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Covering planar graphs with forests

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Abstract

We study the problem of covering graphs with trees and a graph of bounded maximum degree. By a classical theorem of Nash-Williams, every planar graph can be covered by three trees. We show that every planar graph can be covered by two trees and a forest, and the maximum degree of the forest is at most 8. Stronger results are obtained for some special classes of planar graphs.

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1. Introduction

For a graph G, we use $V(G)$ and $E(G)$ to denote the vertex set and edge set of G, respectively. For two subgraphs H and K of a graph, we use $H \cup K$ to denote the union

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of H and K. We say that a graph G can be *covered* by subgraphs $G_1, \dots, G_k$ of G if $\bigcup_{i=1}^{k} G_i = G$.

A well-known theorem of Nash-Williams [5] (based on a result proved independently in [4,7]) states that the edges of a graph G can be covered by t trees if, and only if, for every $A \subseteq V(G)$, $e(A) \leqslant (|A| - 1)t$, where $e(A)$ denotes the number of edges of G with both ends in A. One way to extend this result is to cover graphs with trees (or forests) and a graph with bounded degree. We say that a graph is *(t, D)-coverable* if it can be covered by at most t forests and a graph of maximum degree D.

It is easy to check that if a graph G is (t, D)-coverable, then, for any two disjoint subsets A, B of $V(G)$, $f_t(A) + e(A, B) \leqslant D \cdot |A| + t(|A| + |B| - 1)$, where $e(A, B)$ denotes the number of edges of G with one endpoint in A and the other in B, $f_t(A) = e(A)$ if $e(A) \leqslant t(|A| - 1)$, and $f_t(A) = 2e(A) - t(|A| - 1)$ otherwise. Unfortunately, this condition is not sufficient. For example, by deleting one edge from the Petersen graph, we obtain a graph that satisfies the above inequality with $t = D = 1$, but is not $(1, 1)$-coverable.

It is interesting to know what can be said about planar graphs. The aforementioned theorem of Nash-Williams implies that every planar graph is $(3, 0)$-coverable. As pointed out by Lovász [3] there are infinitely many planar graphs which are not (2,3)-coverable: take a triangle, put a vertex inside and connect it to the vertices of the triangle, and repeat this operation for each new triangle. After repeating this process for a while, we get a graph on n vertices with roughly $2n/3$ vertices of degree 3. This graph does not satisfy the above inequality about $f_t(A)$ (with $t = 2$, $D = 3$, B the set of vertices of degree 3, and A the set of vertices of degree at least 4), and so, it is not (2,3)-coverable. The double wheel on $2D + 4$ vertices (that is, a cycle of length $2D + 2$ plus two vertices and all edges from these two vertices to the cycle) shows that planar graphs need not be $(1, D)$-coverable. However, we believe the following is correct.

Conjecture 1. *Every simple planar graph is* $(2, 4)$*-coverable.*

As evidence for this conjecture, we shall prove that every simple planar graph is $(2, 8)$-coverable. This will be done in Section 3, with the help of a result from Section 2. In Section 4, we shall show that every simple outerplanar graph is $(1, 3)$-coverable, and as a consequence, every 4-connected planar graph is $(2, 6)$-coverable. We shall also consider graphs which are series-parallel or contain no $K_{3,2}$-subdivision. We conclude this section with some notation.

Throughout the remainder of this paper, we shall consider only simple graphs. Let G be a graph. An edge of G with endpoints x and y will be denoted by xy or yx. Paths and cycles in G will be denoted by sequences of vertices of G. For any $x \in V(G)$, let $N_G(x) := \{y \in V(G) : xy \in E(G)\}$, and let $d_G(x) := |N_G(x)|$, the degree of x. When G is known from the context, we shall simply write $N(x)$ and $d(x)$. Let $\Delta(G) := \max\{d(x) : x \in V(G)\}$. For any $S \subseteq V(G)$, we use $G - S$ to denote the graph with vertex set $V(G) - S$ and edge set $\{uv \in E(G) : \{u, v\} \subseteq V(G) - S\}$. For any $S \subseteq E(G)$, we use $G - S$ to denote the graph with vertex set $V(G)$ and edge set $E(G) - S$. When $S = \{s\}$, we shall simply write $G - s$. Let H be a subgraph of G and let $S \subseteq V(G) \cup E(G)$ such that every edge of G in S has both endpoints in $V(H) \cup (S \cap V(G))$, then we use $H + S$ to denote the graph with vertex set $V(H) \cup (S \cap V(G))$ and edge set $E(H) \cup (S \cap E(G))$.

