Three-dimensional microscopic interlaminar analysis of cross-ply laminates based on a homogenization theory

Tetsuya Matsuda a,*, Dai Okumura b, Nobutada Ohno b, Masamichi Kawai a

a Department of Engineering Mechanics and Energy, University of Tsukuba, 1-1-1 Tennodai, Tsukuba 305-8573, Japan
b Department of Mechanical Science and Engineering, Nagoya University, Furo-cho, Chikusa-ku, Nagoya 464-8603, Japan

Received 22 January 2007; received in revised form 4 May 2007; accepted 14 June 2007
Available online 21 June 2007

Abstract

In this study, a method for three-dimensional microscopic interlaminar analysis of cross-ply laminates is developed based on a homogenization theory to analyze microscopic interactions between unidirectional long fiber-reinforced laminae. For this, a unit cell of a cross-ply laminate, which includes interlaminar areas, is defined under the assumption that each lamina in the laminate has a transversely square fiber array. Then, showing that the laminate has a point-symmetric internal structure, the symmetry is utilized to introduce half of the unit cell as the domain of analysis. Moreover, the domain of analysis is divided into substructures using a substructure method combined with the homogenization theory, significantly reducing the computational costs. The present method is then applied to the analysis of interlaminar stress distributions in a carbon fiber/epoxy cross-ply laminate subjected to in-plane uniaxial tension. It is shown that microscopic shear stress noticeably occurs at the interface between the 0°/C176 and 90°/C176 plies. It is also shown that the microscopic interaction between the two plies is observed only in the vicinity of the interface.

Keywords: Interlaminar; Long fiber-reinforced laminate; Homogenization; Point-symmetry; Domain of analysis; Substructure

1. Introduction

Long fiber-reinforced laminates are important engineering materials because of their high specific stiffness and high specific strength, in addition to other advantageous features. Since such laminates are usually manufactured by stacking unidirectional long fiber-reinforced laminae, they have interfaces between the laminae, i.e., interlaminar areas. In these areas, microscopic failures, e.g., matrix cracking and delamination, are apt to occur. Such microscopic failures can result in the macroscopic failure of laminates. Thus, it is essential to analyze the microscopic stress/strain distributions at interlaminar areas.

The microscopic interlaminar analysis mentioned above is different from macroscopic interlaminar analysis in which the laminae are regarded as homogeneous materials. In microscopic interlaminar analysis, laminae...
are not considered homogeneous but as heterogeneous materials that have microscopic internal structures comprised of fibers and matrices. This analysis provides detailed stress/strain distributions at interlaminar areas microscopically. The analysis also elucidates how far the influence of interactions between laminae reaches from interfaces between laminae on a microscopic level. At areas outside the effective range of the microscopic interaction between laminae, it may be possible to treat laminae as homogeneous materials instead of heterogeneous ones. Therefore, performing microscopic interlaminar analysis has significant value.

Finite element method (FEM) based analysis is one of the most useful approaches for microscopic interlaminar analysis, because FEM is capable of modeling the microstructures of laminae explicitly and in detail. The pioneering work of Pagano and Rybicki (1974) analyzed the microscopic interlaminar stress distributions in a unidirectional long fiber-reinforced composite with a free edge. Recently, Raghavan et al. (2001), Ghosh et al. (2001), and Raghavan and Ghosh (2004) have had great success in finding the interlaminar stress distributions in unidirectional composites microscopically using the Voronoi cell FEM (Ghosh and Mukhopadhyay, 1993) in conjunction with the adaptive mesh method (Moorthy and Ghosh, 2000). These studies, however, dealt only with a two-dimensional or a generalized two-dimensional analysis of unidirectional long fiber-reinforced composites. The present authors have more interest in the interlaminar analysis of multidirectional fiber-reinforced laminates, in which the microscopic interlaminar stress/strain must be analyzed three-dimensionally.

