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Banach spaces in various positions ☆

Jesús M.F. Castillo [a,*], Anatolij Plichko [b]

[a] *Departamento de Matemáticas, Universidad de Extremadura, Avenida de Elvas, 06071 Badajoz, Spain*
[b] *Department of Mathematics, Cracow University of Technology, ul. Warszawska 24, 31-155 Cracow, Poland*

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Abstract

We formulate a general theory of positions for subspaces of a Banach space: we define equivalent and isomorphic positions, study the automorphy index $\mathfrak{a}(Y, X)$ that measures how many non-equivalent positions Y admits in X, and obtain estimates of $\mathfrak{a}(Y, X)$ for X a classical Banach space such as $\ell_p, L_p, L_1, C(\omega^\omega)$ or $C[0, 1]$. Then, we study different aspects of the automorphic space problem posed by Lindenstrauss and Rosenthal; namely, does there exist a separable automorphic space different from c_0 or ℓ_2? Recall that a Banach space X is said to be automorphic if every subspace Y admits only one position in X; i.e., $\mathfrak{a}(Y, X) = 1$ for every subspace Y of X. We study the notion of extensible space and uniformly finitely extensible space (UFO), which are relevant since every automorphic space is extensible and every extensible space is UFO. We obtain a dichotomy theorem: Every UFO must be either an $\mathcal{L}_\infty$-space or a weak type 2 near-Hilbert space with the Maurey projection property. We show that a Banach space all of whose subspaces are UFO (called hereditarily UFO spaces) must be asymptotically Hilbertian; while a Banach space for which both X and X^* are UFO must be weak Hilbert. We then refine the dichotomy theorem for Banach spaces with some additional structure. In particular, we show that an UFO with unconditional basis must be either c_0 or a superreflexive weak type 2 space; that a hereditarily UFO Köthe function space must be Hilbert; and that a rearrangement invariant space UFO must be either L_∞ or a superreflexive type 2 Banach lattice.

Keywords: Extension of isomorphisms; Positions in Banach spaces; Automorphic space problem

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* Corresponding author.
E-mail addresses: castillo@unex.es (J.M.F. Castillo), aplichko@usk.pk.edu.pl (A. Plichko).

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1. Introduction

Lindenstrauss and Rosenthal [52] showed that c_0 has the property that every isomorphism between two of its infinite codimensional subspaces can be extended to an automorphism of the whole space and formulated the so-called

Automorphic space problem. Are c_0 and ℓ_2 the only separable Banach spaces with that property?

This paper outgrowths from the study of different aspects of that problem, as we describe now.

In Section 3 we formulate a general theory of positions for subspaces of a Banach space: we define equivalent and isomorphic positions and borrow from [63] the notion of automorphy index $\mathfrak{a}(Y, X)$ that measures how many non-equivalent positions Y admits in X. We also define the automorphy index of X as $\mathfrak{a}(X) = \sup_Y \mathfrak{a}(Y, X)$. A Banach space is said to be automorphic if $\mathfrak{a}(X) = 1$. Thus, the general automorphic space problem is whether there exist automorphic spaces different from $c_0(\Gamma)$ or $\ell_2(\Gamma)$. We obtain some general principles and basic techniques to estimate $\mathfrak{a}(Y, X)$.

In Section 4 we estimate the automorphy indices $\mathfrak{a}(Y, X)$ for classical Banach spaces. We obtain, among other results, the following: $\mathfrak{a}(c_0, X) \in \{0, 1, 2, \aleph_0\}$ for every separable Banach space X; $\mathfrak{a}(Y, \ell_p) = \mathfrak{c}$ for all subspaces of ℓ_p $p \neq 2s$, and $\mathfrak{a}(Y, L_p) = \mathfrak{c}$ for all subspaces of L_p, $p > 2$ not isomorphic to ℓ_2; while $\mathfrak{a}(\ell_2, L_p) = 1$; for $1 < p < 2$ one has $\mathfrak{a}(Y, L_p) = \mathfrak{c}$ for all non-strongly embedded subspaces of L_p; $\mathfrak{a}(Y, L_1) = \mathfrak{c}$ for all nonreflexive subspaces of L_1, while $\mathfrak{a}(\ell_2, L_1) = \mathfrak{c}$; $\mathfrak{a}(Y, C[0, 1]) \in \{1, \mathfrak{c}\}$ for every separable Banach space Y. Examples of spaces admitting just one position in $C[0, 1]$ include the subspaces of c_0 and the weak*-closed subspaces of ℓ_1 – with respect to the duality with c_0; while examples of spaces admitting a continuum of non-equivalent positions include ℓ_p for $1 < p < \infty$ and those Y such that $C[0, 1]/Y$ has separable dual. The results in Sections 3 and 4 support the conjecture that whenever there are two non-isomorphic positions of Y in X then $\mathfrak{a}(Y, X) \in \{0, 1, \mathfrak{c}\}$ for X separable, while $\mathfrak{a}(Y, X) \in \{0, 1, \aleph_0, \mathfrak{c}\}$ for an arbitrary X.

Section 5 is devoted to study the notions of extensible and uniformly finitely extensible space (UFO). ℓ_∞ would be the prototype of extensible non-automorphic space, while every $\mathcal{L}_\infty$ space is an UFO. These notions are relevant since it follows from [19,65] that every automorphic space is extensible and every extensible space is UFO. After establishing some stability properties we obtain a dichotomy theorem: Every UFO must be either an $\mathcal{L}_\infty$-space or a weak type 2 near-Hilbert space.

Section 6 refines the dichotomy theorem for Banach spaces with some additional properties: in particular, if both X and X^* are UFO then X must be weak Hilbert; and if all subspaces of X are UFO (we call this a hereditarily UFO or HUFO) then X must be asymptotically Hilbertian.

Section 7 refines the dichotomy for Banach spaces with some additional structure. In particular, we show that an UFO with unconditional basis must be either c_0 or a superreflexive weak type 2 space; that an HUFO Köthe function space must be a Hilbert space; and that a rearrangement invariant UFO must be either L_∞ or a superreflexive type 2 space.

The dichotomy theorem in Section 5, together with its refined versions in Sections 6 and 7, probably constitute the first sound support for the Lindenstrauss–Rosenthal conjecture.

2. Preliminaries

Throughout the paper we will use standard notation in Banach space theory, see e.g. [54,55]. Unless otherwise stated, all linear subspaces are assumed to be closed and all operators are supposed to be linear and bounded. Given two subspaces E and F of a Banach space X, $E \oplus F$ denotes the algebraic sum of E and F with conditions: $E \cap F = 0$ and $E + F$ is closed. $E \simeq F$ denotes that E is isomorphic to F. By $\operatorname{dist}(E, F)$ we denote the Banach–Mazur distance between Banach spaces E and F, and by d_E, the Banach–Mazur distance from E to a Hilbert space of the same dimension (finite or infinite) as E. The projection constant $\lambda(E, X)$ of a subspace E of a Banach space X is defined as the infimum of the norms of the projections of X onto E. id_X denotes the identity operator in a space X. S_X denotes the unit sphere of X and B_X its closed unit ball. The *distance* between subsets U and V of a Banach space X is defined as $\rho(U, V) = \inf\{\|u - v\|\colon u \in U, v \in V\}$. By $|\Gamma|$ we denote the cardinality of a set Γ.

A Banach space X is said to be of *type* p ($1 \leqslant p \leqslant 2$), respectively *cotype* q ($2 \leqslant q < \infty$) (see e.g. [55, p. 72]) if there exist constants c, C such that, for every elements $(x_i)_1^n$ in X

$$\mathbb{E}\left\|\sum_{i=1}^n \varepsilon_i x_i\right\| \leqslant C\left(\sum_{i=1}^n \|x_i\|^p\right)^{1/p}, \quad \text{resp.} \quad \mathbb{E}\left\|\sum_{i=1}^n \varepsilon_i x_i\right\| \geqslant c\left(\sum_{i=1}^n \|x_i\|^q\right)^{1/q}, \tag{1}$$

where ε_i are independent symmetric Bernoulli variables. We set

$$p(X) = \sup\{p\colon X \text{ is of type } p\} \quad \text{and} \quad q(X) = \inf\{q\colon X \text{ is of cotype } q\}.$$

Kwapien's theorem [47] establishes that a Banach space is isomorphic to a Hilbert space if and only if it has type 2 and cotype 2. A Banach space X is said to have the *Maurey projection property* [20, p. 127] if there is a function $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that for each, not necessarily infinite dimensional, subspace Y of X with $d_Y < \infty$ there exists a projection $P : X \to Y$ with $\|P\| \leqslant f(d_Y)$. Type 2 spaces enjoy the Maurey projection property [20, p. 127]. Let us say that a Banach space is *near-Hilbert* if $p(X) = q(X) = 2$. Szankowski [83] proves that a Banach space all of whose subspaces have the approximation property is near-Hilbert (without giving them a specific name). A Banach space X is said to be *asymptotically Hilbertian* [73] if there is a constant c such that for every n there is a subspace $X_n \subset X$ of finite codimension such that *every* n-dimensional subspace $E \subset X_n$ satisfies $d_E \leqslant c$. A Banach space X is said to have *property upper* (H), respectively *property lower* (H) (Casazza and Nielsen [19]) if there is a function $f(\lambda)$ (resp. $g(\lambda)$), so that for every normalized λ-unconditional basic sequence $(x_i)_1^n$ in X

$$\left\|\sum_1^n x_i\right\| \leqslant f(\lambda)\sqrt{n} \quad \text{resp.} \quad \left\|\sum_1^n x_i\right\| \geqslant g(\lambda)\sqrt{n}. \tag{2}$$

A Banach space X is said to have *property* (H) if it has the properties upper (H) and lower (H) simultaneously.

A Banach space X is said to have *weak type* 2 [74, p. 172] if there is a constant C and a $\delta \in (0, 1)$, so that whenever E is a subspace of X and an operator $T : E \to \ell_2^n$, there is an orthogonal projection P on ℓ_2^n of rank $> \delta n$ and an operator $S : X \to \ell_2^n$ with

$$Sx = PTx \quad \text{for all } x \in E, \quad \text{and} \quad \|S\| \leqslant C\|T\|.$$

Analogously, X is said to have *weak cotype* 2 [74, p. 153] if there is a constant C and a $\delta \in (0, 1)$, so that whenever E is a finite dimensional subspace of X then there is a subspace F of E with

$$\dim F \geqslant \delta \cdot \dim E \quad \text{and} \quad d_F \leqslant C.$$

A space having simultaneously weak type and weak cotype 2 is called a *weak Hilbert* space. A weak type 2 space X verifies $p(X) = 2$ [74, p. 170] while a weak cotype 2 space X verifies $q(X) = 2$ [74, p. 159]. One therefore has the gradation

$$\boxed{\text{weak Hilbert}} \quad \Rightarrow \quad \boxed{\text{property } (H)} \quad \Rightarrow \quad \boxed{\text{asymptotically Hilbertian}} \quad \Rightarrow \quad \boxed{\text{near-Hilbert}}$$

Each asymptotically Hilbertian space with a symmetric basis is isomorphic to Hilbert. Actually, this statement is valid for more general bases (sf. [74, p. 219]). Each minimal (see Section 3 for definition) asymptotically Hilbertian space is isomorphic to a Hilbert space (Johnson [39]). Every weak Hilbert space is asymptotically Hilbertian [73, Section 4].

An *exact sequence*

$$0 \longrightarrow Y \xrightarrow{\ j\ } X \xrightarrow{\ q\ } Z \longrightarrow 0 \tag{3}$$

is a diagram formed by Banach spaces and linear continuous operators in which the kernel of each arrow coincides with the image of the preceding. Two exact sequences $0 \to Y \to X \to Z \to 0$ and $0 \to Y \to X_1 \to Z \to 0$ are said to be *equivalent* if there exists an operator $\tau : X \to X_1$ making commutative the diagram

$$\begin{tikzcd}
0 \arrow[r] & Y \arrow[r] \arrow[d, equal] & X \arrow[r] \arrow[d, "\tau"] & Z \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & Y \arrow[r] & X_1 \arrow[r] & Z \arrow[r] & 0.
\end{tikzcd}$$

The exact sequence (3) is said *to split* if it is equivalent to the trivial sequence $0 \to Y \to Y \oplus Z \to Z \to 0$; this exactly means that $j(Y)$ is complemented in X.

Following the notation and terminology of [24], two positions (i.e. into isomorphisms, see Section 3 for details) $i : Y \to X$ and $j : Y \to X_1$ are said to be *semi-equivalent* if the operator j can be extended to an operator $J : X \to X_1$ through i and the operator i can be extended to an operator $I : X_1 \to X$ through j. Dually, two quotient maps $p : X \to Z$ and $q : X_1 \to Z$ are said to be semi-equivalent if one can be lifted through the other and vice-versa.

Recall from [22,17,24] the identification of exact sequences $0 \to Y \to X \to Z \to 0$ of Banach spaces with *z-linear maps* $F : Z \to Y$; namely, homogeneous maps such that for some constant $K > 0$ and every finite set $x_1, \dots, x_n$ one has $\|F(\sum x_k) - \sum F x_k\| \leqslant K \sum \|x_k\|$. The identification between an exact sequence and a z-linear map will be written as $0 \to Y \to X \to Z \to 0 \equiv F$. Two z-linear maps F, G are said to be *equivalent*, and written $F \equiv G$, when the associated exact sequences are equivalent. Under these identifications, the lower sequence in the diagram

$$\begin{tikzcd}
0 \arrow[r] & Y \arrow[r] & X \arrow[r] & Z \arrow[r] & 0 \equiv F \\
0 \arrow[r] & Y \arrow[r] \arrow[u, equal] & PB \arrow[r] \arrow[u] & Z_1 \arrow[r] \arrow[u, "\tau"'] & 0 \equiv F\tau
\end{tikzcd}$$

is called the *pull-back* sequence of F and τ, and its associated z-linear map is $F\tau$ (standard composition as maps). Dually, the lower sequence in a diagram

$$\begin{tikzcd}
0 \arrow[r] & Y \arrow[r, "j"] \arrow[d, "\tau"'] & X \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d, equal] & 0 \equiv F \\
0 \arrow[r] & Y_1 \arrow[r, "j_1"] & PO \arrow[r] & Z \arrow[r] & 0 \equiv \tau F
\end{tikzcd}$$

is called the *push-out* sequence of τ and F, and its associated z-linear map is τF. The push-out construction enjoys the following universal property: Given operators $\alpha_1 : Y_1 \to M$ and $\alpha : X \to M$ such that $\alpha j = \alpha_1 \tau$ there exists a unique operator $\gamma : PO \to M$ such that $\alpha_1 = \gamma j_1$ and $\alpha = \gamma u$.

3. Positions

We state the following general problem:

Problem. Let Y, X be Banach spaces. How many different positions Y admits in X?

Even if the problem is meaningful in other categories, we restrict ourselves to work within the category of (mainly separable) Banach spaces and linear continuous operators. Let us first give a precise meaning to the words "different positions" and "the same position". An *embedding* $i : Y \to X$ is an into isomorphism, and a *position* of Y in X is defined by an embedding $i : Y \to X$. Unless otherwise stated, all embeddings are assumed to be infinite codimensional; i.e., $X/i(Y)$ is infinite dimensional.

Definition. Two positions $i : Y \to X$ and $j : Y \to X$ are said to be *equivalent*, and represented as $i \sim j$, if there exists an automorphism $\sigma : X \to X$ such that $\sigma i = j$.

This definition has a homological root since $i \sim j$ if and only if there exist isomorphisms σ, σ' making commutative the diagram

$$\begin{tikzcd}
0 \arrow[r] & Y \arrow[r, "i"] \arrow[d, equal] & X \arrow[r] \arrow[d, "\sigma"] & X/i(Y) \arrow[r] \arrow[d, "\sigma'"] & 0 \\
0 \arrow[r] & Y \arrow[r, "j"] & X \arrow[r] & X/j(Y) \arrow[r] & 0.
\end{tikzcd}$$

This definition corresponds to Kalton's notion of "strongly equivalent embeddings" [44], and is consistent with Moreno's notion of automorphy index introduced in [63] (see Section 3.2) as an attempt to quantify the problem of how many positions Y admits in X.