Recall that a plane graph is a graph drawn in the plane with no pairs of edges crossing. A *facial cycle* of a plane graph G is a cycle that bounds a face of G. A *planar triangulation* is a plane graph in which every face is bounded by a triangle.

2. High vertices

In this section, we shall prove the following result about planar graphs. This result will be used in the next section to prove that all planar graphs are $(2, 8)$-coverable. Let G be a graph and $x \in V(G)$. Then x is said to be *high* if $d(x) \geqslant 11$, and *low* otherwise.

Theorem 2. *Every planar graph contains a vertex of degree at most* 5 *which is adjacent to at most two high vertices.*

Proof. Suppose the statement is not true. Then there is a planar triangulation G such that every vertex of degree at most 5 is adjacent to at least three high vertices. Therefore, all vertices of G have degree at least 3.

Let $v \in V(G)$ with $d(v) = 4$. We say that v is *4-independent* if, for any $u \in N(v)$, $d(u) \neq 4$; otherwise, we say that v is *4-dependent*. Let u_1, u_2 be two adjacent 4-dependent vertices. Then $G - \{u_1, u_2\}$ has a facial cycle $v_1v_2v_3v_4v_1$, and v_1, v_2, v_3, v_4 are all high vertices of G. Furthermore, the notation can be chosen so that v_1, v_3 are adjacent to both u_1 and u_2, and v_2 (respectively, v_4) is adjacent with u_1 (respectively, u_2). In this case we say that v_1, v_3 are u_1*-weak* and v_2 is u_1*-strong*, and v_1, v_3 are u_2*-weak* and v_4 is u_2*-strong*.

Next, we define a weight function $\omega : V(G) \to \mathbb{R}$ by making changes to the degree function $d : V(G) \to \mathbb{R}$. For each high vertex v of G, we make changes to $d(v)$ and $d(u)$ for all $u \in N(v)$ with $d(u) \leqslant 5$, according to the following rules:

(R1) If $u \in N(v)$ and $d(u) = 3$, then subtract 1 from $d(v)$ and add 1 to $d(u)$.
(R2) If $u \in N(v)$ and $d(u) = 5$, then subtract $\frac{1}{3}$ from $d(v)$ and add $\frac{1}{3}$ to $d(u)$.
(R3) If $u \in N(v)$ and u is 4-independent, then subtract $\frac{2}{3}$ from $d(v)$ and add $\frac{2}{3}$ to $d(u)$.
(R4) If $u \in N(v)$, u is 4-dependent, and v is u-strong, then subtract 1 from $d(v)$ and add 1 to $d(u)$.
(R5) If $u \in N(v)$, u is 4-dependent, and v is u-weak, then subtract $\frac{1}{2}$ from $d(v)$ and add $\frac{1}{2}$ to $d(u)$.

Let $\omega : V(G) \to \mathbb{R}$ denote the resulting function. For convenience, when we subtract a quantity α from $d(v)$ and add a quantity α to $d(u)$, we shall simply say that v sends *charge* α to u or u receives *charge* α from v.

Clearly,

$$\sum_{x \in V(G)} d(x) = \sum_{x \in V(G)} \omega(x).$$

Since G has $3|V(G)| - 6$ edges, $\sum_{x \in V(G)} d(x) < 6|V(G)|$. Hence there exists a vertex x of G such that $\omega(x) < 6$. We shall derive a contradiction by showing that $\omega(x) \geqslant 6$ for all $x \in V(G)$. Let $x \in V(G)$. We distinguish two cases.

Case 1: x is low.

If $d(x) = 3$ then, since all its neighbors are high, $\omega(x) = d(x) + 3 = 3 + 3 = 6$ by (R1).

If $d(x) = 5$ then, since x has $k \geqslant 3$ high neighbors, $\omega(x) = d(x) + k/3 = 5 + k/3 \geqslant 6$ by (R2).

Now assume $d(x) = 4$. If x is 4-independent then, since x has $k \geqslant 3$ high neighbors, $\omega(x) = d(x) + 2k/3 = 4 + 2k/3 \geqslant 6$ by (R3). If x is 4-dependent then, since x has three high neighbors (two are x-weak and one is x-strong), $\omega(x) = 4 + \frac{1}{2} + \frac{1}{2} + 1 = 6$ by (R4) and (R5).

If $6 \leqslant d(x) \leqslant 10$, then $\omega(x) = d(x) \geqslant 6$.

Case 2: x is high.