In earlier papers (Matsuda et al., 2002, 2003), we analyzed the in-plane elastic–viscoplastic behavior of multidirectional CFRP laminates using the homogenization theory of nonlinear time-dependent composites (Wu and Ohno, 1999; Ohno et al., 2000) combined with the classical lamination theory. The homogenization theory is based on the unit cell problem (Bensoussan et al., 1978; Sanchez-Palencia, 1980; Bakhvalov and Panasenko, 1984) and enables us to analyze not only the macroscopic properties of composites, but also the microscopic distributions of stress and strain in unit cells. The analysis procedure in the previous papers was as follows: First, a unit cell containing fibers and a matrix as a microstructure of each lamina was defined and the microscopic distributions of stress and strain rates in each lamina were computed using the homogenization theory. Next, the microscopic stress and strain rates with respect to the unit cell were averaged, obtaining the macroscopic stress and strain rates of each lamina. Finally, using the macroscopic stress and strain rates of the laminae, the macroscopic constitutive relation of a laminate was derived based on the classical lamination theory. This method, therefore, was able to analyze the microscopic stress/strain distributions at internal areas of laminae, but was not able to analyze those at interlaminar areas.

However, we can perform microscopic interlaminar analysis of laminates using the homogenization theory by assuming the microscopic internal structure of a laminate as illustrated in Fig. 1, and by defining a unit cell of the laminate. The use of such a unit cell allows us to analyze the microscopic stress/strain distributions in the laminate three-dimensionally, as well as the macroscopic behavior of the laminate. By employing this

---

Fig. 1. Microscopic internal structure and unit cell $Y$ of a cross-ply laminate subjected to in-plane off-axis tensile load.
method, therefore, not only the microscopic interlaminar stress/strain distributions, but also the effective range of microscopic interaction between laminae can be investigated.

In this study, a novel method for the three-dimensional microscopic interlaminar analysis of cross-ply laminates is proposed based on the homogenization theory. Utilizing the point-symmetry of the internal structure in the laminate, the domain of analysis is reduced by half based on our previous result (Ohno et al., 2001). A substructure method (Zienkiewicz and Taylor, 2000) is subsequently introduced into the homogenization theory to further reduce computational costs. The proposed method is then applied to the analysis of microscopic stress distributions at an interlaminar area in a carbon fiber/epoxy cross-ply laminate subjected to in-plane uniaxial tension. Finally, the microscopic influence of interactions between laminae in the interlaminar area is discussed.

2. Three-dimensional microscopic interlaminar analysis of cross-ply laminates

In the present study, we consider the cross-ply laminate illustrated in Fig. 1, in which each unidirectional long fiber-reinforced lamina is assumed to have a square fiber array and to possess $2^N$ fibers in the stacking direction ($y_1$-direction). The laminate is subjected to in-plane uniaxial tension with an off-axis angle $\theta$ as shown in Fig. 1, and exhibits macroscopically uniform elastic deformation. The deformation is assumed to be infinitesimal.

2.1. Homogenization theory

To apply the homogenization theory based on the unit cell problem (Bensoussan et al., 1978; Sanchez-Palencia, 1980; Bakhvalov and Panasenko, 1984) to the present analysis, we first define a unit cell $Y$ of the laminate as shown in Fig. 1 so that $Y$ includes the interfaces between laminae. The constituents of $Y$, i.e., fibers and matrix, are assumed to be elastic materials, and obey the following constitutive equation:

$$ r_{ij} = c_{ijkl} e_{kl}, $$

where $r_{ij}$ and $e_{ij}$ denote the microscopic stress and strain, respectively, and $c_{ijkl}$ signifies the elastic stiffness satisfying $c_{ijkl} = c_{iklj} = c_{ijlk} = c_{klij}$. The homogenization theory then gives the field of microscopic stress $\Sigma_{ij}$ in $Y$ and the relation between macroscopic stress $\sigma_{ij}$ and macroscopic strain $E_{ij}$:

$$ \sigma_{ij} = c_{ijpq} \left( \delta_{pi} \delta_{qj} + \chi_{ijkl}^{ij} \right) E_{kl}, $$

$$ \Sigma_{ij} = \left\langle c_{ijpq} \left( \delta_{pi} \delta_{qj} + \chi_{ijkl}^{kl} \right) \right\rangle E_{kl}, $$

