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Boolean Products of MV-Algebras: Hypernormal MV-Algebras*

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INTRODUCTION

MV-algebras were introduced by Chang [7, 8] as the algebraic counterpart of the Łukasiewicz infinite valued propositional logic (see [34, pp. 47–52]). These algebras have appeared in the literature under different names and polynomially equivalent presentations: CN-*algebras* [20], *Wajsberg algebras* [32, 16], *bounded commutative* BCK-*algebras* [37, 27], and *bricks* [5] (see also [4]). In the past few years it was discovered that MV-algebras are naturally related to the Murray–von Neumann order of projections in operator algebras on Hilbert spaces, and that they play an interesting role as invariants of approximately finite-dimensional C*-algebras (see [26, 28, 29, 11, 31]. They are also naturally related to Ulam's searching games with lies [30].

MV-algebras admit a natural lattice reduct (see [7]), and hence a natural order structure. Many important properties can be derived from the fact, established by Chang [8], that nontrivial MV-algebras are subdirect prod-

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ucts of MV-chains, i.e., totally ordered MV-algebras. To prove this fundamental result Chang introduced the notion of a *prime ideal* in an MV-algebra.

The lattice of principal ideals of an MV-algebra **A**, ordered by inclusion, is a *dual completely normal lattice* and hence the poset of prime ideals of an MV-algebra is a spectral root system (cardinal sum of spectral roots). Indeed, as shown in [12], each spectral root system is order isomorphic to the poset of prime ideals of an MV-algebra. Special cases are the MV-algebras whose prime spectra (the set of prime ideals) are cardinal sum of chains.

By using the Stone−Zariski topology and some standard arguments, Chang's subdirect decompositions can be transformed into representations by global sections of sheaves of totally ordered MV-algebras over spectral spaces (see for instance [5, Theorem 3.5, p. 95]). Special cases of representations by global sections are the representations as weak Boolean products [22]. It is natural to try to classify MV-algebras by the order structure of their spectra. The simplest case corresponds to the trivial order, i.e., when prime ideals are maximal. The algebras with this property are called *hyperarchimedean.* This class contains the Boolean algebras, and more generally, all subvarieties of MV-algebras which are generated by a finite number of finite MV-chains. The hyperarchimedean MV-algebras are just the Boolean products of simple MV-algebras [35]. As a natural step further, we investigate the class formed by the MV-algebras such that their spectra are cardinal sums of chains. Following the nomenclature introduced in [25] for lattice theory, we call the algebras in this class *hypernormal*.

The aim of this paper is to give algebraic characterizations of hypernormal MV-algebras and weak Boolean products of MV-chains reminiscent of that given in [36] for Boolean products of MV-chains (see Theorem 3.1). Concretely, we obtain characterizations of hypernormal MV-algebras and weak Boolean products of local MV-algebras and, as a particular case of both, weak Boolean products of MV-chains. These results are obtained in Section 3. By considering MV-algebras of real continuous functions over compact spaces, we show in Section 4 how the conditions established in Section 3 are related to topological separation properties. Using different topological spaces we give examples which show that the classes considered in Section 3 are indeed different.

We include in the paper two preliminary sections. In Section 1, we give the definitions and results of the theory of MV-algebras which are needed in the remainder of the paper. In Section 2, we recall some basic results on the representation of MV-algebras as weak Boolean products. We obtain as corollaries some known results on the representations of MV-algebras as weak Boolean products of totally ordered MV-algebras [35]. We also give a new characterization of the liminary MV-algebras introduced in [11].

1. DEFINITIONS AND FIRST PROPERTIES

An MV-algebra is an algebra $\mathbf{A} = (A, \oplus, \neg, 0)$ of type $(2,1,0)$ fulfilling the following equations:

MV1. $(x \oplus y) \oplus z \approx x \oplus (y \oplus z)$

MV2. $x \oplus y \approx y \oplus x$

MV3. $x \oplus 0 \approx y$

MV4. $\neg(\neg x) \approx x$

MV5. $x \oplus \neg 0 \approx \neg 0$

MV6. $\neg(\neg x \oplus y) \oplus y \approx \neg(x \oplus \neg y) \oplus x$.

By taking $y = \neg 0$ in MV6, we deduce:

MV7. $x \oplus \neg x \approx \neg 0$.

Therefore, if we set $1 = \neg 0$ and $x \odot y = \neg(\neg x \oplus \neg y)$, then $(A, \oplus, \odot, \neg, 0, 1)$ satisfies all the axioms given in [26, Lemma 2.6], and hence the above definition of MV-algebras is equivalent to Chang's definition [7] (cf. [11]).

We denote the set of natural numbers by ω. We define $0x = 0$, $x^o = 1$, and for each $n \in \omega$,$(n+1)x = x \oplus nx$, $x^{n+1} = x \odot x^n$.

In the language of MV-algebras we consider the following terms:

$$x \vee y =_{\text{def}} (x \odot \neg y) \oplus y, \qquad x \wedge y =_{\text{def}} (x \oplus \neg y) \odot y.$$

Then for each MV-algebra $\mathbf{A}$, the reduct $\mathbf{L}(\mathbf{A}) = (A, \wedge, \vee, 0, 1)$ is a distributive lattice, with least element 0 and greatest element 1. The corresponding order relation, which we call the *natural order* of $\mathbf{A}$, is given by $x \leq y$ if and only if $\neg x \oplus y = 1$ (or equivalently, $x \odot \neg y = 0$). Moreover, the following properties hold in any MV-algebra:

1.1. $x \odot y \leq x \wedge y \leq x \vee y \leq x \oplus y$

1.2. $(x \odot \neg y) \wedge (y \odot \neg x) \approx 0$

1.3. $(x \odot \neg y) \oplus (y \odot \neg x) \approx 0$ iff $x \approx y$

1.4. $x \wedge y \approx \neg(\neg x \vee \neg y)$, $x \vee y \approx \neg(\neg x \wedge \neg y)$

1.5. for any $0 < n \in \omega$, $n(a \wedge b) = \bigwedge_{s+t=n} sa \oplus tb$.

An MV-algebra such that its natural order is total is called an *MV-chain*.