Let $d(x) = k$. Then $k \geqslant 11$. Since G is a planar triangulation, $G - x$ has a facial cycle C_k such that $V(C_k) = N(x)$. We partition $V(C_k)$ into the following five sets. Let $A := \{u \in N(x) : d(u) = 3$, or u is 4-dependent and x is u-strong$\}$. Let $B := \{u \in N(x) : u$ is 4-dependent and x is u-weak$\}$. Let $C := \{u \in N(x) : u$ is 4-independent$\}$. Let $D := \{u \in N(x) : d(u) = 5\}$. Finally, let $S := \{u \in N(x) : d(u) \geqslant 6\}$. Because every vertex of degree at most 5 has at least 3 high neighbors, one can easily check that the following statements hold:

(1) if $u \in A$, then u has two neighbors in S, and u receives charge 1 from x (by (R1) and (R4)).
(2) if $u \in B$, then (by planarity) u has one neighbor in B and one neighbor in S, and u receives charge $\frac{1}{2}$ from x (by (R5)).
(3) if $u \in C$, then u has at least one neighbor in S and at most one neighbor in D, and u receives charge $\frac{2}{3}$ from x (by (R3)).
(4) if $u \in D$, then u can have neighbors in $C \cup D \cup S$, and u receives charge $\frac{1}{3}$ from x (by (R2)).
(5) if $u \in S$, then u receives no charge from x.

Therefore, if $S = \emptyset$, then $A = B = C = \emptyset$, and hence, $D = V(C_k)$ and, by (4), $\omega(x) = k - (k/3) \geqslant \frac{22}{3} > 6$.

So assume $S \neq \emptyset$. Let $S = \{s_1, \ldots, s_m\}$ such that $s_1, \ldots, s_m$ occur on C_k in that clockwise order. If $m = 1$, let $S_1 = C_k$ and $s_2 = s_1$. If $m \geqslant 2$, the vertices in S divide C_k into k internally disjoint paths: for $1 \leqslant i \leqslant k$, let S_i denote the clockwise subpath of C_k from s_i to s_{i+1}, where $s_{m+1} = s_1$. Let $S_i' := S_i - \{s_i, s_{i+1}\}$.

We claim that, for each $1 \leqslant i \leqslant m$, one of the following holds:

(a) $|V(S_i')| \leqslant 1$.
(b) $|V(S_i')| = 2$ and $V(S_i') \subseteq B$.
(c) $|V(S_i')| = 2$, $V(S_i') \subseteq C \cup D$ and $V(S_i') \cap D \neq \emptyset$.
(d) $|V(S_i')| \geqslant 3$, $V(S_i') \subseteq C \cup D$ and all internal vertices of S_i' are contained in D.

To prove this claim, assume that $|V(S_i')| \geqslant 2$ (that is, not (a)) and let $S_i = x_0 x_1, \ldots, x_n x_{n+1}$, where $x_0 = s_i$ and $x_{n+1} = s_{i+1}$. Thus, $x_0, x_{n+1} \in S$, $n \geqslant 2$, and $x_1, \ldots, x_n \notin S$. Recall that we allow $x_0 = x_{n+1}$, which occurs when $m = 1$. Then, for each $1 \leqslant j \leqslant n$, $x_j \notin A$; for otherwise, by (1), $\{x_{j-1}, x_{j+1}\} \subseteq S$, contradicting the fact that $x_1, \ldots, x_n \notin S$.

Now assume that there is some $x_j \in B$. Since x_j has at least three high neighbors, one element of $\{x_{j-1}, x_{j+1}\}$ is high. By symmetry we may assume that x_{j-1} is high. Then $x_{j-1} \in S$. Since $x_j \in B$, x is x_j-weak. So $x_{j+1} \in B$, x_{j+2} is high, and $x_{j+2} \in S$. Hence, $x_{j-1} = x_0$ and $x_{j+2} = x_{n+1}$, $n = 2$, and $\{x_1, x_2\} \subseteq B$. That is, $V(S_i')$ consists of exactly two vertices which are in B, and (b) holds. So we may assume that $\{x_1, \ldots, x_n\} \subseteq C \cup D$, that is, $V(S_i') \subseteq C \cup D$. Then, since each x_j has at least three high neighbors, $x_2, \ldots, x_{n-1} \in D$ and, if $n = 2$ then $x_1 \in D$, or $x_n \in D$. So we have (c) and (d).

