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A SIMPLE PROOF OF A THEOREM OF JUNG

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Jung's theorem states that if G is a 1-tough graph on $n \geqslant 11$ vertices such that $d(x)+d(y) \geqslant n-4$ for all distinct nonadjacent vertices x, y, then G is hamiltonian. We give a simple proof of this theorem for graphs with 16 or more vertices.

1. Introduction

A problem of recent interest is that of determining sufficient degree conditions for a 1-tough graph to have a long cycle [1, 2, 3, 4]. A well known result in this area is the following theorem of Jung [8].

Theorem 1. *Let G be a* 1*-tough graph on $n \geqslant 11$ vertices such that $d(x)+d(y) \geqslant n-4$ for all distinct nonadjacent vertices x, y. Then G is hamiltonian.*

Unfortunately, the proof of this theorem in [8] is lengthy and somewhat difficult. In this note we present a simple proof of this theorem for graphs with 16 or more vertices. Our approach allows us to quickly reduce the entire proof to showing that if a longest cycle in G has length $n-1$ and if the vertex of G not on the cycle has degree $(n-3)/2$ or $(n-4)/2$, then G is not 1-tough.

Our terminology and notation are standard except as indicated. A good reference for any undefined terms is [5]. We require a few definitions and some convenient notation. Let $c(G)$ denote the number of components of a graph G. As introduced by Chvátal [7], a graph G is 1-*tough* if for every nonempty subset X of the vertex set V of G we have $c(G-X) \leqslant |X|$, where $G-X$ is the subgraph obtained from G by removing the vertices of X. A cycle C of G is a *dominating cycle* if every edge of G has at least one of its vertices on C. For convenience we will use C for both a cycle, i.e. its vertices and edges, as well as just its vertex set. If C is a cycle of G we denote by $\vec{C}$ the cycle with a given orientation. If $u, v \in C$ then $[\overrightarrow{u, v}]$ denotes the consecutive vertices on C from u to v in the direction specified by $\vec{C}$. The same vertices, in reverse order, are denoted by $[\overleftarrow{v, u}]$. We

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use u^+ to denote the successor of u on $\vec{C}$ and u^- to denote its predecessor. If $v \in V$ then $N(v)$ is the set of all vertices in V adjacent to v. Whenever $A \subseteq C$ we let $A^+ = \{v^+ / v \in A\}$. The sets A^- and A^{++} are defined analogously. Finally, we use $d_C(v)$ to denote the number of vertices of C which are adjacent to v.

The main part of our proof is given in Section 2. First, we present some simple but useful lemmas. In each lemma below we assume C is a longest cycle in G, $v \in V - C$ and $A = N(v) \subseteq C$.

Lemma 1. *$A \cap A^+ = \phi$ and $A^+ \cap A^{++} = \phi$.*

Lemma 2. *$A^+(A^-)$ is an independent set of vertices.*

Lemma 3. *Let $x_1, x_2 \in A$. Then*

(i) *There is no vertex $z \in [\overrightarrow{x_1^+, x_2^+}]$ such that $x_2^+ z, x_1^+ z^+ \in E$,*

(ii) *There is no vertex $w \in [\overrightarrow{x_2^+, x_1^+}]$ such that $x_1^+ w, x_2^+ w^+ \in E$, and*

(iii) *$d_C(x_1^+) + d_C(x_2^+) \leq |C|$.*

We note that (i) and (ii) employ standard arguments and (iii) follows easily from (i) and (ii). An analogous lemma holds if we replace x_1^+ by x_1^- and x_2^+ by x_2^-. This analogous lemma will also be referred to as Lemma 3.