Other forms to define equal and different positions are possible:

Definition. Consider Y, Y' subspaces of X via the canonical embedding. Let us say that Y and Y' have *isomorphic positions* if there is an automorphism σ in X such that $\sigma(Y) = Y'$.

Thus, Y and Y' are in two non-isomorphic positions in X when no automorphism σ of X verifies $\sigma(Y) = Y'$. This definition corresponds to Kalton's notion [44] of "equivalent embeddings": there exist isomorphisms i, j, k making commutative the diagram

$$\begin{tikzcd}
0 \arrow[r] & Y \arrow[r] \arrow[d, "i"'] & X \arrow[r] \arrow[d, "j"] & X/Y \arrow[r] \arrow[d, "k"] & 0 \\
0 \arrow[r] & Y' \arrow[r] & X \arrow[r] & X/Y' \arrow[r] & 0,
\end{tikzcd}$$

namely, the two exact sequences are isomorphically equivalent in the sense of [23].

It is clear that equivalent positions are isomorphic, although isomorphic positions can be non-equivalent (see examples below). We do not know if the fact that all positions of Y in X are isomorphic implies that all positions of Y in X are equivalent.

The following lemma detects non-equivalent positions; its proof is just mimicry of that of [65, Prop. 3.1].

Lemma 3.1. *Let Y, Y' be isomorphic subspaces of a Banach space X such that every isomorphism $Y \to Y'$ can be extended to an automorphism of X. Then every bounded linear operator $Y \to Y'$ can be extended to a bounded linear operator in X.*

Therefore, if one gets two positions $i : Y \to X$ and $j : Y \to X$ in such a way that the operator j cannot be extended to an operator $J : X \to X$ through i, the positions are not equivalent. There is a clean homological way to formulate this: the notion of semi-equivalent positions.

3.1. Semi-equivalent positions and the parallel lines principle

The semi-equivalence of the sequences $0 \to Y \xrightarrow{i} X \to X/i(Y) \to 0$ and $0 \to Y \xrightarrow{j} X_1 \to X_1/j(Y) \to 0$ exactly means that they are one pull-back of the other. Dually, the semi-equivalence of the sequences $0 \to \ker p \to X \xrightarrow{p} Z \to 0$ and $0 \to \ker q \to X_1 \xrightarrow{q} Z \to 0$ exactly means that they are one push-out of the other. One has:

Proposition 3.2. *Assume one has the following commutative diagram*

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & Y \arrow[r, equal] \arrow[d, "h"'] & Y \arrow[d, "i"] & & \\
0 \arrow[r] & X \arrow[r, "f"] \arrow[d, "q_h"'] & X_1 \arrow[r, "q_f"] \arrow[d, "q_i"] & Z_2 \arrow[r] \arrow[d, equal] & 0 \equiv F \\
0 \arrow[r] & Z_1 \arrow[r, "g"'] \arrow[d] & Z_2 \arrow[r, "q_g"'] \arrow[d] & X'' \arrow[r] & 0 \equiv G \\
 & 0 \arrow[d, phantom, sloped, "\equiv"] & 0 \arrow[d, phantom, sloped, "\equiv"] & & \\
 & H & I. & &
\end{tikzcd}
\tag{4}
$$

Then the exact sequences F and G are semi-equivalent if and only if the sequences H and I are semi-equivalent.

Proof. This result is part of a general principle regarding couples of exact sequences involved in a pull-back/push-out diagram. Indeed, there are three possible situations; one is as described, and the other two are

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & Y \arrow[r] \arrow[d] & X \arrow[r, "q_f"] \arrow[d] & Z \arrow[r] \arrow[d, equal] & 0 \equiv F \\
0 \arrow[r] & X' \arrow[r] \arrow[d, "q_h"'] & P \arrow[r, "q_g"'] \arrow[d, "q_i"] & Z \arrow[r] & 0 \equiv G \\
 & Z' \arrow[r, equal] \arrow[d] & Z' \arrow[d] & & \\
 & 0 \arrow[d, phantom, sloped, "\equiv"] & 0 \arrow[d, phantom, sloped, "\equiv"] & & \\
 & H & I & &
\end{tikzcd}
\tag{5}
$$

and

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & Y' \arrow[r, equal] \arrow[d, "h"'] & Y' \arrow[d, "i"] & \\
0 \arrow[r] & Y \arrow[r, "f"] \arrow[d, equal] & P \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d] & 0 \equiv F \\
0 \arrow[r] & Y \arrow[r, "g"'] & X \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d] & 0 \equiv G \\
 & & 0 \arrow[d, equiv] & 0 \arrow[d, equiv] & \\
 & & H & I. &
\end{tikzcd} \tag{6}$$

In each of them, the sequences F, G are semi-equivalent if and only if I, H are semi-equivalent. A unifying proof for the three assertions is as follows: From [24, Lem. 1] we know that the semi-equivalence of the couple (F, G) is equivalent to $0 = Fq_g = Gq_f$ (in the first and second diagram, and to $0 = gF = fG$ in the third); while the semi-equivalence of the couple (I, H) corresponds to $0 = iH = gH$ (in the first and third diagram, and to $0 = Hq_g = Gq_H$ in the second). But $Fq_g = jI$ since this is the diagonal sequence $0 \to X \to Z \oplus X_1 \to Z_1 \to 0$ (in the second diagram the equality is $Fq_g = Hq_I$ while in the third diagram is $fG = hI$). $\square$

Remark. A classical proof of the necessity in the situation described in diagram (4), which is the case we will mostly consider, is as follows:

Proof. Assume there is an extension T of h through i. We prove that there is a lifting ν of q_g through q_f; i.e., $q_f\nu = q_g$. To this end, let T be the extension mentioned in the hypothesis. The operator $id_{X_1} - fT$ verifies $(id_{X_1} - fT)fh = fh - fTfh = fh - fh = 0$ and so there is an operator $\nu : Z_2 \to X_1$ such that $id_{X_1} - fT = \nu q_i$. This operator ν is a lifting of q_g through q_f because $q_f\nu q_i = q_f(id_{X_1} - fT) = q_f - q_f hT = q_f = q_g q_i$; since q_i is surjective, $q_f\nu = q_g$. $\square$

Corollary 3.3. *Let X be a separable Banach space containing an uncomplemented copy of itself $j : X \to X$ and let $i : Y \to X$ be a position of Y in X. Assume that one of the following conditions holds*

(1) $X/i(Y) \simeq c_0$;
(2) *$X/j(X)$ is an $\mathcal{L}_1$-space and $X/i(Y)$ is either complemented in some dual space or a subspace of c_0.*

Then the positions i and ji are not equivalent.

Proof. Consider the commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & Y \arrow[r, equal] \arrow[d, "i"'] & Y \arrow[d, "ji"] & & \\
0 \arrow[r] & X \arrow[r, "j"] \arrow[d, "a"'] & X \arrow[r, "q"] \arrow[d, "p"] & X/j(X) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & X/i(Y) \arrow[r, "b"'] \arrow[d] & A \arrow[r, "Q"'] \arrow[d] & B \arrow[r] & 0 \\
 & 0 & 0. & &
\end{tikzcd}$$

In each case the lower sequence splits (Sobczyk's theorem [54, p. 106] in (1), Lindenstrauss lifting principle [49] in the first part of (2) and the vector valued version of Sobczyk's theorem [26] in the second part of (2)) but the middle sequence does not, they cannot be semi-equivalent. Hence i cannot extend through ji. □

3.2. *The automorphy index*

Following [63], we define the automorphy index of Y in X and the automorphy index of X as follows. Recall that the *density character* of a Banach space X, denoted by $\operatorname{dens} X$, is defined as the smallest cardinal of a dense subset in E. Let Y, X be Banach spaces and α a cardinal. Let $\mathfrak{i}_\alpha(Y, X)$ be the set of all (infinite codimensional) embeddings $i : Y \to X$ with $\operatorname{dens} X/i(Y) = \alpha$. The elements of the quotient space $\mathfrak{i}_\alpha(Y, X)/\sim$ will be called the space of *α-automorphy classes* of Y into X. We agree that it is empty when Y cannot be embedded into X with the condition $\operatorname{dens} X/i(Y) = \alpha$.

Definition. The *automorphy index* of Y in X is defined as the number of automorphy classes:

$$\mathfrak{a}(Y, X) = \sup_\alpha \left|\mathfrak{i}_\alpha(Y, X)/\sim\right|.$$

The *automorphy index* of X is defined as

$$\mathfrak{a}(X) = \sup_Y \mathfrak{a}(Y, X).$$

Thus, the automorphy index of Y in X measures in how many different forms Y can be embedded into X. Since the number of isomorphic embeddings of a separable space into a separable superspace is $\mathfrak{c}$ one always has $\mathfrak{a}(Y, X) \leqslant \mathfrak{c}$ for separable X. A Banach space X is said to be Y*-automorphic* if $\mathfrak{a}(Y, X) = 1$. A Banach space X is said to be *automorphic* (see [24]) if $\mathfrak{a}(X) = 1$. It is clear that $\mathfrak{a}(Y, X)$ is an isomorphic invariant; which we formulate for later use as:

Lemma 3.4. *If $\mathfrak{a}(Y,X)=\mathfrak{m}$ and Y' is isomorphic to Y then $\mathfrak{a}(Y',X)=\mathfrak{m}$. In particular, if Y is isomorphic to its hyperplanes then Y and each of its finite codimensional subspaces have the same automorphy index in X.*

This justifies the initial restriction of considering infinite codimensional positions only. One has

Proposition 3.5. *Let Y be a Banach space.*

- *For a set Γ with uncountable cardinal $|\Gamma|=\aleph_\alpha$ for which α is not a limit ordinal and such that $|\Gamma|>\operatorname{dens} Y$ one has $\mathfrak{a}(Y,\ell_\infty(\Gamma))=1$.*
- *There exists a Banach superspace X for which $\mathfrak{a}(Y,X)>1$.*

Proof. We prove the first assertion. Let $j:Y\to\ell_\infty(\Gamma)$ be an isometric embedding. Let $\Gamma_n\{\gamma\in\Gamma\colon\ \rho(e_\gamma,j(Y))>\frac{1}{n}\}$, where (e_γ) are the standard unit vectors of $\ell_\infty(\Gamma)$. Since $|\Gamma|=|\bigcup_n\Gamma_n|+|\{\gamma\colon e_\gamma\in j(Y)\}|$, for some n one has $|\Gamma_n|=|\Gamma|$. By Rosenthal [75], the quotient map $q:\ell_\infty(\Gamma)\to\ell_\infty(\Gamma)/j(Y)$ fixes a copy of $c_0(\Gamma)$, hence of $\ell_\infty(\Gamma)$. Thus, the following sequences are isomorphically equivalent:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Y & \xrightarrow{j} & \ell_\infty(\Gamma) & \longrightarrow & \ell_\infty(\Gamma)/j(Y) & \longrightarrow & 0\\
 & & \| & & & & & & \\
0 & \longrightarrow & Y & \xrightarrow{(j,0)} & \ell_\infty(\Gamma)\oplus\ell_\infty(\Gamma) & \longrightarrow & \ell_\infty(\Gamma)/j(Y)\oplus\ell_\infty(\Gamma) & \longrightarrow & 0.
\end{array}$$

Let now $i:Y\to\ell_\infty(\Gamma)$ be another embedding. Since the two sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Y & \xrightarrow{j} & \ell_\infty(\Gamma) & \longrightarrow & \ell_\infty(\Gamma)/j(Y) & \longrightarrow & 0\\
 & & \| & & & & & & \\
0 & \longrightarrow & Y & \xrightarrow{i} & \ell_\infty(\Gamma) & \longrightarrow & \ell_\infty(\Gamma)/i(Y) & \longrightarrow & 0
\end{array}$$

are semi-equivalent, the diagonal principles [24] yield that the following exact sequences are isomorphically equivalent:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Y & \xrightarrow{(j,0)} & \ell_\infty(\Gamma)\oplus\ell_\infty(\Gamma) & \longrightarrow & \ell_\infty(\Gamma)/j(Y)\oplus\ell_\infty(\Gamma) & \longrightarrow & 0\\
 & & \| & & & & & & \\
0 & \longrightarrow & Y & \xrightarrow{(i,0)} & \ell_\infty(\Gamma)\oplus\ell_\infty(\Gamma) & \longrightarrow & \ell_\infty(\Gamma)/i(Y)\oplus\ell_\infty(\Gamma) & \longrightarrow & 0,
\end{array}$$

which concludes the proof.

We prove now the second assertion. If Y is injective, i.e. complemented in any superspace then, by Rosenthal's theorem [75], it contains a complemented subspace isomorphic to ℓ_∞, hence $Y=\ell_\infty\oplus E=\ell_\infty\oplus\ell_\infty\oplus E$, and thus Y has in $X=Y\oplus\ell_2$ two evident different positions: one that gives ℓ_2 as the quotient and the other that gives $\ell_\infty\oplus\ell_2$ as the quotient. If Y is not injective then there exists a superspace Z in which Y is uncomplemented. So, Y has in $X = Y \oplus Z$ at least two different positions: one, complemented; and the other, uncomplemented. $\square$

When Y is separable, the superspace X for which $\mathfrak{a}(Y, X) > 1$ can be chosen separable: indeed, if $Y \simeq c_0$ then $Y = Z \oplus Z$ has in $X = Y \oplus \ell_2$ two evident different positions: Z and $Z \oplus Z$. If $Y \not\simeq c_0$ then, by the well-known Zippin theorem [85], there exists a separable superspace Z in which Y is uncomplemented. So, Y has in $X = Y \oplus Z$ at least two different positions: one, complemented; and the other, uncomplemented.

The technique shown in the previous proof can be isolated to detect equal positions:

Definition. Let us say that a position $i : Y \to X$ of Y into X is *small* if the corresponding quotient operator $q : X \to X/i(Y)$ is an isomorphism on a complemented copy of X. The subspace Y of X is said to be *small* if all positions of Y into X are small.

In particular, every infinite codimensional subspace of c_0 is small [2]. If X is a Banach space with unconditional basis and containing ℓ_1, then the kernel of every surjection $q : \ell_1 \to X$ is embedded in a small form since a Banach space with an unconditional basis containing ℓ_1 must also contain a complemented copy of ℓ_1 [30]. If the dual to $C[0,1]/Y$ is nonseparable then Y is small [78, p. 766].

The argument of the first part of Proposition 3.5 also shows that, under the hypothesis on the cardinal of Γ, all subspaces of $\ell_\infty(\Gamma)$ with density character strictly smaller than $|\Gamma|$ are small, as well as all separable subspaces of ℓ_∞.

Moreover, we have

Proposition 3.6. *Let X be a Banach space such that every operator $Z \to X$ from a subspace Z of X can be extended to the whole X (these will be called* extensible *in Section* 5*). If Y is a small subspace of X then X is Y-automorphic.*

Therefore, we have

Corollary 3.7. *(See [53, p. 235].)* $\mathfrak{a}(Y, \ell_\infty) = 1$ *for every separable subspace* $Y \subset \ell_\infty$.

Returning to the general situation, let us recall a few notions. H. Rosenthal defined (see [7]) a Banach space X to be *minimal* if each of its infinite dimensional subspaces contains a copy of X, and *complementably minimal* if each of its infinite dimensional subspaces contains a complemented (in X) copy of X. We will say that the space Y is *fully complemented* in X if every copy of Y in X is complemented. A Banach space X is *prime* [7] if each of its complemented subspaces (finite codimensional too) is isomorphic to X. The spaces Y, X are said to be *totally incomparable* (Rosenthal [20, p. 95]) if they have no isomorphic subspaces. For instance, the space ℓ_p is complementably minimal (Pełczyński [68]). All subspaces of ℓ_p are minimal. The Tsirelson's space $\mathcal{T}$ fails to have a minimal subspace. Its dual $\mathcal{T}^*$ is minimal [20, pp. 54–59], but not complementably minimal. Hence, every subspace of $\mathcal{T}^*$ is minimal. The arbitrarily distortable Schlumprecht space S is complementably minimal. This space is also 'partially' prime (Androulakis and Schlumprecht [7]). The space c_0 is the only space that is fully complemented in every separable superspace. The spaces ℓ_2 and c_0 are minimal, prime and fully complemented in themselves [68]. We do not know other spaces with these properties. The definition of fully complemented subspace is a reformulation of problem 2° in [68]. Fully complemented subspaces of L_p are discussed by Rosenthal [78, p. 770]. The spaces ℓ_p are prime for $1 \leqslant p \leqslant \infty$ as well as c_0 and every indecomposable space which is isomorphic to its hyperplanes (such spaces exist (Gowers and Maurey [36])). There are known no other prime spaces [84].

