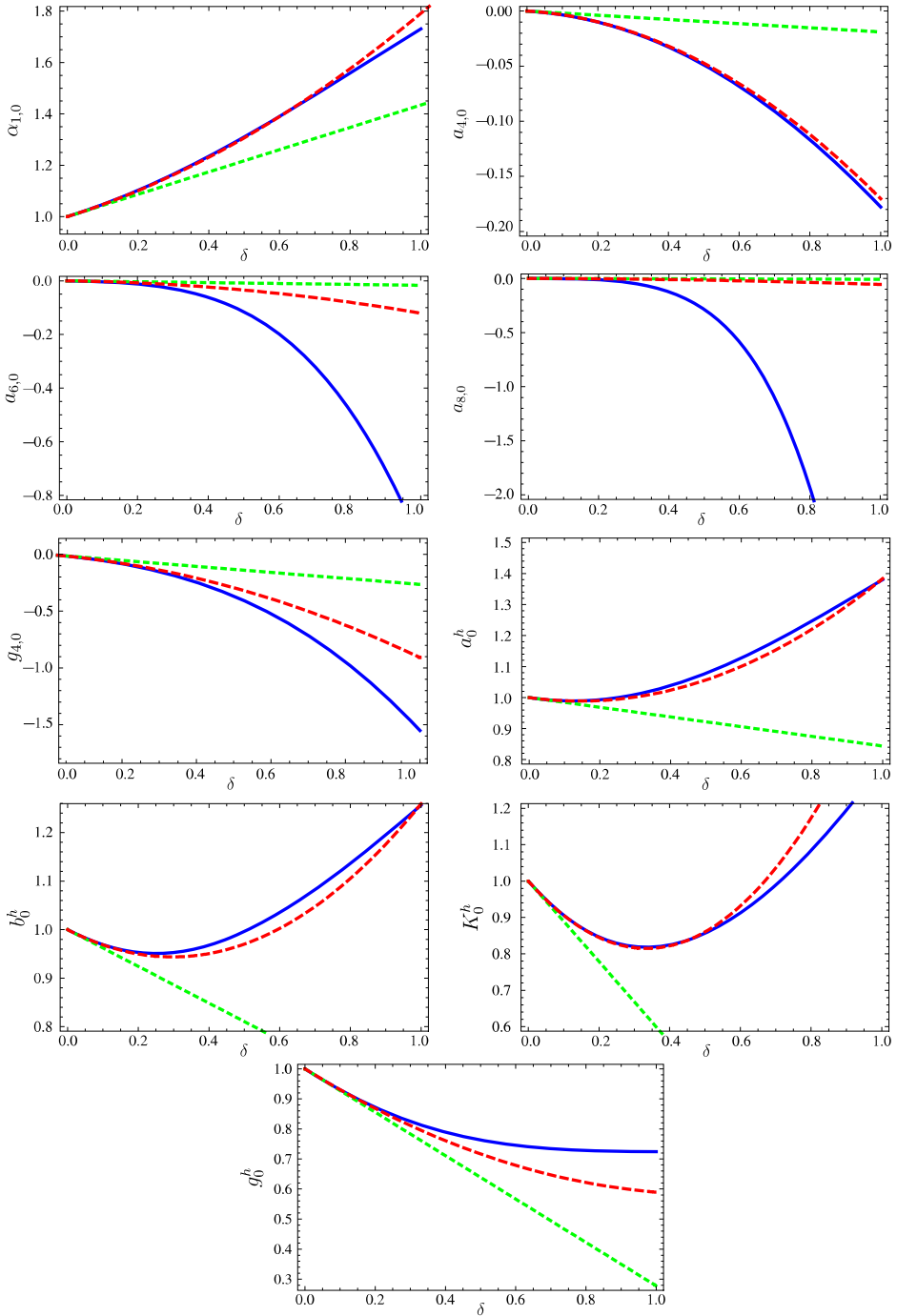


Fig. 1. (Colour online.) Comparison of values of UV parameters  $\{\alpha_{1,0}, a_{4,0}, a_{6,0}, a_{8,0}, g_{4,0}\}$  and IR parameters  $\{a_0^h, b_0^h, K_0^h, g_0^h\}$  (see (3.57)) in the range  $\delta \in [0, 1]$  (blue curves) with their perturbative predictions (3.74)–(3.75) at first (green dotted) and second order (red dashed) in  $\delta$ .

### 4. Cascading gauge theory on $dS_4$ with spontaneously broken chiral symmetry

#### 4.1. $R^{1,3} \rightarrow dS_4$ deformation of Klebanov–Strassler state of cascading gauge theory

$\mathcal{N} = 1$  supersymmetric ground state of cascading gauge theory on  $R^{3,1}$  — referred to as Klebanov–Strassler state — spontaneously breaks chiral symmetry [8]. A natural route to construct a  $\chi$ SB state of the theory on  $dS_4$  is to “deform” Klebanov–Strassler state:  $R^{1,3} \rightarrow dS_4$ . We explain now how to achieve this in a “continuous” fashion.

Consider the five-dimensional metric of the type:

$$ds_5^2 = g_{\mu\nu}(y) dy^\mu dy^\nu = c_1^2 \left( -dt^2 + \frac{1}{H^2} \cosh^2(Ht) (dS^3)^2 \right) + c_3^2 (d\rho)^2, \tag{4.1}$$

where  $c_i = c_i(\rho)$ . We will be interested in  $\chi$ SB states of cascading gauge theory on  $dS_4$  with a Hubble scale  $H$ . One can derive equations of motion from (2.1). Alternatively, we can construct an effective 1-dimensional action<sup>7</sup> from (2.1), by restricting to the metric ansatz (4.1), and the  $\rho$ -only dependence of the scalar fields  $\{\Phi, h_i, \Omega_i\}$ :

$$S_5[g_{\mu\nu}, \Omega_i, h_i, \Phi] \implies S_1[c_i, \Omega_i, h_i, \Phi]. \tag{4.2}$$

It can be verified that equations of motion obtained from  $S_1$  coincide with those obtained from (2.1), provided we vary<sup>8</sup>  $S_1$  with respect to  $c_3$ , treating it as an unconstrained field. The 1-dimensional effective action approach makes it clear that the only place where the information about  $dS_4$  enters is through the evaluation of  $R_5$  in (2.2):

$$R_5 = -\frac{8c_1''}{c_3^2 c_1} + \frac{8c_1' c_3'}{c_3^3 c_1} - \frac{12(c_1')^2}{c_3^2 c_1^2} + \frac{12\kappa}{c_1^2}, \tag{4.3}$$

where derivatives are with respect to  $\rho$ , and  $\kappa = H^2$ .

#### 4.2. Equations of motion

As in (3.1) and (2.8) we denote

$$\begin{aligned} c_1 &= h^{-1/4} \rho^{-1}, & c_3 &= h^{1/4} \rho^{-1}, & \Phi &= \ln g, \\ h_1 &= \frac{1}{P} \left( \frac{K_1}{12} - 36\Omega_0 \right), & h_2 &= \frac{P}{18} K_2, & h_3 &= \frac{1}{P} \left( \frac{K_3}{12} - 36\Omega_0 \right), \\ \Omega_1 &= \frac{1}{3} f_c^{1/2} h^{1/4}, & \Omega_2 &= \frac{1}{\sqrt{6}} f_a^{1/2} h^{1/4}, & \Omega_3 &= \frac{1}{\sqrt{6}} f_b^{1/2} h^{1/4}. \end{aligned} \tag{4.4}$$

The equations of motion obtained from  $S_1[c_i, \Omega_i, h_i, \Phi]$  are

$$\begin{aligned} 0 &= f_c'' - \frac{3f_c'}{\rho} - 3hf_c\kappa - \frac{(f_c')^2}{2f_c} + \frac{5f_c}{\rho^2} + \frac{f_c(g')^2}{8g^2} + \frac{3f_b'f_c'}{4f_b} + \frac{63f_a}{16f_b\rho^2} \\ &+ \frac{63f_b}{16f_a\rho^2} + \frac{3f_c}{f_a\rho^2} - \frac{f_c(f_a')^2}{8f_a^2} + \frac{3f_a'f_c'}{4f_a} + \frac{f_c(h')^2}{8h^2} - \frac{f_c(f_b')^2}{8f_b^2} \end{aligned}$$

<sup>7</sup> Effectively, in obtaining  $S_1$  we perform Kaluza–Klein-like reduction of the effective action  $S_5$  on  $dS_4$ .

<sup>8</sup> This produces the first order constraint similar to (3.10).

$$\begin{aligned}
& + \frac{3f_c}{f_b\rho^2} - \frac{63}{8\rho^2} - \frac{K_1^2}{8f_a^2h^2f_b^2\rho^2} + \frac{3gP^2}{2f_a^2h\rho^2} \\
& - \frac{f_c f_a f_b'}{2f_a f_b} - \frac{27K_1K_3}{32f_a h f_b g P^2 \rho^2} - \frac{K_2^2 K_1^2}{32f_a^2 h^2 f_b^2 \rho^2} \\
& + \frac{K_2 K_1^2}{8f_a^2 h^2 f_b^2 \rho^2} - \frac{K_2^2 K_3^2}{32f_a^2 h^2 f_b^2 \rho^2} - \frac{3f_c (K_1')^2}{32hf_b^2 g P^2} \\
& - \frac{3f_c (K_3')^2}{32f_a^2 h g P^2} + \frac{3gP^2 K_2^2}{8hf_b^2 \rho^2} + \frac{3gP^2 K_2^2}{8f_a^2 h \rho^2} - \frac{3gP^2 K_2}{2f_a^2 h \rho^2} \\
& - \frac{9f_c^2}{f_a f_b \rho^2} + \frac{f_c h'}{h\rho} + \frac{K_2^2 K_1 K_3}{16f_a^2 h^2 f_b^2 \rho^2} \\
& - \frac{K_2 K_1 K_3}{8f_a^2 h^2 f_b^2 \rho^2} - \frac{gP^2 f_c (K_2')^2}{12f_a h f_b} + \frac{27K_1^2}{64f_a h f_b g P^2 \rho^2} + \frac{27K_3^2}{64f_a h f_b g P^2 \rho^2}, \tag{4.5}
\end{aligned}$$