Let $\mathbf{G} = (G, +, 0, \leq)$ be an *abelian lattice ordered* group and u a strictly positive element in G. Then $\Gamma(\mathbf{G}, u) = ([0, u], \oplus, \neg, 0)$ is an MV-algebra (see [26] and [23]), where $[0, u] = \{b \in G : 0 \leq b \leq u\}$, $x \oplus y = u \wedge (x + y)$, and $\neg x = u - x$. Moreover, $x \odot y = 0 \vee (x + y - u)$, $\neg 0 = u$ and the natural order of this algebra is the restriction of the order of $\mathbf{G}$. In particular, $\Gamma(\mathbf{R}, 1)$, where $\mathbf{R}$ denotes the additive ordered group of the reals, corresponds essentially to the matrix used by Łukasiewicz to define an infinite-valued propositional calculus (see [34, pp. 47–52]).

Let $\mathbf{A}$ be a MV-algebra. A subset I of A is called an *ideal* provided that:

(I1) $0 \in I$,

(I2) $a \in I$ and $b \in I$ imply $a \oplus b \in I$, and

(I3) $a \leq b$ and $b \in I$ imply $a \in I$.

By 1.1, any ideal of $\mathbf{A}$ is a lattice ideal of $\mathbf{L}(\mathbf{A})$. An ideal I of $\mathbf{A}$ is called *prime* provided that it is prime as an ideal of $\mathbf{L}(\mathbf{A})$: $I \neq A$, and $a \wedge b \in I$ implies $a \in I$ or $b \in I$.

The set $\mathscr{I}(\mathbf{A})$ of all proper ideals of $\mathbf{A}$, ordered by inclusion, is the universe of an algebraic lattice, which we denote by $\mathfrak{I}(\mathbf{A})$. If $\mathbf{A}$ denotes either an MV-algebra or a bounded distributive lattice, then Spec $\mathbf{A}$ will denote the set of prime ideals of $\mathbf{A}$ and **Spec A** the poset (Spec A, $\subseteq$).

Let Con($\mathbf{A}$) be the algebraic lattice of all congruence relations of $\mathbf{A}$. The correspondence

$$\theta \mapsto J(\theta) = 0/\theta = \{a \in A : (a, 0) \in \theta\}$$

establishes an isomorphism J from **Con**($\mathbf{A}$) onto $\mathfrak{I}(\mathbf{A})$. The inverse of J is given by

$$J^{-1}(I) = \{(a, b) \in A^2 : (a \odot \neg b) \oplus (b \odot \neg a) \in I\}$$

for each ideal I (see [7]). For any ideal I of $\mathbf{A}$, we write $\mathbf{A}/I$ in place of $\mathbf{A}/J^{-1}(I)$.

It was shown in [13] and in [16] that the variety of MV-algebras is arithmetical, i.e., each MV-algebra $\mathbf{A}$ is congruence-permutable and the lattice Con($\mathbf{A}$) is distributive. Therefore $\mathfrak{I}(\mathbf{A})$ is a distributive lattice.

Given an MV-algebra $\mathbf{A}$ and $a \in A$, $\langle a \rangle$ denotes the principal ideal generated by a in $\mathbf{A}$, $\langle a \rangle = \{b \in A : b \leq na$ for some $n \in \omega\}$. It follows from the definition of an ideal that for any $a, b \in A$:

1.6. $\langle a \rangle \vee \langle b \rangle = \langle a \vee b \rangle = \langle a \oplus b \rangle$.

Moreover, by an argument dual to the one used in [16, Theorem 14], from 1.5 we deduce (see also [2, Lemma 1]):

1.7. $\langle a \rangle \cap \langle b \rangle = \langle a \wedge b \rangle$.

Hence the family of all principal ideals of **A** is a sublattice of $\mathfrak{I}(\mathbf{A})$. We denote by $\mathfrak{I}\mathfrak{p}(\mathbf{A})$ this distributive lattice.

If **A**, **B** are MV-algebras, then any homomorphism $h: \mathbf{A} \to \mathbf{B}$ induces a homomorphism $\mathfrak{I}\mathfrak{p}(h): \mathfrak{I}\mathfrak{p}(\mathbf{A}) \to \mathfrak{I}\mathfrak{p}(\mathbf{B})$ by the prescription $\mathfrak{I}\mathfrak{p}(h)(\langle a \rangle) = \langle h(a) \rangle$, for each $a \in A$. It also follows from the above remarks that the compact elements of the algebraic closure system $\mathscr{I}(\mathbf{A})$ are just the principal ideals. Hence the algebraic lattice $\mathfrak{I}(\mathbf{A})$ is isomorphic to the lattice of all ideals of the distributive lattice $\mathfrak{I}\mathfrak{p}(\mathbf{A})$ (see [17, p. 80]). Therefore we have:

LEMMA 1.8. *$\mathfrak{I}\mathfrak{p}$ is a covariant functor from the category of MV-algebras to the category of bounded distributive lattices. For each MV-algebra* **A**, *the posets* **Spec A** *and* **Spec** $\mathfrak{I}\mathfrak{p}(\mathbf{A})$ *are isomorphic.*

It follows from [10, Theorem 1.9] that the functor $\mathfrak{I}\mathfrak{p}$ coincides with the one defined by Belluce in [1].

It follows from 1.3 (see [7]) that an ideal I of an MV-algebra **A** is prime if and only if $\mathbf{A}/I$ is an MV-chain. Thus *the partially ordered set of all prime ideals containing a given prime ideal, ordered by inclusion, is a chain.* Bounded distributive lattices **L** such that **Spec L** satisfies this property are known in the literature as *dual completely normal lattices* (see [25, 24, and 9]). Then, by Lemma 1.8, the range of the functor $\mathfrak{I}\mathfrak{p}$ is contained in the class of dual completely normal lattices (cf. [10, Corollary 1.7]).

For any MV-algebra **A**, $\mathbf{B}(\mathbf{A})$ denotes the Boolean algebra of all complemented elements in $\mathbf{L}(\mathbf{A})$. Since for any $a \in A$ and $b \in B(\mathbf{A})$, $a \oplus b = a \vee b$ and $a \odot b = a \wedge b$, $\mathbf{B}(\mathbf{A})$ is a subalgebra of **A** [7, 32] in which $\neg b$ is the complement of b. We recall that a *Stone ideal* of a bounded distributive lattice $\mathbf{L} = (L, \wedge, \vee, 0, 1)$ is a lattice ideal I generated by $I \cap B(\mathbf{L})$, i.e., given $a \in I$ there is a complemented element b in I such that $a \leq b$. A *Stone ultraideal* of **L** is a Stone ideal I such that $I \cap B(\mathbf{L})$ is a prime ideal of $\mathbf{B}(\mathbf{L})$. Let **A** be an MV-algebra, then $B(\mathbf{A}) = B(\mathbf{L}(\mathbf{A}))$. It is easy to check that Stone ideals of $\mathbf{L}(\mathbf{A})$ are ideals of **A**. Moreover, given a Stone ideal I of $\mathbf{L}(\mathbf{A})$ the associated congruence relation is given by (see [32, p. 81]):

1.9. $J^{-1}(I) = \{(a, b) \in A \times A$: there is $c \in I \cap B(\mathbf{A})$, such that $a \vee c = b \vee c\}$.