Lemma 3.8. *Let Y be a minimal Banach space fully complemented in X. Then one of the following alternatives holds*:

(i) *The complement of each copy of Y in X contains a* (*complemented*) *copy of Y.*
(ii) *$X \simeq Y \oplus Z$ with Z totally incomparable with Y.*

Hence, alternative (i) leads to the following characterization of when X is Y-automorphic (see also [8, Lem. 1.38]).

Proposition 3.9. *Let Y be a separable Banach space isomorphic to its square and fully complemented in X. Then X is Y-automorphic* (*i.e.* $\mathfrak{a}(Y, X) = 1$) *if and only if every complement Z of Y in X contains a copy of Y.*

Proof. Let us show the sufficiency. Indeed, $X \simeq Y \oplus Z$ and

$$Z \simeq Y \oplus Z' \simeq (Y \oplus Y) \oplus Z' \simeq Y \oplus \big(Y \oplus Z'\big) \simeq Y \oplus Z \simeq X.$$

Therefore, if $Y_1 \simeq Y_2 \simeq Y$ are subspaces of X then $X = Y_1 \oplus Z_1$, $X = Y_2 \oplus Z_2$ and $Z_1 \simeq Z_2 \simeq X$. So one can extend the isomorphism $Y_1 \simeq Y_2$ to an automorphism in X.

As for the necessity, since $X \simeq Y \oplus Z \simeq Y \oplus Y \oplus Z$ and X is Y-automorphic one gets $Z \simeq Y \oplus Z$. $\square$

Proposition 3.10. *Let X be a separable Banach space. Assume $X = Y \oplus Z$ with Y and Z totally incomparable and Y is minimal, prime and fully complemented in itself.*

(1) *If Z is isomorphic to its hyperplanes then $\mathfrak{a}(Y, X) = 2$.*
(2) *If Z is not isomorphic to its hyperplanes then $\mathfrak{a}(Y, X) = \aleph_0$.*

Proof. There are two evident different positions of Y in X: $Y_1 = Y \subset Y \oplus Z$ and $Y_2 \subset Y_1$, $Y_2 \simeq Y$, $\dim Y_2 = \dim Y_1/Y_2 = \infty$ (since Y is minimal, such Y_2 exists).

Let $V \subset X$, $V \simeq Y$. By Lemma 3.4, we may pay no attention to finite dimensional subspaces and assume that

$$V \cap Z = 0 \quad \text{and} \quad V + Z \text{ is closed.}$$

Let Y' be a subspace of Y such that $V/Z = Y'/Z$. Then Y' is isomorphic to Y and, since Y is fully complemented in itself, has a complement E in Y. So

$$X = (V \oplus E) \oplus Z = V \oplus (E \oplus Z).$$

There are tree possibilities.

a) $\dim E < \infty$ and Z is isomorphic to its hyperplanes. Then $E \oplus Z \simeq Z$. So one can extend the isomorphism $V \simeq Y_1$ to an automorphism in X.

b) $\dim E = n$ and Z is not isomorphic to its hyperplanes. Then, since Y is prime, $Y \simeq Y \oplus E$, and for different n, the positions of $Y \oplus E$ are different.

c) $\dim E = \infty$. Then, since Y is prime, $E \oplus Z \simeq Y \oplus Z$. So one can extend the isomorphism $V \simeq Y_2$ to an automorphism in X. □

Corollary 3.11. *Let Y be isomorphic to its square and assume that $X \simeq Y \oplus Z$ with $Z \not\simeq X$. Then $\mathfrak{a}(Y, X) > 1$.*

Proof. Indeed, from $X \simeq Y \oplus Z$ and $X \simeq (Y \oplus Y) \oplus Z \simeq Y \oplus (Y \oplus Z) \simeq Y \oplus X$ we obtain two positions of Y in X. □

Corollary 3.12. *Let Y be a Banach space isomorphic to its square, minimal, prime and fully complemented in X. Then $\mathfrak{a}(Y, X) \in \{1, 2, \aleph_0\}$.*

Proof. By Lemma 3.8, either (i) holds in which case, by Proposition 3.9, $\mathfrak{a}(Y, X) = 1$; or (ii) holds in which case, by Proposition 3.10, $\mathfrak{a}(Y, X) = 2$ or $= \aleph_0$. □

In order to estimate the automorphy index $\mathfrak{a}(Y, X)$, observe that the simplest way to get two different positions of Y in X is to have one complemented (with infinite dimensional complement) and the other uncomplemented. We present two versions of this observation.

Lemma 3.13.

(1) *Assume that a Banach space Y has in a Banach space X two positions i and j such that no isomorphism of $i(Y)$ onto $j(Y)$ can be extended to a bounded linear operator in X. If X is isomorphic to its square, then $\mathfrak{a}(Y, X) \geqslant \aleph_0$.*
(2) *If, moreover, X is isomorphic to $\ell_p(X)$ for some $1 \leqslant p \leqslant +\infty$ or to $c_0(X)$, then $\mathfrak{a}(Y, X) \geqslant \mathfrak{c}$.*

Proof. We prove (1). Since X is isomorphic to its square, we can consider, for each n, $X^n = X_1 \oplus \cdots \oplus X_n$ with $X_k = X$ instead of X. Let p_k be the natural projection of X^n onto X_k. Denote by $i_k : Y \to X_k$ and $j_k : Y \to X_k$ the copies of $i(Y)$ and $j(Y)$ in X_k, $k = 1, \ldots, n$. No isomorphism of $i_k(Y)$ onto $j_k(Y)$ can be extended to a bounded linear operator in X_k. Each $\theta = (\theta_1, \ldots, \theta_n)$, with $\theta_k \in \{i_k, j_k\}$, is a position of Y in X^n. We show that two different positions θ and θ' are not equivalent.

Assume that, say, $\theta_m = i_m$ and $\theta'_m = j_m$ for some m. The operator $\sigma : \theta(Y) \to \theta'(Y)$ defined by $\sigma(\theta_1(y), \ldots, \theta_n(y)) = (\theta'_1(y), \ldots, \theta'_n(y))$ is an isomorphism, because for every k

$$\sigma_k := \sigma|_{\theta_k(Y)} = \theta'_k \theta_k^{-1}.$$

If there were an extension of σ to an automorphism Θ of X^n, then $p_m S|_{X_m}$ is an extension of the isomorphism $\sigma_m : i_m(Y) \to j_m(Y)$ to a bounded linear operator in X_m since

$$p_m \Theta|_{X_m} i_m = \sigma_m i_m = \sigma \theta_m = \theta'_m = j_m,$$

which is a contradiction

To prove (2) we repeat the argument with some variations. Since $X \simeq \ell_p(X)$, we can consider $\ell_p(X_k)$ with $X_k = X$ instead of X. The meaning of $i_k : Y \to X_k$ and $j_k : Y \to X_k$

is as before, as well as p_k. For every sequence $(\theta_1, \theta_2, \ldots)$, with $\theta_k \in \{i_k, j_k\}$, the operator $\theta = (2^{-1}\theta_1, \ldots, 2^{-k}\theta_k, \ldots)$ is a position of Y in $\ell_p(X_k)$ and there is a continuum of such different sequences. Let us show that any two of them θ and θ' are not equivalent.

Assume that for some m one has, say, $\theta_m = i_m$ and $\theta'_m = j_m$. The operator $\sigma : \theta(Y) \to \theta'(Y)$ defined by

$$\sigma\left(2^{-1}\theta_1, \ldots, 2^{-k}\theta_k, \ldots\right) = \left(2^{-1}\theta'_1, \ldots, 2^{-k}\theta'_k, \ldots\right)$$

is an isomorphism since, for every k,

$$\sigma_k := \sigma|_{\theta_k(Y)} = 2^{-k}\theta'_k 2^k \theta_k^{-1}.$$

If there were an extension of σ to an automorphism Θ in $\ell_p(X_k)$ then we get a contradiction with the fact that $p_m\Theta|_{X_m}$ is an extension of $\sigma_m : i_m(Y) \to j_m(Y)$ to a bounded linear operator in X_m:

$$p_m\Theta|_{X_m} i_m = \sigma_m i_m = \sigma\theta_m = \theta'_m(y) = j_m. \qquad \square$$

Remark. The condition $X \simeq \ell_p(X)$ in (2) can be replaced by the assumption that X is isomorphic to a countable unconditional sum of Banach spaces X_k, where $X \simeq X_k$ for each k. For example, X can be an arbitrary space with symmetric basis or any r.i. function space with absolutely continuous norm.

The second version provides a lower estimate for the automorphy index:

Lemma 3.14. *Let V be a complemented subspace of X. Then $\mathfrak{a}(V, V) \leqslant \mathfrak{a}(Y, X)$ for every $V \subset Y \subset X$.*

Proof. What we actually show is the following.

Claim. *If Y is a subspace of a Banach space X and V is a subspace of Y that is complemented in X, in such a way that the following condition is satisfied*:

(∗) *There are subspaces $\{V\gamma\}_{\gamma\in\Gamma}$ of V, isomorphic to V, where Γ is a set of ordinals, and isomorphisms $\tau_{\gamma\delta} : V_\gamma \to V_\delta$, $\gamma < \delta$, which cannot be extend to any bounded linear operators in V*

then $\mathfrak{a}(Y, X) \geqslant |\Gamma|$.

This is enough since the largest cardinal of such a set Γ is precisely $\mathfrak{a}(V, V)$.

Proof of the Claim. Let P be projection of X onto V and $U = Y \cap \ker P$. Of course, each $U \oplus V_\gamma$ is isomorphic to Y. Define the isomorphism $\sigma_{\gamma\delta} : U \oplus V_\gamma \to U \oplus V_\delta$, $\gamma, \delta \in \Gamma$, by

$$\sigma_{\gamma\delta}(u + v) = u + \tau_{\gamma\delta}v, \quad u \in U,\ v \in V_\gamma.$$

This isomorphism cannot be extended to an automorphism of X; because if an extension $S_{\gamma\delta}$ exists then the restriction $PS_{\gamma\delta}|_V$ is a bounded linear operator in V extending $\tau_{\gamma\delta}$. $\square$

Sometimes a local version of the argument can be given. Recall the well-known fact that for every $1 \leqslant p < \infty$, $p \neq 2$, there is a sequence of subspaces $E_k \subset \ell_p^n$, $k = k(n)$ (and $n = n(k)$), uniformly isomorphic to ℓ_p^k, such that projection constants $\lambda(E_k, \ell_p^n) \to \infty$ as $k \to \infty$ [11,13,76]. Fix p and let c be the mentioned constant of uniform isomorphism.

Proposition 3.15. *In the space ℓ_p, $1 \leqslant p < \infty$, $p \neq 2$ there are uncomplemented subspaces Y_i, $i = 1, 2, \ldots,$ uniformly isomorphic to ℓ_p, and isomorphisms $\tau_{ij} : X_i \to X_j$, $1 \leqslant i < j < \infty$ which cannot be extended to bounded linear operators in ℓ_p.*

Proof. Let us construct two non-equivalent uncomplemented positions of ℓ_p in itself.

1. Write $X = \ell_p$ in a form $X = (X_1 \oplus X_2)_p$, where X_1 and X_2 are isometric to ℓ_p. Denote by $X_1 = \sum_{n=1}^{\infty} \ell_p^n$ the natural decomposition of X_1 into sum of n-dimensional subspaces. Let E_k be the mentioned k-dimensional subspaces of ℓ_p^n, $n = n(k)$, which are c-isomorphic to ℓ_p^k and whose projection constants $\lambda(E_k, \ell_p^n) \to \infty$ as $k \to \infty$. Put $Y_1 = \sum_k E_k$. Of course, Y_1 is c-isomorphic to ℓ_p.

Choose an increasing sequence $i(k)$, $k = 1, 2, \ldots,$ of positive integers such that

$$\frac{\lambda(E_k, \ell_p^{n(k)})}{\lambda(E_{i(k)}, \ell_p^{n(i(k))})} \to \infty \quad \text{as } k \to \infty.$$

Let for every k, F_k be $(k - i(k))$-dimensional subspaces spanned by the consecutive standard basic vectors of X_2 $(= \ell_p)$. Take $G_k = E_{i(k)} \oplus F_k$. Of course, G_k are c-isomorphic to ℓ_p^k. Put $Y_2 = \sum_{n=1}^{\infty} G_k$; then Y_2 is c-isomorphic to ℓ_p. Let $\tau_{1,2} : Y_1 \to Y_2$ be the natural linear operator which maps c^2-isomorphically E_k onto G_k. Then $\tau_{1,2}$ is c^2-isomorphism. Since ℓ_p^n is 1-complemented in X,

$$\lambda(E_k, X) = \lambda\big(E_k, \ell_p^{n(k)}\big); \qquad \lambda(E_{i(k)}, X)\lambda\big(E_{i(k)}, \ell_p^{n(i(k))}\big) \quad \text{and} \quad \lambda(G_k, X)\lambda(E_{i(k)}, X).$$

Therefore,

$$\frac{\lambda(E_k, X)}{\lambda(G_k, X)} = \frac{\lambda(E_k, X)}{\lambda(E_{i(k)}, X)} \frac{\lambda(E_k, \ell_p^{n(k)})}{\lambda(E_{i(k)}, \ell_p^{n(i(k))})} \to \infty \quad \text{as } k \to \infty.$$

So, by [65, Th. 4.4], $\tau_{1,2}$ cannot be extended to any bounded linear operator in X.

2. Write now X in a form $X = (X_1 \oplus Y_2 \oplus X_3)_p$, where X_1, X_2 and X_3 are isometric to ℓ_p. Let (E_k) and (G_k) be the subspaces from $(X_1 \oplus X_2)_p$ in the item **1**. Similarly as we constructed (G_k) by (E_k), we can, starting from (G_k), to construct in X a sequence of subspaces H_k which are c-isomorphic to ℓ_p^k, have intersected supports, and such that

$$\frac{\lambda(G_k, X)}{\lambda(H_k, X)} \to \infty \quad \text{as } k \to \infty \quad \left(\text{and, hence, } \quad \frac{\lambda(E_k, X)}{\lambda(H_k, X)} \to \infty\right).$$

Put $Y_3 = \sum_{n=1}^{\infty} H_k$; then Y_3 is c-isomorphic to ℓ_p. Let $\tau_{2,3} : Y_2 \to Y_3$ be the natural linear operator which maps c^2-isomorphically G_k onto H_k. Then $\tau_{2,3}$ is c^2-isomorphism and, by [65, Th. 4.4], it cannot be extended to any bounded linear operator in X. The same arguments work for the natural linear operator $\tau_{1,3} : Y_1 \to Y_3$ mapping c^2-isomorphically E_k onto H_k. $\square$

Corollary 3.16. *Every subspace X of ℓ_p, $1 \leqslant p < \infty$, $p \neq 2$, and every subspace $Y \subset X$ verify* $\mathfrak{a}(Y, X) = \mathfrak{c}$.

Proof. Let X be a subspace of ℓ_p and Y be a subspace of X. There exists a subspace V of Y which is isomorphic to ℓ_p and complemented in ℓ_p (Pełczyński [68]). So, we obtain the proof by combination of Lemma 3.14 and Proposition 3.15. □

This dashes the hope that every Banach space could enjoy a partially automorphic character, as some earlier results (see [8,9]) might suggest.

4. Positions in classical Banach spaces

4.1. Positions of c_0 and ℓ_2

Propositions 3.9, 3.10 and Sobczyk's theorem allow us to present a rather complete description of all possible positions of c_0 in a separable superspace. Recall that a Banach space X is *Y-saturated* (Rosenthal [80]) if every closed infinite dimensional subspace of X contains a copy of Y. Isomorphically polyhedral spaces [31] and subspaces of $C(\alpha)$ for countable α [81, p. 1571] are c_0-saturated. In particular, Schreier-like spaces (see [3]), all its subspaces and all its quotients are c_0-saturated. For other c_0-saturated spaces see Leung [48] and Gasparis [34,35]. For us the following definition is more natural. We say that a Banach space X is *complementably Y-saturated* if X contains isomorphically Y and, moreover, any infinite dimensional complement of a copy of Y in X contains a complemented copy of Y. Every $C(K)$ space with metrizable K is complementably c_0-saturated [68]. See also [33] for additional information.