$$\begin{aligned}
0 = & f_a'' - \frac{45f_a^2}{16f_c f_b \rho^2} + \frac{f_a h'}{h\rho} + \frac{gP^2 (K_2')^2}{36hf_b} + \frac{5(K_3')^2}{32f_a h g P^2} \\
& - \frac{f_a f_b' f_c'}{4f_c f_b} - \frac{(f_a')^2}{8f_a} + \frac{5f_a}{\rho^2} - \frac{3f_a'}{\rho} \\
& - \frac{K_2^2 K_1^2}{32f_c f_a h^2 f_b^2 \rho^2} + \frac{K_2 K_1^2}{8f_c f_a h^2 f_b^2 \rho^2} - \frac{K_2^2 K_3^2}{32f_c f_a h^2 f_b^2 \rho^2} - \frac{3gP^2 K_2}{2f_c f_a h \rho^2} + \frac{3gP^2 K_2^2}{8f_c f_a h \rho^2} \\
& - \frac{9K_3^2}{64f_c h f_b g P^2 \rho^2} - \frac{9K_1^2}{64f_c h f_b g P^2 \rho^2} + \frac{3f_a}{f_b \rho^2} + \frac{3f_c}{f_b \rho^2} \\
& + \frac{9K_1 K_3}{32f_c h f_b g P^2 \rho^2} + \frac{K_2^2 K_1 K_3}{16f_c f_a h^2 f_b^2 \rho^2} \\
& - \frac{K_2 K_1 K_3}{8f_c f_a h^2 f_b^2 \rho^2} - \frac{5f_a g P^2 K_2^2}{8f_c h f_b^2 \rho^2} - \frac{K_1^2}{8f_c f_a h^2 f_b^2 \rho^2} + \frac{3gP^2}{2f_c f_a h \rho^2} \\
& - \frac{3f_a (K_1')^2}{32hf_b^2 g P^2} - \frac{9}{\rho^2} + \frac{f_a (g')^2}{8g^2} \\
& - 3f_a h k + \frac{f_a f_b'}{2f_b} + \frac{f_c' f_a'}{4f_c} - \frac{f_a (f_b')^2}{8f_b^2} + \frac{9f_a}{8f_c \rho^2} + \frac{f_a (h')^2}{8h^2} + \frac{27f_b}{16f_c \rho^2}, \tag{4.6}
\end{aligned}$$

$$\begin{aligned}
0 = & f_b'' - \frac{3f_b'}{\rho} - \frac{(f_b')^2}{8f_b} + \frac{5f_b}{\rho^2} - \frac{45f_b^2}{16f_c f_a \rho^2} + \frac{f_b h'}{h\rho} - \frac{K_1^2}{8f_c h^2 f_a^2 f_b \rho^2} - \frac{3f_b (K_3')^2}{32hg f_a^2 P^2} \\
& - \frac{K_2^2 K_1^2}{32f_c h^2 f_a^2 f_b \rho^2} + \frac{K_2 K_1^2}{8f_c h^2 f_a^2 f_b \rho^2} - \frac{K_2^2 K_3^2}{32f_c h^2 f_a^2 f_b \rho^2} \\
& - \frac{9K_1^2}{64f_c h g f_a P^2 \rho^2} + \frac{3gP^2 K_2^2}{f_c h f_b \rho^2} \\
& - \frac{9K_3^2}{64f_c h g f_a P^2 \rho^2} - \frac{5g f_b P^2}{2f_c h f_a^2 \rho^2} + \frac{3f_b}{f_a \rho^2} + \frac{3f_c}{f_a \rho^2}
\end{aligned}$$



$$\begin{aligned}
 & -\frac{f_b f'_c f'_a}{4 f_c f_a} + \frac{5(K'_1)^2}{32 h g f_b P^2} + \frac{g P^2 (K'_2)^2}{36 h f_a} \\
 & -\frac{9}{\rho^2} + \frac{27 f_a}{16 f_c \rho^2} + \frac{9 f_b}{8 f_c \rho^2} + \frac{K_2^2 K_1 K_3}{16 f_c h^2 f_a^2 f_b \rho^2} \\
 & -\frac{K_2 K_1 K_3}{8 f_c h^2 f_a^2 f_b \rho^2} + \frac{5 g f_b P^2 K_2}{2 f_c h f_a^2 \rho^2} - \frac{5 g f_b P^2 K_2^2}{8 f_c h f_a^2 \rho^2} \\
 & + \frac{9 K_1 K_3}{32 f_c h g f_a P^2 \rho^2} + \frac{f_b (g')^2}{8 g^2} - 3 h f_b \kappa \\
 & + \frac{f'_a f'_b}{2 f_a} - \frac{f_b (f'_a)^2}{8 f_a^2} + \frac{f'_b f'_c}{4 f_c} + \frac{f_b (h')^2}{8 h^2}, \tag{4.7}
 \end{aligned}$$

$$\begin{aligned}
 0 = h'' & + \frac{K_2^2 K_1^2}{4 f_c f_a^2 f_b^2 h \rho^2} - \frac{K_2 K_1^2}{f_c f_a^2 f_b^2 h \rho^2} + \frac{K_2^2 K_3^2}{4 f_c f_a^2 f_b^2 h \rho^2} \\
 & + \frac{9 K_1^2}{16 f_c f_a f_b \rho^2 g P^2} + \frac{9 K_3^2}{16 f_c f_a f_b \rho^2 g P^2} \\
 & + \frac{2 h f'_c}{f_c \rho} + \frac{4 h f'_b}{f_b \rho} + \frac{4 h f'_a}{f_a \rho} + \frac{(K'_1)^2}{8 f_b^2 g P^2} + \frac{(K'_3)^2}{8 f_a^2 g P^2} + \frac{g P^2 K_2^2}{2 f_c f_b^2 \rho^2} \\
 & + \frac{g P^2 K_2^2}{2 f_c f_a^2 \rho^2} - \frac{2 g P^2 K_2}{f_c f_a^2 \rho^2} + \frac{f'_c h'}{2 f_c} \\
 & + \frac{h' f'_b}{f_b} + \frac{h' f'_a}{f_a} - \frac{16 h}{\rho^2} - \frac{(h')^2}{h} + 12 h^2 \kappa - \frac{K_2^2 K_1 K_3}{2 f_c f_a^2 f_b^2 h \rho^2} \\
 & + \frac{K_2 K_1 K_3}{f_c f_a^2 f_b^2 h \rho^2} + \frac{K_1^2}{f_c f_a^2 f_b^2 h \rho^2} \\
 & + \frac{2 g P^2}{f_c f_a^2 \rho^2} + \frac{g P^2 (K'_2)^2}{9 f_a f_b} - \frac{9 K_1 K_3}{8 f_c f_a f_b \rho^2 g P^2} - \frac{3 h'}{\rho}, \tag{4.8}
 \end{aligned}$$

$$\begin{aligned}
 0 = K_1'' & - \frac{g K_2^2 K_1 P^2}{f_c f_a^2 h \rho^2} + \frac{g K_2^2 K_3 P^2}{f_c f_a^2 h \rho^2} + \frac{4 g K_2 K_1 P^2}{f_c f_a^2 h \rho^2} - \frac{2 g K_2 K_3 P^2}{f_c f_a^2 h \rho^2} \\
 & - \frac{9 f_b K_1}{2 f_c f_a \rho^2} + \frac{9 f_b K_3}{2 f_c f_a \rho^2} \\
 & - \frac{4 g K_1 P^2}{f_c f_a^2 h \rho^2} + \frac{K'_1 f'_c}{2 f_c} - \frac{K'_1 g'}{g} - \frac{K'_1 h'}{h} + \frac{f'_a K'_1}{f_a} - \frac{3 K'_1}{\rho} - \frac{K'_1 f'_b}{f_b}, \tag{4.9}
 \end{aligned}$$

$$\begin{aligned}
 0 = K_3'' & + \frac{g K_2^2 K_1 P^2}{f_c f_b^2 h \rho^2} - \frac{g K_2^2 K_3 P^2}{f_c f_b^2 h \rho^2} - \frac{2 g K_2 K_1 P^2}{f_c f_b^2 h \rho^2} + \frac{9 f_a K_1}{2 f_c f_b \rho^2} - \frac{9 f_a K_3}{2 f_c f_b \rho^2} + \frac{K'_3 f'_c}{2 f_c} \\
 & - \frac{K'_3 g'}{g} + \frac{f'_b K'_3}{f_b} - \frac{K'_3 h'}{h} - \frac{3 K'_3}{\rho} - \frac{K'_3 f'_a}{f_a}, \tag{4.10}
 \end{aligned}$$

$$\begin{aligned}
0 = & K_2'' - \frac{9f_b K_2}{2f_c f_a \rho^2} - \frac{9f_a K_2}{2f_c f_b \rho^2} + \frac{9f_b}{f_c f_a \rho^2} - \frac{9K_2 K_1^2}{8f_c g P^2 h f_b f_a \rho^2} + \frac{9K_2 K_1 K_3}{4f_c g P^2 h f_b f_a \rho^2} \\
& - \frac{9K_2 K_3^2}{8f_c g P^2 h f_b f_a \rho^2} + \frac{9K_1^2}{4f_c g P^2 h f_b f_a \rho^2} - \frac{9K_1 K_3}{4f_c g P^2 h f_b f_a \rho^2} + \frac{K_2' f_c'}{2f_c} \\
& + \frac{K_2' g'}{g} - \frac{K_2' h'}{h} - \frac{3K_2'}{\rho}, \tag{4.11}
\end{aligned}$$

$$\begin{aligned}
0 = & g'' - \frac{g^2 P^2 K_2^2}{2f_c f_a^2 h \rho^2} - \frac{g^2 P^2 K_2^2}{2f_c f_b^2 h \rho^2} + \frac{2g^2 P^2 K_2}{f_c f_a^2 h \rho^2} \\
& + \frac{9K_1^2}{16f_c f_a f_b h \rho^2 P^2} + \frac{9K_3^2}{16f_c f_a f_b h \rho^2 P^2} \\
& - \frac{(g')^2}{g} - \frac{9K_1 K_3}{8f_c f_a f_b h \rho^2 P^2} + \frac{(K_3')^2}{8f_a^2 h P^2} + \frac{(K_1')^2}{8f_b^2 h P^2} \\
& - \frac{2g^2 P^2}{f_c f_a^2 h \rho^2} - \frac{g^2 P^2 (K_2')^2}{9f_a f_b h} + \frac{g' f_c'}{2f_c} \\
& + \frac{g' f_a'}{f_a} + \frac{g' f_b'}{f_b} - \frac{3g'}{\rho}. \tag{4.12}
\end{aligned}$$