where $\langle \rangle_j$ stands for the differentiation with respect to $y_j$, $\delta_{ij}$ indicates Kronecker’s delta, and $\langle \rangle$ designates the volume average in $Y$ defined as $\langle \# \rangle = |Y|^{-1} \int_Y \# dY$, in which $|Y|$ signifies the volume of $Y$. Moreover, $\chi_{ijkl}^{ij}$ in Eqs. (2) and (3) denotes the characteristic function obtained by solving the following boundary value problem with the $Y$-periodic boundary condition:

$$ \int_Y c_{ijpq} \chi_{ijkl}^{kl} v_{ij} dY = - \int_Y c_{ijkl} v_{ij} dY, $$

where $v_{ij}$ is an arbitrary field of perturbed displacement satisfying the $Y$-periodicity. Since the above problem (4) is generally solved by FEM, we rewrite Eq. (4) in the finite element discretized form:

$$ K^{kl} \chi_{ijkl}^{kl} = F^{kl} \quad (kl = 11, 22, \ldots, 31), $$

where $\chi_{ijkl}^{kl}$ denotes the nodal vector of $\chi_{ijkl}^{ij}$, and $K$ and $F^{kl}$ are expressed as

$$ K = \int_Y B^T C B dY, \quad F^{kl} = - \int_Y B^T C^{kl} dY. $$

Here, $B$ denotes the transformation matrix from nodal displacement to strain, $C$ is the elastic stiffness matrix representing $c_{ijkl}$, $T$ stands for the transpose, and $C^{kl} = \{ c_{11kl} \ c_{22kl} \ c_{33kl} \ c_{12kl} \ c_{23kl} \ c_{31kl} \}^T$. 
By solving Eq. (5) for $\chi^{kl}$, the microscopic stress field in $Y$ can be determined based on Eq. (2) and the macroscopic elastic behavior of the laminate can be evaluated using Eq. (3). Adopting the present framework of analysis, therefore, both the microscopic interlaminar stress distribution and the effective range of microscopic interaction between 0°- and 90°-plies can be investigated under macroscopic loading.

2.2. Semunit cell

In the previous section, the framework of microscopic interlaminar analysis of cross-ply laminates was described based on the homogenization theory. The analysis, however, involves a huge amount of computational resources because of the large size of $K$ in Eq. (5), which is due to employing the large-scale unit cell containing a lot of fibers and matrix. Thus, in this section, the domain of analysis is reduced by half utilizing the point-symmetry of the internal structure of the laminate.

Let us consider half of the unit cell, as shown in Fig. 2, which hereafter is referred to as a semunit cell $\bar{Y}$. A close look at Fig. 2 reveals that the internal structure of the laminate has a point-symmetry with respect to the centers of the left and right lateral boundary facets of $\bar{Y}$, $C_A$ and $C_B$. Consequently, the distribution of $\chi^{kl}$ also satisfies the point-symmetry with respect to these points. Using the point-symmetry as a boundary condition on the left and right lateral boundary facets of $\bar{Y}$, we are able to employ $\bar{Y}$ instead of $Y$ as the domain of analysis, leading to the following boundary value problem with respect to $\bar{Y}$ (Ohno et al., 2001):

$$\int_{\bar{Y}} c_{ijpq} \chi^{kl}_{pq} v_{ij} d\bar{Y} = - \int_{\bar{Y}} c_{ijkl} v_{ij} d\bar{Y}. \quad (7)$$

Since the above boundary value problem (7) has the same form as Eq. (4), the problem can be solved in the same manner using FEM as in Eq. (4). Thus, Eq. (7) is rewritten into the finite element discretized form:

$$\tilde{K}^{kl} = \tilde{F}^{kl}, \quad (kl = 11, 22, \ldots, 31), \quad (8)$$

where $\tilde{K}$ and $\tilde{F}^{kl}$ are expressed as follows:

$$\tilde{K} = \int_{\bar{Y}} B^T C B d\bar{Y}, \quad \tilde{F}^{kl} = - \int_{\bar{Y}} B^T C^{kl} d\bar{Y}. \quad (9)$$

It is noted that, when solving Eq. (8), not the $Y$-periodic but the point-symmetric boundary condition with respect to $C_A$ and $C_B$ is imposed on $\chi^{kl}$, on the left and right lateral boundary facets of $\bar{Y}$. Whereas, on the other boundary facets of $\bar{Y}$, the $Y$-periodic boundary condition is imposed on $\chi^{kl}$.
It is emphasized that the use of the semiunit cell $\tilde{Y}$ reduces the degrees of freedom (the number of nodes) in the boundary value problem by almost half, compared with the analysis using the whole unit cell $Y$ mentioned in the previous section. This yields a considerable reduction in computational load.