Lemma 4. *Let $x_1, x_2 \in A$, where $x_2 = x_1^{+++}$. Then*

(i) *there is no vertex $z \in [\overrightarrow{x_2, x_1^-}]$ such that $x_1^{++} z, x_1^+ z^+ \in E$, and*

(ii) *ther e is no vertex $z \in [\overrightarrow{x_2, x_1^-}]$ such that $x_1^+ z, x_1^{++} z^+ \in E$.*

Proof. If (i) is not satisfied the cycle $C' : x_1^+ [\overrightarrow{z^+, x_1}] v [\overrightarrow{x_2, z}] x_1^{++} x_1^+$ is a longer cycle then C. If (ii) is not satisfied then $C'' : x_1^+ [\overleftarrow{z, x_2}] v [\overleftarrow{x_1, z^+}] x_1^{++} x_1^+$ is a longer cycle than C. □

Lemma 5. *Let $x_1^+ \in A^+ \cap A^-$ and $z \in N(x_1^+) \cap C$. Then*

(i) *$\{z^+\} \cup A^+$ is an independent set of vertices, and*

(ii) *$\{z^-\} \cup A^-$ is an independent set of vertices.*

Proof. We prove (i); the proof of (ii) uses an analogous argument. Let $z^+ x_j^+ \in E$, where $x_j \in A$. If $x_j^+ \in A^+ \cap [\overrightarrow{x_1^{++}, z}]$ then $C' : x_j^+ [\overrightarrow{z^+, x_1}] v [\overleftarrow{x_j, x_1^+}][\overleftarrow{z, x_j^+}]$ is a longer cycle than C. If $x_j^+ \in A^+ \cap [\overrightarrow{z^{++}, x_1^+}]$, then $C'' : [\overrightarrow{x_j^+, x_1^+}][\overleftarrow{z, x_1^{++}}] v [\overleftarrow{x_j, z^+}] x_j^+$ is a longer cycle than C. □

We will also use a corollary of the following theorem of Bondy [6].

Theorem 2. *Let G be a 2-connected graph on n vertices such that $d(x) + d(y) + d(z) \geq n + 2$ for all independent sets of vertices x, y, z. Then every longest cycle in G is a dominating cycle.*

Corollary 3. *Let G be a 2-connected graph on n vertices such that $d(x) + d(y) \geqslant 2n/3 + 1$ for all distinct nonadjacent vertices x, y. Then every longest cycle in G is a dominating cycle.*

2. The proof

Let G be a 1-tough graph on $n \geqslant 16$ vertices such that $d(x) + d(y) \geqslant n - 4$ for all distinct nonadjacent vertices x, y and suppose G is not hamiltonian. Let C be a longest cycle in G. By Corollary 3, C is a dominating cycle. Let v_0 be a vertex of largest degree in $V - C$, $A = N(v_0)$ and $H = (V - C) \cup A^+$. We first assert the following:

H is an independent set of vertices.

Let $v_1 \in V - C$ and $w \in A$. By Lemma 2 and Corollary 3, it suffices to show that $v_1 w^+ \notin E$. Suppose otherwise. Clearly, $v_1 \neq v_0$ and thus $|C| \leqslant n - 2$. We claim that v_1 is not adjacent to any vertex in $(A^+ - \{w^+\}) \cup A^{++}$. Clearly $v_1 w^{++} \notin E$. If $v_1 s^+ \in E$, where $s^+ \in A^+ - \{w^+\}$, then $C' : v_1[\overrightarrow{s^+, w}]v_0[\overleftarrow{s, w^+}]v_1$ is a longer cycle than C and if $v_1 s^{++} \in E$, where $s^{++} \in A^{++}$, then $C'' : v_1[\overrightarrow{s^{++}, w}]v_0[\overleftarrow{s, w^+}]v_1$ is a longer cycle than C. Since $(A^+ - \{w^+\}) \cap A^{++} = \phi$ by Lemma 1, we have $d(v_1) \leqslant |C| - 2d(v_0) + 1$. Since $d(v_0) + d(v_1) \geqslant n - 4$ and $d(v_0) \geqslant d(v_1)$ it follows that $d(v_0) \geqslant (n-4)/2$. Hence $d(v_0) + d(v_1) \leqslant (n-2) - (n-4)/2 + 1 = (n+2)/2 < n - 4$, a contradiction since $n \geqslant 16$. This proves the assertion. □