Additionally, we have the first order constraint

$$\begin{aligned}
0 = & \frac{8}{9} g^2 (K_2')^2 f_b f_a P^4 + (K_3')^2 f_b^2 + (K_1')^2 f_a^2 - \frac{4g^2 K_2^2 f_a^2 P^4}{f_c \rho^2} + \frac{4g f_a^2 f_b^2 P^2 (h')^2}{h} \\
& + \frac{4h (g')^2 f_a^2 f_b^2 P^2}{g} + \frac{96hg f_a^2 f_b P^2}{\rho^2} + \frac{96hg f_a f_b^2 P^2}{\rho^2} \\
& - \frac{96hg f_a^2 f_b^2 P^2}{\rho^2} - \frac{4g K_1^2 P^2}{f_c h \rho^2} \\
& + 96h^2 g f_a^2 f_b^2 P^2 \kappa + \frac{9K_1 K_3 f_b f_a}{f_c \rho^2} + \frac{32g f_a^2 f_b^2 P^2 h'}{\rho} \\
& + \frac{16g^2 K_2 f_b^2 P^4}{f_c \rho^2} - \frac{4g^2 K_2^2 f_b^2 P^4}{f_c \rho^2} \\
& - \frac{g K_2^2 K_1^2 P^2}{f_c h \rho^2} + \frac{4g K_2 K_1^2 P^2}{f_c h \rho^2} - \frac{g K_2^2 K_3^2 P^2}{f_c h \rho^2} \\
& + \frac{64hg f_a^2 f_b P^2 f_b'}{\rho} + \frac{64hg f_a f_b^2 P^2 f_a'}{\rho} \\
& - 16hg f_a f_b P^2 f_a' f_b' - \frac{32f_c h g f_a f_b P^2}{\rho^2} - \frac{18hg f_a f_b^3 P^2}{f_c \rho^2} \\
& - \frac{18hg f_a^3 f_b P^2}{f_c \rho^2} + \frac{36hg f_a^2 f_b^2 P^2}{f_c \rho^2} - \frac{9K_3^2 f_b f_a}{f_c \rho^2} \\
& - 4hg f_b^2 P^2 (f_a')^2 - 4hg f_a^2 P^2 (f_b')^2 - \frac{16g^2 f_b^2 P^4}{f_c \rho^2} + \frac{2g K_2^2 K_1 K_3 P^2}{f_c h \rho^2}
\end{aligned}$$

$$\begin{aligned}
 & - \frac{4gK_2K_1K_3P^2}{f_c h \rho^2} - \frac{8hg f_b^2 f_a P^2 f'_c f'_a}{f_c} \\
 & + \frac{32hg f_a^2 f_b^2 P^2 f'_c}{f_c \rho} - \frac{8hg f_a^2 f_b P^2 f'_b f'_c}{f_c} - \frac{9K_1^2 f_b f_a}{2f_c \rho^2}.
 \end{aligned} \tag{4.13}$$

We explicitly verified that for any value  $\kappa$  the constraint (4.13) is consistent with (4.5)–(4.9). Moreover, with

$$f_c = f_2, \quad f_a = f_b = f_3, \quad K_1 = K_3 = K, \quad K_2 = 1, \tag{4.14}$$

Eqs. (4.5)–(4.13) are equivalent to (3.5)–(3.10).

### 4.3. UV asymptotics

The general UV (as  $\rho \rightarrow 0$ ) asymptotic solution of (4.5)–(4.13) describing the phase of cascading gauge theory with spontaneously broken chiral symmetry takes the form

$$\begin{aligned}
 f_c &= 1 - \alpha_{1,0}\rho + \left( -\frac{3}{8}g_0P^2 - \frac{1}{4}K_0 + \frac{1}{4}\alpha_{1,0}^2 + \frac{1}{2}P^2g_0 \ln \rho \right) \rho^2 \\
 &+ \frac{1}{4}P^2\alpha_{1,0}g_0\rho^3 + \sum_{n=4}^{\infty} \sum_k f_{c,n,k} \rho^n \ln^k \rho,
 \end{aligned} \tag{4.15}$$

$$\begin{aligned}
 f_a &= 1 - \alpha_{1,0}\rho + \left( -\frac{1}{2}g_0P^2 - \frac{1}{4}K_0 + \frac{1}{4}\alpha_{1,0}^2 + \frac{1}{2}P^2g_0 \ln \rho \right) \rho^2 + f_{a,3,0}\rho^3 \\
 &+ \sum_{n=4}^{\infty} \sum_k f_{a,n,k} \rho^n \ln^k \rho,
 \end{aligned} \tag{4.16}$$

$$\begin{aligned}
 f_b &= 1 - \alpha_{1,0}\rho + \left( -\frac{1}{2}g_0P^2 - \frac{1}{4}K_0 + \frac{1}{4}\alpha_{1,0}^2 + \frac{1}{2}P^2g_0 \ln \rho \right) \rho^2 \\
 &+ \left( \frac{1}{2}P^2\alpha_{1,0}g_0 - f_{a,3,0} \right) \rho^3 + \sum_{n=4}^{\infty} \sum_k f_{b,n,k} \rho^n \ln^k \rho,
 \end{aligned} \tag{4.17}$$

$$\begin{aligned}
 h &= \frac{1}{8}g_0P^2 + \frac{1}{4}K_0 - \frac{1}{2}P^2g_0 \ln \rho + \left( -P^2g_0 \ln \rho + \frac{1}{2}K_0 \right) \alpha_{1,0}\rho \\
 &+ \left( \left( -\frac{1}{4}g_0P^2 - \frac{5}{4}P^2g_0 \ln \rho + \frac{5}{8}K_0 \right) \alpha_{1,0}^2 + \frac{119}{576}P^4g_0^2 \right. \\
 &+ \left. \frac{31}{96}P^2g_0K_0 + \frac{1}{8}K_0^2 + \frac{1}{2}P^4g_0^2 \ln \rho^2 - \frac{31}{48}P^4g_0^2 \ln \rho - \frac{1}{2} \ln \rho P^2g_0K_0 \right) \rho^2 \\
 &+ \left( \left( -\frac{5}{4}P^2g_0 \ln \rho - \frac{11}{24}g_0P^2 + \frac{5}{8}K_0 \right) \alpha_{1,0}^3 \right. \\
 &+ \left. \left( \frac{3}{2}P^4g_0^2 \ln \rho^2 - \frac{23}{16}P^4g_0^2 \ln \rho + \frac{19}{64}P^4g_0^2 - \frac{3}{2} \ln \rho P^2g_0K_0 + \frac{23}{32}P^2g_0K_0 \right. \right. \\
 &+ \left. \left. \frac{3}{8}K_0^2 \right) \alpha_{1,0} \right) \rho^3 + \sum_{n=4}^{\infty} \sum_k h_{n,k} \rho^n \ln^k \rho,
 \end{aligned} \tag{4.18}$$

$$\begin{aligned}
K_1 = & K_0 - 2P^2 g_0 \ln \rho - P^2 \alpha_{1,0} g_0 \rho \\
& + \left( -\frac{1}{4} P^2 \alpha_{1,0}^2 g_0 - \frac{1}{4} P^4 g_0^2 \ln \rho + \frac{9}{16} P^4 g_0^2 + \frac{1}{8} P^2 g_0 K_0 \right) \rho^2 \\
& + \left( -\frac{1}{12} \alpha_{1,0}^3 g_0 P^2 + \frac{1}{48} g_0 P^2 (-36 P^2 g_0 \ln \rho + 13 P^2 g_0 + 6 K_0) \alpha_{1,0} \right. \\
& \left. + \frac{1}{48} g_0 P^2 (96 f_{a,3,0} \ln \rho + 32 f_{a,3,0} + 32 k_{2,3,0}) \right) \rho^3 \\
& + \sum_{n=4}^{\infty} \sum_k k_{1,n,k} \rho^n \ln^k \rho,
\end{aligned} \tag{4.19}$$

$$\begin{aligned}
K_2 = & 1 + \left( k_{2,3,0} - \frac{3}{4} \alpha_{1,0} P^2 g_0 \ln \rho + 3 f_{a,3,0} \ln \rho \right) \rho^3 \\
& + \sum_{n=4}^{\infty} \sum_k k_{2,n,k} \rho^n \ln^k \rho,
\end{aligned} \tag{4.20}$$

$$\begin{aligned}
K_3 = & K_0 - 2P^2 g_0 \ln \rho - P^2 \alpha_{1,0} g_0 \rho \\
& + \left( -\frac{1}{4} P^2 \alpha_{1,0}^2 g_0 - \frac{1}{4} P^4 g_0^2 \ln \rho + \frac{9}{16} P^4 g_0^2 + \frac{1}{8} P^2 g_0 K_0 \right) \rho^2 \\
& + \left( -\frac{1}{12} \alpha_{1,0}^3 g_0 P^2 + \frac{1}{48} g_0 P^2 (12 P^2 g_0 \ln \rho + 29 P^2 g_0 + 6 K_0) \alpha_{1,0} \right. \\
& \left. - \frac{1}{48} g_0 P^2 (96 f_{a,3,0} \ln \rho + 32 f_{a,3,0} + 32 k_{2,3,0}) \right) \rho^3 \\
& + \sum_{n=4}^{\infty} \sum_k k_{3,n,k} \rho^n \ln^k \rho,
\end{aligned} \tag{4.21}$$

$$g = g_0 \left( 1 - \frac{1}{2} P^2 g_0 \rho^2 - \frac{1}{2} \alpha_{1,0} P^2 g_0 \rho^3 + \sum_{n=4}^{\infty} \sum_k g_{n,k} \rho^n \ln^k \rho \right). \tag{4.22}$$

It is characterized by 11 parameters:

$$\{K_0, H, g_0, \alpha_{1,0}, k_{2,3,0}, f_{c,4,0}, f_{a,3,0}, f_{a,6,0}, f_{a,7,0}, f_{a,8,0}, g_{4,0}\}. \tag{4.23}$$

In what follows we developed the UV expansion to order  $\mathcal{O}(\rho^{10})$  inclusive.

#### 4.4. IR asymptotics

As in Section 3.3, we use a radial coordinate  $\rho$  that extends to infinity, see (3.4). The crucial difference between the IR boundary conditions for a chirally symmetric phase discussed in Section 3.3 and the IR boundary conditions for a  $\chi$ SB phase discussed here is that in the former case the manifold  $\mathcal{M}_5$  geodesically completes with (a smooth) shrinking to zero size of  $dS_4 \subset \mathcal{M}_5$ , while in the latter case, much like in supersymmetric Klebanov–Strassler state of cascading gauge theory [8], the 10-dimensional uplift of  $\mathcal{M}_5$ ,

$$\mathcal{M}_5 \rightarrow \mathcal{M}_{10} = \mathcal{M}_5 \times X_5, \tag{4.24}$$

geodesically completes with (a smooth) shrinking of a 2-cycle in the compact manifold  $X_5$  [8].  
Introducing

$$y \equiv \frac{1}{\rho}, \quad h^h \equiv y^{-4}h, \quad f_{a,b,c}^h \equiv y^2 f_{a,b,c}, \tag{4.25}$$

the general IR (as  $y \rightarrow 0$ ) asymptotic solution of (4.5)–(4.13) describing the  $\chi$ SB phase of cascading gauge theory takes form