Given $S \subset B(\mathbf{A})$, we shall write $\theta(S)$ instead of $J^{-1}(\langle S \rangle)$.

2. BOOLEAN PRODUCTS OF MV-ALGEBRAS

A *weak Boolean product* of a family $(\mathbf{A}_x : x \in X)$ of algebras over a Boolean space X is a subdirect product **A** of the given family such that the following conditions hold:

(a) if $a, b \in A$, then $[\![a = b]\!] = \{x \in X : a(x) = b(x)\}$ is open;

(b) if $a, b \in A$ and Z is a clopen in X, then $a|_Z \cup b|_{X \setminus Z} \in A$.

By requiring in condition (a) that $[\![a = b]\!]$ be *clopen* we obtain the notion of a *Boolean product*.

A (*weak*) *Boolean representation* of an MV-algebra $\mathbf{A}$ is an isomorphism from A onto a (weak) Boolean product of MV-algebras. Without loss of generality, we may assume that weak Boolean representations are *proper*, i.e., all the algebras have at least two elements (cf. [14]). Note that by 1.3, condition (a) can be replaced by:

(a′) If $a \in A$, then $[\![a = 0]\!]$ is open (resp. clopen).

Given a subset B of A, $\langle B \rangle$ will represent the ideal of $\mathbf{A}$ generated by B. Then if B is an ideal of $\mathbf{B}(\mathbf{A})$, $\langle B \rangle$ is a Stone ideal. We recall that the cardinal sum of a family of posets is the poset whose universe is the disjoint union of the universes of the members of the family and whose partial order is the disjoint union of the orders of the members of the family.

THEOREM 2.1. *Let* $\mathbf{A}$ *be a nontrivial MV-algebra. For each subalgebra* $\mathbf{C}$ *of* $\mathbf{B}(\mathbf{A})$ *we have that* $\mathbf{A}$ *is representable as the weak Boolean product of the family*

$$(\mathbf{A}/\theta(P) : P \in \operatorname{Spec} \mathbf{C})$$

over the Boolean space Spec $\mathbf{C}$, *and* **Spec A** *is order isomorphic to the cardinal sum of the posets*

$$(\mathbf{Spec}\, \mathbf{A}/\theta(P) : P \in \operatorname{Spec} \mathbf{C}).$$

Moreover, there is a one-to-one correspondence between the equivalence classes of weak Boolean representations of a nontrivial MV-algebra $\mathbf{A}$ *and the subalgebras of* $\mathbf{B}(\mathbf{A})$.

Proof. Assume that $\mathbf{C}$ is a subalgebra of $\mathbf{B}(\mathbf{A})$. If $P \in \operatorname{Spec} \mathbf{C}$, then $\theta(P)$ is the congruence relation associated to the Stone ideal generated by P. Hence the claimed Boolean decomposition follows from 1.9 (cf. [21, Sect. 8.4; 14, p. 85; and 22, 4.34]). Let $\pi_J \colon A \to A/\theta(J)$ be the natural projection associated with $J \in \operatorname{Spec} \mathbf{C}$, and $T_J = \{P \in \operatorname{Spec} \mathbf{A} : P \cap C = J\}$. Then the correspondence $Q \mapsto \pi_J^{-1}(Q)$ defines and order isomorphism from $\mathbf{Spec}\, \mathbf{A}/\theta(J)$ onto $(T_J, \subseteq)$, and it is plain that **Spec A** is the cardinal sum of the posets $((T_J, \subseteq) : J \in \operatorname{Spec} \mathbf{C})$.

On the other hand, if α is a representation of $\mathbf{A}$ as a weak Boolean product of a family $(\mathbf{A}_x : x \in X)$ over a Boolean space X, then it is not hard to see that $C = \{a \in A : \alpha a(x) \in \{0,1\}$ for each $x \in X\}$ is the universe of a subalgebra $\mathbf{C}$ of $\mathbf{B}(\mathbf{A})$. The correspondence $x \mapsto P_x = \{c \in C : \alpha c(x) = 0\}$ is a homeomorphism from X onto Spec $\mathbf{C}$, and for each $x \in X$, $\mathbf{A}_x \cong \mathbf{A}/\theta(P_x)$. ∎

Let $\mathbf{A}$ be an MV-algebra. If $b \in B(\mathbf{A}) \setminus \{0,1\}$, then $\langle b \rangle \vee \langle \neg b \rangle = A$ and $\langle b \rangle \cap \langle \neg b \rangle = \{0\}$. Since $\mathbf{A}$ is congruence-permutable, by [6, Theorem 7.5], $\mathbf{A}$ is isomorphic to direct product $\mathbf{A}/\langle b \rangle \times \mathbf{A}/\langle \neg b \rangle$ (see [32] for a direct description of the quotients). Conversely, if $\mathbf{A}_1$ and $\mathbf{A}_2$ are MV-algebras, then $(1,0) \in B(\mathbf{A}_1 \times \mathbf{A}_2) \setminus \{(0,0),(1,1)\}$. Hence an MV-algebra $\mathbf{A}$ is (directly) *indecomposable* if and only if $B(\mathbf{A}) = \{0,1\}$. Thus an MV-algebra $\mathbf{A}$ is indecomposable if and only if the Stone ideals of $\mathbf{A}$ are $\{0\}$ and A.