Now, let us calculate $\omega(x)$ by finding out how much charge x sends to vertices of S_i'. Suppose (a) holds for S_i'. If $|V(S_i')| = 1$ then the charge that x sends to S_i' is at most $1 = \lfloor \frac{|V(S_i')|+1}{2} \rfloor$. If $|V(S_i')| = 0$ then the charge that x sends to S_i' is $0 = \lfloor \frac{|V(S_i')|+1}{2} \rfloor$. If (b) holds for S_i', then by (2), the charge that x sends to vertices of S_i' is $\frac{1}{2} + \frac{1}{2} = 1 = \lfloor |V(S_i')|/2 \rfloor$. Now assume (c) or (d) holds for S_i'. If $|V(S_i')| = 2$ then by (c) at least one vertex of S_i' is in D, and by (3) and (4), the charge that x sends to vertices of S_i' is at most $\frac{2}{3} + \frac{1}{3} = 1 = \lfloor |V(S_i')|/2 \rfloor$. If $|V(S_i')| \geqslant 3$, then by (d), all internal vertices of S_i' are in D, and by (3) and (4), the charge that x sends to S_i' is at most $(n-2)/3 + \frac{2}{3} + \frac{2}{3} = (n+2)/3 \leqslant \lfloor (n+1)/2 \rfloor = \lfloor (|V(S_i')| + 1)/2 \rfloor$ (because $n = |V(S_i')| \geqslant 3$). By (5), x sends no charge to vertices in S. Hence, the total charge that x sends to its neighbors is at most

$$\sum_{i=1}^{m} \left\lfloor \frac{|V(S_i')| + 1}{2} \right\rfloor \leqslant \left\lfloor \frac{(\sum_{i=1}^{m} |V(S_i')|) + m}{2} \right\rfloor = \lfloor d(x)/2 \rfloor.$$

So $\omega(x) \geqslant d(x) - \lfloor d(x)/2 \rfloor$. Since $d(x) \geqslant 11$, $\omega(x) \geqslant 6$. □

Theorem 2 no longer holds if we define high vertices as those of degree 10 or more. Consider a planar triangulation with vertices of degrees 6 and 5. Put into each triangle a vertex and join it to all vertices of the triangle. We get a planar triangulation with vertices of degrees 3, 10, 12, and each vertex has at least 3 neighbors of degree at least 10.

3. Covering with forests

In this section we prove that every planar graph is (2,8)-coverable. In fact, we prove the following stronger result.

Theorem 3. *For each planar graph G, there exist forests T_1, T_2, and T_3 such that $G = T_1 \cup T_2 \cup T_3$ and $\Delta(T_3) \leqslant 8$.*

The proof is by way of contradiction. Suppose Theorem 3 is not true. Let G be a counter example with $|V(G)|$ minimum. Without loss of generality, we may assume that G is a planar triangulation. Hence the minimum degree of G is at least 3. We shall derive a contradiction to Theorem 2 by showing that every vertex of G with degree at most 5 has at least three high neighbors.

Lemma 4. *If $x \in V(G)$ and $d(x) = 3$, then all three neighbors of x are high.*

Proof. Consider the graph $G' := G - x$. By the choice of G, G' can be covered by three forests T_1', T_2', and T_3' such that $\Delta(T_3') \leqslant 8$. Without loss of generality, we may further assume that T_1', T_2', T_3' are edge disjoint, and subject to this, $|E(T_3')|$ is minimum. Therefore, for any $u \in V(T_3')$, $d_{T_i'}(u) \geqslant 1$ for $i = 1, 2$. Hence, $d_{T_3'}(v) \leqslant d_{G'}(v) - 2$ for every vertex v of G'.

Suppose some neighbor of x is not high, say y. Then $d_G(y) \leqslant 10$. So $d_{G'}(y) \leqslant 9$, and $d_{T_3'}(y) \leqslant d_{G'}(y) - 2 \leqslant 7$. Let v, w be the other two neighbors of x. Let $T_1 := T_1' + \{x, xv\}$, $T_2 := T_2' + \{x, xw\}$, and let $T_3' := T_3 + \{x, xy\}$. It is easy to check that T_1, T_2, T_3 are forests and cover G. Note that $d_{T_3}(y) = d_{T_3'}(y) + 1 \leqslant 8$ and, for any $u \in V(T_3) - \{y\}$, $d_{T_3}(u) = d_{T_3'}(u) \leqslant 8$. So $\Delta(T_3) \leqslant 8$. Hence, the existence of T_1, T_2, T_3 contradicts the choice of G. So all neighbors of x are high. □

Lemma 5. *If $x \in V(G)$ and $d(x) = 4$, then at least three neighbors of x are high.*

Proof. Let u, y, v and z denote the neighbors of x, occurring in that clockwise order around x. Since G is planar, $uv \notin E(G)$ or $yz \notin E(G)$. Without loss of generality we may assume that $yz \notin E(G)$. Then $G' := (G - x) + yz$ is a planar triangulation. By the choice of G, G' can be covered by three forests T_1', T_2', T_3' such that $\Delta(T_3') \leqslant 8$. We may further assume that T_1', T_2', T_3' are edge disjoint, and subject to this, $|E(T_3')|$ is minimum. Therefore, $d_{T_3'}(v) \leqslant d_{G'}(v) - 2$ for every vertex v of G'.