$$\begin{aligned} f_c^h = & \frac{3}{4} f_{a,0}^h + \left( -\frac{19(k_{2,2}^h)^2 P^2 g_0^h}{540 h_0^h} - \frac{3}{4} f_{a,0}^h h_0^h \kappa - \frac{3 f_{a,0}^h k_{2,4}^h}{2 k_{2,2}^h} - \frac{13 P^2 g_0^h}{15 (f_{a,0}^h)^2 h_0^h} + \frac{6}{5} \right. \\ & + \frac{f_{a,0}^h (k_{1,3}^h)^2}{64 P^2 g_0^h h_0^h} - \frac{27}{5 f_{a,0}^h k_{2,2}^h} + \frac{19 (k_{3,1}^h)^2}{320 P^2 f_{a,0}^h g_0^h h_0^h} + \left. \frac{3 k_{1,3}^h k_{3,1}^h}{20 k_{2,2}^h P^2 f_{a,0}^h g_0^h h_0^h} \right) y^2 \\ & + \sum_{n=2}^{\infty} f_{c,n}^h y^{2n}, \end{aligned} \tag{4.26}$$

$$\begin{aligned} f_a^h = & f_{a,0}^h + \left( \frac{17 (k_{2,2}^h)^2 P^2 g_0^h}{405 h_0^h} + 2 f_{a,0}^h h_0^h \kappa + \frac{f_{a,0}^h k_{2,4}^h}{k_{2,2}^h} \right. \\ & - \frac{4 P^2 g_0^h}{45 (f_{a,0}^h)^2 h_0^h} + \frac{11}{5} + \frac{f_{a,0}^h (k_{1,3}^h)^2}{48 P^2 g_0^h h_0^h} \\ & + \left. \frac{18}{5 f_{a,0}^h k_{2,2}^h} - \frac{17 (k_{3,1}^h)^2}{240 P^2 f_{a,0}^h g_0^h h_0^h} - \frac{k_{1,3}^h k_{3,1}^h}{10 k_{2,2}^h P^2 f_{a,0}^h g_0^h h_0^h} \right) y^2 \\ & + \sum_{n=2}^{\infty} f_{a,n}^h y^{2n}, \end{aligned} \tag{4.27}$$

$$f_b = 3 y^2 + \sum_{n=2}^{\infty} f_{b,n}^h y^{2n}, \tag{4.28}$$

$$\begin{aligned} h^h = & h_0^h + \left( -\frac{g_0^h P^2 (k_{2,2}^h)^2}{27 f_{a,0}^h} - 2 \kappa (h_0^h)^2 - \frac{4 g_0^h P^2}{9 (f_{a,0}^h)^3} - \frac{(k_{1,3}^h)^2}{48 g_0^h P^2} - \frac{(k_{3,1}^h)^2}{16 g_0^h P^2 (f_{a,0}^h)^2} \right) y^2 \\ & + \sum_{n=2}^{\infty} h_n^h y^{2n}, \end{aligned} \tag{4.29}$$

$$K_1 = k_{1,3}^h y^3 + \sum_{n=2}^{\infty} k_{1,n}^h y^{2n+1}, \tag{4.30}$$

$$K_2 = k_{2,2}^h y^2 + k_{2,4}^h y^4 + \sum_{n=3}^{\infty} k_{2,n}^h y^{2n}, \tag{4.31}$$

$$\begin{aligned}
 K_3 = & k_{3,1}^h y + \left( \frac{41 P^2 g_0^h (k_{2,2}^h)^2 k_{3,1}^h}{810 f_{a,0}^h h_0^h} + \frac{4 P^2 g_0^h k_{2,2}^h k_{1,3}^h}{135 f_{a,0}^h h_0^h} + \frac{7}{10} h_0^h k_{3,1}^h \kappa - \frac{1}{5} k_{1,3}^h + \frac{k_{2,4}^h k_{3,1}^h}{k_{2,2}^h} \right. \\
 & + \frac{2 P^2 g_0^h k_{3,1}^h}{15 (f_{a,0}^h)^3 h_0^h} + \frac{4 k_{3,1}^h}{5 f_{a,0}^h} + \frac{(k_{1,3}^h)^2 k_{3,1}^h}{480 P^2 g_0^h h_0^h} + \frac{18 k_{3,1}^h}{5 (f_{a,0}^h)^2 k_{2,2}^h} - \frac{41 (k_{3,1}^h)^3}{480 P^2 (f_{a,0}^h)^2 g_0^h h_0^h} \\
 & \left. - \frac{k_{1,3}^h (k_{3,1}^h)^2}{10 P^2 (f_{a,0}^h)^2 g_0^h h_0^h k_{2,2}^h} \right) y^3 + \sum_{n=2}^{\infty} k_{3,n}^h y^{2n+1}, \tag{4.32}
 \end{aligned}$$

$$\begin{aligned}
 g = & g_0^h \left( 1 + \left( \frac{P^2 g_0^h (k_{2,2}^h)^2}{27 f_{a,0}^h h_0^h} + \frac{4 P^2 g_0^h}{9 (f_{a,0}^h)^3 h_0^h} - \frac{(k_{1,3}^h)^2}{48 P^2 h_0^h g_0^h} - \frac{(k_{3,1}^h)^2}{16 P^2 (f_{a,0}^h)^2 h_0^h g_0^h} \right) y^2 \right. \\
 & \left. + \sum_{n=2}^{\infty} g_n^h y^{2n} \right). \tag{4.33}
 \end{aligned}$$

Notice that the prescribed IR boundary conditions imply

$$\lim_{y \rightarrow 0} \Omega_3^2 = \lim_{y \rightarrow 0} \frac{1}{6} f_b h^{1/2} = \lim_{y \rightarrow 0} \frac{y^2}{6} f_b (h^h)^{1/2} = 0, \tag{4.34}$$

with all the other warp factors in (2.4) being finite. Moreover, see (2.4),

$$\lim_{y \rightarrow 0} (\Omega_1^2 g_5^2 + \Omega_2^2 [g_3^2 + g_4^2]) = \frac{1}{6} f_{a,0}^h (h_0^h)^{1/2} \left( \frac{1}{2} g_5^2 + g_3^2 + g_4^2 \right), \tag{4.35}$$

which is the metric of the round  $S^3$  which stays of finite size in the deep infrared as the 2-cycle fibered over it (smoothly) shrinks to zero size (4.34). Asymptotic solution (4.26)–(4.33) is characterized by 7 additional parameters:

$$\{ f_{a,0}^h, h_0^h, k_{1,3}^h, k_{2,2}^h, k_{2,4}^h, k_{3,1}^h, g_0^h \}. \tag{4.36}$$

In what follows we developed the IR expansion to order  $\mathcal{O}(y^{10})$  inclusive.

#### 4.5. Symmetries and numerical procedure

The background geometry (4.4) dual to a phase of cascading gauge theory with spontaneously broken chiral symmetry on  $dS_4$  enjoys all the symmetries, properly generalized, discussed in Section 3.4:

- $P \rightarrow \lambda P, \quad g \rightarrow \frac{1}{\lambda} g,$   
 $\{\rho, f_{a,b,c}, h, K_{1,2,3}\} \rightarrow \{\rho, f_{a,b,c}, h, K_{1,2,3}\},$  (4.37)

- $P \rightarrow \lambda P, \quad \rho \rightarrow \frac{1}{\lambda} \rho,$   
 $\{h, K_{1,3}\} \rightarrow \lambda^2 \{h, K_{1,3}\}, \quad \{f_{a,b,c}, K_2, g\} \rightarrow \{f_{a,b,c}, K_2, g\},$  (4.38)

- $\rho \rightarrow \lambda \rho, \quad H \rightarrow \frac{1}{\lambda} H,$   
 $\{P, f_{a,b,c}, h, K_{1,2,3}, g\} \rightarrow \{P, f_{a,b,c}, h, K_{1,2,3}, g\},$  (4.39)

$$\bullet \quad \begin{pmatrix} P \\ \rho \\ h \\ f_{a,b,c} \\ K_{1,2,3} \\ g \end{pmatrix} \Rightarrow \begin{pmatrix} \hat{P} \\ \hat{\rho} \\ \hat{h} \\ \hat{f}_{a,b,c} \\ \hat{K}_{1,2,3} \\ \hat{g} \end{pmatrix} = \begin{pmatrix} P \\ \rho/(1 + \alpha\rho) \\ (1 + \alpha\rho)^4 h \\ (1 + \alpha\rho)^{-2} f_{a,b,c} \\ K_{1,2,3} \\ g \end{pmatrix}, \quad \alpha = \text{const.} \quad (4.40)$$

• Thus, much like in Section 3.4, we can set

$$g_0 = 1, \quad H = 1, \quad \frac{K_0}{P^2} = \ln \frac{1}{\Lambda^2 P^2} \equiv \frac{1}{\delta}, \quad (4.41)$$

The residual diffeomorphisms (4.40) are actually completely fixed once we insist on the IR asymptotics as in (4.26)–(4.33).

The numerical procedure for solving the background equations (4.5)–(4.13), subject to the boundary conditions (4.15)–(4.22) and (4.26)–(4.33) is identical to the one described earlier, see Section 3.6. Given (4.41), for a fixed  $\delta$ , the gravitational solution is characterized by 8 parameters in the UV and 7 parameters in the IR:

$$\begin{aligned} \text{UV: } & \{\alpha_{1,0}, k_{2,3,0}, f_{c,4,0}, f_{a,3,0}, f_{a,6,0}, f_{a,7,0}, f_{a,8,0}, g_{4,0}\}, \\ \text{IR: } & \{f_{a,0}^h, h_0^h, k_{1,3}^h, k_{2,2}^h, k_{2,4}^h, k_{3,1}^h, g_0^h\}. \end{aligned} \quad (4.42)$$

Notice that  $8 + 7 = 15$  is precisely the number of integration constants needed to specify a solution to (4.5)–(4.13) — we have 8 second order differential equations and a single first order differential constraint:  $2 \times 8 - 1 = 15$ .

In practice, we replace the second-order differential equation for  $f_c$  (4.5) with the constraint equation (4.13), which we use to algebraically eliminate  $f'_c$  from (4.6)–(4.12). The solution is found using the “shooting” method as detailed in [16].

Ultimately, we are interested in the solution at  $\kappa = H^2 = 1$ . Finding such a “shooting” solution in 15-dimensional parameter space (4.42) is quite challenging. Thus, we start with the analytic result for  $\kappa = 0$  (the Klebanov–Strassler state of cascading gauge theory), and a fixed value of  $\delta$ , and slowly increase  $\kappa$  to  $\kappa = 1$ . We further use the obtained solution as a starting point to explore other values of  $\delta$ .