THEOREM 2.2. *Let α be a representation of an MV-algebra $\mathbf{A}$ as a weak Boolean product of the family of MV-algebras $(\mathbf{A}_x : x \in X)$ over the Boolean space X, and let $\mathbf{C}$ be the subalgebra of $\mathbf{B}(\mathbf{A})$ associated to α. Then all algebras $\mathbf{A}_x$ are indecomposable if and only if $\mathbf{C} = \mathbf{B}(\mathbf{A})$. Hence each nontrivial MV-algebra can be represented as a weak Boolean product of indecomposable MV-algebras. Moreover, all such representations are equivalent.*

Proof. Suppose that $\mathbf{A}_x$ is not indecomposable; then by the above remarks there is b_x in $B(\mathbf{A}_x) \setminus \{0,1\}$. Since α is a subdirect representation, there is b in A such that $\alpha b(x) = b_x$, and hence there is $c \in C$ such that $x \in [\![\alpha c = 0]\!] \subseteq [\![\alpha(b \wedge \neg b) = 0]\!]$. Let $d \in A$ be such that $\alpha d = \alpha b|_{[\![\alpha c = 0]\!]} \cup 0|_{X \setminus [\![\alpha c = 0]\!]}$. Then $d \in B(\mathbf{A}) \setminus C$. Conversely, suppose $b \in B(\mathbf{A}) \setminus C$. Then there is $x \in X$ such that $\alpha b(x) \notin \{0,1\}$, and since $\alpha b(x) \in B(\mathbf{A}_x)$, it follows that $B(\mathbf{A}_x) \neq \{0,1\}$ and $\mathbf{A}_x$ is not indecomposable. ∎

An MV-algebra is said to be *local* [3] provided it has only one maximal ideal. Let $\mathbf{A}$ be an MV-algebra, and suppose $b \in B(\mathbf{A}) \setminus \{0,1\}$. Then there are maximal ideals M_1 and M_2 of $\mathbf{A}$ such that $b \in M_1$ and $\neg b \in M_2$, and since $b \oplus \neg b = b \vee \neg b = 1$, $M_1 \neq M_2$. Therefore, all local MV-algebras are indecomposable (cf. [3]). The next result is an immediate consequence of Theorems 2.1 and 2.2:

COROLLARY 2.3. *A nontrivial MV-algebra $\mathbf{A}$ is a weak Boolean product of local MV-algebras if and only if each prime ideal of $\mathbf{B}(\mathbf{A})$ is contained in a unique maximal ideal of $\mathbf{A}$, and if and only if each Stone ultraideal of $\mathbf{L}(\mathbf{A})$ is contained in a unique maximal ideal of $\mathbf{A}$.*

Given an MV-algebra $\mathbf{A}$ and $P \in$ Spec $\mathbf{B}(\mathbf{A})$, $\mathbf{A}/\theta(P)$ is an MV-chain if and only if $\langle P \rangle$ is a prime ideal of $\mathbf{A}$. Since the MV-chains are local, we obtain:

THEOREM 2.4 [35, Theorem 4]. *A nontrivial MV-algebra* **A** *is a weak Boolean product of MV-chains if and only if the Stone ultraideals of* $\mathbf{L}(\mathbf{A})$ *are prime ideals of* **A**. *Any two representations of* **A** *as weak Boolean products of MV-chains are equivalent*.

Particular cases of MV-chains are the simple MV-algebras, i.e., the subalgebras of $\Gamma(\mathbf{R}, 1)$ (see [8]). Since for any MV-algebra **A**, A/J is simple if and only if J is maximal ideal, it follows from Theorem 2.2 that

COROLLARY 2.5. *An MV-algebra* **A** *is a weak Boolean product of simple algebras if and only if each Stone ultraideal of* $\mathbf{L}(\mathbf{A})$ *is a maximal ideal of* **A**.

Remark. Suppose that **A** is a weak Boolean product of simple MV-algebras, and let $a \in A$ and $P \in \operatorname{Spec} \mathbf{B}(\mathbf{A})$ be such that $P \in [\![a \neq 0]\!]$. Then a does not belong to the ideal $\langle P \rangle$ generated by P in $\mathbf{L}(\mathbf{A})$, and since by Corollary 2.5 $\langle P \rangle$ is a maximal ideal of **A**, there are $c \in \langle P \rangle$ and $n \in \omega$ such that $c \oplus na = 1$, i.e., $\neg\, na \leq c$. Hence there is $b \in P$ such that $\neg\, na \leq b$, and this implies that $P \in [\![b = 0]\!] \subseteq [\![a \neq 0]\!]$. Therefore $[\![a = 0]\!]$ is clopen for each $a \in A$, and we have that *weak Boolean products of simple MV-algebras are in fact Boolean products* (cf. [35]).

It is proved in [35, Theorem 10] that the (weak) Boolean products of simple MV-algebras are precisely the *hyperarchimedean* MV-algebras (called *archimedean* in [32, 16, and 35]). These algebras have several characterizations (see [10, Theorem 2.2 and references given there]). For instance, **A** *is hyperarchimedean if and only if each prime ideal of* **A** *is maximal*, hence *the prime ideals of* **A** *are just the Stone ultraideals of* **A**.

An MV-algebra **A** is called *liminary* provided that for any $P \in \operatorname{Spec} \mathbf{A}$, $\mathbf{A}/P$ is *finite*. These algebras correspond to the liminary C*-algebras with Boolean primitive spectra, and they have the property that the MV-structure is uniquely determined by their order structure (see [11] for details). Let **A** be a liminary MV-algebra. Since A/J is a finite MV-chain for each prime ideal J, we have that all prime ideals are maximal. Hence **A** is hyperarchimedean.

THEOREM 2.6. *Let* **A** *be an MV-algebra. Then* **A** *is liminary if and only if* **A** *is representable as a* (*weak*) *Boolean product of finite MV-chains*.

Proof. Suppose that **A** is liminary. Since it is hyperarchimedean, it is representable as a Boolean product of simple MV-algebras given by the quotients $\mathbf{A}/\mathscr{U}$, where $\mathscr{U}$ are Stone ultraideals. Since the Stone ultraideals are prime ideals of **A**, the quotients $\mathbf{A}/\mathscr{U}$ are finite chains.

Conversely, suppose that **A** is a weak Boolean product of finite MV-chains. Then, by Theorem 2.4, the prime ideals of **A** are the Stone ultraideals. By hypothesis, the quotient of **A** by a Stone ultraideal is a finite MV-chain. ∎

3. HYPERNORMAL MV-ALGEBRAS

The following theorem is proved in [36]:

THEOREM 3.1. *The following are equivalent conditions for each MV-algebra* **A**:

(i) **A** *is representable as a Boolean product of MV-chains.*

(ii) *For all* $a \in A$, *there is* $b \in B(\mathbf{A})$ *such that for every* $c \in A$: $a \wedge c = 0$ *if and only if* $c \leq b$.

Boolean products of MV-chains are, in particular, weak Boolean products of MV-chains, as well as weak Boolean products of local MV-algebras. They also have the property that their spectra are cardinal sums of (spectral) chains. Our aim in this section is to characterize the classes of MV-algebras determined by each of the above properties by means of algebraic relations of the kind given in Theorem 3.1(ii).