2.3. Substructure method

In the previous section, the semiunit cell $\tilde{Y}$ was introduced so that we could reduce the domain of analysis by half. But, even $\tilde{Y}$ is still considerably large-scale for computation. In this section, therefore, the substructure method (Zienkiewicz and Taylor, 2000) is introduced into the homogenization theory to solve the boundary value problem (8).

First, note that the semiunit cell $\tilde{Y}$ consists of cubic cells $A_i (i = 1, 2, \ldots, N)$ and $B_i (i = 1, 2, \ldots, N)$ for the $0^\circ$- and $90^\circ$-plies, respectively, as shown in Fig. 3. Hence, we divide $\tilde{Y}$ into $A_i$ and $B_i$ as substructures, and then derive the boundary value problems for the substructures in finite element discretized form as follows:

$$K_A \chi_{Ai}^{kl} = F_{Ai}^{kl} \quad (i = 1, 2, \ldots, N),$$

$$K_B \chi_{Bi}^{kl} = F_{Bi}^{kl} \quad (i = 1, 2, \ldots, N),$$

where $\chi_{Ai}^{kl}$ and $\chi_{Bi}^{kl}$ denote the nodal vectors of characteristic function in $A_i$ and $B_i$, respectively, and $K_A$, $F_{Ai}^{kl}$, $K_B$, and $F_{Bi}^{kl}$ have the following expressions:

$$K_A = \int_{A_i} B^T C B dA_i, \quad F_{Ai}^{kl} = - \int_{A_i} B^T C^{kl} dA_i,$$

$$K_B = \int_{B_i} B^T C B dB_i, \quad F_{Bi}^{kl} = - \int_{B_i} B^T C^{kl} dB_i.$$

It is noteworthy that all $A_i$ have common $K_A$ and $F_{Ai}^{kl}$ because the geometry and material properties of all $A_i$ are the same. For the same reason, all $B_i$ have common $K_B$ and $F_{Bi}^{kl}$, which are easily obtained by rotating $A_i$ by $90^\circ$ with respect to the $y_1$-direction. It is therefore enough for us to calculate $K_A$, $F_{Ai}^{kl}$, $K_B$, and $F_{Bi}^{kl}$ only once.

Next, the components of $\chi_{Ai}^{kl}$ are divided into two parts, $\chi_{A|\Gamma}^{kl}$ and $\chi_{A|\Gamma'}^{kl}$, which represent the characteristic functions at the internal and the boundary nodes of $A_i$, respectively. The components of $\chi_{Bi}^{kl}$ are also divided...
into $\mathbf{\xi}^{k(\Gamma)}_{B_k}$ and $\mathbf{\xi}^{k(\Gamma)}_{B_k}$. Then, the boundary value problems for $A_i$ and $B_i$, Eqs. (10) and (11), are rewritten into the following equations, respectively:

$$
\begin{align}
\begin{bmatrix}
K_A^{(\Omega)} & K_A^{(\Gamma)} \\
K_A^{(\Gamma)} & K_A^{(\Gamma)} \\
\end{bmatrix}
\begin{bmatrix}
\mathbf{\xi}^{k(\Omega)}_{A_i} \\
\mathbf{\xi}^{k(\Gamma)}_{A_i} \\
\end{bmatrix}
&=
\begin{bmatrix}
F_A^{k(\Omega)} \\
F_A^{k(\Gamma)} \\
\end{bmatrix},
\end{align}
$$