If $yz \in E(T_3')$, we let $T_1 := T_1' + \{x, ux\}$, $T_2 := T_2' + \{x, vx\}$ and $T_3 := (T_3' - yz) + \{x, yx, xz\}$. It is easy to see that T_1, T_2, T_3 are forests and cover G. Note that $d_{T_3}(x) = 2$ and, for any $w \in V(T_3) - \{x\}$, $d_{T_3}(w) = d_{T_3'}(w) \leqslant 8$. So $\Delta(T_3) \leqslant 8$. Hence, the existence of T_1, T_2, T_3 contradicts the choice of G.

So $yz \notin E(T_3')$. Then $yz \in E(T_1') \cup E(T_2')$. By symmetry, we may assume that $yz \in E(T_1')$.

We claim that u must be high. For, suppose u is low. Then $d_{G'}(u) = d_G(u) - 1 \leqslant 9$ and $d_{T_3'}(u) \leqslant d_{G'}(u) - 2 \leqslant 7$. Let $T_1 := (T_1' - yz) + \{x, xy, xz\}$, $T_2 := T_2' + \{x, xv\}$, and $T_3 := T_3' + \{x, xu\}$. Then T_1, T_2, T_3 are forests and cover G. Note that $d_{T_3}(x) = 1$ and $d_{T_3}(u) = d_{T_3'}(u) + 1 \leqslant 8$, and for any $w \in V(T_3) - \{u, x\}$, $d_{T_3}(w) = d_{T_3'}(w) \leqslant 8$. So $\Delta(T_3) \leqslant 8$. Hence the existence of T_1, T_2, T_3 contradicts the choice of G.

By a symmetric argument, we can show that v is also high.

Next we show that y is high or z is high. Suppose both y and z are low. Since T_1' is a forest and $yz \in E(T_1')$, $T_1' - yz$ does not contain both a y–v path and a z–v path. By symmetry, we may assume that $T_1' - yz$ contain no y–v path. Let $T_1 := (T_1' - yz) + \{x, v, yx, xv\}$, $T_2 := T_2' + \{x, ux\}$ and $T_3 := T_3' + \{x, xz\}$. Then T_1, T_2, T_3 are forests and cover G. Note that $d_{T_3}(x) = 1$ and, for any $w \in V(T_3) - \{x\}$, $d_{T_3}(w) = d_{T_3'}(w) \leqslant 8$. So $\Delta(T_3') \leqslant 8$. Hence, the existence of T_1, T_2, T_3 contradicts the choice of G.

Therefore, at least three neighbors of x are high. □

Lemma 6. *Let $x \in V(G)$ with $d(x) = 5$, and let x_0, x_1, x_2, x_3 and x_4 denote the neighbors of x which occur around x in that clockwise order. For any $0 \leqslant i \leqslant 4$, if $x_i x_{i+2} \notin E(G)$ and $x_i x_{i-2} \notin E(G)$, then both x_{i-1} and x_{i+1} are high. (Subscripts are taken modulo 5.)*

Proof. Since G is a planar triangulation, $x_0x_1x_2x_3x_4x_0$ is a facial cycle of $G - x$. Suppose $0 \leqslant i \leqslant 4$, $x_i x_{i+2} \notin E(G)$, and $x_i x_{i-2} \notin E(G)$. Then by the choice of G, $G' = (G - x) +$

$\{x_i x_{i+2}, x_i x_{i-2}\}$ can be covered by three forests T_1', T_2', T_3', with $\Delta(T_3') \leqslant 8$. We may further assume that T_1', T_2', T_3' are edge disjoint, and subject to this, $|E(T_3')|$ is minimum. Therefore, $d_{T_3'}(v) \leqslant d_{G'}(v) - 2$ for every vertex v of G'.

Case 1: $\{x_i x_{i+2}, x_i x_{i-2}\} \subseteq E(T_3')$.

Let $T_1 := T_1' + \{x, xx_{i+1}\}$, $T_2 := T_2' + \{x, xx_{i-1}\}$ and $T_3 := (T_3' - \{x_i x_{i+2}, x_i x_{i-2}\}) + \{x, xx_{i+2}, xx_{i-2}, xx_i\}$. Clearly, T_1, T_2, T_3 are forests and cover G. Note that $d_{T_3}(x) = 3$ and, for any $w \in V(T_3) - \{x\}$, $d_{T_3}(w) = d_{T_3'}(w) \leqslant 8$. So $\Delta(T_3) \leqslant 8$. Hence the existence of T_1, T_2, T_3 contradicts the choice of G.