#### 4.6. $\kappa$ -deformation of Klebanov–Strassler state

We begin with mapping the Klebanov–Strassler solution [8] to a  $\kappa = 0$  solution of (4.5)–(4.13). We set

$$g_0 = 1, \quad P = 1. \quad (4.43)$$

$\mathcal{N} = 1$  supersymmetric Klebanov–Strassler solution takes form<sup>9</sup>:

$$\begin{aligned} ds_5^2 &= H_{\text{KS}}^{-1/2} (-dt^2 + dx_1^2 + dx_2^2 + dx_3^2) + H_{\text{KS}}^{1/2} \omega_{1,\text{KS}}^2 dr^2, \\ \Omega_i &= \omega_{i,\text{KS}} H_{\text{KS}}^{1/2}, \quad h_i = h_{i,\text{KS}}, \end{aligned} \quad (4.44)$$

<sup>9</sup> See Eqs. (2.22) and (2.34) in [18].

$$\begin{aligned}
 h_{1,KS} &= \frac{\cosh r - 1}{18 \sinh r} \left( \frac{r \cosh r}{\sinh r} - 1 \right), & h_{2,KS} &= \frac{1}{18} \left( 1 - \frac{r}{\sinh r} \right), \\
 h_{3,KS} &= \frac{\cosh r + 1}{18 \sinh r} \left( \frac{r \cosh r}{\sinh r} - 1 \right), & g &= 1, \\
 \omega_{1,KS} &= \frac{\epsilon^{2/3}}{\sqrt{6} \hat{K}_{KS}}, & \omega_{2,KS} &= \frac{\epsilon^{2/3} \hat{K}_{KS}^{1/2}}{\sqrt{2}} \cosh \frac{r}{2}, \\
 \omega_{3,KS} &= \frac{\epsilon^{2/3} \hat{K}_{KS}^{1/2}}{\sqrt{2}} \sinh \frac{r}{2}, & &
 \end{aligned} \tag{4.45}$$

with

$$\begin{aligned}
 \hat{K}_{KS} &= \frac{(\sinh(2r) - 2r)^{1/3}}{2^{1/3} \sinh r}, \\
 H'_{KS} &= \frac{16((9h_{2,KS} - 1)h_{1,KS} - 9h_{3,KS}h_{2,KS})}{9\epsilon^{8/3} \hat{K}_{KS}^2 \sinh^2 r}, \\
 \Omega_0 &= 0,
 \end{aligned} \tag{4.46}$$

where now  $r \rightarrow \infty$  is the boundary and  $r \rightarrow 0$  is the IR. Above solution is parametrized by a single constant  $\epsilon$  which will be mapped to  $K_0$ , and which in turn will determine all the parameters in (4.42) once  $\kappa = 0$ .

Comparing the metric ansatz in (4.44) and (4.1), (4.4) we identify

$$\frac{(d\rho)^2}{\rho^4} = (w_{1,KS}(r))^2 (dr)^2. \tag{4.47}$$

Introducing

$$z \equiv e^{-r/3}, \tag{4.48}$$

we find from (4.47)

$$\frac{1}{\rho} = \frac{\sqrt{6}(2\epsilon)^{2/3}}{4} \int_1^z du \frac{u^6 - 1}{u^2(1 - u^{12} + 12u^6 \ln u)^{1/3}}. \tag{4.49}$$

In the UV,  $r \rightarrow \infty$ ,  $z \rightarrow 0$  and  $\rho \rightarrow 0$  we have

$$\begin{aligned}
 e^{-r/3} \equiv z &= \frac{\sqrt{6}(2\epsilon)^{2/3}}{4} \rho \left( 1 + Q\rho + Q^2\rho^2 + Q^3\rho^3 + Q^4\rho^4 + Q^5\rho^5 \right. \\
 &+ \left( \frac{27}{80}\epsilon^4 \ln 3 + Q^6 + \frac{27}{800}\epsilon^4 - \frac{9}{16}\epsilon^4 \ln 2 + \frac{9}{20}\epsilon^4 \ln \epsilon + \frac{27}{40}\epsilon^4 \ln \rho \right) \rho^6 \\
 &+ \left( -\frac{63}{16}\epsilon^4 Q \ln 2 + \frac{189}{80}\epsilon^4 Q \ln 3 + Q^7 \right. \\
 &+ \left. \frac{729}{800} Q \epsilon^4 + \frac{63}{20}\epsilon^4 Q \ln \epsilon + \frac{189}{40} Q \epsilon^4 \ln \rho \right) \rho^7 \\
 &+ \left( \frac{2403}{400}\epsilon^4 Q^2 - \frac{63}{4}\epsilon^4 Q^2 \ln 2 + \frac{189}{20}\epsilon^4 Q^2 \ln 3 \right. \\
 &+ \left. \frac{63}{5}\epsilon^4 Q^2 \ln \epsilon + Q^8 + \frac{189}{10}\epsilon^4 Q^2 \ln \rho \right) \rho^8
 \end{aligned}$$



$$\begin{aligned}
 & + \left( \frac{189}{5} \epsilon^4 \mathcal{Q}^3 \ln \epsilon + \frac{9729}{400} \epsilon^4 \mathcal{Q}^3 - \frac{189}{4} \epsilon^4 \mathcal{Q}^3 \ln 2 \right. \\
 & \left. + \frac{567}{20} \epsilon^4 \mathcal{Q}^3 \ln 3 + \mathcal{Q}^9 + \frac{567}{10} \epsilon^4 \mathcal{Q}^3 \ln \rho \right) \rho^9 + \mathcal{O}(\rho^{10} \ln \rho),
 \end{aligned} \tag{4.50}$$

where

$$\begin{aligned}
 \mathcal{Q} &= \frac{\sqrt{6}(2\epsilon)^{2/3}}{4} \left\{ \int_0^1 du \left( \frac{1-u^6}{u^2(1-u^{12}+12u^6 \ln u)^{1/3}} - \frac{1}{u^2} \right) - 1 \right\} \\
 &= -\frac{\sqrt{6}(2\epsilon)^{2/3}}{4} \times 0.839917(9).
 \end{aligned} \tag{4.51}$$

In the IR,  $r \rightarrow 0$ ,  $z \rightarrow 1_-$  and  $\frac{1}{\rho} \rightarrow 0$  we have

$$r = \frac{\sqrt{6} \cdot 2^{1/3}}{3^{1/3} \cdot \epsilon^{2/3}} y \left( 1 - \frac{2^{2/3} \cdot 3^{1/3}}{15 \cdot \epsilon^{4/3}} y^2 + \frac{71 \cdot 3^{2/3} \cdot 2^{1/3}}{2625 \cdot \epsilon^{8/3}} y^4 + \mathcal{O}(y^6) \right). \tag{4.52}$$

Using (4.50) and (4.52), and the exact analytic solution describing the Klebanov–Strassler state of cascading gauge theory (4.45), (4.46) we can identify parameters<sup>10</sup> (4.42)

$$\begin{aligned}
 K_0 &= -\ln 3 + \frac{5}{3} \ln 2 - \frac{4}{3} \ln \epsilon - \frac{2}{3}, & a_{1,0} &= 2\mathcal{Q}, \\
 k_{2,3,0} &= \frac{3\sqrt{6}}{8} \epsilon^2 (3 \ln 3 - 5 \ln 2 + 4 \ln \epsilon), & f_{c,4,0} &= 0, & f_{a,3,0} &= \frac{3\sqrt{6}}{4} \epsilon^2, \\
 f_{a,6,0} &= \left( -\frac{27}{16} \ln 2 + \frac{81}{50} + \frac{81}{80} \ln 3 + \frac{27}{20} \ln \epsilon \right) \epsilon^4 + \frac{3\sqrt{6}}{4} \mathcal{Q}^3 \epsilon^2, \\
 f_{a,7,0} &= \frac{3}{800} \mathcal{Q} (2268 - 1800 \ln 2 + 1440 \ln \epsilon + 1080 \ln 3) \epsilon^4 + \frac{3\sqrt{6}}{4} \epsilon^2 \mathcal{Q}^4, \\
 f_{a,8,0} &= \frac{3}{32} \mathcal{Q}^2 (270 - 180 \ln 2 + 108 \ln 3 + 144 \ln \epsilon) \epsilon^4 + \frac{3\sqrt{6}}{4} \mathcal{Q}^5 \epsilon^2, \\
 g_{4,0} &= 0,
 \end{aligned} \tag{4.53}$$

in the UV, and

$$\begin{aligned}
 f_{a,0}^h &= 2^{1/3} \cdot 3^{2/3} \cdot \epsilon^{4/3}, & h_0^h &= \epsilon^{-8/3} \times 0.056288(0), \\
 k_{1,3}^h &= \frac{4\sqrt{6}}{9 \cdot \epsilon^2}, & k_{2,2}^h &= \frac{2^{2/3}}{3^{2/3} \cdot \epsilon^{4/3}}, & k_{2,4}^h &= -\frac{11 \cdot 2^{1/3} \cdot 3^{2/3}}{45 \cdot \epsilon^{8/3}}, \\
 k_{3,1}^h &= \frac{4\sqrt{6} \cdot 2^{1/3} \cdot 3^{2/3}}{27 \cdot \epsilon^{2/3}}, & g_0^h &= 1,
 \end{aligned} \tag{4.54}$$

in the IR. Notice that inverting the first identification in (4.53),  $\epsilon = \epsilon(K_0)$ , we obtain a prediction for all the parameters (4.42) as a function of  $K_0$ .

Figs. 2 and 3 compare the results of select UV and IR parameters in (4.42) obtained numerically (blue dots) with analytic predictions (red curves) (4.53) and (4.54) for the supersymmetric

<sup>10</sup> We matched the asymptotic expansions (4.15)–(4.22) and (4.26)–(4.33) with the exact solution (4.45) to the order we developed them:  $\mathcal{O}(\rho^{10})$  and  $\mathcal{O}(y^{10})$  correspondingly.

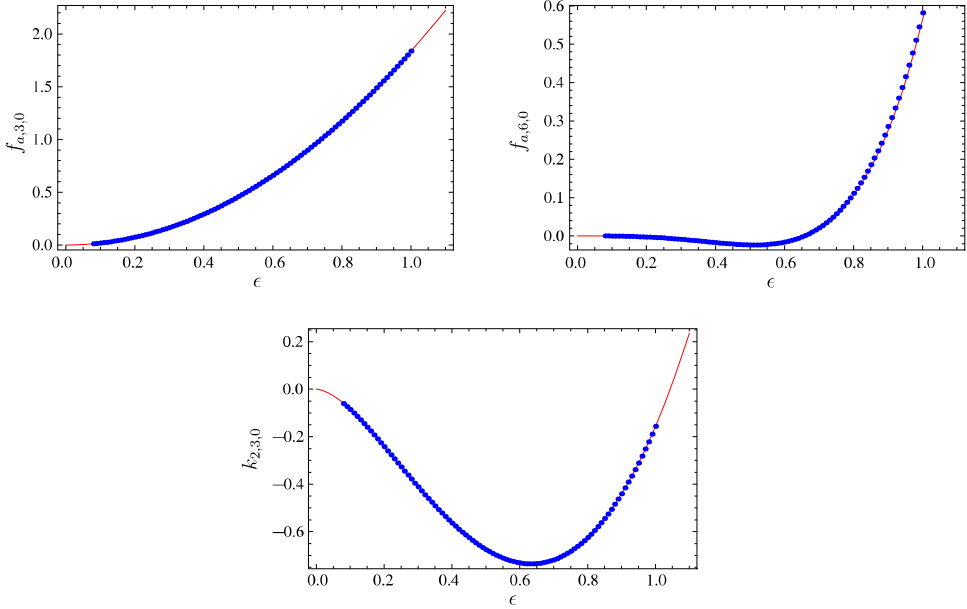


Fig. 2. (Colour online.) Comparison of values of select UV parameters  $\{f_{a,3,0}, f_{a,6,0}, k_{2,3,0}\}$  of Klebanov–Strassler state obtained numerically (blue dots) with the analytic prediction (red curves), see (4.53).