(14)

$$
\begin{align}
\begin{bmatrix}
K_B^{(\Omega)} & K_B^{(\Gamma)} \\
K_B^{(\Gamma)} & K_B^{(\Gamma)} \\
\end{bmatrix}
\begin{bmatrix}
\mathbf{\xi}^{k(\Omega)}_{B_i} \\
\mathbf{\xi}^{k(\Gamma)}_{B_i} \\
\end{bmatrix}
&=
\begin{bmatrix}
F_B^{k(\Omega)} \\
F_B^{k(\Gamma)} \\
\end{bmatrix},
\end{align}
$$

(15)

and we obtain

$$
\begin{align}
\mathbf{\xi}^{k(\Omega)}_{A_i} &= \left(K_A^{(\Omega)}\right)^{-1} \left(F_A^{k(\Omega)} - K_A^{(\Gamma)} \mathbf{\xi}^{k(\Gamma)}_{A_i}\right),
\end{align}
$$

(16)

$$
\begin{align}
\mathbf{\xi}^{k(\Omega)}_{B_i} &= \left(K_B^{(\Omega)}\right)^{-1} \left(F_B^{k(\Omega)} - K_B^{(\Gamma)} \mathbf{\xi}^{k(\Gamma)}_{B_i}\right).
\end{align}
$$

(17)

The elimination of $\mathbf{\xi}^{k(\Omega)}_{A_i}$ and $\mathbf{\xi}^{k(\Omega)}_{B_i}$ from Eqs. (14) and (15) using the above equations, respectively, yields

$$
\begin{align}
\bar{K}_A^{(\Gamma)} \mathbf{\xi}^{k(\Omega)}_{A_i} &= \bar{F}_A^{k(\Gamma)},
\end{align}
$$

(18)

$$
\begin{align}
\bar{K}_B^{(\Gamma)} \mathbf{\xi}^{k(\Omega)}_{B_i} &= \bar{F}_B^{k(\Gamma)},
\end{align}
$$

(19)

where $\bar{K}_A^{(\Gamma)}$, $\bar{K}_A^{(\Gamma)}$, $\bar{K}_B^{(\Gamma)}$, and $\bar{F}_B^{k(\Gamma)}$ are expressed as follows:

$$
\begin{align}
\bar{K}_A^{(\Gamma)} &= K_A^{(\Gamma)} - K_A^{(\Gamma)} \left(K_A^{(\Omega)}\right)^{-1} K_A^{(\Omega)},
\bar{F}_A^{k(\Gamma)} &= F_A^{k(\Gamma)} - K_A^{(\Omega)} \left(K_A^{(\Omega)}\right)^{-1} F_A^{k(\Omega)},
\end{align}
$$

(20)

$$
\begin{align}
\bar{K}_B^{(\Gamma)} &= K_B^{(\Gamma)} - K_B^{(\Gamma)} \left(K_B^{(\Omega)}\right)^{-1} K_B^{(\Omega)},
\bar{F}_B^{k(\Gamma)} &= F_B^{k(\Gamma)} - K_B^{(\Gamma)} \left(K_B^{(\Omega)}\right)^{-1} F_B^{k(\Omega)}.
\end{align}
$$

(21)

Finally, Eqs. (18) and (19) are assembled into one equation, which is a boundary value problem with respect to just the boundary nodes of all substructures, which the joint nodes of adjacent substructures belong to. Thus, we have

$$
\begin{align}
\bar{K}^{(\Gamma)} \mathbf{\xi}^{k(\Gamma)} &= \bar{F}^{k(\Gamma)},
\end{align}
$$

(22)

where $\bar{K}^{(\Gamma)}$ stands for the matrix consisting of $\bar{K}_A^{(\Gamma)}$ and $\bar{K}_B^{(\Gamma)}$, $\bar{F}^{k(\Gamma)}$ indicates the vector consisting of $\bar{F}_A^{k(\Gamma)}$ and $\bar{F}_B^{k(\Gamma)}$, and $\mathbf{\xi}^{k(\Gamma)}$ denotes the nodal vector of the characteristic function at the boundary nodes of substructures. The characteristic function $\mathbf{\xi}^{k(\Gamma)}$ is determined by solving Eq. (22) with appropriate boundary conditions, i.e., the point-symmetric and the $Y$-periodic conditions stated in Section 2.2, and the continuity condition at the joint nodes of adjacent substructures. Then, the characteristic functions at the internal nodes, $\mathbf{\xi}^{k(\Omega)}_{A_i}$ and $\mathbf{\xi}^{k(\Omega)}_{B_i}$, are calculated using Eqs. (16) and (17).