We call an MV-algebra **A** *hypernormal* if and only if **Spec A** is a cardinal sum of (spectral) chains, i.e., if and only if $\Im\mathfrak{p}(\mathbf{A})$ is a hypernormal lattice in the sense of Monteiro [25] or a perfect lattice in the terminology of [15]. Since a bounded distributive lattice is hypernormal if and only if it is simultaneously completely normal and dual completely normal, by the remarks following Lemma 1.8 we have that *an MV-algebra* **A** *is hypernormal if and only if* $\Im\mathfrak{p}(\mathbf{A})$ *is a completely normal lattice*. The following result gives an algebraic characterization of hypernormal MV-algebras.

THEOREM 3.2. *The following are equivalent conditions for each MV-algebra* **A**:

(i) **A** *is hypernormal.*

(ii) *For any* $a, b \in A$, *there exists* $t \in A$ *such that* $\langle a \rangle \cap \langle t \rangle \subseteq \langle b \rangle$ *and* $\langle b \rangle \cap \langle \neg t \rangle \subseteq \langle a \rangle$.

(iii) *For any* $a, b \in A$, $a \wedge b = 0$ *implies that there exists* $t \in A$ *such that* $a \wedge t = 0$ *and* $b \wedge \neg t = 0$.

Proof. (i) ⇒ (ii): If **A** is hypernormal, then, in particular, $\Im\mathfrak{p}(\mathbf{A})$ is a completely normal lattice. Thus, by the results of [25] (see also [24] and [9]), for any $a, b \in A$, there exist $c, d \in A$ such that $\langle a \rangle \cap \langle c \rangle \subseteq \langle b \rangle$, $\langle b \rangle \cap \langle d \rangle \subseteq \langle a \rangle$, and $\langle c \rangle \vee \langle d \rangle = A$. If $\langle c \rangle \vee \langle d \rangle = A$, then there exists $n < \omega$ such that $\neg(nc) \in \langle d \rangle$. Thus $\langle \neg(nc) \rangle \subset \langle d \rangle$, and hence $\langle b \rangle \cap \langle \neg(nc) \rangle \subseteq \langle a \rangle$. On the other hand, since $\langle nc \rangle = \langle c \rangle$, we have $\langle a \rangle \cap \langle nc \rangle \subseteq \langle b \rangle$. Thus $t = nc$ satisfies (ii).

(ii) ⇒ (i): Because for any $t \in A$, $\langle t \rangle \vee \langle \neg t \rangle = A$.

(ii) ⇒ (iii): If $a \wedge b = 0$, then, by (ii), there is $t \in A$ such that $\langle a \rangle \cap \langle t \rangle \subseteq \langle b \rangle$ and $\langle b \rangle \cap \langle \neg t \rangle \subseteq \langle a \rangle$, and hence $\langle a \rangle \cap \langle t \rangle \subset \langle a \rangle \cap \langle b \rangle = \{0\}$. Thus, $a \wedge t = 0$. Similarly, we obtain $b \wedge \neg t = 0$.

(iii) ⇒ (ii): Given $a, b \in A$, by 1.2, $(a \odot \neg b) \wedge (b \odot \neg a) = 0$. Hence there is $t \in A$ such that $(a \odot \neg b) \wedge t = 0$ and $(b \odot \neg a) \wedge \neg t = 0$. Then

$$\begin{aligned} \langle a \rangle \cap \langle t \rangle \subseteq \langle b \vee a \rangle \cap \langle t \rangle &= \langle b \oplus (a \odot \neg b) \rangle \cap \langle t \rangle \\ &= (\langle b \rangle \vee \langle (a \odot \neg b) \rangle) \cap \langle t \rangle \\ &= (\langle b \rangle \cap \langle t \rangle) \vee (\langle a \odot \neg b \rangle \cap \langle t \rangle) \\ &= \langle b \rangle \cap \langle t \rangle \subseteq \langle b \rangle. \end{aligned}$$

Similarly, we obtain $\langle b \rangle \cap \langle \neg t \rangle \subseteq \langle a \rangle$. ∎

Our next result is an algebraic characterization of weak Boolean products of local MV-algebras (see Corollary 2.3).

THEOREM 3.3. *The following are equivalent conditions for each MV-algebra* $\mathbf{A}$:

(i) *Each Stone ultraideal of* $\mathbf{L}(\mathbf{A})$ *is contained in a unique maximal ideal of* $\mathbf{A}$.

(ii) *Given* a, b *in* A *with* $a \vee b = 1$ *there are* $n < \omega$ *and* $z \in B(\mathbf{A})$ *such that* $z \leq na$ *and* $\neg z \leq nb$ (*or equivalently*, $\neg z \vee na = 1$ *and* $z \vee nb = 1$).

(iii) *Given* a, b *in* A *with* $a \wedge b = 0$, *there are* $n < \omega$ *and* $z \in B(\mathbf{A})$ *such that* $a^n \leq z$ *and* $b^n \leq \neg z$ (*or equivalently*, $\neg z \wedge a^n = 0$ *and* $z \wedge b^n = 0$).

Proof. (i) ⇒ (ii): Suppose that $a, b \in A$ are such that $a \vee b = 1$, and let $J = \langle a \rangle \cap B(\mathbf{A})$ and $F = \{z \in B(\mathbf{A})$: there exists $n \in \omega$ such that $z \vee nb = 1\}$. If $z \in J \cap F$, then there are $k, l \in \omega$ such that $z \leq ka$ and $\neg z \leq lb$, and by taking $n = \max(k, l)$, we have $z \leq na$ and $\neg z \leq nb$. Therefore to complete the proof we need to show that $J \cap F \neq \emptyset$. Suppose not, i.e., $J \cap F = \emptyset$. It is plain that J and F are respectively an ideal and a filter of $B(\mathbf{A})$, and hence there is a prime ideal P of $B(\mathbf{A})$ such that $J \subseteq P$ and $P \cap F = \emptyset$. Then if $\langle a, P \rangle$ represents the ideal of $\mathbf{A}$ generated by $\{a\} \cup P$, we have that $b \notin \langle a, P \rangle$ and $a \notin \langle b, P \rangle$. Indeed, if $b \in \langle a, P \rangle$ then there would be $n \in \omega$ and $p \in P$ such that $b \leq na \oplus p = na \vee p$. Hence $1 = a \vee b \leq na \vee b \leq na \vee p$, and $\neg p \leq na$, i.e., $\neg p \in \langle a \rangle \cap B(\mathbf{A}) = J \subset P$, and P would not be proper. If $a \in \langle b, P \rangle$, then there would be $n \in \omega$ and $p \in P$ such that $a \leq nb \vee p$. Then $1 = a \vee b \leq nb \vee p$, and we would have $p \in P \cap F = \emptyset$. Therefore, $\langle a, P \rangle$ and

$\langle b, P \rangle$ are proper ideals of **A**, and hence there are maximal ideals M_1 and M_2 such that $\langle a, P \rangle \subseteq M_1$ and $\langle b, P \rangle \subseteq M_2$. Since $a \vee b = 1$, $b \notin M_1$, $a \notin M_2$, and $M_1 \neq M_2$. But $P \subseteq M_1 \cap M_2$. Therefore if (i) holds, $J \cap F \neq \emptyset$, i.e., (i) implies (ii).