Klebanov–Strassler state. In this numerical computation we must set  $\kappa = 0$ . Notice that in Klebanov–Strassler state the string coupling is identically constant, *i.e.*,  $g = 1$ . The latter in particular implies that  $g_{4,0} = 0$  and  $g_0^h = 1$ . To find our numerical solutions, we set those values as constants and eliminate the second order equation (4.12) for  $g$ , finding excellent agreement between the expected and the numerical result.

As we mentioned earlier, we are after the states of cascading gauge theory with broken chiral symmetry on  $dS_4$ , *i.e.*, the deformations of Klebanov–Strassler states at  $\kappa = 1$ . In practice we start with numerical Klebanov–Strassler state at  $K_0 = 0.25$  ( $P = 1$ ) and increase  $\kappa$  in increments of  $\delta\kappa = 10^{-3}$  up to  $\kappa = 1$ . The resulting state is then used as a starting point to explore the states of cascading gauge theory on  $dS_4$  with  $\chi$ SB for other values of  $K_0 \neq 0.25$ .

### 5. Ground state of cascading gauge theory on $dS_4$

Recall that effective potential  $\mathcal{V}_{eff}$  of a theory on  $dS_4$  is defined (by analogy with the free energy density in thermodynamics) via

$$e^{-V_4^E \mathcal{V}_{eff}} = \mathcal{Z}_E, \tag{5.1}$$

where  $\mathcal{Z}_E$  is a Euclidean partition function of the theory on  $dS_4$ , and  $V_4^E$  is a volume of the analytically continued de Sitter,  $dS_4 \rightarrow S^4$ ,

$$V_4^E = \frac{8\pi^2}{3H^4}. \tag{5.2}$$

For a cascading gauge theory with a dual gravitational action given by (2.1), the effective potential is

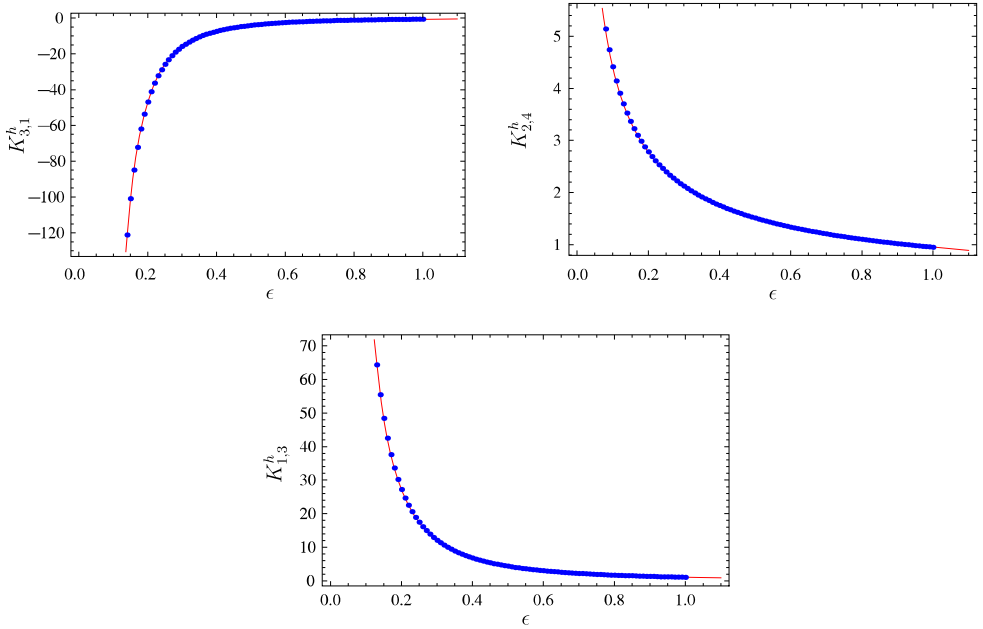


Fig. 3. (Colour online.) Comparison of values of select IR parameters  $\{K_{3,1}^h, K_{2,4}^h, K_{1,3}^h\}$  of Klebanov–Strassler state obtained numerically (blue dots) with the analytic prediction (red curves), see (4.54).

$$\mathcal{V}_{\text{eff}} = \int_{\rho_{\text{UV}}}^{\infty} d\rho \mathcal{L}_E, \tag{5.3}$$

where  $\mathcal{L}_E$  is the Euclidean one-dimensional Lagrangian density corresponding to the state, and  $\rho_{\text{UV}}$  is the UV cut-off, regularizing the Euclidean gravitational action in (5.3). Briefly, holographic renormalization of the theory modifies the effective potential

$$\int_{\rho_{\text{UV}}}^{\infty} d\rho \mathcal{L}_E \rightarrow \int_{\rho_{\text{UV}}}^{\infty} d\rho \mathcal{L}_E + S_{\text{GH}}^{\rho_{\text{UV}}} + S_{\text{counterterms}}^{\rho_{\text{UV}}}, \tag{5.4}$$

to include the Gibbons–Hawking and the local counterterms at the cut-off boundary  $\rho = \rho_{\text{UV}}$  in a way that would render the renormalized effective potential finite in the limit  $\rho_{\text{UV}} \rightarrow 0$ .

Here, we have to distinguish two states of cascading gauge theory: with broken (we use the superscript  $^b$ ) and the unbroken (we use the superscript  $^s$ ) chiral symmetry. These states are constructed (numerically) in Sections 4 and 3 correspondingly. Given a cascading gauge theory on  $dS_4$ , *i.e.*, having fixed its strong coupling scale  $\Lambda$ , the dilaton asymptotic value  $g_0$ , the rank offset parameter  $P$ , and the Hubble scale  $H$ , the true ground state of the theory minimizes the effective potential  $\mathcal{V}_{\text{eff}}$ .

We now present some computational details of  $\mathcal{V}_{\text{eff}}^b$  — the effective potential of the state of cascading gauge theory on  $dS_4$  with (spontaneously) broken chiral symmetry. Using the equations of motion (4.5)–(4.13), it is possible to show that the on-shell gravitational Lagrangian (2.1) takes form

$$\begin{aligned} \mathcal{L}_E^b &= \frac{108}{16\pi G_5} \times \left( \frac{d}{d\rho} \left( \frac{2c_1^3 c_1' \Omega_1 \Omega_2^2 \Omega_3^2}{c_3} \right) - 6\kappa c_1^2 c_3 \Omega_1 \Omega_2^2 \Omega_3^2 \right) \\ &= -\frac{108}{16\pi G_5} \times \left( \frac{d}{d\rho} \left( \frac{f_c^{1/2} f_a f_b (\rho h' + 4h)}{216h\rho^4} \right) + \frac{\kappa}{18} \frac{h f_a f_b f_c^{1/2}}{\rho^3} \right), \end{aligned} \tag{5.5}$$

leading to

$$\begin{aligned} \frac{16\pi G_5}{108} \mathcal{V}_{eff}^b &= -\left( \frac{f_c^{1/2} f_a f_b (\rho h' + 4h)}{216h\rho^4} \right) \Big|_{\rho=\rho_{UV}}^\infty - \frac{\kappa}{18} \int_{\rho_{UV}}^\infty d\rho \frac{h f_a f_b f_c^{1/2}}{\rho^3} \\ &= -\left( \frac{f_c^{1/2} f_a f_b (\rho h' + 4h)}{216h\rho^4} \right) \Big|_{\rho=\rho_{UV}}^\infty - \frac{\kappa}{18} \int_{\rho_{UV}}^\infty d\rho \frac{h f_a f_b f_c^{1/2}}{\rho^3}, \end{aligned} \tag{5.6}$$

where we used the fact that (see (4.26)–(4.29))

$$\lim_{\rho \rightarrow \infty} \frac{f_c^{1/2} f_a f_b (\rho h' + 4h)}{216h\rho^4} = -\lim_{y \rightarrow 0} \frac{(f_c^h)^{1/2} f_a^h f_b^h (h^h)'}{216h^h} = 0. \tag{5.7}$$

Both terms in (5.6) are divergent as  $\rho_{UV} \rightarrow 0$ . First, using the asymptotic expansion (4.15)–(4.18), we isolate the divergence of the integral in (5.6):

$$\mathcal{I}_{\rho_{UV}}^b \equiv -6\kappa \int_{\rho_{UV}}^1 d\rho \frac{f_a f_b f_c^{1/2} h}{\rho^3} \equiv \mathcal{I}_{finite}^b + \mathcal{I}_{\rho_{UV},divergent}^b + \mathcal{O}(\rho_{UV} \ln^2 \rho_{UV}), \tag{5.8}$$

$$\begin{aligned} \mathcal{I}_{finite}^b &= -6\kappa \int_0^1 d\rho \left( \frac{f_a f_b f_c^{1/2} h}{\rho^3} - \mathcal{J}_{divergent}^b \right), \\ \mathcal{J}_{divergent}^b &= \frac{1}{\rho^3} \left( \frac{1}{8} g_0 P^2 + \frac{1}{4} K_0 - \frac{1}{2} P^2 g_0 \ln \rho \right) \\ &\quad + \frac{1}{\rho^2} \left( \frac{1}{4} \alpha_{1,0}^b g_0 P^2 \ln \rho - \frac{1}{16} \alpha_{1,0}^b (5g_0 P^2 + 2K_0) \right) \\ &\quad + \frac{1}{\rho} \left( -\frac{1}{8} P^4 \kappa g_0^2 \ln^2 \rho + \left( \frac{1}{8} K_0 P^2 \kappa g_0 + \frac{5}{48} P^4 \kappa g_0^2 \right) \ln \rho \right) \\ &\quad + \frac{1}{16} (\alpha_{1,0}^b)^2 P^2 g_0 + \frac{67}{1152} P^4 \kappa g_0^2 - \frac{5}{96} K_0 P^2 \kappa g_0 - \frac{1}{32} K_0^2 \kappa, \end{aligned} \tag{5.9}$$

$$\begin{aligned} \mathcal{I}_{\rho_{UV},divergent}^b &= -6\kappa \int_{\rho_{UV}}^1 d\rho \mathcal{J}_{divergent}^b \\ &= \frac{1}{\rho_{UV}^2} \left( \frac{3}{2} \kappa g_0 P^2 \ln \rho_{UV} - \frac{3}{8} \kappa (-g_0 P^2 + 2K_0) \right) \\ &\quad + \frac{1}{\rho_{UV}} \left( -\frac{3}{2} \kappa \alpha_{1,0}^b g_0 P^2 \ln \rho_{UV} + \frac{3}{8} \kappa \alpha_{1,0}^b (g_0 P^2 + 2K_0) \right) \end{aligned}$$