Proposition 4.1. *Let X be a separable Banach space. Then*

(1) $\mathfrak{a}(c_0, X) \in \{0, 1, 2, \aleph_0\}$.
(2) *If X is complementably c_0-saturated, then* $\mathfrak{a}(c_0, X) = 1$.
(3) *If X contains c_0 but some quotient X/c_0 does not then*:
  (a) $\mathfrak{a}(c_0, X) = 2$ *if X/c_0 is isomorphic to its hyperplanes.*
  (b) $\mathfrak{a}(c_0, X) = \aleph_0$ *if X/c_0 is not isomorphic to its hyperplanes.*

The same argument works for other Banach spaces X in which every copy of c_0 is complemented, such as WCG or $C(\alpha)$-spaces for α an ordinal. There exist other (nonseparable) spaces, say ℓ_∞, in which c_0 has a unique position (Lindenstrauss and Rosenthal [52]; see also Corollary 3.7). We do not know a Banach space X for which $2 < \mathfrak{a}(c_0, X) < \aleph_0$. Note that part (1) of Lemma 3.13 requires to look for X where all copies of c_0 are complemented, or all uncomplemented, or an X not isomorphic to its square; in other words, that a natural example to get a finite number of positions of c_0 such as $c_0 \oplus \ell_\infty$ actually verifies $\mathfrak{a}(c_0, c_0 \oplus \ell_\infty) \geqslant \aleph_0$. Also, the Lindenstrauss–Pełczyński theorem [50], see also [24], yields that every subspace Y of c_0 has in a separable $C(K)$-space exactly one position. The paper [25] characterizes the Banach spaces with this property.

The Hilbert space ℓ_2 is the other separable automorphic space currently known. The theory of its positions is much more complicated than that of c_0. Our previous approach plainly works for Banach spaces in which every copy of ℓ_2 is complemented; one therefore has

Proposition 4.2. *Let X be a Banach space in which ℓ_2 is fully complemented. Then*

(1) $\mathfrak{a}(\ell_2, X) \in \{1, 2, \aleph_0\}$.
(2) *If X is complementably ℓ_2-saturated, then* $\mathfrak{a}(\ell_2, X) = 1$.
(3) *If X contains ℓ_2 but some quotient X/ℓ_2 does not then*:
   (a) $\mathfrak{a}(\ell_2, X) = 2$ *if X/ℓ_2 is isomorphic to its hyperplanes.*
   (b) $\mathfrak{a}(\ell_2, X) = \aleph_0$ *if X/ℓ_2 is not isomorphic to its hyperplanes.*

The proof of this proposition is similar to that of Proposition 4.1. Banach spaces X in which ℓ_2 is fully complemented include those with the Maurey projection property. Thus, the space $L_p(0,1)$, $p \geqslant 2$ is complementably ℓ_2-saturated [41].

There are ℓ_2-saturated spaces where no copy of ℓ_2 is complemented such as the Bourgain–Delbaen $\mathcal{L}_\infty$-space constructed in [14]. Examples of non-Hilbert ℓ_2-saturated space where all copies of ℓ_2 are complemented are the spaces $\ell_2(\ell_{p_n}^{m_n})$ when $p_n \downarrow 2$, or the weak-Hilbert space constructed by Androulakis, Casazza and Kutzarova [6]. A natural example of an ℓ_2-saturated space which contains both complemented and uncomplemented ℓ_2 is Bernstein's space, which can be described as follows. A finite subset $N = \{n_1 < n_2 < \cdots < n_k\}$ of natural numbers is said to be *admissible* if $k < n_1$. The family of admissible sets will be denoted $\mathcal{A}$. If N and M are finite non-void subsets of $\mathbb{N}$, we write $N < M$ for $\max N < \min M$. We write Nx to mean $x \cdot 1_N$, where 1_N is the characteristic function of N. The *Bernstein's space* $\mathcal{B}$ [20] is defined as the completion of the space of finitely supported sequences with respect to the norm

$$\|x\|_{\mathcal{B}} = \sup\left\{\sqrt{\sum_{k=1}^{n} \|N_k x\|_{\ell_1}^2}:\ N_k \in \mathcal{A} \text{ and } N_1 < N_2 < \cdots < N_n,\ n = 1, 2, \ldots\right\}.$$

This space is ℓ_2-saturated [20, p. 7] and, moreover:

Proposition 4.3. $\mathfrak{a}(\ell_2, \mathcal{B}) = \mathfrak{c}$.

Proof. (*A sketch*). Take first the subspace Z of $\mathcal{B}$ spanned by a sequence of subspaces $\{N_k x\colon x \in \mathcal{B}\}$ where $N_k \in \mathcal{A}$, $N_1 < N_2 < \cdots < N_k < \cdots$ and the gaps between N_k go to infinity very fast. This subspace Z is isomorphic to $(\sum_k \ell_1^{n_k})_{\ell_2}$ where $n_k = |N_k|$. If $\sup_k n_k = \infty$ then Z contains both complemented and uncomplemented subspaces isomorphic to ℓ_2 [13]. Choosing Z so that, moreover, $Z \simeq \ell_2(Z)$ we get, by Lemma 3.13(2), $\mathfrak{a}(\ell_2, Z) = \mathfrak{c}$. This Z is complemented in $\mathcal{B}$, because is spanned by a subsequence of the standard (unconditional!) basis of $\mathcal{B}$. Moreover, repeating the proof of Lemma 3.14, with appropriate modifications, we get the result. □

The following problem has been posed in [63]:

Problem. Does there exist a non-automorphic Banach space X such that $\mathfrak{a}(X) < \infty$?

Note that the identity $\mathfrak{a}(Y, X) = k$ means that X contains exactly k subspaces $Y_1, \ldots, Y_k$, each isomorphic to Y, such that for every $1 \leqslant m < n \leqslant k$ there exists an isomorphism $\tau_{mn} : Y_m \to Y_n$ which cannot be extended to an automorphism of X. Since $\mathfrak{a}(c_0) = 1 = \mathfrak{a}(\ell_2)$ a reasonable candidate to have finite automorphy index is $c_0 \oplus \ell_2$. However, we only have:

Proposition 4.4. *Let Z be a subspace of $c_0 \oplus \ell_2$.*

(1) $\mathfrak{a}(c_0 \oplus \ell_2, c_0 \oplus \ell_2) = 3$.
(2) *If Z contains no copies of c_0, then it is isomorphic to ℓ_2 and complemented in $c_0 \oplus \ell_2$.*
(3) *If Z contains no copies of ℓ_2, then it is isomorphic to a subspace of c_0.*
(4) $\mathfrak{a}(c_0, c_0 \oplus \ell_2) = \mathfrak{a}(\ell_2, c_0 \oplus \ell_2) = 2$.

Proof. *Proof of* (1). Denote $X := c_0 \oplus \ell_2$ and let Y be a subspace of X which is isomorphic to X. Pick a subspace $F \subset c_0$, isometric to c_0, with $\dim c_0/F = \infty$. Then F has in c_0 a complement isometric to c_0. Pick a subspace $G \subset \ell_2$ with $\dim G = \dim \ell_2/G = \infty$. Then G has in ℓ_2 a complement isometric to ℓ_2. So, there are three evident different positions of Y in X: $c_0 \oplus G$, $F \oplus \ell_2$ and $F \oplus G$. Let $Y = Y_0 \oplus Y_1$, $Y_0 \simeq c_0$ and $Y_1 \simeq \ell_2$. First we will show that there are subspaces E_0 and E_1 of X such that: else $E_0 \simeq c_0$ or it is finite dimensional; else $E_1 \simeq \ell_2$ or it is finite dimensional; and

$$X = (Y_0 \oplus Y_1) \oplus (E_0 \oplus E_1). \tag{7}$$

Since the Hilbert space Y_1 is incomparable with c_0 and with Y_0, there is a finite codimensional subspace Y_1' of Y_1 such that

$$c_0 \cap Y_1' = 0 \quad \text{and} \quad Y_0 \cap Y_1' = 0.$$

Because Y_1' is incomparable with c_0, there exists a bounded projection $c_0 \oplus Y_1' \to c_0$ along Y_1'. By the Sobczyk theorem, one can find a superspace $Y_2 \supset Y_1'$ such that

$$c_0 \oplus Y_2 = X.$$

Denote by E_1 a complement of Y_1' in Y_2. Of course, either $E_1 \simeq \ell_2$ or it is finite dimensional.

By incomparability of Y_0 and Y_2,

$$\dim(Y_0 \cap Y_2) < \infty.$$

So there is a finite codimensional subspace Y_0' of Y_0 such that

$$Y_0' \cap Y_2 = 0.$$

Moreover, by the incomparability of Y_0' and Y_2,

$$Y_0'/Y_2 \simeq Y_0 \simeq c_0.$$

Since $c_0/Y_2 = X/Y_2$, there is a subspace Z of c_0 such that $Z/Y_2 = Y_0'/Y_2$. The subspace Z ($\simeq Z/Y_2$) is isomorphic to c_0, hence has a complement E_0 in c_0. So, $E_0/Y_2 \oplus Y_0'/Y_2 c_0/Y_2 \simeq c_0$. Therefore, $Y_0' \oplus E_0$ is the complement of Y_2 in X. Of course, either $E_0 \simeq c_0$ or it is finite dimensional.

We thus get the decomposition

$$X = \left(Y_0' \oplus E_0\right) \oplus \left(Y_1' \oplus E_1\right) \simeq \left(Y_0' \oplus Y_1'\right) \oplus (E_0 \oplus E_1).$$

To obtain the decomposition (7), one has to note that Y is isomorphic to its finite codimensional subspaces and use Lemma 3.4.

There are three possibilities:

(1) $\dim E_0 < \infty$ and $\dim E_1 = \infty$, hence $E_0 \oplus E_1 \simeq \ell_2$; in which case $Y_0 \oplus Y_1 \simeq c_0 \oplus G$,
(2) $\dim E_0 = \infty$ and $\dim E_1 < \infty$, hence $E_0 \oplus E_1 \simeq c_0$; in which case $Y_0 \oplus Y_1 \simeq F \oplus \ell_2$,
(3) $\dim E_0 = \infty$ and $\dim E_1 = \infty$, hence $E_0 \oplus E_1 \simeq c_0 \oplus \ell_2$; in which case $Y_0 \oplus Y_1 \simeq F \oplus G$,

and the extension to an automorphism in X is clear.

Proof of (2). Since Z and c_0 are totally incomparable, then $Z \cap c_0$ is finite dimensional and $Z + c_0$ is closed. Let Z_0 be a complement of $Z \cap c_0$ in Z. We consider the projection $P : x + y \in c_0 \oplus \ell_2 \to y \in \ell_2$. Then $P|_{Z_0}$ is an isomorphism and $P(Z_0)$ is complemented in ℓ_2. If E is a complement of $P(Z_0)$ in ℓ_2, then $Z = Z_0 \oplus P^{-1}(E)$; hence Z is isomorphic to ℓ_2 and complemented in $c_0 \oplus \ell_2$.

Proof of (3). Recall that an operator $T : X \to Y$ is *upper semi-Fredholm* if its kernel $\ker T$ is finite dimensional and its range $T(X)$ is closed. Let J denote the embedding of Z into $X = c_0 \oplus \ell_2$ and $P : x + y \in c_0 \oplus \ell_2 \to y \in \ell_2$. Since PJ is strictly singular, $(id_X - P)J$ is upper semi-Fredholm. Since $(id_X - P)(Z) \subset c_0$, Z is isomorphic to a subspace of c_0.

(4) follows from (2), (3), Sobczyk's theorem and Propositions 4.1 and 4.2. □

It is conceivable that $c_0 \oplus \ell_2$ contains subspaces Y with $\mathfrak{a}(Y, c_0 \oplus \ell_2) \geqslant \aleph_0$. We know no subspace having $4, 5, 6, \ldots$ positions.

4.2. Positions in ℓ_p and L_p

We have already shown in Corollary 3.16 that all subspaces Y of ℓ_p, $1 \leqslant p < \infty$, $p \neq 2$, verify $\mathfrak{a}(Y, \ell_p) = \mathfrak{c}$. The situation for subspaces of L_p is different.

Proposition 4.5.

(1) *A separable Banach space Y is Hilbert if and only $\mathfrak{a}(Y, L_p) = 1$ for some (all) $2 < p < \infty$.*
(2) *Every complemented subspace of L_p, $1 \leqslant p < \infty$ has only one* complemented *position in L_p. In particular, the spaces ℓ_p and L_p have only one complemented position in L_p. The space ℓ_2 has only one complemented position in L_p, $1 < p < \infty$.*
(3) *Let Y be a non-strongly embedded subspace of L_p, $1 \leqslant p < 2$. Then $\mathfrak{a}(Y, L_p) = \mathfrak{c}$.*
(4) *Let Y be a subspace of L_p, $Y \not\simeq \ell_2$, $2 < p < \infty$. Then $\mathfrak{a}(Y, L_p) = \mathfrak{c}$.*
(5) *Let Y be a nonreflexive subspace of L_1. Then $\mathfrak{a}(Y, L_1) = \mathfrak{c}$.*
(6) *Let Y be a subspace of L_1 which contains a complemented copy of ℓ_2. Then $\mathfrak{a}(Y, L_1) = \mathfrak{c}$. In particular, $\mathfrak{a}(L_p, L_1) = \mathfrak{c}$ for $1 < p < 2$.*

Proof. (1). Since L_p, $2 \leqslant p < \infty$, has type 2, the first part of the item (1) follows from Proposition 4.2. If Y is not isomorphic to ℓ_2, then, by Kadec and Pełczyński [41, Cor. 3], it contains a complemented (in L_p) subspace isomorphic to ℓ_p, and we can use Lemma 3.14.

(2). The item follows from Alspach, Enflo and Odell [4] (see also [59]) asserting that L_p is primary (i.e. if $L_p \simeq E \oplus F$ then either $E \simeq L_p$ or $F \simeq L_p$).

(3). Recall that a subspace Y of L_p is *non-strongly embedded* if and only if contains a copy of ℓ_p complemented in L_p [1, Th. 6.4.7]. Thus, Y contains a complemented copy of ℓ_p. Since ℓ_p contains an uncomplemented copy of ℓ_p [11], the same occurs to L_p, and thus we get from Lemma 3.14 the estimate $\mathfrak{a}(Y, L_p) \geqslant \mathfrak{a}(\ell_p, L_p) = \mathfrak{c}$.

(4). By [41, Cor. 3], Y either is isomorphic to ℓ_2 or contains a complemented subspace, isomorphic to ℓ_p. Then apply the same reasoning as in (3).

(5). By [41, Th. 6], Y contains a complemented subspace, isomorphic to ℓ_1; Lemma 3.14 then yields $\mathfrak{c} = \mathfrak{a}(\ell_1, \ell_1) \leqslant \mathfrak{a}(Y, L_1)$.

(6). We first prove that $\mathfrak{a}(\ell_2, L_1) \geqslant 2$. Let (r_n) be the Rademacher sequence and (γ_n) be the sequence of standard Gaussian independent random variables on $[0, 1]$. The L_1-closed linear spans $[r_n]_1^\infty$ and $[\gamma_n]_1^\infty$ are isomorphic to ℓ_2 [54, p. 66] and [74, p. 14].

Lemma 4.6. *(Essentially N. Kalton and A. Pełczyński; see also [23].) In the space L_1 does not exist any bounded linear operator isomorphically sending $[r_n]_1^\infty$ into $[\gamma_n]_1^\infty$. In particular, ℓ_2 has in L_1 at least two non-isomorphic positions.*

Proof. Suppose, such operator T exists. Then $\xi_n = T(r_n)$ are independent random variables in $[\gamma_n]_1^\infty$ and there exists $a > 0$ such that $\|\xi_n\| \geqslant a$ for each n. Multiplying T, if necessary, by $1/a$ one can suppose $\|\xi_n\| \geqslant 1$ for each n. By the well-known properties of the Gaussian variables, (ξ_n) is a sequence of Gaussian variables with 0 expectation.