Thus H is an independent set of vertices and similarly $H' = (V - C) \cup A^-$ is independent. Since G is 1-tough it follows that $|V - C| + |A^+| \leqslant n/2$. If $V - C = \{v_0\}$ then $|C| = n - 1$. Otherwise $V - C$ contains a vertex $v_1 \neq v_0$, and thus $d(v_0) \geqslant (n-4)/2$ as before. Hence $|C| \geqslant n/2 + |A^+| = n/2 + d(v_0) \geqslant n/2 + (n-4)/2 = n - 2$, and equality holds only if $d(v_0) = d(v_1) = (n-4)/2$.

Case 1. $|C| = n - 2$.

Case 1*a*. *There exists $w \in A$ such that $w^{++}, w^{+++} \notin A$.*

Let $t^+ \in A^+ \cap A^-$. By Lemma 2, $N(t^+) \subseteq A \cup \{w^{++}\}$. But then $G - (A \cup \{w^{++}\})$ has at least $n/2$ components and G would not be 1-tough.

Case 1*b*. *There exist $u, w \in A$ such that $u^{++}, w^{++} \notin A$.*

If $t^+ \in A^+ \cap A^-$ then by Lemma 2, $N(t^+) \subseteq A$. Hence $G - A$ has at least $(n-2)/2$ components, and G would not be 1-tough.

Case 2. $|C| = n-1$.

We first show that it suffices to consider only the cases $d(v_0) = (n-3)/2$ or $(n-4)/2$. If $d(v_0) > (n-1)/2 = |C|/2$ then G is clearly hamiltonian. If $d(v_0) = (n-1)/2$ or $(n-2)/2$ then $G - A$ has more than $d(v_0)$ components. Let $x_1, x_2 \in A$. If $d(v_0) < (n-7)/2$ then $d(x_1^+), d(x_2^+) > (n-1)/2$, contradicting Lemma 3. Suppose $(n-7)/2 \leq d(v_0) \leq (n-5)/2$. We now show that G contains another cycle C' with $|C'| = n-1$, $w_0 \in V - C'$ and $d(w_0) \geq (n-3)/2$. Suppose otherwise. Let $x^+ \in A^+$ and $w^{++} \in A^{++} - \{x^{++}\}$. If $x^+w^{++} \in E$ then $C' : x^+\overrightarrow{[w^{++}, x]}v_0\overleftarrow{[w, x^+]}$ has $|C'| = n-1$, $w^+ \in V - C'$ and since $w^+v_0 \notin E$, $d(w^+) \geq (n-3)/2$. Thus we may assume $x^+w^{++} \notin E$ for all $w^{++} \in A^{++} - \{x^{++}\}$. Since $x^+v_0 \notin E$ it now follows from Lemmas 1 and 2 that $d(x^+) \leq (n-1) - 2(d(v_0) - 1) - 1 = n - 2d(v_0)$. Since $d(v_0) + d(x^+) \geq n-4$ we conclude $d(v_0) \leq 4$. However $d(v_0) \geq (n-7)/2$, a contradiction for $n \geq 16$.

Case 2*a*. $d(v_0) = (n-3)/2$.

Case 2*ai*. *There exists* $z \in A$ *such that* $z^{++}, z^{+++} \notin A$.

Let $t^+ \in A^+ - \{z^+\}$. By Lemma 2, t^+z^+, $t^+z^{+++} \notin E$. If $t^+z^{++} \in E$ then by Lemma 3, $z^+z^{+++} \notin E$ and thus $G - (A \cup \{z^{++}\})$ has $(n+1)/2$ components, and G would not be 1-tough. If $t^+z^{++} \notin E$ for any $t^+ \in A^+ - \{z^+\}$ then $G - A$ has $(n-1)/2$ components and again G would not be 1-tough.