Case 2: $\{x_i x_{i+2}, x_i x_{i-2}\} \subseteq E(T_1')$ or $\{x_i x_{i+2}, x_i x_{i-2}\} \subseteq E(T_2')$.

By symmetry, we may assume that $\{x_i x_{i+2}, x_i x_{i-2}\} \subseteq E(T_1')$. We show that both x_{i+1} and x_{i-1} are high. For, assume by symmetry that x_{i-1} is low. Then $d_{G'}(x_{i-1}) = d_G(x_{i-1}) - 1 \leqslant 9$ and $d_{T_3'}(x_{i-1}) \leqslant d_{G'}(x_{i-1}) - 2 \leqslant 7$. Let $T_1 := (T_1' - \{x_i x_{i+2}, x_i x_{i-2}\}) + \{x, xx_{i+2}, xx_{i-2}, xx_i\}$, $T_2 := T_2' + \{x, x_{i+1}x\}$, $T_3 := T_3' + \{x, xx_{i-1}\}$. Then T_1, T_2, T_3 are forests and cover G. Note that $d_{T_3}(x) = 1$, $d_{T_3}(x_{i-1}) \leqslant 8$ and, for any $w \in V(T_3) - \{x, x_{i-1}\}$, $d_{T_3}(w) = d_{T_3'}(w) \leqslant 8$. So $\Delta(T_3) \leqslant 8$. Hence the existence of T_1, T_2, T_3 contradicts the choice of G.

Case 3: One element of $\{x_i x_{i+2}, x_i x_{i-2}\}$ is in $E(T_3')$ and the other is in $E(T_1') \cup E(T_2')$.

By symmetry, we may assume that $x_i x_{i+2} \in E(T_1')$ and $x_i x_{i-2} \in E(T_3')$. We consider five subcases.

Subcase 3.1: $T_1' - x_i x_{i+2}$ contains an x_i-x_{i+1} path. Then $T_1' - x_i x_{i+2}$ contains no x_{i+1}-x_{i+2} path. In this case, let $T_1 := (T_1' - x_i x_{i+2}) + \{x, xx_{i+2}, xx_{i+1}\}$, $T_2 := T_2' + \{x, xx_{i-1}\}$ and $T_3 := (T_3' - x_i x_{i-2}) + \{x, xx_i, xx_{i-2}\}$. Then T_1, T_2, T_3 are forests and cover G. Note that $d_{T_3}(x) = 2$ and, for any $w \in V(T_3) - \{x\}$, $d_{T_3}(w) = d_{T_3'}(w) \leqslant 8$. So $\Delta(T_3) \leqslant 8$. Hence the existence of T_1, T_2, T_3 contradicts the choice of G.

Subcase 3.2: $T_1' - x_i x_{i+2}$ contains an x_i-x_{i-1} path. Then $T_1' - x_i x_{i+2}$ contains no x_{i-1}-x_{i+2} path. In this case, let $T_1 := (T_1' - x_i x_{i+2}) + \{x, xx_{i+2}, xx_{i-1}\}$, $T_2 := T_2' + \{x, xx_{i+1}\}$ and $T_3 := (T_3' - x_i x_{i-2}) + \{x, xx_i, xx_{i-2}\}$. Then T_1, T_2, T_3 are forests and cover G. Note that $d_{T_3}(x) = 2$ and, for any $w \in V(T_3) - \{x\}$, $d_{T_3}(w) = d_{T_3'}(w) \leqslant 8$. So $\Delta(T_3) \leqslant 8$. Hence the existence of T_1, T_2, T_3 contradicts the choice of G.

Subcase 3.3: $T_1' - x_i x_{i+2}$ contains neither an x_i-x_{i+1} path nor an x_i-x_{i-1} path, and $x_i x_{i-1} \in E(T_3')$. Let $T_1 := (T_1' - x_i x_{i+2}) + \{x, x_{i-1}, x_i x_{i-1}, xx_{i+2}\}$, $T_2 := T_2' + \{x, xx_{i+1}\}$, and $T_3 := (T_3' - \{x_i x_{i-2}, x_i x_{i-1}\}) + \{x, xx_i, xx_{i-1}, xx_{i-2}\}$. Then T_1, T_2, T_3 are forests and cover G. Note that $d_{T_3}(x) = 3$, $d_{T_3}(x_i) = d_{T_3'}(x_i) - 1$, and for any $w \in V(T_3') - \{x, x_i\}$, $d_{T_3}(w) = d_{T_3'}(w) \leqslant 8$. So $\Delta(T_3) \leqslant 8$. Hence the existence of T_1, T_2, T_3 contradicts the choice of G.