In general, the total number of boundary nodes of all substructures is much less than the number of all nodes in the domain of analysis, resulting in a significant reduction of computational memory and time. Incidentally, Okumura et al. (2004) applied the substructure method to an in-plane buckling analysis of hexagonal honeycombs using the homogenization theory of finite deformation (Ohno et al., 2002).

3. Analysis

In this section, the present method is applied to the microscopic interlaminar analysis of a carbon fiber/epoxy cross-ply laminate subjected to an in-plane uniaxial tensile load.

3.1. Cross-ply laminate and macroscopic boundary condition

Considering that the carbon fiber/epoxy cross-ply laminate is made of general use prepreg sheets, each lamina is assumed to have 16 fibers in the stacking direction ($N = 8$). The volume fraction of fibers is taken to be 56%, as in previous studies (Matsuda et al., 2002, 2003). The laminate is subjected to an in-plane tensile load,
which is either on-axis ($\theta = 0^\circ$) or $45^\circ$ off-axis ($\theta = 45^\circ$). The macroscopic strain in the loading direction, $E_{\theta}$, is prescribed to be 0.5%. The present analysis is performed under the macroscopic plane stress condition.

3.2. Substructures and finite element discretization

As mentioned in Section 2.3, the semiunit cell $\tilde{Y}$ is divided into cubic substructures $A_i$ ($i = 1, 2, \ldots, 8$) and $B_i$ ($i = 1, 2, \ldots, 8$). Moreover, $A_i$ and $B_i$ are discretized into eight-node isoparametric elements as depicted in Fig. 4. The finite element meshes of $A_i$ and $B_i$ have 4320 elements and 5005 nodes, respectively.

If we employed the whole unit cell $Y$ as the domain of analysis, we would be forced to solve the boundary value problem (5) with 464,763 degrees of freedom because the unit cell $Y$ would have 154,921 nodes. By contrast, with the present method, the degrees of freedom in the boundary value problem (22) are only 62,304 because the number of boundary nodes of each substructure is 1298. This reduction in degrees of freedom demonstrates the efficiency of the present method.

3.3. Material properties

The carbon fibers are regarded as transversely isotropic elastic materials, while the epoxy matrix as an isotropic elastic material. The material constants used in the present analysis are listed in Table 1 (Matsuda et al., 2002, 2003). In the table, the subscripts L and T indicate the longitudinal and the transverse directions of fibers, respectively.

3.4. Results of analysis

First, let us discuss the microscopic stresses caused by the $45^\circ$ off-axis tensile load. Fig. 5 shows the vector distributions of resultant shear stress $[(\sigma_{12})^2 + (\sigma_{13})^2]^{1/2}$ at three parts in the laminate, i.e., the midsection of the $0^\circ$-ply (left lateral surface of $A_8$) (Fig. 5(a)), the vicinity of the interlaminar plane (interface between $A_1$ and $A_2$) (Fig. 5(b)), and the interlaminar plane (interface between $A_1$ and $B_1$) (Fig. 5(c)). In the figure, only the stress distributions in $0^\circ$-ply are depicted because the stress distributions in the $0^\circ$- and $90^\circ$-plies are symmetrical. First, as seen in Fig. 5(c), a considerably high shear stress occurs at the interlaminar plane microscopically. This is caused by the rotation of fibers in the $0^\circ$- and $90^\circ$-plies toward the loading direction. The maximum resultant shear stress is about 9 MPa, which reaches 21% of the macroscopic tensile stress (73 MPa) in terms of von Mises equivalent stress. By contrast, such a shear stress disappears in the vicinity of the interlaminar plane as shown in Fig. 5(b). Additionally, no shear stress occurs at the midsection of the $0^\circ$-ply (Fig. 5(a)). These results suggest that the microscopic interaction between $0^\circ$- and $90^\circ$-plies is local.