(ii) ⇒ (i): Suppose (ii) holds and let M_1 and M_2 be distinct maximal ideals of **A**. Let $a \in M_1$ and $a \notin M_2$. Then there is $b \in M_2$ such that $\langle a \vee b \rangle = \langle a \oplus b \rangle = A$, and there is $n \in \omega$ such that $1 = n(a \vee b) = na \vee nb$. By (ii) there are $m \in \omega$ and $z \in B(\mathbf{A})$ such that $z \leq m(nc)$ and $\neg z \leq m(nb)$. Therefore $z \in M_1 \cap B(\mathbf{A})$ and $\neg z \in M_2 \cap B(\mathbf{A})$. Thus there is no Stone ultraideal contained in $M_1 \cap M_2$.

(ii) ⇔ (iii): By duality. ∎

Now we can look at MV-algebras which are representable as a weak Boolean product of MV-chains as a particular case of both hypernormal MV-algebras and of those representable as a weak Boolean product of local MV-algebras. Using Theorems 3.2 and 3.3 we can give an algebraic characterization of these algebras.

THEOREM 3.4. *Let* **A** *be an MV-algebra. Then the following are equivalent*:

(i) **A** *is representable as a weak Boolean product of MV-chains*.

(ii) *For any* $a, b \in A$, $a \wedge b = 0$ *implies that there exists* $t \in B(\mathbf{A})$ *such that* $a \wedge t = 0$ *and* $b \wedge \neg t = 0$.

(iii) **A** *is a hypernormal MV-algebra which is representable as a weak Boolean product of local MV-algebras*.

Proof. (i) ⇒ (ii): If **A** is a weak Boolean product of the family $(\mathbf{A}_x : x \in X)$ of MV-chains, then $a \wedge b = 0$ implies $[\![a \neq 0]\!] \cap [\![b \neq 0]\!] = \emptyset$. Since the space X is Boolean, and $[\![a \neq 0]\!]$, $[\![b \neq 0]\!]$ are disjoint closed sets, they are separable by a clopen set. Hence, there exists $t \in B(A)$ such that $[\![a \neq 0]\!] \subset [\![t = 0]\!]$ and $[\![b \neq 0]\!] \subset [\![\neg t = 0]\!]$. Clearly, $a \wedge t$ and $b \wedge \neg t$ belong to the intersection of the family of all Stone ultraideals and hence $a \wedge t = b \wedge \neg t = 0$.

(ii) ⇒ (iii): (ii) is a particular case of both Theorem 3.2(iii) and also of Theorem 3.3(iii).

(iii) ⇒ (i): Let $\mathscr{U}$ be a Stone ultraideal of $\mathbf{L}(A)$. By Corollary 2.3 there is only one maximal ideal of **A**, say M, such that $\mathscr{U} \subseteq M$. Since $\mathscr{U}$ is an ideal of **A**, it is an intersection of prime ideals, and each prime ideal which contains $\mathscr{U}$ is contained in M. Therefore $\mathscr{U}$ is the intersection of a chain of prime ideals. Then $\mathscr{U}$ must be a prime ideal of **A**, and by Theorem 2.4, property (i) holds. ∎

Remark. Both statements (iii) in Theorem 3.3 and (ii) in Theorem 3.4 remain true if $\wedge$ is replaced by $\odot$. This is not the case in Theorem 3.2.

4. MV-ALGEBRAS OF REAL VALUED CONTINUOUS FUNCTIONS

Let X be a topological space, and let $\mathbf{I} = [0,1]$ be the closed unit interval of the real line with the usual topology. Set

$$W(X) = \{h\colon X \to \mathbf{I} : h \text{ continuous}\}.$$

In $W(X)$ we define $f \oplus g$ and $\neg f$ as follows:

$$(f \oplus g)(x) = \min(1, f(x) + g(x)), \qquad (\neg f)(x) = 1 - f(x).$$

Then $W(X)$ is closed under $\oplus$ and $\neg$, and hence it is the universe of a subalgebra of the MV-algebra $\Gamma(\mathbf{R}, 1)^X$. Hence, $\mathbf{W}(X) = (W(X), \oplus, \neg, \mathbf{0})$ is an MV-algebra, where $\mathbf{0}$ denotes the constant function associated with $0 \in \mathbf{I}$.

As in the case of rings of real-valued continuous functions (see [18]), we can show that for any topological space X there exists a Tychonoff space (i.e., a completely regular and Hausdorff space) Y, such that $\mathbf{W}(X) \cong \mathbf{W}(Y)$. Hence we can assume without loss of generality that all topological spaces considered are Tychonoff. On the other hand, as in [33] (for the lattice ordered group of real-valued functions) and [19, Theorem 1] (for the lattice of real-valued continuous functions) we can show that for compact and Hausdorff spaces, $\mathbf{W}(X)$ determines X.

Given a topological space X, a *zero-set* of X is $[\![h = \mathbf{0}]\!]_X = \{x \in X | h(x) = 0\}$ for some $h \in W(X)$. A *cozero-set* is $[\![h \neq \mathbf{0}]\!]_X = \{x \in X | h(x) \neq 0\}$ for some $h \in W(X)$. Clearly, zero-sets are closed in X and cozero-sets are open.