Subcase 3.4: $T_1' - x_i x_{i+2}$ contains neither an x_i-x_{i+1} path nor an x_i-x_{i-1} path, and $x_i x_{i+1} \in E(T_3')$. Let $T_1 := (T_1' - x_i x_{i+2}) + \{x, x_{i+1}, x_i x_{i+1}, xx_{i+2}\}$, $T_2 := T_2' + \{x, xx_{i-1}\}$, and $T_3 := (T_3' - \{x_i x_{i-2}, x_i x_{i+1}\}) + \{x, xx_i, xx_{i+1}, xx_{i-2}\}$. Then T_1, T_2, T_3 are forests and cover G. Note that $d_{T_3}(x) = 3$, $d_{T_3}(x_i) = d_{T_3'}(x_i) - 1$, and for any $w \in V(T_3') - \{x, x_i\}$, $d_{T_3}(w) = d_{T_3'}(w) \leqslant 8$. So $\Delta(T_3) \leqslant 8$. Hence the existence of T_1, T_2, T_3 contradicts the choice of G.

Subcase 3.5: $T_1' - x_i x_{i+2}$ contains neither an x_i-x_{i+1} path nor an x_i-x_{i-1} path, and $x_i x_{i-1}, x_i x_{i+1} \notin E(T_3')$. Then $x_i x_{i-1}, x_i x_{i+1} \in E(T_2')$. Let $T_1 := (T_1' - x_i x_{i+2}) + \{x, x_{i+1}, x_i x_{i+1}, xx_{i+2}\}$, $T_2 := (T_2' - x_i x_{i+1}) + \{x, xx_{i-1}, xx_{i+1}\}$, and $T_3 := (T_3' - x_i x_{i-2}) + \{x, xx_i, xx_{i-2}\}$. Then T_1, T_2, T_3 are forests and cover G. Note that $d_{T_3}(x) = 2$ and, for any

$w \in V(T_3') - \{x\}$, $d_{T_3}(w) = d_{T_3'}(w) \leqslant 8$. So $\Delta(T_3) \leqslant 8$. Hence the existence of T_1, T_2, T_3 contradicts the choice of G.

Case 4: One element of $\{x_i x_{i+2}, x_i x_{i-2}\}$ is in $E(T_1')$ and the other is in $E(T_2')$.

Without loss of generality, we may assume that $x_i x_{i+2} \in E(T_1')$ and $x_i x_{i-2} \in E(T_2')$. Then, up to symmetry, it suffices to check the following six subcases.

Subcase 4.1: $T_1' - x_i x_{i+2}$ contains neither an x_i-x_{i-1} path nor an x_i-x_{i+1} path, and $T_2' - x_i x_{i-2}$ contains neither an x_i-x_{i-1} path nor an x_i-x_{i+1} path.

Then $\{x_i x_{i+1}, x_i x_{i-1}\} \subseteq E(T_3')$. Let $T_1 := (T_1' - x_i x_{i+2}) + \{x, x_{i+1}, x_i x_{i+1}, x x_{i+2}\}$, $T_2 := (T_2' - x_i x_{i-2}) + \{x, x_{i-1}, x_i x_{i-1}, x x_{i-2}\}$, and $T_3 := (T_3' - \{x_i x_{i+1}, x_i x_{i-1}\}) + \{x, x x_{i+1}, x x_i, x x_{i-1}\}$. Then T_1, T_2, T_3 are forests and cover G. Note that $d_{T_3}(x) = 3$, $d_{T_3}(x_i) = d_{T_3'}(x_i) - 1$ and, for any $w \in V(T_3) - \{x, x_i\}$, $d_{T_3}(w) = d_{T_3'}(w) \leqslant 8$. So $\Delta(T_3) \leqslant 8$. Hence the existence of T_1, T_2, T_3 contradicts the choice of G.

Subcase 4.2: $T_1' - x_i x_{i+2}$ contains both an x_i-x_{i-1} path and an x_i-x_{i+1} path, or $T_2' - x_i x_{i-2}$ contains both an x_i-x_{i-1} path and an x_i-x_{i+1} path.