Case 2*aii*. *There exist vertices* $z, w \in A$ *such that* $z^{++}, w^{++} \notin A$.

We reach a contradiction by showing that z^+w^{++}, $z^{++}w^+ \notin E$ and hence $G - A$ has $(n-1)/2$ components, and G would not be 1-tough. Suppose $z^+w^{++} \in E$. If $z^{+++} \neq w$ then $w^{--} \in N(v_0)$. By Lemma 2, $N(w^-) \subseteq A$ and since $v_0w^- \notin E$, $d(w^-) \geq (n-5)/2$. If $z \neq w^{+++}$ then either w^-z or $w^-w^{+++} \in E$, contradicting Lemma 3. If $z = w^{+++}$ and all vertices in $A^+ - \{z^+, w^+\}$ are not adjacent to z, then each such vertex has degree at most $(n-5)/2$. But then $d(x^+) + d(y^+) \leq n-5$ for every pair of vertices $x^+, y^+ \in A^+ - \{z^+, w^+\}$, a contradiction. Hence we may assume $z^{+++} = w$. By Lemma 4, $w^+z^{++} \notin E$. Since $d(w^+) + d(z^{++}) \geq n-4$ and n is odd we may assume, without loss of generality, $d(w^+) \geq (n-3)/2$. Clearly $N(w^+) \subseteq A \cup \{w^{++}\}$. Hence either $w^+z \in E$ or $w^+w^{+++} \in E$. However $w^+w^{+++} \in E$ contradicts Lemma 3 and $w^+z \in E$ contradicts Lemma 4. Thus we conclude that $z^+w^{++} \notin E$. An analogous argument shows that $z^{++}w^+ \notin E$.

Case 2*b*. $d(v_0) = (n-4)/2$.

Case 2*bi*. *There exists* $z \in A$ *such that* $z^{++}, z^{+++}, z^{++++} \notin A$.

Let t^+ be any vertex in $A^+ - \{z^+\}$. If $t^+z^{++} \in E$ then by Lemma 5, $A^+ \cup \{z^{+++}\}$ is an independent set. Also $z^+z^{++++} \notin E$ by Lemma 3. Hence $G - (A \cup \{z^{++}\})$ has $n/2$ components, and G would not be 1-tough. Thus $t^+z^{++} \notin E$ and similarly $t^+z^{+++} \notin E$. But this implies that $G - A$ has $(n-2)/2$ components, contradicting that G is 1-tough.

Case 2bii. There exist vertices $z, w \in A$ such that $z^{++}, w^{++}, w^{+++} \notin A$.

Let t^+ be any vertex in $A^+ - \{z^+, w^+\}$ and suppose $t^+w^{++} \in E$. Then by Lemma 5, $A^+ \cup \{w^{+++}\}$ and $A^- \cup \{w^+\}$ are both independent sets of vertices. Thus $G - (A \cup \{w^{++}\})$ has $n/2$ components, a contradiction. Hence $t^+w^{++} \notin E$. It now follows easily that $N(t^+) = A$. Next we show that $w^+z^{++} \notin E$. Suppose otherwise. If $z \neq w^{++++}$ then $w^+z^{++}, z^-w \in E$ contradicts Lemma 3. Thus we may assume $z = w^{++++}$. Since $z^+v_0 \notin E$, $d(z^+) \geq (n-4)/2$. Thus z^+ must be adjacent to either w, w^{++}, w^{+++} or z^{+++}. However if either z^+z^{+++} or $z^+w^{+++} \in E$ we contradict Lemma 3 and if either z^+w or $z^+w^{++} \in E$ we contradict Lemma 4. Hence $w^+z^{++} \notin E$. Using an analogous argument we conclude $z^+w^{+++} \notin E$. We may now conclude that G is not 1-tough. For if $w^+w^{+++} \notin E$ then $G - (A \cup \{w^{++}\})$ has $n/2$ components and if $w^+w^{+++} \in E$, then by Lemma 3 $z^+w^{++}, z^{++}w^{++} \notin E$ and $G - A$ has $(n-2)/2$ components.