$$\begin{aligned}
& -\frac{1}{4}\kappa^2 P^4 g_0^2 \ln^3 \rho_{UV} - \frac{1}{192}\kappa(-72K_0 P^2 \kappa g_0 - 60P^4 \kappa g_0^2) \ln^2 \rho_{UV} \\
& - \frac{1}{192}\kappa(-72(\alpha_{1,0}^b)^2 P^2 g_0 - 67P^4 \kappa g_0^2 + 60K_0 P^2 \kappa g_0 + 36K_0^2 \kappa) \ln \rho_{UV} \\
& + \left\{ -\frac{1}{192}\kappa(-72\alpha_{1,0}^b g_0 P^2 - 144K_0 - 144\alpha_{1,0}^b K_0 + 72g_0 P^2) \right\}, \quad (5.10)
\end{aligned}$$

where in the last line we separated the finite piece coming from the upper limit of integration in  $\mathcal{I}_{\rho_{UV},divergent}^b$ . The superscript  $b$  in the UV parameter  $\alpha_{1,0}$  is used to indicate that it is computed in the phase with broken chiral symmetry. Combining the divergent terms in (5.10) with divergences of the boundary term in (5.6) we find

$$\begin{aligned}
\frac{16\pi G_5}{108} \mathcal{V}_{eff}^b = & \left\{ \mathcal{V}_{eff,-4}^b \frac{1}{\rho^4} + \mathcal{V}_{eff,-3}^b \frac{1}{\rho^3} + \mathcal{V}_{eff,-2}^b \frac{1}{\rho^2} + \mathcal{V}_{eff,-1}^b \frac{1}{\rho} + \mathcal{V}_{eff,0}^b \right. \\
& \left. + \mathcal{O}(\rho^0) \right\} \Big|_{\rho=\rho_{UV}}, \quad (5.11)
\end{aligned}$$

with

$$\mathcal{V}_{eff,-4}^b = \frac{K_0 - 2 \ln \rho}{27(1 + 2K_0 - 4 \ln \rho)}, \quad (5.12)$$

$$\begin{aligned}
\mathcal{V}_{eff,-3}^b = & \frac{\alpha_{1,0}^b}{27(1 + 2K_0 - 4 \ln \rho)^2} \\
& \times (16 \ln \rho^2 - (4(1 + 4K_0)) \ln \rho + 1 + 2K_0 + 4K_0^2), \quad (5.13)
\end{aligned}$$

$$\begin{aligned}
\mathcal{V}_{eff,-2}^b = & -\frac{1}{3888(1 + 2K_0 - 4 \ln \rho)^3} (6912 \ln \rho^4 - (192(37 + 72K_0 - 36(\alpha_{1,0}^b)^2)) \ln \rho^3 \\
& + (32(43 + 333K_0 - 108(\alpha_{1,0}^b)^2 + 324K_0^2 - 324K_0(\alpha_{1,0}^b)^2)) \ln \rho^2 \\
& - (4(-97 + 344K_0 - 360(\alpha_{1,0}^b)^2 + 1332K_0^2 - 864K_0(\alpha_{1,0}^b)^2 + 864K_0^3 \\
& - 1296K_0^2(\alpha_{1,0}^b)^2)) \ln \rho - 99 - 194K_0 + 36(\alpha_{1,0}^b)^2 + 344K_0^2 - 720K_0(\alpha_{1,0}^b)^2 \\
& + 888K_0^3 - 864K_0^2(\alpha_{1,0}^b)^2 + 432K_0^4 - 864K_0^3(\alpha_{1,0}^b)^2), \quad (5.14)
\end{aligned}$$

$$\begin{aligned}
\mathcal{V}_{eff,-1}^b = & \frac{\alpha_{1,0}^b}{3888(1 + 2K_0 - 4 \ln \rho)^4} \left( -27648 \ln \rho^5 + (1536(32 + 45K_0 - 6(\alpha_{1,0}^b)^2)) \ln \rho^4 \right. \\
& - (64(413 + 1536K_0 - 108(\alpha_{1,0}^b)^2 + 1080K_0^2 - 288K_0(\alpha_{1,0}^b)^2)) \ln \rho^3 \\
& + (48(161 + 826K_0 - 88(\alpha_{1,0}^b)^2 + 1536K_0^2 - 216K_0(\alpha_{1,0}^b)^2 + 720K_0^3 \\
& - 288K_0^2(\alpha_{1,0}^b)^2)) \ln \rho^2 - (16(134 + 483K_0 + 21(\alpha_{1,0}^b)^2 + 1239K_0^2 \\
& - 264K_0(\alpha_{1,0}^b)^2 + 1536K_0^3 - 324K_0^2(\alpha_{1,0}^b)^2 + 540K_0^4 - 288K_0^3(\alpha_{1,0}^b)^2)) \ln \rho \\
& + 301 + 1072K_0 - 300(\alpha_{1,0}^b)^2 + 1932K_0^2 + 168K_0(\alpha_{1,0}^b)^2 + 3304K_0^3 \\
& - 1056K_0^2(\alpha_{1,0}^b)^2 + 3072K_0^4 - 864K_0^3(\alpha_{1,0}^b)^2 \\
& \left. + 864K_0^5 - 576K_0^4(\alpha_{1,0}^b)^2 \right), \quad (5.15)
\end{aligned}$$

$$\mathcal{V}_{eff,0}^b = -\frac{1}{432} \ln \rho^3 + \frac{1}{3456} (13 + 12K_0) \ln \rho^2 - \frac{1}{82944} (103 + 312K_0 - 576(\alpha_{1,0}^b)^2 + 144K_0^2) \ln \rho, \tag{5.16}$$

where we set  $P = 1$ ,  $g_0 = 1$ ,  $\kappa = 1$ , and used (4.15)–(4.18). Turns out that all the divergences are removed once we include the generalized<sup>11</sup> Gibbons–Hawking term, see [23],

$$S_{GH}^{\rho_{UV}} = \frac{108}{8\pi G_5 c_3} (c_1^4 \Omega_1 \Omega_2^2 \Omega_3^2) \Big|_{\rho=\rho_{UV}} = \frac{1}{8\pi G_5} \frac{\rho}{h^{1/4}} \left( \frac{h^{1/4} f_c^{1/2} f_a f_b}{\rho^4} \right) \Big|_{\rho=\rho_{UV}}, \tag{5.17}$$

and the local counter-terms obtained in [23] with the following obvious modifications:

$$K^{KT} = \frac{1}{2} K_1 + \frac{1}{2} K_3, \quad \Omega_1^{KT} = 3\Omega_1, \quad \Omega_2^{KT} = \frac{\sqrt{6}}{2} (\Omega_2 + \Omega_3). \tag{5.18}$$

We find

$$\begin{aligned} 16\pi G_5 \mathcal{V}_{eff}^b &= 3f_{c,4,0} + \frac{9}{32} (\alpha_{1,0}^b)^2 + \frac{3}{16} K_0 (\alpha_{1,0}^b)^2 + \frac{59}{48} K_0 + \frac{805}{1152} - \frac{3}{4} \alpha_{1,0}^b K_0 \\ &\quad - \frac{3}{8} \alpha_{1,0}^b - \frac{1}{8} K_0^2 + \mathcal{I}_{finite}^b + \int_0^1 dy (-6h^h f_a^h f_b^h (f_c^h)^{1/2}) + \mathcal{V}_{ambiguity}^b, \\ \mathcal{V}_{ambiguity}^b &= -36\kappa_1^b K_0^2 - 36\kappa_2^b K_0 - 36\kappa_3^b, \end{aligned} \tag{5.19}$$

where  $\mathcal{V}_{ambiguity}^b$  comes from the renormalization scheme ambiguities  $\{\kappa_i^b\}$ , see [23]. Note that the ambiguities are completely specified by the gauge theory parameters, *i.e.*,  $\{K_0, P, g_0\}$  and the Hubble scale  $H$  (the non-normalizable coefficients of the holographic gravitational dual).

Identical analysis for the symmetric phase leads to

$$\begin{aligned} 16\pi G_5 \mathcal{V}_{eff}^s &= 3a_{4,0} + \frac{805}{1152} - \frac{3}{8} \alpha_{1,0}^s + \frac{59}{48} K_0 + \frac{9}{32} (\alpha_{1,0}^s)^2 - \frac{3}{4} \alpha_{1,0}^s K_0 + \frac{3}{16} (\alpha_{1,0}^s)^2 K_0 \\ &\quad - \frac{1}{8} K_0^2 + \mathcal{I}_{finite}^s + \int_0^1 dy (-6h^h (f_3^h)^2 (y f_2^h)^{1/2}) + \mathcal{V}_{ambiguity}^s, \\ \mathcal{I}_{finite}^s &= -6 \int_0^1 d\rho \left( \frac{f_3^2 f_2^{1/2} h}{\rho^3} - \mathcal{J}_{divergent}^s \right), \\ \mathcal{J}_{divergent}^s &= \frac{1}{\rho^3} \left( \frac{1}{8} + \frac{1}{4} K_0 - \frac{1}{2} \ln \rho \right) + \frac{1}{\rho^2} \left( \frac{1}{4} \alpha_{1,0}^s \ln \rho - \frac{1}{16} \alpha_{1,0}^s (5 + 2K_0) \right) \\ &\quad + \frac{1}{\rho} \left( -\frac{1}{8} \ln^2 \rho + \left( \frac{1}{8} K_0 + \frac{5}{48} \right) \ln \rho + \frac{1}{16} (\alpha_{1,0}^s)^2 \right) \\ &\quad + \frac{67}{1152} - \frac{5}{96} K_0 - \frac{1}{32} K_0^2, \\ \mathcal{V}_{ambiguity}^s &= -36\kappa_1^s K_0^2 - 36\kappa_2^s K_0 - 36\kappa_3^s. \end{aligned} \tag{5.20}$$

<sup>11</sup> “Generalized” five-dimensional Gibbons–Hawking term is just a dimensional reduction of the 10-dimensional Gibbons–Hawking term corresponding to (2.4).