The Rademacher sequence is obviously order bounded. Let us show that (ξ_n) is order unbounded. Suppose $|\xi_n(t)| \leqslant \varphi(t)$ for each n, where $\varphi(t)$ is some measurable function. Given $\varepsilon > 0$, there is $c > 0$ such that the Lebesgue measure $\mu\{t \in [0, 1]\colon \varphi(t) < c\} > 1 - \varepsilon$. However, since (ξ_n) are independent,

$$\mu\bigl\{t\colon \bigl|\xi_i(t)\bigr| > c \text{ for some } i \in \{1, \ldots, n\}\bigr\} = 1 - \prod_1^n \mu\bigl\{t\colon \bigl|\xi_i(t)\bigr| \leqslant c\bigr\}$$
$$\geqslant 1 - \bigl(\mu\bigl\{t\colon \|\xi_1\|^{-1}\bigl|\xi_1(t)\bigr| \leqslant c\bigr\}\bigr)^n \to 1 \quad \text{as } n \to \infty.$$

So, the sequence (ξ_n) is order unbounded.

By [82, p. 232, Th. 1.5(ii)], every bounded linear operator in L_1 is order bounded, i.e. sends an order bounded sequence into order bounded one. So, does not exist any such operator T in L_1 which translates $(r_n)_1^\infty$ onto $(\xi_n)_1^\infty$. □

We need the following version of Lemma 3.14.

Lemma 4.7. *Let X be isomorphic to its square and let Z be a complemented subspace of a subspace $Y \subset X$. Let Z' be isomorphic to Z and such that no isomorphism $Z \to Z'$ can be extended to an automorphism in X. Then $\mathfrak{a}(Y, X) > 1$.*

Proof. Let $X = V \oplus V'$, $X \simeq V \simeq V'$. Without loss of generality one can assume $Y \subset V$ and there is a subspace $Y' \subset V'$, $Y' \simeq Y$, $Z' \subset Y'$ and complemented in Y'. It is easy to construct an isomorphism $\tau : Y \to Y'$ which cannot be extended to an automorphism in X. □

It follows from Lemma 4.6 that ℓ_2 admits two non-isomorphic positions in L_1 and, in particular, $\mathfrak{a}(\ell_2, L_1) \geqslant 2$. By Lemma 3.13(2) one gets $\mathfrak{a}(l_2, L_1) = \mathfrak{c}$. So, by Lemma 4.7, one has $\mathfrak{a}(Y, L_1) = \mathfrak{c}$. Since L_1 contains isomorphically L_p for $1 < p < 2$ (see e.g. [16]), and these contain complemented copies of ℓ_2, we have $\mathfrak{a}(L_p, L_1) = \mathfrak{c}$ for $1 < p \leqslant 2$. □

We conjecture that $\mathfrak{a}(Y, L_p) = \mathfrak{c}$ for $1 \leqslant p < \infty$, $p \neq 2$, and every subspace $Y \subset L_p$ not containing ℓ_2.

4.3. Positions in $C(K)$

Separable $C(K)$-spaces are isomorphic either $C(\alpha)$ for a countable ordinal α or $C[0,1]$ [69, §8]. Except when $\alpha < \omega^\omega$, in which case $C(\alpha) \simeq c_0$ is automorphic, the problem of calculating $\mathfrak{a}(Y, C(K))$ is mostly open. In what follows, $\delta : Y \to C(B_{Y^*})$ represents the canonical embedding of a separable Banach space Y into $C(B_{Y^*})$.

Proposition 4.8. *For every separable Banach space Y one has*

$$\mathfrak{a}\big(Y, C[0,1]\big) \in \{1, \mathfrak{c}\}.$$

Moreover,

(1) $\mathfrak{a}(Y, C[0,1]) = \mathfrak{c}$ *holds for*:
  (a) *any Banach space containing a complemented copy of any $C(K)$-space different from c_0*;
  (b) *ℓ_p for $1 < p < \infty$*;
  (c) *a Banach space Y such that $C[0,1]/Y$ does not contain ℓ_1*;
(2) $\mathfrak{a}(Y, C[0,1]) = 1$ *holds for*:
  (a) *subspaces of c_0*;
  (b) *weak*-closed subspaces of ℓ_1*;
  (c) *$c_0(Y)$, when $\mathfrak{a}(Y, C[0,1]) = 1$*;
  (d) *twisted sums of two spaces X, Y with $\mathfrak{a}(X, C[0,1]) = 1 = \mathfrak{a}(Y, C[0,1])$.*

Proof. Let K be a compact metric space such that $C(K) \not\simeq c_0$. Since $C[0,1]$ contains complemented and uncomplemented copies of $C(K)$ ([81, p. 1554], [5]) and $C(K) \simeq c_0(C(K))$ [81, pp. 1553, 1564], Lemma 3.13 implies that $\mathfrak{a}(C(K), C[0,1]) = \mathfrak{c}$. Let 1 represent a complemented embedding and let L represent an uncomplemented embedding. We already know that all elements of $\{1, L\}^{\mathbb{N}}$ represent non-equivalent embeddings $c_0(C[0,1]) \to c_0(C[0,1])$.

By Milutin's theorem, we will consider δ as an embedding $\delta : Y \to C[0,1]$. Now, assume Y has in $C[0,1]$ at least two positions. A combination of [24, Prop. 4.6] and [44, Th. 2.8] shows that a position $j : Y \to C[0,1]$ is equivalent to $\delta : Y \to C[0,1]$ if and only if they are semi-equivalent. Then there is $j : Y \to C[0,1]$ a position not semi-equivalent to $\delta : Y \to C[0,1]$. Form the embeddings $\Delta : Y \to c_0(C[0,1])$ defined as $\Delta(y)(n) = n^{-1}\delta(y)$ and $J : Y \to c_0(C[0,1])$ defined as $J(y)(n) = n^{-1} j(y)$.

Now, two different $u, v \in \{1, L\}^{\mathbb{N}}$ yield non-equivalent embeddings $u\Delta$ and vJ (or $v\Delta$ and uJ): since at some coordinate n one has, say, $u(n) = 1$ and $v(n) = L$, the existence of some operator σ verifying $\sigma v J = u\Delta$ means the existence of a certain operator T verifying $Tj = \delta$ something that does not occur.

Assertion (1.a) follows from the first sentence of this proof and Lemma 3.14.

Assertion (1.b) follows from Kalton [44], where he showed that $\mathfrak{a}(\ell_p, C[0,1]) \geqslant 2$.

We prove now assertion (1.c) following an idea taken from Kalton [44]. Recall that given a subspace $Y \subset C[0,1]$ the quotient space $C[0,1]/Y$ does not contain ℓ_1 if and only if it has separable dual [77]; see also [81]. Let $j: Y \to C[0,1]$ be an embedding in such a way that $C[0,1]/j(Y)$ does not contain ℓ_1. Let $\mathbf{D}$ denote Cantor ternary set. $\iota: \mathbf{D} \to [0,1]$ will be Lebesgue's map $\iota((\varepsilon_n)) = \sum 2\varepsilon_n 3^{-n}$. It is well known that the induced exact sequence

$$0 \longrightarrow C[0,1] \xrightarrow{\iota^\circ} C(\mathbf{D}) \xrightarrow{q} c_0 \longrightarrow 0$$

does not split. A little less known is that if $[p_n, q_n]$ denotes the sequence of different intervals $[1/3, 2/3]$, $[1/9, 2/9]$, $[7/9, 8/9], \ldots$ generating the Cantor set $\mathbf{D}$ then no lifting of no subsequence (e_{p_n}) of the canonical basis of c_0, for which the set of indices (p_n) is dense in $\mathbf{D}$, can be weakly Cauchy (see also [18]). To show this, let $f_n \in C(\mathbf{D})$ be a lifting of the canonical basis (e_n); this means that $|f_n(q_n) - f_n(p_n)| = 1$. Take f_1. Choose p_1 or q_1 – say, p_1 – and set an open interval I_1 of it on which f_1 has oscillation lesser than or equal to $1/4$. By the denseness, there is some $[p_{n_2}, q_{n_2}] \subset I_1$. Take f_{n_2} and observe that at one of the points p_{n_2}, q_{n_2} one has – say p_{n_2} again –

$$\left| f_{n_2}(p_{n_2}) - f_1(p_1) \right| \geqslant 1/2$$

and choose some new interval I_2 of p_{n_2} where f_{n_2} has oscillation at most $1/8$. Continues in this way obtaining a nested sequence I_k, and points p_{n_k} in such a way that

$$\left| f_{n_{k+1}}(p_{n_{k+1}}) - f_{n_k}(p_{n_k}) \right| \geqslant 1/2$$

while f_{n_k} has oscillation at most $1/2^k$. Take $p \in \bigcap I_k$. One has

$$\left| f_{n_{k+1}}(p) - f_{n_k}(p) \right| \geqslant 1/4$$

which is enough to make it non-weakly Cauchy.

We show now that the exact sequences

$$\begin{tikzcd}
0 \arrow[r] & Y \arrow[r, "j"] \arrow[d, equal] & C[0,1] \arrow[r] & C[0,1]/j(Y) \arrow[r] & 0 \\
0 \arrow[r] & Y \arrow[r, "\iota^\circ j"] & C(\mathbf{D}) \arrow[r] & \iota^\circ j(Y) \arrow[r] & 0
\end{tikzcd}$$

are not isomorphic, showing that the embeddings j and $\iota^\circ j$ are not equivalent. By Milyutin's theorem [62], we identify $C[0,1]$ and $C(\mathbf{D})$. Consider the commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & Y \arrow[r, equal] \arrow[d, "j"'] & Y \arrow[d, "\iota^\circ j"] & & \\
0 \arrow[r] & C[0,1] \arrow[r, "\iota^\circ"] \arrow[d, "p"'] & C[0,1] \arrow[r, "q"] \arrow[d] & c_0 \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & C[0,1]/j(Y) \arrow[r, "b"'] \arrow[d] & C[0,1]/\iota^\circ j(Y) \arrow[r, "Q"'] \arrow[d] & c_0 \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

and let us show that the two horizontal sequences are not semi-equivalent; namely, that no operator $\nu : C[0,1]/\iota^\circ j(Y) \to C[0,1]$ can exist such that $q\nu = Q$. Since the dual of $C[0,1]/j(Y)$ is separable, let us number a dense set of functionals (x_k^*) and a basis (B_n) for the topology of $[0,1]$ to get

$$\forall n \exists u_n, v_n \in B_n\colon \forall 1 \leqslant k \leqslant n\colon \quad x_k^*(pf_{u_n} - pf_{v_n}) \leqslant 2^{-k}$$

which simply follows because the map $(x_1^* p, \ldots, x_n^* p) : C[0,1] \to \mathbb{R}^n$ is finite dimensional, and thus from the images of each sequence (f_j) (with indices $j \in B_n$) one can extract a convergent subsequence. The existence of the operator ν transforms this into a weakly convergent lifting of $(e_{u_n} - e_{v_n})$, which we have already shown cannot exist.

Assertion (2.a) follows from Lindenstrauss–Pełczyński theorem [51] ($C(K)$-valued operators defined on subspaces of c_0 can be extended to c_0), which shows that all positions of a subspace H of c_0 in $C[0,1]$ are semi-equivalent to the canonical one $\delta : H \to C(B_{H^*})$. Assertion (2.b) was proved by Kalton in [43] and [44, Th. 5.2], and (2c) and (2.d) in [44]. □

Proposition 4.8 completes the results of Moreno [63] who showed that there were at least $\aleph_1$ different mutually non-isomorphic positions of $C[0,1]$ inside $C[0,1]$. Kalton asked in [42] if $\mathfrak{a}(Y, C[0,1]) = 1$ can hold for a superreflexive space Y. For countable compacta one has:

Proposition 4.9. *For every Banach space Y and $\omega^\omega \leqslant \alpha < \omega_1$ one has*

$$\mathfrak{a}\big(Y, C(\alpha)\big) \in \{0, 1, \mathfrak{c}\}.$$

Moreover,

(1) $\mathfrak{a}(Y, C(\alpha)) = \mathfrak{c}$ *holds for any Banach space containing complemented copies of $C(\beta)$, $\beta \geqslant \omega^\omega$;*
(2) $\mathfrak{a}(Y, C(\alpha)) = 1$ *holds for subspaces Y of c_0.*

A distinguished class of subspaces of the spaces of continuous functions on countable compacta is formed by the Schreier-like spaces. The *Schreier space* $\mathcal{S}$ constructed over the family $\mathcal{A}$ of admissible sets is the completion of the space of finitely supported sequences with respect to the norm $\|x\|_{\mathcal{S}} = \sup_{N\in\mathcal{A}} \|Nx\|_{\ell_1}$. The family $\mathcal{A}$ is countable and forms a closed, hence compact, subspace of $\{0,1\}^{\mathbb{N}}$; in fact, it is homeomorphic to ω^ω. On the other hand, $\mathcal{S}$ is a subspace of $C(\mathcal{A})$ through the canonical embedding $\delta : \mathcal{S} \to C(\mathcal{A})$ given by $\delta(x)(A) = \sum_{j\in A} x_j$. The space $\mathcal{S}$ is therefore c_0-saturated, hence $\mathfrak{a}(c_0, \mathcal{S}) = 1$ by Proposition 4.1.

Since $\mathcal{S}$ contains uniformly complemented ℓ_1^n, and also contains ℓ_1^n but not uniformly complemented (since it contains c_0) it cannot be an UFO (for definition see the next section). Since $\mathcal{S} = c_0(\mathcal{S})$, the space $\mathcal{S}$ contains a complemented copy of $c_0(\ell_1^n)$ and also an uncomplemented one through the embedding $c_0(\ell_1^n) \to c_0(c_0) = c_0 \to \mathcal{S}$; therefore $\mathfrak{a}(c_0(\ell_1^n), \mathcal{S}) = \mathfrak{c}$. This and Lemma 3.14 immediately yield $\mathfrak{a}(\mathcal{S}, \mathcal{S}) = \mathfrak{c}$. We feel tempted to conjecture that $\mathfrak{a}(\mathcal{S}, C(\omega^\omega)) = 1 = \mathfrak{a}(\mathcal{S}, C[0,1])$.

Among embeddings between $C(K)$-spaces, a special role is played by isometric embeddings of the form $\varphi^\circ : C(L) \to C(K)$, where $\varphi : K \to L$ is a continuous surjection, and $\varphi^\circ(f) = f \circ \varphi$. The following apparently open problem was posed by Pełczyński [70]:

Problem. Is every exact sequence

$$0 \longrightarrow C[0,1] \xrightarrow{\ j\ } C[0,1] \longrightarrow V \longrightarrow 0$$

isomorphically equivalent to an exact sequence

$$0 \longrightarrow C[0,1] \xrightarrow{\ \varphi^\circ\ } C[0,1] \longrightarrow W \longrightarrow 0\,?$$

However, even if the answer to the previous problem was to be no, an analogous argument to that of Proposition 3.13 shows that there exists a continuum of different positions of $C(\mathbf{D})$ inside $C(\mathbf{D})$ of the form φ°. Indeed, let K, L be compact Hausdorff spaces. Two continuous surjections $\varphi : K \to L$ and $\psi : K \to L$ will be called *equivalent* if the embeddings φ° and ψ° are equivalent.

We will show:

Proposition 4.10. *There is a continuum of mutually non-equivalent continuous surjections* $\mathbf{D} \to \mathbf{D}$.

Proof. Let us consider $\varphi : \mathbf{D} \to \mathbf{D}$ a continuous surjection such that φ° is an uncomplemented position and $\psi : \mathbf{D} \to \mathbf{D}$ a continuous surjection such that ψ° is a complemented position. So φ° and ψ° are not semi-equivalent. Let us show that $\{\psi, \varphi\}^{\mathbb{N}}$ is a continuum of mutually non-isomorphic continuous surjections $\eta = (\eta_n) : \mathbf{D}^{\mathbb{N}} \to \mathbf{D}^{\mathbb{N}}$ given by $(\eta_n)((x_n))(\eta_n x_n)$. Take two different elements $\eta, \mu \in \{\psi, \varphi\}^{\mathbb{N}}$; let k be a coordinate where $\eta_k = \varphi$ and $\mu_k = \psi$, and let $p_k : \mathbf{D}^{\mathbb{N}} \to \mathbf{D}$ be projection onto the kth-coordinate. If some isomorphism $\sigma : C(\mathbf{D}^{\mathbb{N}}) \to C(\mathbf{D}^{\mathbb{N}})$ exists such that $\sigma\eta^\circ = \mu^\circ$, then $\sigma\eta^\circ p_k^\circ = \mu^\circ p_k^\circ$.