LEMMA 4.1. *For any topological space X and for all $f, g \in W(X)$ we have*

1. $[\![f \wedge g = \mathbf{0}]\!]_X = [\![f = \mathbf{0}]\!]_X \cup [\![g = \mathbf{0}]\!]_X$, $[\![f \wedge g \neq \mathbf{0}]\!]_X = [\![f \neq \mathbf{0}]\!]_X \cap [\![g \neq \mathbf{0}]\!]_X$.
2. $[\![f \vee g = \mathbf{0}]\!]_X = [\![f = \mathbf{0}]\!]_X \cap [\![g = \mathbf{0}]\!]_X$, $[\![f \vee g \neq \mathbf{0}]\!]_X = [\![f \neq \mathbf{0}]\!]_X \cup [\![g \neq \mathbf{0}]\!]_X$.
3. *A subset $N \subset X$ is clopen if and only if it is the zero-set of a Boolean element in $\mathbf{W}(X)$.*
4. *For any $h \in W(X)$ and any $f \in B(\mathbf{W}(X))$, $h \leq f$ iff $[\![f = \mathbf{0}]\!]_X \subseteq [\![h = \mathbf{0}]\!]_X$.*
5. *For any $h \in W(X)$, $[\![h = \mathbf{0}]\!]_X \cap [\![\neg h = \mathbf{0}]\!]_X = \emptyset$.*
6. *Suppose that X is compact. Then $[\![f = \mathbf{0}]\!]_X \cap [\![g = \mathbf{0}]\!]_X = \emptyset$, if and only if there is $h \in W(X)$, such that $[\![f = \mathbf{0}]\!]_X \subseteq [\![h = \mathbf{0}]\!]_X$ and $[\![g = \mathbf{0}]\!]_X \subseteq [\![\neg h = \mathbf{0}]\!]_X$.*

Proof. We are going to prove Lemma 4.1.6. The other facts require simple verification.

6. Let $\langle f \rangle$ and $\langle g \rangle$ be the principal ideals generated by f and g respectively. We claim:

$$[\![f = \mathbf{0}]\!]_X \cap [\![g = \mathbf{0}]\!]_X = \emptyset \qquad \text{if and only if } \langle f \rangle \vee \langle g \rangle = W(X).$$

Assume that $[\![f = \mathbf{0}]\!]_X \cap [\![g = \mathbf{0}]\!]_X = [\![f \vee g = \mathbf{0}]\!]_X = \emptyset$. Then for any $x \in X$, $(f \vee g)(x) \neq 0$. Since X is compact, there is $x_o \in X$ such that for any $x \in X, (f \vee g)(x_o) \leq (f \vee g)(x)$. Since $\Gamma(\mathbf{R}, 1)$ is a simple MV-algebra, there is $n < \omega$ such that $n(f \vee g)(x_o) = 1$, hence for any $x \in X$, we have $n(f \vee g)(x) = 1$, and $n(f \vee g) = \mathbf{1}$. That is, $W(X) = \langle f \vee g \rangle = \langle f \rangle \vee \langle g \rangle$. Conversely, if $\langle f \vee g \rangle = \langle f \rangle \vee \langle g \rangle = W(X)$, then there is $n < \omega$ such that $n(f \vee g) = \mathbf{1}$, and hence for any $x \in X$, $f(x) = 0$ implies $g(x) \neq 0$. This completes the proof of the claim.

Now, if $\langle f \rangle \vee \langle g \rangle = W(X)$, then there is $n \in \omega$ such that $ng \oplus nf = \mathbf{1}$. Take $h = ng$. Then $\neg h = \neg(ng) \leq nf$, and $[\![f = \mathbf{0}]\!]_X \subseteq [\![\neg h = \mathbf{0}]\!]_X$. Moreover, since $\langle ng \rangle = \langle g \rangle$, we have $[\![g = \mathbf{0}]\!]_X = [\![h = \mathbf{0}]\!]_X$.

The converse is an immediate consequence of 4.1.5. ∎

Note that X is a connected space if and only if $\mathbf{W}(X)$ is indecomposable.

A topological space X is called an *F-space* provided that disjoint cozero-sets are completely separable (i.e., they are separated by disjoint zero-sets) (see [24] and [18]).

LEMMA 4.2. *If X is a compact F-space, then $\mathbf{W}(X)$ is a hypernormal MV-algebra.*

Proof. Let $f, g \in W(X)$ be such that $f \wedge g = \mathbf{0}$. Then, by 4.1.1 we have $[\![f \neq \mathbf{0}]\!]_X \cap [\![g \neq \mathbf{0}]\!]_X = \emptyset$. Since X is an F-space, $[\![f \neq \mathbf{0}]\!]_X$ and $[\![g \neq \mathbf{0}]\!]_X$ are separated by disjoint zero-sets. Hence, by 4.1.6, there exists $h \in W(X)$ such that $[\![f \neq \mathbf{0}]\!]_X \subseteq [\![h = \mathbf{0}]\!]_X$ and $[\![g \neq \mathbf{0}]\!]_X \subseteq [\![\neg h = \mathbf{0}]\!]_X$. It is straightforward to see that $f \wedge h = \mathbf{0}$ and $g \wedge \neg h = \mathbf{0}$. So, by Theorem 3.2, $\mathbf{W}(X)$ is hypernormal. ∎

THEOREM 4.3. *A topological space X is an F-space if and only if $\mathbf{W}(X)$ is a hypernormal MV-algebra.*

Proof. Let βX be the Stone–Čech compactification of X. Since X is dense in βX we have $\mathbf{W}(X) \cong \mathbf{W}(\beta X)$. Moreover, X is an F-space if and only if βX is an F-space (see [18, 14.25]). Then by Lemma 4.2, for each F-space X, $\mathbf{W}(X)$ is hypernormal. Conversely, assume that $\mathbf{W}(X)$ is hypernormal. Let $f, g \in W(X)$ be such that $[\![f \neq \mathbf{0}]\!]_X \cap [\![g \neq \mathbf{0}]\!]_X = \emptyset$. Thus $[\![f \wedge g \neq \mathbf{0}]\!]_X = \emptyset$, hence $[\![f \wedge g = \mathbf{0}]\!]_X = X$ and $f \wedge g = \mathbf{0}$. Then, by

Theorem 3.2, there exists $h \in W(X)$ such that $f \wedge h = \mathbf{0}$ and $g \wedge \neg h = \mathbf{0}$; this implies $[\![f \neq \mathbf{0}]\!]_X \subseteq [\![h = \mathbf{0}]\!]_X$ and $[\![g \neq \mathbf{0}]\!]_X \subseteq [\![\neg h = \mathbf{0}]\!]_X$. Thus by 4.1.5, $[\![f \neq \mathbf{0}]\!]_X$ and $[\![g \neq \mathbf{0}]\!]_X$ are completely separable. ∎

We say that a topological space X is a *strong F-space* if and only if disjoint cozero-sets are completely separable by clopen sets. Since each clopen subset is a zero-set, any strong F-space is an F-space. The converse is not true, as we will show by giving an example.