Case 2biii. There exist vertices $u, w, z \in A$ such that $u^{++}, w^{++}, z^{++} \notin A$.

It suffices to show that $z^{++}w^+, z^+w^{++}, z^{++}u^+, z^+u^{++}, w^{++}u^+, w^+u^{++} \notin E$ since then $G - A$ has $(n-2)/2$ components and G would not be 1-tough. We show that $z^{++}w^+$ and $z^+w^{++} \notin E$; symmetric arguments will complete the proof. We assume, without loss of generality, that $u^+ \in \overrightarrow{[w^+, z^+]}$. Suppose $z^{++}w^+ \in E$. If $w = z^{+++}$ consider any distinct pair of vertices $x^+, y^+ \in A^+ \cap A^-$. Since $N(x^+), N(y^+) \subseteq A - \{w\}$, $d(x^+) + d(y^+) < n - 4$, a contradiction. If $w \neq z^{+++}$, then since $z^+v_0 \notin E$ we have $d(z^+) \geq (n-4)/2$ and by Lemma 2, z^+ must be adjacent to at least one of w, w^{++}, z^{+++} and u^{++}. However $z^+w, z^+w^{++} \notin E$ by Lemma 4 and $z^+z^{+++} \notin E$ by Lemma 3. If $z^+u^{++} \in E$, then $C' : z^+\overleftarrow{[u^{++}, w^+]}\overrightarrow{[z^{++}, w]}v_0\overrightarrow{[u^{+++}, z^+]}$ is a hamiltonian cycle. Hence $z^{++}w^+ \notin E$. Now suppose $z^+w^{++} \in E$ and consider w^+. Reasoning as above, w^+ must be adjacent to at least one of z, z^{++}, w^{+++} and u^{++}. However $w^+z, w^+z^{++} \notin E$ by Lemma 4 and $w^+w^{+++} \notin E$ by Lemma 3. Hence $w^+u^{++} \in E$. In the same way we conclude $z^{++}u^+ \in E$. Since $n \geq 16$, $A^+ - \{u^+, w^+, z^+\} \neq \phi$. Without loss of generality suppose $w^- \in A^+ - \{u^+, w^+, z^+\}$. Clearly $N(w^-) = A$. Thus $w^-w^{+++} \in E$. However, since $z^+w^{++} \in E$ we contradict Lemma 3. This completes the proof. $\square$

3. Concluding remarks

It is not difficult to prove directly (i.e. without Theorem 2) that under the hypothesis of Jung's theorem every longest cycle is a dominating cycle when $n \geq 16$. The proof of Theorem 2 is straightforward, however, and we elected to use it. We also remark that our observation that $H = (V - C) \cup A^+$ is an independent set of vertices leads to a number of interesting results concerning long cycles in 1-tough graphs with $\delta \geq n/3$ [3]. Finally, it is possible to give a proof of Jung's theorem, i.e. for $n \geq 11$, along the lines of the proof presented here by using the following recent result [3].

Theorem 4. *Let G be a 1-tough graph on n vertices such that $d(x) + d(y) + d(z) \geqslant n$ for all independent sets of vertices x, y, z. Then every longest cycle in G is a dominating cycle.*

This establishes that under the hypothesis of Jung's theorem every longest cycle is a dominating cycle. Our methods quickly reduce the proof to the case where a longest cycle C has length $n - 1$. More detailed arguments are required to reduce the proof to the cases $d(v_0) = (n - 3)/2$ or $(n - 4)/2$, where $v_0 \in V - C$.

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