It is therefore enough to show that the surjections $\psi \oplus \varphi : \mathbf{D} \times \mathbf{D} \to \mathbf{D} \times \mathbf{D}$ and $\varphi \oplus \psi : \mathbf{D} \times \mathbf{D} \to \mathbf{D} \times \mathbf{D}$ are not equivalent because the embeddings $C(\mathbf{D}) \oplus C(\mathbf{D}) \to C(\mathbf{D}) \oplus C(\mathbf{D})$ given by $(\psi^\circ, \varphi^\circ)$ and $(\varphi^\circ, \psi^\circ)$ are not semi-equivalent. To show this, assume that there could exist an extension $\left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right) : C(\mathbf{D}) \oplus C(\mathbf{D}) \to C(\mathbf{D}) \oplus C(\mathbf{D})$ of $(\psi^\circ, \varphi^\circ)$ through $(\varphi^\circ, \psi^\circ)$. This means that for all $x, x' \in C(\mathbf{D})$ one has

$$\left(\alpha\varphi^\circ x + \beta\psi^\circ x', \gamma\varphi^\circ x + \delta\psi^\circ x'\right) = \left(\psi^\circ x, \varphi^\circ x'\right).$$

Setting $x' = 0$ implies $\alpha\varphi^\circ x = \psi^\circ x$. Therefore also φ° admits a projection, against the hypothesis.

That is enough to prove the assertion about $C(\mathbf{D})$ since $\mathbf{D}$ is homeomorphic to $\mathbf{D}^{\mathbb{N}}$, and if $h : K \to L$ is a homeomorphism between compact spaces and $q : K \to M$ is a continuous surjection then:

- q° admits a projection if and only if qh admits a projection: indeed, if P verifies $P(qh)^\circ = 1$ then $(Ph^\circ)q^\circ = 1$ and if Q verifies $Qq^\circ = 1$ then $Q(h^\circ)^{-1}(qh)^\circ = 1$.
- Two arbitrary continuous surjections φ and ϕ are equivalent if and only if φh and ϕh are equivalent: if σ is the automorphism so that $\sigma\varphi^\circ = \phi^\circ$ then $h^\circ\sigma(h^\circ)^{-1}$ verifies $h^\circ\sigma(h^\circ)^{-1}(\varphi h)^\circ = (\phi h)^\circ$. □

Problem. Determine when two continuous surjections $\varphi, \psi : [0, 1] \to [0, 1]$ are equivalent. Does there exist a continuum of mutually non-equivalent continuous surjections $[0, 1] \to [0, 1]$?

5. A dichotomy for extensible Banach spaces

Extensible Banach spaces were introduced in [65] after the observation in Lemma 3.1 that if X is Y-automorphic then all operators $Y \to X$ can be extended to X.

Definition. A Banach space X is said to be *extensible* if for every subspace $Y \subset X$ every operator $\tau : Y \to X$ can be extended to an operator $T : X \to X$. If there is a $\lambda > 0$ such that some extension exists verifying $\|T\| \leqslant \lambda\|\tau\|$ then we will say that X is λ-extensible. The space X is said to be *uniformly extensible* if it is λ-extensible for some λ (see also [64]).

Automorphic spaces are extensible [65, Th. 3.2]. The converse does not hold since ℓ_∞ (injective spaces in general) is extensible and not automorphic. It was proved in [40] (resp. [65]) that the spaces $c_0(\Gamma)$ are extensible (resp. automorphic). Obviously, each subspace of a Hilbert space is extensible, while a subspace of c_0 is extensible if and only if it is c_0 [56,65].

Lemma 5.1. *(See [19,65].) A Banach space X containing an uncomplemented and a complemented copy of a space Y cannot be extensible.*

Proof. Let $i : Y \to X$ be an uncomplemented position and let $j : Y \to X$ be a complemented one with projection $P : X \to Y$; i.e., $Pj = id_Y$. If X were extensible, there would be an extension $J : X \to X$ of j through i, i.e., $Ji = j$. Therefore PJ would be a projection through i since $PJi = Pj = id_Y$. □

Extensible spaces do enjoy very few stability properties:

Proposition 5.2.

(1) *A complemented subspace of an extensible space is extensible.*
(2) *The product of extensible spaces is not necessarily extensible*; *hence the class of extensible spaces does not have the* 3-*space property.*
(3) *The quotient of two extensible spaces is not necessarily extensible.*

Proof. Assertion (1) is clear. The example $c_0 \oplus \ell_\infty$ proves (2); which moreover exhibits a separably injective non-extensible space. Even the product of incomparable extensible spaces need not be extensible, as the space $c_0 \oplus \ell_2$ shows: since $\ell_\infty(\ell_2^n)$ contains complemented copies of ℓ_2 (an explicit proof can be seen in [21]), there exists an operator $\ell_\infty(\ell_2^n) \to \ell_2$ which is not 2-summing and cannot therefore be extended to ℓ_∞. Since all operators $X \to \ell_2$ are 2-summing if and only if all operators $X^{**} \to \ell_2$ are 2-summing, there necessarily exists an operator $c_0(\ell_2^n) \to \ell_2$ that is not 2-summing and cannot therefore be extended to an operator $c_0 \to \ell_2$.

We prove now (3): recall that calling $\mathbb{N}^* = \beta\mathbb{N} - \mathbb{N}$ one has the identification $C(\mathbb{N}^*) = \ell_\infty/c_0$.

Proposition 5.3. *Under the continuum hypothesis* (*in short* ***CH***), *the space* $C(\mathbb{N}^*)$ *is not extensible.*

Proof. It is enough if we prove that under **CH**, $C(\mathbb{N}^*)$ contains an uncomplemented subspace isometric to $C(\mathbb{N}^*)$. We refine Amir's proof [5] that $C(\mathbb{N}^*)$ is not complemented in $\ell_\infty(2^{\mathbb{N}^*})$ to show that *there exists a Banach space* X *of density character* $\mathfrak{c}$ *that contains an uncomplemented copy of* $C(\mathbb{N}^*)$. Following Amir's paper [5], let Σ be a family of subsets of $\mathbb{N}^*$ that contains a basis for the topology of $\mathbb{N}^*$, and which is closed under complementation, finite union and the closure operation. We can consider the Banach space $B(\Sigma)$, sitting as $C(\mathbb{N}^*) \subset B(\Sigma) \subset \ell_\infty(\mathbb{N}^*)$ defined as the subspace of $\ell_\infty(\mathbb{N}^*)$ generated by the characteristic functions of the elements of Σ. Let also D_Σ be the union of the boundaries of all open sets living in Σ. By [5, Cor. 1], if $C(\mathbb{N}^*)$, is complemented in $B(\Sigma)$, then D_Σ is nowhere dense in $\mathbb{N}^*$. We indicate now how to construct such a family Σ of cardinality $\mathfrak{c}$ and with D_Σ dense in $\mathbb{N}^*$, so that the space $X = B(\Sigma)$ is as stated in the theorem. For every clopen subset A of $\mathbb{N}^*$, choose $U_A \subset A$ be an open not closed set. Consider then Σ the least family of subsets of $\mathbb{N}^*$ that contains all clopen A and all open sets U_A and that is closed under complementation, finite union and the closure operation.

Now, it is a consequence of Parovičenko's theorem [12] that $\mathbb{N}^*$ can be mapped onto every compact space of weight at most $\aleph_1$. Therefore, every Banach space of density character at most $\aleph_1$ is isometric to a subspace of $C(\mathbb{N}^*)$. Applying this to the space X constructed above yields the result. □

This example also show that ultrapowers need not be extensible: Bankston [10] proved under **CH** that if $\mathcal{U}$ is a free ultrafilter on the integers and **D** denotes the Cantor set then the ultrapower $\mathbf{D}_{\mathcal{U}}$ is homeomorphic to $\mathbb{N}^*$. Therefore, under **CH**, one has $C[0,1]_{\mathcal{U}} = C(\mathbb{N}^*)$ which proves the claim. In [8] it is proved that infinite dimensional ultrapowers $X_{\mathcal{U}}$ are never injective. We conjecture they are never extensible (apart from Hilbert).

Two problems about extensible spaces were posed in [65] and remain unsolved.

Extensible space problem. Do there exist separable extensible spaces different from c_0 and ℓ_2?

Uniformity problem. Is every extensible space uniformly extensible?

The extensible space problem can be considered as an approach to the automorphic space problem, in combination with the remaining question: Must a separable extensible space be automorphic? Let us present a partial positive solution to the uniformity problem.

Lemma 5.4. *Let X be an extensible Banach space such that $X = Y_1 \oplus Z_1$ and that for each n one has $Z_n = Y_{n+1} \oplus Z_{n+1}$. Then all except a finite number of the Y_n must be uniformly extensible.*

Proof. In the proof we use ideas of [19, Th. 1.1] and [65, Lem. 4.1]. Observe that there is a simple construction of a sequence of bounded projections $P_n : X \to \operatorname{lin}(Y_i)_1^n$ verifying $\ker P_n \supset \operatorname{lin}(Y_i)_{n+1}^\infty$, because $X = Y_1 \oplus \cdots \oplus Y_n \oplus Z_n$, $(Y_i)_{n+1}^\infty \subset Z_n$.

Assume now that each Y_n is not uniformly extensible. Then there are subspaces E_n of Y_n and operators $\tau_n : E_n \to Y_n$ such that $\|\tau_n\| = 1$ for each n, and the norm of every extension of τ_n to an operator $Y_n \to Y_n$ is greater than $2^n(\|P_n\| + \|P_{n-1}\|)n$. Define the operator $\tau : \operatorname{lin}(E_n)_1^\infty \to X$ by

$$\tau\Bigl(\sum x_n\Bigr) = \sum 2^{-n}\tau_n x_n, \quad x_n \in E_n.$$

By construction, $\|\tau\| \leqslant 1$. Suppose that there exists an extension $T : X \to X$ of τ. Put $S_n = 2^n(P_n - P_{n-1})T|_{Y_n}$. Then S_n is an extension of τ_n to an operator $Y_n \to Y_n$ and

$$\|S_n\| \leqslant 2^n\bigl(\|P_n\| + \|P_{n-1}\|\bigr)\|T\|;$$

which is impossible for large n. $\square$

Theorem 5.5. *An extensible space isomorphic to its square is uniformly extensible.*

We conjecture that a separable extensible space that is isomorphic to its square is automorphic.

The notion of uniformly extensible space can be localized as follows:

Definition. (See [65].) A Banach space X is said to be *uniformly finitely extensible* (an UFO, in short) if there exists a $\lambda \geqslant 1$ such that for every finite dimensional subspace $E \subset X$ each linear operator $\tau : E \to X$ can be extended to a linear operator $T : X \to X$ with $\|T\| \leqslant \lambda\|\tau\|$.

The two following results were proved in [19, Th. 1.1] and [65, Prop. 4.2] and are essential for our purposes:

Lemma 5.6.

(1) *Every extensible space is an UFO.*
(2) *Every μ-uniformly finitely extensible space that is λ-complemented in its bidual is $\lambda\mu$-extensible.*

Every $\mathcal{L}_\infty$-space is an UFO [53, p. 334], so an UFO does not have to be extensible. Let us recall that a subspace Y of a Banach space X is *locally complemented* if there exists a constant $\lambda \geqslant 1$ such that whenever F is a finite dimensional subspace of X and $\varepsilon > 0$, there is a linear operator $S : F \to Y$ with $Sy = y$ for all $y \in F \cap Y$ and $\|S\| \leqslant \lambda + \varepsilon$. The principle of local reflexivity [53, Th. 3.1] tells us, in particular, that any Banach space is locally complemented in its bidual.

Lemma 5.7. *A locally complemented subspace of an UFO is an UFO.*

Proof. Let E be a finite dimensional subspace of Y, a locally complemented subspace of an UFO X. Let $\tau : E \to Y$ be an operator and let $T : X \to X$ be an extension of it. By the definition of locally complemented subspace, for the subspace $F = \tau(E)$ there exists an operator $S : F \to Y$ such that $S|_{F\cap Y} = id_{F\cap Y}$ and $\|S\| \leqslant \lambda + \varepsilon$. Then $ST|_Y$ is an extension for τ since for $e \in E$ one has $ST(e) = S\tau e = \tau e$ and $\|ST\| \leqslant (\lambda + \varepsilon)\|T\|$. $\square$

Proposition 5.8. *If X is an UFO and $\mathcal{U}$ is an ultrafilter on a set I then the ultrapower $X_{\mathcal{U}}$ is an UFO.*

Proof. Let λ be the constant with which X is an UFO and let $\varepsilon > 0$. Let E be a subspace of dimension N of an ultrapower $X_{\mathcal{U}}$ of X, and let $\phi : E \to X_{\mathcal{U}}$ be an operator. Let $\eta_1, \ldots, \eta_N$ be a basis for E and let $\nu_k = \phi\eta_k$ be with $\eta_k = [x_i^k]$ and $\nu_k = [y_i^k]$. It is clear that there is some $B \in \mathcal{U}$ such that for all $i \in B$ the space $[x_i^k]_k$ is of dimension N.

Let $(\lambda_1, \ldots, \lambda_n(\varepsilon))$ be a δ-net for the unit ball of E. For a fixed λ_l one has

$$\lim_{\mathcal{U}}\left\|\sum_k \lambda_l(k)y_i^k\right\| \leqslant \|\phi\| \lim_{\mathcal{U}}\left\|\sum_k \lambda_l(k)x_i^k\right\|$$

which implies that there must be a set $A_l \in \mathcal{U}$ so that

$$\left\|\sum_k \lambda_l(k)y_i^k\right\| \leqslant (1+\varepsilon)\|\phi\|\left\|\sum_k \lambda_l(k)x_i^k\right\|$$

for all $i \in A_l$. Let $A = B \cap \bigcap_{1\leqslant l\leqslant n(\varepsilon)} A_l \in \mathcal{U}$. Thus, for $j \in A$ we set $E_j = [x_j^k\colon 1 \leqslant k \leqslant N]$ and define an operator $\tau_j : E_j \to X$ as $\tau_j(x_j^k) = y_j^k$. Given a point $\sum_k \lambda(k)x_j^k$ in the unit ball of E_j there is some λ_l such that

$$\begin{aligned}\left\|\tau_j\Big(\sum \lambda(k)x_j^k\Big)\right\| &= \left\|\sum \lambda(k)y_j^k\right\| \\ &\leqslant \left\|\sum\big(\lambda - \lambda_l(k)\big)y_j^k\right\| + \left\|\sum \lambda_l(k)y_j^k\right\| \\ &\leqslant \delta\,\mathrm{dist}\big(E, \ell_1^N\big) + (1+\varepsilon)\|\phi\|.\end{aligned}$$

This yields $\|\tau_j\| \leqslant (1+2\varepsilon)\|\phi\|$, with the proper choice $\delta \leqslant \varepsilon\,\mathrm{dist}(E, \ell_1^N)$. Let $T_j : X \to X$ be an extension of τ_j with norm at most $\lambda(1+2\varepsilon)\|\phi\|$. Let $T : X_{\mathcal{U}} \to X_{\mathcal{U}}$ be the ultrapower operator $T[x_i] = [R_i x_i]$ with $R_i = T_i$ if $i \in A$ and $R_i = 0$ if $i \notin A$. One has $T\eta_k = T[x_i^k] = [R_i x_i^k]$; thus, when $i \in A$ we get $R_i x_i^k = T_i x_i^k = \tau_i x_i^k y_i^k$, hence $T_{|E} = \phi$. $\square$

Corollary 5.9. *If X is an UFO then X^{**} is uniformly extensible.*

Proof. If X is an UFO, the ultrapower $X_{\mathcal{U}}$ is UFO. Since X^{**} is a complemented subspace of some ultrapower of X, it must also be an UFO, hence uniformly extensible. □

Theorem 5.10. *An UFO Banach space X is either an $\mathcal{L}_\infty$-space or a weak type 2 near-Hilbert space with the Maurey projection property.*

Proof. Assume X is an UFO containing ℓ_1^n uniformly. The ultrapower $X_{\mathcal{U}}$ via a free ultrafilter on $\mathbb{N}$ is an UFO and contains $(\ell_1^n)_{\mathcal{U}}$, which in turn contains ℓ_1. So, its bidual $E = (X_{\mathcal{U}})^{**}$ is extensible (Lemma 5.7 and Proposition 5.8) and contains ℓ_1. We show below that an extensible space containing ℓ_1 must be separably injective, hence an $\mathcal{L}_\infty$-space. Therefore $X_{\mathcal{U}}$ must also be an $\mathcal{L}_\infty$-space, as well as X.