THEOREM 4.4. *A topological space X is a strong F-space if and only if $\mathbf{W}(X)$ is representable as a weak Boolean product of MV-chains.*

Proof. Since the clopens of X are determined by the Boolean elements of $\mathbf{W}(X)$, the proof of the result is obtained as the proof of Theorem 4.3 by taking $[\![h = \mathbf{0}]\!]_X$ and $[\![\neg h = \mathbf{0}]\!]_X$ clopens. ∎

We recall that a topological space is called *basically disconnected* if and only if the closure of any cozero-set is open (and hence it is clopen). Every basically disconnected space is an F-space; the converse fails (see [18, 14N]). We recall that every basically disconnected Tychonoff space has a basis of clopen sets (see [38, 14C.2] and [18, 4K.8]).

THEOREM 4.5. *Let X be a Tychonoff topological space. Then $\mathbf{W}(X)$ is representable as a Boolean product of MV-chains if and only if X is basically disconnected.*

Proof. Assume that $\mathbf{W}(X)$ is representable as a Boolean product of MV-chains. Given $h \in W(X)$, the pseudocomplement of h, sh, exists and $sh \in B(\mathbf{A})$. Then $[\![h \neq \mathbf{0}]\!]_X \subset [\![sh = \mathbf{0}]\!]_X$. Since X is a Tychonoff space, the clopens form a basis for closed sets. For each $f \in B(\mathbf{W}(X))$, $[\![h \neq \mathbf{0}]\!]_X \subset [\![f = \mathbf{0}]\!]_X$ implies $h \wedge f = \mathbf{0}$. Hence $f \leq sh$ and, by 4.1.4, $[\![sh = \mathbf{0}]\!]_X \subseteq [\![f = \mathbf{0}]\!]_X$. Thus $\mathrm{cl}[\![h \neq \mathbf{0}]\!]_X = [\![sh = \mathbf{0}]\!]_X$. Conversely, assume that X is basically disconnected. Let $h \in W(X)$ and $f \in B(\mathbf{W}(X))$ such that $[\![f = \mathbf{0}]\!]_X$ is the closure of $[\![h \neq \mathbf{0}]\!]_X$. Then $[\![h \neq \mathbf{0}]\!]_X \subseteq [\![f = \mathbf{0}]\!]_X$ implies $h \wedge f = \mathbf{0}$. On the other hand, if $g \in W(X)$ is such that $h \wedge g = \mathbf{0}$, then $[\![h \neq \mathbf{0}]\!]_X \subseteq [\![g = \mathbf{0}]\!]_X$. Hence $[\![f = \mathbf{0}]\!]_X \subseteq [\![g = \mathbf{0}]\!]_X$, and by 4.1.4 $g \leq h$. Thus f is the pseudocomplement of h. ∎

We can now give an example of a hypernormal and indecomposable MV-algebra which is not representable as a weak Boolean product of MV-chains.

Let R^+ be the space of nonnegative reals with the topology induced by the usual topology of $\mathbf{R}$, and let βR^+ be the Stone−Čech compactification of R^+. The topological space $\beta R^+ \setminus R^+$ is a compact and connected F-space (see [18, p. 211]). Thus $\mathbf{W}(\beta R^+ \setminus R^+)$ is a hypernormal and indecomposable MV-algebra. Moreover, by an argument similar to that used in [9, 2.5], if $\mathbf{W}(\beta R^+ \setminus R^+)$ were representable as a weak Boolean product of MV-chains, then $\beta R^+ \setminus R^+$ would have only one element. But $\beta R^+ \setminus R^+$ has $2^{\mathfrak{c}}$ elements (see [18, p. 211]). Therefore, $\mathbf{W}(\beta R^+ \setminus R^+)$ is not representable as a weak Boolean product of MV-chains. By 4.4, $\beta R^+ \setminus R^+$ is not a strong F-space.

We are now going to give an example of a weak Boolean product of MV-chains which is not representable as Boolean product of MV-chains. To obtain the example we will exhibit a strong F-space which is not basically disconnected.

We consider the topological space $\mathbf{M}$ defined in [24, p. 84] as follows: Let ω_1 be the first uncountable ordinal, and S be a countable set disjoint from $\omega_1 \cup \{\omega_1\}$. Consider the set $\mathbf{M} = \omega_1 \cup \{\omega_1\} \cup S$ and let $\mathscr{G}$ be a nonprincipal ultrafilter on S (an ultrafilter containing the filter of all cofinite subsets). The topology on $\mathbf{M}$ is defined by taking as a basis of neighborhoods the following sets:

—If $x \in \omega_1 \cup S$, we have $\{\{x\}\}$. That is the points of $\omega_1 \cup S$ are isolated.

—For ω_1 we take the subsets of the form $[\alpha, \omega_1] \cup E$, where $\alpha < \omega_1$, and $[\alpha, \omega_1] = \{\sigma | \alpha \leq \sigma \leq \omega_1\}$ and $E \in \mathscr{G}$.

In [24] it is shown that the space $\mathbf{M}$ is an F-space. By analyzing Mandelker's proof, we obtain the following fact:

LEMMA 4.6. *Disjoint cozero-sets of* $\mathbf{M}$ *are separable by clopen sets. That is,* $\mathbf{M}$ *is a strong F-space.*

COROLLARY 4.7. $\mathbf{W}(\mathbf{M})$ *is representable as a weak boolean product of MV-chains.*

THEOREM 4.8. $\mathbf{M}$ *is not basically disconnected.*

Proof. Since S is not closed, it cannot be a zero-set. But S is a cozero-set. Indeed, it is the cozero-set of the function $g \in W(\mathbf{M})$ defined as follows:

$$g(x) = \begin{cases} 0 & \text{if } x \in \omega_1 \cup \{\omega_1\} \\ 1/n & \text{if } x = s_n, \end{cases}$$

where $S = \{s_n : n \in \omega\}$ is an enumeration of S. Observe that for $h \in B(\mathbf{W}(\mathbf{M}))$, $g \wedge h = \mathbf{0}$ if and only if $S = [\![g \neq \mathbf{0}]\!]_{\mathbf{M}} \subseteq [\![h = \mathbf{0}]\!]_{\mathbf{M}}$. Now, any clopen N containing S contains its closure $S \cup \{\omega_1\}$. Since N is open and contains a neighborhood of ω_1, it follows that N contains a set of the form $[\alpha, \omega_1] \cup S$, with $\alpha < \omega_1$. Clearly, there is β, $\alpha < \beta < \omega_1$, and $T = [\beta, \omega_1] \cup S$ is also clopen. Consequently, $\operatorname{cl} S = S \cup \{\omega_1\}$ is not open. Thus $\mathbf{M}$ is not basically disconnected. ∎

COROLLARY 4.9. **W(M)** *is not representable as a Boolean product of MV-chains*.

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