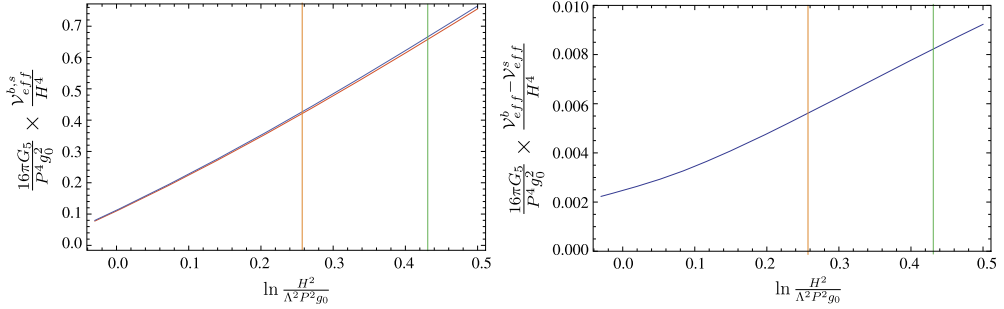


Fig. 4. (Colour online.) Left panel: effective potentials of the chirally symmetric ( $V_{eff}^s$ , red) and the broken phase ( $V_{eff}^b$ , blue) of the cascading gauge theory on  $dS_4$ . Right panel: the difference ( $V_{eff}^b - V_{eff}^s$ ). The vertical lines represent the first order chiral symmetry breaking phase transitions of cascading gauge theory on  $S^3$  [21] (green line) and at finite temperature [16] (orange line).

We can now compare the effective potentials of a chirally symmetric state and a state spontaneously breaking chiral symmetry for a cascading gauge theory on  $dS_4$  (we restored the full  $\{P, g_0, H\}$  dependence)

$$\begin{aligned}
 & 16\pi G_5 (\mathcal{V}_{eff}^b - \mathcal{V}_{eff}^s) \\
 &= 3(f_{c,4,0} - H^4 a_{4,0}) \\
 &+ \frac{3}{16} (-3P^2 \alpha_{1,0}^b g_0 + 2P^2 g_0 - 2K_0 \alpha_{1,0}^b + 4K_0) H^2 (H \alpha_{1,0}^s - \alpha_{1,0}^b) \\
 &- \frac{3}{32} (3P^2 g_0 + 2K_0) H^2 (H \alpha_{1,0}^s - \alpha_{1,0}^b)^2 + (\mathcal{I}_{finite}^b - \mathcal{I}_{finite}^s) \\
 &+ H^2 \left( \int_0^1 dy (-6h^h f_a^h f_b^h (f_c^h)^{1/2}) - \int_0^1 dy (-6h^h (f_3^h)^2 (y f_2^h)^{1/2}) \right), \tag{5.21}
 \end{aligned}$$

where we used the *same* renormalization scheme for computing both  $\mathcal{V}_{eff}^b$  and  $\mathcal{V}_{eff}^s$ , *i.e.*, we set

$$H^{-4} \kappa_i|^b = H^{-4} \kappa_i|^s, \quad i = 1, 2, 3. \tag{5.22}$$

Fig. 4 presents effective potentials (and their difference) between the state with spontaneously broken chiral symmetry,  $\mathcal{V}_{eff}^b$ , and the chirally symmetric state,  $\mathcal{V}_{eff}^s$ , of cascading gauge theory on  $dS_4$  as a function of  $\ln \frac{H^2}{\Lambda^2}$ . Over the range of  $\frac{H}{\Lambda}$  studied,<sup>12</sup>

$$\frac{16\pi G_5}{P^4 g_0^2} \times \frac{\mathcal{V}_{eff}^b - \mathcal{V}_{eff}^s}{H^4} > 0, \quad \ln \frac{H^2}{\Lambda^2} \geq -0.03, \tag{5.23}$$

implying that chirally symmetric phase is a true ground state of cascading gauge theory on  $dS_4$ . For comparison, the vertical green and orange lines indicate the first order chiral symmetry breaking phase transitions of cascading gauge theory on  $S^3$  [21] and at finite temperature [16].

<sup>12</sup> It is difficult to keep our current numerical procedure stable for smaller values of  $\frac{H}{\Lambda}$ .

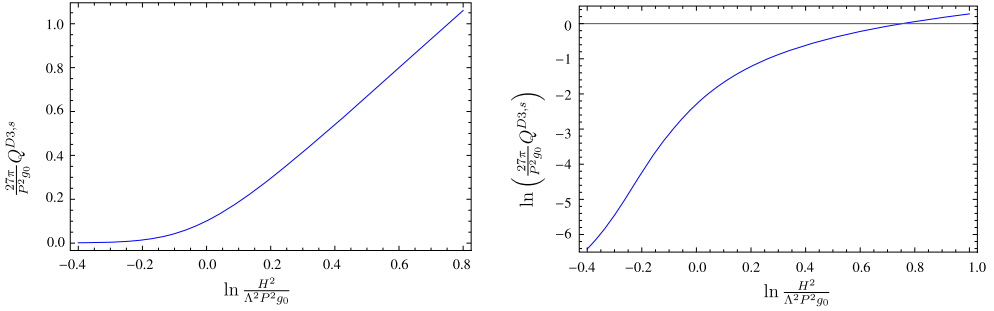


Fig. 5. (Colour online.) Left panel: D3 brane charge at the tip of the conifold of the  $dS_4$  deformed KT throat geometry,  $Q^{D3,s}$ , as a function of  $\frac{H}{\Lambda}$ . Right panel: logarithm of D3 brane charge at the tip of the conifold of the  $dS_4$  deformed KT throat geometry,  $Q^{D3,s}$ , as a function of  $\frac{H}{\Lambda}$ .

### 6. Properties of $dS_4$ deformed KT/KS geometries

Given numerical constructions of  $dS_4$  deformed KT/KS geometries as in Section 3, we can compute the D3 brane charge at the tip of the conifold. Following [10], we find (see (3.21))

$$Q^{D3,s} = \frac{1}{27\pi} \lim_{y \rightarrow 0} K(y) = \frac{K_0^h}{27\pi}, \tag{6.1}$$

and (see (4.30)–(4.32))

$$Q^{D3,b} = \frac{1}{54\pi} \lim_{y \rightarrow 0} (K_1(y)(2 - K_2(y)) + K_2(y)K_3(y)) = 0, \tag{6.2}$$

where we use superscripts  $b$  and  $s$  to denote chiral symmetry broken (deformed KS) and chiral symmetry unbroken (deformed KT) phases.

Fig. 5 presents D3 brane charge at the tip of the conifold of the  $dS_4$  deformed KT throat geometry,  $Q^{D3,s}$ , as a function of  $\frac{H}{\Lambda}$ . Note that over all the range of parameters accessible with our numerical code  $Q^{D3,s} > 0$ .

### 7. Properties of $S^3$ deformed KT/KS geometries

Using numerical constructions of  $S^3$  deformed KT/KS geometries presented in [21], we can compute the D3 brane charge at the tip of the conifold. Following [10], we find (see Eq. (3.24) of [21])

$$Q^{D3,s} = \frac{1}{27\pi} \lim_{y \rightarrow 0} K(y) = \frac{K_0^h}{27\pi}, \tag{7.1}$$

and (see Eqs. (5.34)–(5.36) of [21])

$$Q^{D3,b} = \frac{1}{54\pi} \lim_{y \rightarrow 0} (K_1(y)(2 - K_2(y)) + K_2(y)K_3(y)) = 0, \tag{7.2}$$

where we use superscripts  $b$  and  $s$  to denote chiral symmetry broken (deformed KS) and chiral symmetry unbroken (deformed KT) phases.



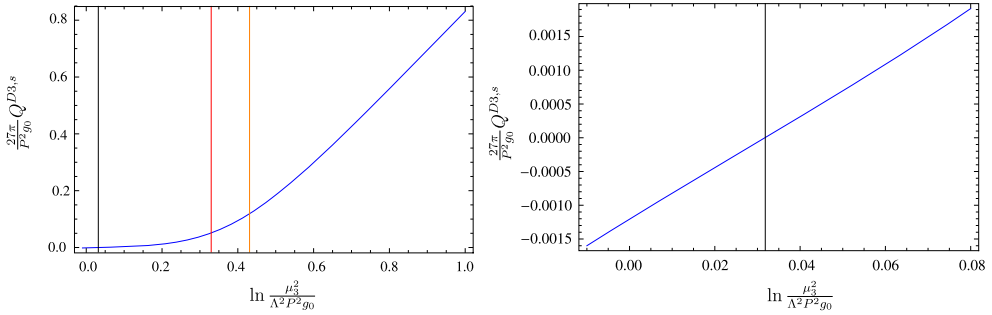


Fig. 6. (Colour online.) D3 brane charge at the tip of the conifold of the  $S^3$  deformed KT throat geometry,  $Q^{D3,s}$ , as a function of  $\frac{\mu_3}{\Lambda}$ . The vertical orange line represents the value of the compactification scale  $\mu_{3,\chi\text{SB}}$  below which it becomes energetically favourable to tunnel to  $S^3$  deformed KS throat geometry, with spontaneous breaking of chiral symmetry. The vertical red line represents the value of the compactification scale  $\mu_{3,tachyon}$  below which some of the linearized fluctuations (spontaneously breaking the chiral symmetry) become tachyonic. The vertical black lines denote the value of the compactification scale  $\mu_{3,negative}$  below which  $Q^{D3,s} < 0$ .

Fig. 6 presents D3 brane charge at the tip of the conifold of the  $S^3$  deformed KT throat geometry,  $Q^{D3,s}$ , as a function of  $\frac{\mu_3}{\Lambda}$ . Here, unlike the  $dS_4$  deformed KT throat geometry, we find that  $Q^{D3,s}$  can become negative! This happens whenever

$$\mu_3 < \mu_{3,negative}, \quad \ln \frac{\mu_{3,negative}^2}{\Lambda^2 P^2 g_0} = 0.0318(3), \tag{7.3}$$

which is represented by black vertical lines in Fig. 6. However, these negative values of  $Q^{D3,s}$  are not physical. The issue is that prior we reach the compactification scale  $\mu_{3,negative}$ , namely at  $\mu_{3,\chi\text{SB}}$  [21]

$$\mu_{3,\chi\text{SB}} > \mu_{3,negative}, \quad \ln \frac{\mu_{3,\chi\text{SB}}^2}{\Lambda^2 P^2 g_0} = 0.4309(8), \tag{7.4}$$

chirally symmetric phase of cascading gauge theory on  $S^3$  undergoes a first order phase transition to a symmetry broken phase (deformed KS geometry), where  $Q^{D3,b} = 0$ , see (7.2). This first order transition is further enhanced by perturbative tachyonic instabilities in chirally symmetric phase which arise at a slightly lower value of  $\mu_3$ , namely at  $\mu_{3,tachyon}$  [21]

$$\mu_{\chi\text{SB}} > \mu_{3,tachyon} > \mu_{3,negative}, \quad \ln \frac{\mu_{3,tachyon}^2}{\Lambda^2 P^2 g_0} = 0.3297(3). \tag{7.5}$$

Thus, a correct behaviour of the D3 charge at the tip of the conifold in  $S^3$  deformed throat geometries is

$$Q^{D3} = \begin{cases} Q^{D3,s} > 0, & \mu_3 > \mu_{3,\chi\text{SB}}; \\ Q^{D3,b} = 0, & \mu_3 \leq \mu_{3,\chi\text{SB}}. \end{cases} \tag{7.6}$$

Once again, the D3 charge at the tip of the conifold is never negative.

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