Thus, let Z be an extensible space containing ℓ_1. Let B be a separable space and let $i: A \to B$ be a subspace. Pick an exact sequence $0 \to K \xrightarrow{k} \ell_1 \xrightarrow{q} B/A \to 0$ and then form the commutative push-out diagram

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r, "k"] \arrow[d, "\phi"'] & \ell_1 \arrow[r, "q"] \arrow[d] & B/A \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & A \arrow[r, "i"'] & B \arrow[r] & B/A \arrow[r] & 0.
\end{tikzcd}$$

Let $j: \ell_1 \to Z$ be an embedding, and let $\tau: A \to Z$ be an operator. Let $\widehat{\tau\phi}: Z \to Z$ be an extension of $\tau\phi$ through jk. Since $\widehat{\tau\phi} jk = \tau\phi$, by the universal property of the push-out, there exists an operator $\nu: B \to Z$ making commutative the diagram

$$\begin{tikzcd}
K \arrow[r, "k"] \arrow[d, "\phi"'] & \ell_1 \arrow[r, "j"] \arrow[d] & Z \arrow[dd, "\widehat{\tau\phi}"] \\
A \arrow[r, "i"'] \arrow[d, equal] & B \arrow[dr, "\nu"] & \\
A \arrow[rr, "\tau"'] & & Z,
\end{tikzcd}$$

which in particular means $\nu i = \tau$.

If X does not contain ℓ_1^n uniformly then it contains ℓ_2^n uniformly complemented [72]. The ultrapower $X_{\mathcal{U}}$ contains ℓ_2 complemented; hence, its bidual $(X_{\mathcal{U}})^{**}$ also contains ℓ_2 via some embedding $i: \ell_2 \to (X_{\mathcal{U}})^{**}$ and complemented via a projection p. Let $\delta: X \to (X_{\mathcal{U}})^{**}$ be the canonical embedding. The following diagram

$$\begin{tikzcd}
E \arrow[r, "j"] \arrow[d, "\phi"'] & X \arrow[r, "\delta"] & (X_{\mathcal{U}})^{**} \arrow[d] \\
\ell_2 \arrow[rr, shift right, "i"'] & & (X_{\mathcal{U}})^{**} \arrow[ll, shift right, "p"']
\end{tikzcd}$$

shows that every operator $\phi : E \to \ell_2$ from a subspace $j : E \to X$ into a Hilbert space can be extended to X. A theorem of Milman and Pisier [61, Th. 10] establishes that X must have weak type 2 (see comments below). By the Maurey–Pisier theorem [60], X contains uniformly $\ell^n_{p(X)}$ and $\ell^n_{q(X)}$. So, if $p(X) < 2$ then, by [11], X contains a sequence of subspaces E_n, uniformly isomorphic to ℓ^n_2, for which $\lambda(E_n, X) \to \infty$ as $n \to \infty$. So, by [65, Th. 4.4], X is non-UFO unless $p(X) = 2$. Now let $q(X) > 2$. If X has exactly cotype $q(X)$, then there exist uniformly complemented copies of $\ell^n_{q(X)}$ in X; otherwise there exist such copies, $O(n^\varepsilon)$-complemented in X, where $\varepsilon > 0$ is arbitrary. By Rosenthal's theorem [76] (see also Lemma 6.11 below), for $q > 2$, ℓ_q contains a sequence of subspaces E_n for which $\lambda(E_n, \ell_q) \geqslant 2^{-1} n^{\frac{q-2}{2q}}$. So, we can apply [65, Th. 4.4] once more.

To show that X has Maurey projection property observe that since X is UFO there is a uniform constant C so that all subspaces ℓ^n_2 are C-complemented (otherwise there would be a sequence of ℓ^n_2 with n increasing not uniformly complemented, and X could not be UFO). So $\lambda(E, X) \leqslant C d_E$. Passing to the limit, the same occurs to infinite dimensional E. □

Corollary 5.11. *An UFO complemented in its bidual is either injective or near-Hilbert.*

Corollary 5.12. *A separable Banach space containing ℓ_1 cannot be extensible.*

Proof. An extensible space containing ℓ_1 must be separably injective; but Zippin showed [85] that c_0 is the only separable separably injective space. □

Remarks. Let us say that a couple (Y, X) of Banach spaces is an *UFO pair* if there exists $C \geqslant 1$ such that for every finite dimensional subspace E of Y and every linear operator $\tau : E \to X$, there exists a linear extension $T : Y \to X$ with $\|T\| \leqslant C\|\tau\|$. This definition has been modelled upon [19,65]. If (Y, X) is an UFO pair and X' is locally complemented subspace of X, $X' \supset Y$, then (Y, X') is an UFO pair. The proof of this statement repeats the proof of Lemma 5.7. In [19, Cor. 1.2] the following assertion was proved for a dual space Y. A few variations in the proof yield that if a Banach space Y' is finitely representable in a Banach space Y and X is a complemented subspace of its dual. If (Y, X) is an UFO pair then (Y', X) is an UFO pair too. Moreover, if every operator from a subspace of Y to X extends to an operator from the whole Y to X, then every operator from a subspace of Y' to X extends to an operator from the whole Y' to X. The spaces Y for which (Y, ℓ_p) is an UFO pair were investigated in [19] under the name M_p *spaces*, and Maurey's extension theorem (see e.g. [86]) can be reformulated in this language as: *Each type* 2 *space is* M_2. It is an open problem whether the converse also holds. A partial solution for this problem is precisely the already mentioned Milman–Pisier theorem [61, Th. 10]: *Each* M_2 *space has weak type* 2. The M_2 spaces are closely connected with the spaces possessing the Maurey projection property. We do not know whether these properties are equivalent or whether the Maurey projection property is equivalent to type 2.

6. Extensible spaces with additional properties

In Theorem 5.10 we have shown that an UFO X must be either an $\mathcal{L}_\infty$-space or a weak type 2 near-Hilbert space with the Maurey projection property. Thus, the automorphic space problem has been transformed in two problems:

- Is a separable automorphic $\mathcal{L}_\infty$ space isomorphic to c_0?

- Is a separable automorphic near-Hilbert space with Maurey projection property isomorphic to ℓ_2?

Let us explore both possibilities.

6.1. On automorphic $\mathcal{L}_\infty$ spaces

Which known $\mathcal{L}_\infty$-spaces could be automorphic? After the results of [9], amongst $C(K)$ spaces we can only still consider very large $C(K)$ spaces. Then, an automorphic space containing ℓ_1 must be separably injective, hence it cannot be separable, since Zippin's theorem asserts that c_0 is the only separable separably injective space. This excludes all Bourgain–Pisier exotic $\mathcal{L}_\infty$-spaces constructed in [15]. If the space contains ℓ_∞ then it must enjoy the property that every separable subspace is contained in a copy of ℓ_∞ contained in the space, hence it must be universally separably injective, in the language of [8]. $\mathcal{L}_\infty$-spaces with unconditional basis must also be excluded by part (3) of Theorem 7.1. One moreover has

Proposition 6.1. *Let X be a separable automorphic $\mathcal{L}_\infty$-space different from c_0. Then X cannot be isomorphic to its square, every copy of X inside X is complemented and if some infinite codimensional copy of X inside X exists then $X \simeq X \oplus c_0$.*

Proof. Recall that every separable $\mathcal{L}_\infty$-space has a quotient isomorphic to c_0. So, [24, Prop. 5.2] shows that if X is isomorphic to its square, c_0 must contain a complemented copy of X, so $X \simeq c_0$. The second item is valid for any Banach space: if $Y \subset X$ is an uncomplemented copy of X then the isomorphism $\tau : Y \to X$ cannot be extended to a bounded operator $T : X \to X$. So, X is not extensible, hence is not automorphic. Set $X = X \oplus Y$; the existence of the exact sequences $0 \to E \to X \to c_0 \to 0$ and $0 \to E \to X \oplus Y \to c_0 \oplus Y \to 0$ yields $c_0 \simeq c_0 \oplus Y$, hence $Y \simeq c_0$. In particular, X must contain c_0. $\square$

6.2. On near-Hilbert extensible spaces

Besides being automorphic, Hilbert spaces enjoy two additional properties: the dual space is also automorphic, and all their subspaces are automorphic. Let us show that an UFO with any of these properties is very close to be a Hilbert space. One has

Corollary 6.2. *If X is a Banach space such that both X and X^* are UFO then it is a weak Hilbert space.*

Proof. Since $\mathcal{L}_1$-spaces cannot be UFO, X must be a weak type 2 space, as well as X^*. So both have weak type 2 and weak cotype 2 i.e., are weak Hilbert spaces [73]. $\square$

Definition. A Banach space is said to be *hereditarily UFO* (an HUFO, in short) if each of its subspaces is an UFO.

Theorem 6.3. *Every hereditarily UFO space is asymptotically Hilbertian.*

Proof. For every n, X contains subspaces F_n^k, $k = 1, 2, \dots$, of the same dimension for which

$$\rho\left(S_{F_n^k}, F_n^l\right) > 1 - \varepsilon \quad \text{for } k < l, \text{ and } d_{F_n^k} > n.$$

By a compactness argument, among them there are, for every $\varepsilon > 0$, two $(1+\varepsilon)$-isometric. Denote them by E_n and E_n'. Thus, we can construct a sequence of subspaces which satisfy conditions of the following Lemma 6.4, and therefore X contains a non-UFO subspace. □

Lemma 6.4. *Let $a > 0$ and $b \geqslant 1$. Let a Banach space X contain a sequence of finite dimensional subspaces E_n, E_n', $n = 1, 2, \ldots$, such that*:

(i) $\rho(S_{E_n}, E_n') > a$ *for each* n;
(ii) $d_{E_n} \to \infty$ *as* $n \to \infty$;
(iii) $\operatorname{dist}(E_n, E_n') \leqslant b$ *for each* n.

Then X contains a non-UFO subspace.

Proof. Let $J_n : E_n \to E_n'$ be the isomorphisms given by the condition (iii). We formulate the fist step as a sublemma:

Sublemma 6.4.1. *Let (E_n, E_n') be a sequence of subspaces verifying* (i), (ii), (iii) *from Lemma* 6.4. *Then for every positive integer k and every finite codimensional subspace Z of X there is $n = n(k)$ and subspaces $F \subset E_n \cap Z$ and $F' \subset E_n' \cap Z$ such that $d_F > k$ and* $\operatorname{dist}(F, F') \leqslant b$.

Indeed, by (ii) one has $d_{E_n} \to \infty$, hence

$$d_{E_n \cap Z} \to \infty \quad \text{as } n \to \infty,$$

moreover, $d_{J_n(E_n \cap Z)} \to \infty$ and $d_{J_n(E_n \cap Z) \cap Z} \to \infty$. Thus, for sufficiently large n, the subspaces

$$F' = J_n(E_n \cap Z) \cap Z \quad \text{and} \quad F = J_n^{-1}\big(J_n(E_n \cap Z) \cap Z\big)$$

satisfy the conditions of the sublemma.

Following [65, Lem. 4.1] it can be shown that given subspaces (E_n, E_n') verifying (i)–(iii) one can construct subspaces F_k, F_k' of X, $k = 1, 2, \ldots$ in such a way that

(iv) $F_1, F_1', \ldots, F_k, F_k', \ldots$ form a finite dimensional Schauder decomposition in its closed linear span;
(v) $\dim F_k = \dim F_k'$;
(vi) $d_{F_k} \to \infty$ as $k \to \infty$, $\operatorname{dist}(F_k, F_k') \leqslant b$.

Indeed, let $\varepsilon > 0$. Set $F_1 = E_1$ and $F_1' = E_1'$. Take Φ_1 a finite subset of the unit sphere S_{X^*} which $(1-\varepsilon)$-norms $F_1 + F_1'$ and let $\Phi_1^\top \subset X$ be its (finite codimensional) annihilator. By Sublemma 6.4.1, there exists a positive integer n_2 and subspaces F_2 of $E_{n_2} \cap \Phi_1^\top$ and F_2' of $E_{n_2}' \cap \Phi_1^\top$ such that $d_{F_2} > 2$ and $\operatorname{dist}(F_2, F_2') \leqslant b$. Take Φ_2 a finite subset of S_{X^*} which $(1-\varepsilon)$-norms $F_1 \oplus F_1' + F_2 + F_2'$ and continue inductively.

Let us now recall a result of [29, Th. 6.7] as sublemma.

Sublemma 6.4.2. *There is a function $\lambda \to f(\lambda)$ so that if E is a Banach space with* $\dim E = n$ *such that for every $F \subset E$ there is a projection of norm $\leqslant \lambda$ from E onto F then $d_E \leqslant f(\lambda)$. One can take $f(\lambda) = c\lambda^{32}$ for a suitable constant c.*

Using this Sublemma 6.4.2, we can obtain subspaces G_k in F_k for which

$$\lambda(G_k, F_k) \to \infty.$$

Denote by Y the closed linear span of the subspaces G_k, F_k', $k = 1, 2, \ldots$. Then G_k are uniformly complemented in Y. On the other hand, the F_k' contain subspaces G_k', uniformly isomorphic to G_k, for which

$$\lambda\bigl(G_k', Y\bigr) \geqslant \lambda\bigl(G_k', F_k'\bigr) \to \infty.$$

By [65, Th. 4.4], this implies that Y is not UFO. □

One could also consider the notion of *hereditarily extensible* space: a Banach space in which every subspace is extensible. It is clear that every hereditarily extensible space is HUFO, and Theorem 6.3 implies that the converse also holds: HUFO are asymptotically Hilbert, hence reflexive (see [74, p. 220]), and reflexive UFO are extensible. Therefore

Corollary 6.5. *Hereditarily UFO and hereditarily extensible spaces coincide.*

Also, the uniformity problem has a positive answer for HUFO: since hereditarily extensible spaces are reflexive, Lemma 5.6(2) yields:

Corollary 6.6. *If a Banach space is hereditarily extensible there exists a* $\lambda > 0$ *such that every subspace is* λ*-extensible.*

All this suggests whether every HUFO space must be isomorphic to a Hilbert space. The converse of Theorem 6.3, however, does not hold:

Example 6.7. There exist weak Hilbert spaces that are not hereditarily UFO.

The example is *Tsirelson's* 2*-convexified space* $\mathcal{T}_2$, which can be obtained as follows. Define inductively a sequence of norms $\| \ \|_i$ on c_{00} as follows $\|x\|_0 = \|x\|_{c_0}$ and for $i > 0$

$$\|x\|_{i+1} = \max\left(\|x\|_i, \frac{1}{2}\sup\left\{\sqrt{\sum_{k=1}^{n}\|N_k x\|_i^2}\colon\ N_k \in \mathcal{A};\ N_1 < \cdots < N_n,\ n = 1, 2, \ldots\right\}\right).$$

It is easy to see that $\|x\|_i \leqslant \|x\|_{\ell_2}$ for every i and thus $\lim_i \|x\|_i =: \|x\|$ exists for every $x \in c_{00}$. The completion of c_{00} with respect to the limiting norm is denoted by $\mathcal{T}_2$. The space $\mathcal{T}_2$ is a weak Hilbert space by [74, p. 205]. To show it contains a non-UFO subspace, we state a result presented without proof in [20, p. 117]; it can be proved in a similar way as for the usual Tsirelson's space.

Lemma 6.8. *Denote by* X_1 (X_2) *the subspace of* $\mathcal{T}_2$ *spanned by odd* (*resp. even*) *unit basic vectors* e_k *and by* $S : X_1 \to X_2$ *the shift operator*: $Se_k = e_{k+1}$. *Then* S *is an isomorphism from* X_1 *onto* X_2.