Fig. 6 shows the distributions of out-of-plane normal stress $\sigma_{11}$ induced by the $45^\circ$ off-axis load. These figures indicate that the out-of-plane tensile and compressive stresses take place in the laminate microscopically,
Table 1  
Material constants (Matsuda et al., 2002, 2003)  

<table>
<thead>
<tr>
<th>Material</th>
<th>$E_{LL}$</th>
<th>$E_{TT}$</th>
<th>$G_{LT}$</th>
<th>$v_{TT}$</th>
<th>$v_{LT}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Carbon fiber</td>
<td>240 GPa</td>
<td>15.5 GPa</td>
<td>24.7 GPa</td>
<td>0.49</td>
<td>0.28</td>
</tr>
<tr>
<td>Epoxy</td>
<td>3.5 GPa</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Fig. 5. Distributions of resultant shear stress $[(σ_{12})^2 + (σ_{13})^2]^{1/2}$ at (a) the midsection of the $0^\circ$-ply (left lateral facet of $A_8$), (b) the vicinity of the interlaminar plane (interface between $A_1$ and $A_2$), and (c) the interlaminar plane (interface between $A_1$ and $B_1$) at $E_θ = 0.5\%$ ($θ = 45^\circ$).

Fig. 6. Distributions of out-of-plane normal stress $σ_{11}$ at (a) the midsection of the $0^\circ$-ply (left lateral facet of $A_8$), (b) the vicinity of the interlaminar plane (interface between $A_1$ and $A_2$), and (c) the interlaminar plane (interface between $A_1$ and $B_1$) at $E_θ = 0.5\%$ ($θ = 45^\circ$).
although the laminate is subjected to only an in-plane tensile load macroscopically. The interlaminar normal stress is lower than the interlaminar shear stress, but the same tendency is observed, i.e., the stress distribution at the interlaminar plane is markedly different from those at the other two parts. As illustrated in Fig. 6(c), at the interlaminar plane, the tensile and compressive stresses occur along the $h = 45^\circ$ and the $h = -45^\circ$ directions, respectively, which is attributable to the rotation of fibers in $0^\circ$ and $90^\circ$-plies toward the loading direction. By contrast, in the vicinity of the interlaminar plane, $r_{11}$ distributes uniformly with respect to the $y_3$-direction as shown in Fig. 6(b). This distribution of $r_{11}$ is almost the same as that at the midsection of the $0^\circ$-ply (Fig. 6(a)), showing the local interaction of the two plies.

Figs. 7 and 8 show the microscopic stresses caused by the on-axis tensile load ($\theta = 0^\circ$). The vector distributions of the resultant shear stress $[ (\sigma_{12})^2 + (\sigma_{13})^2 ]^{1/2}$ are depicted in Fig. 7(a)–(c), while the distributions of out-of-plane normal stress $\sigma_{11}$ are illustrated in Fig. 8(a)–(c). It can be seen from Fig. 7(c) that, even in the on-axis loading case, a resultant shear stress occurs macroscopically at the interlaminar plane although the stress level is much lower than that under the off-axis load. Such a shear stress, however, vanishes in the vicinity of the interlaminar plane (Fig. 7(b)) as well as at the midsection of the $0^\circ$-ply (Fig. 7(a)). Then, from Fig. 8, it can also be ascertained that out-of-plane tensile and compressive stresses take place and the stress level is higher than that under the off-axis load especially at the interlaminar plane (Fig. 8(c)). As shown in Fig. 8(c), compressive stress is observed in the central region of the plane, whereas there is tensile stress in the peripheral region of the plane. By contrast, in the vicinity of the interlaminar plane (Fig. 8(b)), $\sigma_{11}$ has a uniform distribution with respect to the $y_3$-direction and its distribution is almost the same as that at the midsection of the $0^\circ$-ply (Fig. 8(a)). These results indicate that the interlaminar stress distributions under the on-axis load are different from those under the off-axis load, but the local interaction between the two plies is similar to that under the off-axis load.