Now, since the unit vectors form an unconditional basis of $\mathcal{T}_2$, the existence of a non-UFO subspace follows from the previous lemma and the following

Proposition 6.9. *Assume that X contains a subspace of the form $Y \oplus Y'$, $Y \simeq Y'$ for some $Y \not\simeq \ell_2$. Then X contains a non-UFO subspace.*

Proof. Since $Y \not\simeq \ell_2$, it contains a sequence of finite dimensional subspaces E_n verifying condition (ii) of Lemma 6.4 [50, Section 7]. Since $Y \simeq Y'$, the subspace Y' contains a sequence of subspaces E_n', uniformly isomorphic to E_n. The condition $X \supset Y \oplus Y'$ includes the demand $\rho(S_Y, Y') > 0$, hence $\inf_n \rho(S_{E_n}, E_n') > 0$. So, by Lemma 6.4, X contains a non-UFO subspace. □

We do not know if the space $\mathcal{T}_2$ is itself an UFO (in which case it would be the first reflexive extensible non-Hilbert space). In fact we do not know if a weak Hilbert space must be UFO. We show that asymptotically Hilbertian spaces need not be UFO.

Example 6.10. Examples of asymptotically Hilbertian non-UFO spaces.

They are provided by Johnson's spaces (see e.g. [55, p. 112]) of the form

$$Z = \ell_2\big(\ell_{p_n}^{k_n}\big) \quad \text{with } p_n \downarrow 2 \text{ and } k_n \uparrow \infty \tag{8}$$

under some additional conditions. By the Gurariĭ–Kadec–Macaev formula [37], the space Z is not isomorphic to a Hilbert space if and only if

$$\sup_n k_n^{\frac{|p_n-2|}{2p_n}} \sup_n d_{\ell_{p_n}^{k_n}} = \infty. \tag{9}$$

Let us show that the space Z of the form (8) with the condition (9) is non-UFO. Observe that Z contains uniformly complemented subspaces $\ell_{p_n}^{k_n}$ satisfying the condition (9) and, by Dvoretzky's theorem, uniformly Euclidean subspaces too. So, the claim follows from [65, Th. 4.4] once proved that the projection constants tend to infinity:

Lemma 6.11. *The space $E = (\ell_p^n \oplus \ell_2^n)_\infty$, $p > 2$, contains a subspace F which is isometric to ℓ_p^n and such that*

$$\lambda(F, E) \geqslant 2^{-1} n^{\frac{p-2}{2p}}. \tag{10}$$

Proof. We use an idea of [76, Lem. 2A]. Take standard bases $(e_i)_1^n$ and $(h_i)_1^n$ of ℓ_p^n and ℓ_2^n, and put

$$f_i = e_i + n^{-\frac{p-2}{2p}} h_i, \quad i = 1, \ldots, n.$$

By the Hölder inequality, for all scalars (a_i)

$$\sum_1^n |a_i|^2 \leqslant \left(\sum_1^n |a_i|^{2\frac{p}{2}}\right)^{\frac{2}{p}} n^{1-\frac{2}{p}},$$

so

$$\left\|\sum_1^n a_i h_i\right\| \leqslant \left\|\sum_1^n a_i e_i\right\| n^{\frac{p-2}{2p}}.$$

Hence, the subspace $F = \mathrm{lin}(f_i)_1^n$ is isometric to ℓ_p^n. □

There is a common ingredient in Example 6.10 and the second part of Theorem 5.10 that guarantees the non-UFO character.

Definition. We say that a Banach space X has the *property* $\mathcal{P}$ if it contains a sequence of uniformly complemented subspaces E_n, uniformly isomorphic to $\ell_{p_n}^{k_n}$, for which (p_n) is bounded and verifying the condition (9).

Proposition 6.12. *A Banach space X with the property $\mathcal{P}$ is non-UFO.*

Proof. Let us consider two possibilities.

(i). There exists a subsequence $(p_{n_k}) \subset (p_n)$ with $p_{n_k} > 2$ for every k and verifying the condition (9). Then we repeat the proof of Proposition 6.10.

(ii). There exists a subsequence $(p_{n_k}) \subset (p_n)$ with $p_{n_k} < 2$ and verifying the condition (9). Without loss of generality one may suppose $1 < a < p_n < 2$ for some a and each n. By [29, Ex. 3.1], there exists a constant $c \in (0, 1)$ so that every ℓ_p^k, $1 < p < 2$, contains a subspace E of dimension ck with $d_E \leqslant 2$. Let p' be such that $1/p + 1/p' = 1$. If $P : \ell_p^k \to E_k$ is a projection then $\ell_{p'}^k$ contains a subspace $F_k = P^*(\ell_{p'}^k)$ with $d_F \leqslant 2\|P\|$. Moreover, by [11, p. 182], there exists an universal constant b (connected with the Khintchine inequality) so that for every subspace F of $\ell_{p'}^k$

$$d_F \geqslant bk^{1/2}(\dim F)^{-1/p'}.$$

Hence,

$$\|P\| \geqslant \frac{1}{2}bk^{1/2}(ck)^{-1/p'}\frac{b}{2}c^{-1/p'}k^{1/2-1/p'} \geqslant \frac{b}{2c}k^{\frac{2-p}{2p}}.$$

Thus, our hypotheses yield that $\ell_{p_n}^{k_n}$ contain subspaces E_n, $\dim E_n = ck_n$ with $d_{E_n} \leqslant 2$ and projection constants $\lambda(E_n, \ell_{p_n}^{k_n}) \to \infty$ as $n \to \infty$. On the other hand, since $1 < a < p_n < 2$, X contains uniformly complemented $\ell_2^{ck_n}$. The result [65, Th. 4.4] ensures that X is not UFO. □

Observe that a Banach space with the property (H) cannot contain a sequence of subspaces E_n, uniformly isomorphic to $\ell_{p_n}^{k_n}$ with the condition (9). In particular, spaces with property $\mathcal{P}$ cannot have property (H) or be weak Hilbert spaces. Analogously, spaces with the upper (resp. lower) property (H) cannot contain a sequence of subspaces E_n, uniformly isomorphic to $\ell_{p_n}^{k_n}$, with the condition (9) and $p_n > 2$ (resp. $p_n < 2$).

7. Automorphic spaces with lattice structure

Further partial solutions to the automorphic space problem can be obtained for Banach spaces with an additional lattice or unconditional structure. Recall that Theorem 5.10 establishes that an UFO is either an $\mathcal{L}_\infty$-space or a weak type 2 near-Hilbert space. One moreover has (see below for unexplained notation and definitions):

Theorem 7.1.

(1) *An UFO space X with unconditional basis is either lattice isomorphic to c_0 or a superreflexive weak type* 2 *near-Hilbert space with Boyd indices $\alpha(X) = \beta(X) = 2$.*
(2) *An UFO Banach lattice X with p.l.u.st. is either an $\mathcal{L}_\infty$-lattice or a superreflexive weak type* 2 *near-Hilbert space.*
(3) *An UFO r.i. function space X on the interval* $(0, 1)$ *is either L_∞ or a superreflexive type* 2 *near-Hilbert space.*
(4) *If X is an r.i. function space on the interval* $(0, 1)$ *and both X and X^* are UFO then $X = L_2$.*
(5) *An HUFO Köthe function space on* $(0, 1)$ *is lattice isomorphic to L_2.*

Proof. *Assertion* (1). Let us show that an $\mathcal{L}_\infty$ UFO with unconditional basis is lattice isomorphic to c_0: let X be an $\mathcal{L}_\infty$ UFO with unconditional basis (e_n). So, X contains ℓ_∞^n uniformly. Since the spaces $E_n = [e_1, \dots, e_n]$ are uniformly complemented with uniformly bounded unconditional constants, if they are not uniformly isomorphic to the corresponding ℓ_∞^n it follows from [65, Cor. 4.7] that X cannot be an UFO. But if they are uniformly isomorphic to ℓ_∞^n the sequence (e_n) must be weakly 1-summable, hence equivalent to the canonical basis of c_0 (see e.g., [27, Cor. V.7]).

The assertion about Boyd indices can be proved as follows: if X does not contain ℓ_∞^n then one can apply [79] to obtain that ℓ_p is block finite represented in the unconditional basis (e_i) of X for every $\alpha(X) \leqslant p \leqslant \beta(X)$ with

$$\alpha(X) = \varliminf_{n\to\infty} \frac{\log n}{\log \|\sum_1^n e_i\|}; \qquad \beta(X) = \varlimsup_{n\to\infty} \frac{\log n}{\log \|\sum_1^n e_i\|}.$$

So, by [24, Cor. 4.6], unless $\alpha = \beta = 2$ the space X is not extensible.

This result partially responds to a question of Galego [32], who asked whether a space with unconditional basis different from c_0 or ℓ_2 can be automorphic. It is quite tempting to conjecture that if X is a reflexive UFO then also X^* is an UFO; or, what is the same, if X is a reflexive extensible space then also X^* is extensible. It is not hard to see that if X is a reflexive UFO then X^* is a co-UFO (with the obvious meaning that operators into finite dimensional quotients can be uniformly lifted). So the question is whether a reflexive UFO must also be a co-UFO. If this were true, by (1) we would get a positive answer to the Lindenstrauss–Rosenthal conjecture for spaces with unconditional basis. It would be enough to show that if X is a reflexive UFO then also $\ell_2(X)$ is an UFO. Recall that this is false when X is not reflexive since $\ell_2(c_0)$ is not an UFO.

We pass to assertion (2). Denote by $u(F)$ the *unconditional basic constant* of a finite dimensional space F. Recall (see e.g. [38]) that a Banach space X has the (Dubinsky–Pełczyński–Rosenthal-)*local unconditional structure* (l.u.st.) provided there is a constant C such that for every finite dimensional subspace E in X there is a finite dimensional subspace $E \subset F \subset X$ such that $u(F) \leqslant C$.

Definition. We will say that X has the *projectional local unconditional structure* (p.l.u.st.) if, in addition to l.u.st., the estimate $\lambda(F, X) \leqslant C$ holds for the projection constant.

Spaces with unconditional bases as well as $\mathcal{L}_p$-spaces, $1 \leqslant p \leqslant \infty$, have p.l.u.st. [53, p. 328]. Banach lattices have l.u.st. [38, p. 227], but since there are (superreflexive) Banach lattices without the approximation property [55, Ch. 1.g], not every Banach lattice has p.l.u.st. It seems possible to develop for spaces with p.l.u.st. a theory similar to the existing one for spaces with l.u.st. as in [28], but we currently know very few publications about p.l.u.st. One has

Proposition 7.2. *An r.i. function space X on $(0, 1)$ has p.l.u.st.*

Proof. It is consequence of the fact that a finite set $(x_i)_1^n$ of characteristic functions of disjoint sets admits a contractive projection $P : X \to \mathrm{lin}(x_i)_1^n$ [55, p. 122] (for the definition of rearrangement invariant (r.i.) space see [55, p. 118]). □

Let us say that a Banach lattice X is an $\mathcal{L}_\infty$-*lattice* provided there is a constant C such that for every finite dimensional subspace E of X and every $\varepsilon > 0$ there exists a finite collection $(x_i)_1^n \subset X$ of pairwise disjoint elements C-equivalent to the standard basis of ℓ_∞^n, such that $\rho(S_E, \mathrm{lin}(x_i)_1^n) < \varepsilon$. We do not know whether this definition has already appeared in the literature, or whether a lattice which is an $\mathcal{L}_\infty$-space must also be an $\mathcal{L}_\infty$-lattice.

For the proof of (2) recall from [38] that a Banach space X with l.u.st. either contains uniformly ℓ_∞^n, or uniformly complemented ℓ_1^n or is superreflexive. There is an analogue for the p.l.u.st., whose proof is implicit in [38, Cor. III.5] and, more explicit, in [58,71], [46, Th. II.1], [79, Prop. 3.1].

Lemma 7.3. *Let X be a Banach lattice. Then X contains either a lattice copy of ℓ_∞^n or uniformly complemented ℓ_1^n or is superreflexive.*

The second alternative (containing ℓ_1^n uniformly complemented) cannot hold in an UFO.

Proof of (3). For any order continuous Banach lattice X we can denote an associated Banach lattice $X(\ell_2)$ (using the Krivine calculus [55, pp. 40–42]) as the space of sequences (x_n) in X such that $(\sum_1^n |x_k|^2)^{1/2}$ is order bounded (and hence is a convergent sequence) in X. $X(\ell_2)$ becomes an order continuous Banach lattice when normed by $\|(x_n)\| = \|(\sum_1^\infty |x_n|^2)^{1/2}\|$. If X has nontrivial cotype then $X(\ell_2)$ is naturally isomorphic to the space $\mathrm{Rad}\, X$ which is the subspace of $L_2(X)$ of functions of the form $\sum_{n=1}^\infty x_n r_n$ where (r_n) is the sequence of Rademacher functions. For the definition of Köthe function space see [55, p. 28].

Proposition 7.4. *(See [19, Th. 3.10].) Let X be a Köthe function space on $(0, 1)$ with an unconditional basis. If X is an M_2 space then it has type 2.*

Proof. Every M_2 space has the weak type 2 [61, Th. 10 and Remark 11]. Next, a Banach space has type 2 if and only if $\mathrm{Rad}\, X$ (which is equal to $X(\ell_2)$ for Banach lattice with nontrivial cotype) has weak type 2 [74, p. 174]. Finally, by [45], X, as an order continuous, nonatomic Banach lattice with an unconditional basis, is isomorphic as a Banach space to $X(\ell_2)$ $(= \mathrm{Rad}\, X)$. □

Corollary 7.5. *Let X be an UFO (superreflexive) r.i. Banach space on $(0, 1)$ which is not isomorphic to L_∞. Then X has type 2 and $q(X) = 2$.*

Proof. Since X contains complemented ℓ_2 [55, p. 135], it cannot be an $\mathcal{L}_\infty$-space, and the dichotomy Theorem 5.10 yields all the properties except type 2. This is provided by Proposition 7.4 since X has an unconditional basis [55, p. 156]. □

The proof of (3) follows from this and the following, probably known, result:

Proposition 7.6. *If an r.i. function space X on $(0,1)$ is an $\mathcal{L}_\infty$-lattice then it is lattice isomorphic to L_∞.*

(*Hint*: Let C be from the definition of $\mathcal{L}_\infty$-lattice. We consider the natural embedding $L_\infty \subset X$ and note that it is a C-isomorphism on the linear span of finitely valued functions. This span is dense in both spaces.)

To prove (4) recall that, by Corollary 7.5, X and X^* have type 2. Hence X has type and cotype 2; which, by Kwapień theorem [47], makes it L_2.

Our *proof for assertion* (5) is consequence of Theorem 6.3 and the following result:

Proposition 7.7. *If X is a Köthe function space on $(0,1)$, which is asymptotically Hilbertian, then X is lattice isomorphic to L_2.*

Proof. Let c be the constant from the definition of asymptotically Hilbertian space. We wish to show that every finite sequence of normalized and mutually disjoint elements $(x_i)_1^n$ of X is c-equivalent to the unit vector basis of a Euclidean space. It then follows from [55, Th. 1.b.13] that X is lattice isomorphic to L_2.

Let $E = (x_i)_1^n$ and X_n be a finite codimensional subspace of X from the definition of asymptotically Hilbertian space. The asymptotically Hilbertian space is reflexive [74, p. 220], so its norm and the norm of its dual are absolutely continuous. If we will pay no attention to ε, we can assume that there exists a decomposition of $[0,1]$ into disjoint sum of intervals $(I_k)_1^m$ so that the annihilator $X_n^\perp \subset X^*$ belongs to the linear span $\operatorname{lin}(z_k)_1^m$ of characteristic functions z_k of I_k and $(x_i)_1^n$ is a block-basis of $(z_k)_1^m$. Let the function z'_k be equal 1 on a half of the interval I_k and be equal to -1 on the second half of I_k. Then $\operatorname{lin}(z'_k)_1^m \subset (\operatorname{lin}(z_k)_1^m)^\top \subset X_n$ hence is c-isomorphic to a Euclidean space. But the spaces $\operatorname{lin}(z_k)_1^m$ and $\operatorname{lin}(z'_k)_1^m$ are isometric and the basis $(z_k)_1^m$ is 1-unconditional. Therefore $(z_k)_1^m$ is c-equivalent to an orthogonal basis of a Euclidean space. Hence, its block-basis $(x_i)_1^n$ is c-equivalent to an orthogonal basis of a Euclidean space too. □

A similar statement, under the stronger assumption that X is weak Hilbert, was proved by Nielsen [66] using some complicated calculations of [67]. Previously, Mascioni [57] proved a similar statement for Orlicz spaces.

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