![Fig. 7. Distributions of resultant shear stress $[ (\sigma_{12})^2 + (\sigma_{13})^2 ]^{1/2}$ at (a) the midsection of the $0^\circ$-ply (left lateral facet of $A_b$), (b) the vicinity of the interlaminar plane (interface between $A_1$ and $A_2$), and (c) the interlaminar plane (interface between $A_1$ and $B_1$) at $E_0 = 0.5\% (\theta = 0^\circ)$.](image1)

![Fig. 8. Distributions of out-of-plane normal stress $\sigma_{11}$ at (a) the midsection of the $0^\circ$-ply (left lateral facet of $A_b$), (b) the vicinity of the interlaminar plane (interface between $A_1$ and $A_2$), and (c) the interlaminar plane (interface between $A_1$ and $B_1$) at $E_0 = 0.5\% (\theta = 0^\circ)$.](image2)
Actually, a case can exist where the loading direction does not coincide perfectly with the $0^\circ$ direction (fiber axis direction), causing a small off-axis angle. In such a case, the fibers will rotate through the small angle and align with the loading direction. The interlaminar shear stress, therefore, will become slightly higher than that under an on-axis load.

We emphasize though that such microscopic interlaminar stress distributions as discussed in here can be only found by microscopic analysis that explicitly takes into account the microstructures in laminae.

4. Concluding remarks

In this study, the distributions of microscopic interlaminar stress in a CFRP cross-ply laminate subjected to an in-plane uniaxial tensile load were analyzed three-dimensionally using a newly proposed method based on the homogenization theory outlined in Section 2.1. In the proposed method, the domain of analysis was reduced by half using the point-symmetry of the internal structure of a laminate, resulting in a marked reduction of computational cost. As well, a substructure method was introduced into the homogenization theory, further increasing computational efficiency. The analysis was performed in two cases, $45^\circ$ off-axis and on-axis in-plane tensile loads. The analysis results showed that the maximum value of interlaminar resultant shear stress under the $45^\circ$ off-axis tensile load reached more than 20% of the macroscopic tensile stress applied to the laminate in terms of von Mises equivalent stress. By contrast, such microscopic shear stress disappeared at a distance of about a fiber diameter away from the interface between the $0^\circ$- and $90^\circ$-plies, indicating that the microscopic interaction between the two plies was local. This distance of local interaction was also observed under the on-axis tensile load. It therefore can be said that it is necessary to consider the microscopic structure consisting of fibers and matrix around the interface of laminae, while the microscopic structure at a distance of more than a fiber diameter away from the interface may be replaced by the equivalent homogeneous material.

As the present analysis showed, interlaminar shear stress occurred because the fibers in the $0^\circ$- and $90^\circ$-plies rotated toward the loading direction. Obviously, this fiber rotation should take place irrespective of the type of fiber arrays assumed in laminae. Therefore, even if other types of fiber arrays, such as a hexagonal fiber array, are employed instead of the square fiber array, the results of analysis remain qualitatively similar to those for the square fiber array analyzed in the present study. The present results also provide useful suggestions for angle-ply laminates. Let us consider, for example, $[\pm 30^\circ]$, $[\pm 45^\circ]$, and $[\pm 60^\circ]$ angle-ply laminates, which are subjected to an in-plane uniaxial load in the $0^\circ$ direction. Needless to say, the results for the $[\pm 45^\circ]$ angle-ply laminate will coincide with those for the cross-ply laminate subjected to the $45^\circ$ off-axis load as shown in Section 3.4. But, even for the $[\pm 30^\circ]$ and $[\pm 60^\circ]$ angle-ply laminates, the results of analysis can be similar to those for the cross-ply laminate because the main mechanism causing the interlaminar stress, i.e., the fiber rotation toward the loading direction, is common to these laminates. Quantitative discussion, however, remains to be presented in our future work in respect of the various types of fiber arrays and laminate configurations. Moreover, microscopic stress analysis at free edges of laminates (Pagano and Rybicki, 1974; Ghosh et al., 2001; Raghavan et al., 2001), which is outside the scope of the present study, remains for future investigation.

Acknowledgment

The authors acknowledge support, in part, from the Ministry of Education, Culture, Sports, Science and Technology, Japan under a Grants-in-Aid for Young Scientists (B) (No. 18760075).

References