By symmetry, we may assume that $T_1' - x_i x_{i+2}$ contains an x_i-x_{i-1} path and an x_i-x_{i+1} path. Then $T_1' - x_i x_{i+2}$ contains no x_{i+1}-x_{i+2} path. Let $T_1 := (T_1' - x_i x_{i+2}) + \{x, x x_{i+1}, x x_{i+2}\}$, $T_2 := (T_2' - x_i x_{i-2}) + \{x, x x_i, x x_{i-2}\}$, and $T_3 := T_3' + \{x, x x_{i-1}\}$. Then T_1, T_2, T_3 are forests and cover G. Note that $d_{T_3}(x) = 1$ and, for any $w \in V(T_3) - \{x, x_{i-1}\}$, $d_{T_3}(w) = d_{T_3'}(w) \leqslant 8$. If x_{i-1} is low, then $d_{T_3'}(x_{i-1}) \leqslant d_{G'}(x_{i-1}) - 2 = d_G(x_{i-1}) - 3 \leqslant 7$, and so, $d_{T_3}(x_{i-1}) \leqslant 8$ and $\Delta(T_3) \leqslant 8$. Hence the existence of T_1, T_2, T_3 contradicts the choice of G. So x_{i-1} must be high.

Similarly, the forests $T_1 := (T_1' - x_i x_{i+2}) + \{x, x x_{i-1}, x x_{i+2}\}$, $T_2 := (T_2' - x_i x_{i-2}) + \{x, x x_i, x x_{i-2}\}$, and $T_3 := T_3' + \{x, x x_{i+1}\}$ allow us to conclude that x_{i+1} must be high.

Subcase 4.3: There is an x_i-x_{i+1} path in $T_1' - x_i x_{i+2}$, and there are no x_i-x_{i-1} paths in $T_1' - x_i x_{i+2}$ and $T_2' - x_i x_{i-2}$.

Then $x_i x_{i-1} \in E(T_3')$ and $T_1' - x_i x_{i+2}$ contains no x_{i+1}-x_{i+2} path. Let $T_1 := (T_1' - x_i x_{i+2}) + \{x, x x_{i+1}, x x_{i+2}\}$, $T_2 := (T_2' - x_i x_{i-2}) + \{x, x_i x_{i-1}, x x_{i-2}\}$, and $T_3 := (T_3' - x_i x_{i-1}) + \{x, x x_{i-1}, x x_i\}$. Then T_1, T_2, T_3 are forests and cover G. Note that $d_{T_3}(x) = 2$ and, for any $w \in V(T_3) - \{x\}$, $d_{T_3}(w) = d_{T_3'}(w) \leqslant 8$. So $\Delta(T_3) \leqslant 8$. Hence the existence of T_1, T_2, T_3 contradicts the choice of G.

Subcase 4.4: There is an x_i-x_{i-1} path in $T_1' - x_i x_{i+2}$, and there are no x_i-x_{i+1} paths in $T_1' - x_i x_{i+2}$ and $T_2' - x_i x_{i-2}$.

Then $x_i x_{i+1} \in E(T_3')$ and $T_1' - x_i x_{i+2}$ contains no x_{i-1}-x_{i+2} path. Let $T_1 := (T_1' - x_i x_{i+2}) + \{x, x x_{i-1}, x x_{i+2}\}$, $T_2 := (T_2' - x_i x_{i-2}) + \{x, x_i x_{i+1}, x x_{i-2}\}$, and $T_3 := (T_3' - x_i x_{i+1}) + \{x, x x_{i+1}, x x_i\}$. Then T_1, T_2, T_3 are forests and cover G. Note that $d_{T_3}(x) = 2$ and, for any $w \in V(T_3) - \{x\}$, $d_{T_3}(w) = d_{T_3'}(w) \leqslant 8$. So $\Delta(T_3) \leqslant 8$. Hence the existence of T_1, T_2, T_3 contradicts the choice of G.

Subcase 4.5: There is an x_i-x_{i+1} path in $T_1' - x_i x_{i+2}$, there is no x_i-x_{i-1} path in $T_1' - x_i x_{i+2}$, there is an x_i-x_{i-1} path in $T_2' - x_i x_{i-2}$, and there is no x_i-x_{i+1} path in $T_2' - x_i x_{i-2}$.

Then $T_1' - x_i x_{i+2}$ contains no x_{i+1}-x_{i+2} path, and $T_2' - x_i x_{i-2}$ contains no x_{i-1}-x_{i-2} path.

Let $T_1 := (T_1' - x_i x_{i+2}) + \{x, x x_{i+1}, x x_{i+2}\}$, $T_2 := (T_2' - x_i x_{i-2}) + \{x, x x_i, x x_{i-2}\}$, and $T_3 := T_3' + \{x, x x_{i-1}\}$. Then T_1, T_2, T_3 are forests and cover G. Note that $d_{T_3}(x) = 1$ and, for any $w \in V(T_3) - \{x, x_{i-1}\}$, $d_{T_3}(w) = d_{T_3'}(w) \leqslant 8$. If x_{i-1} is low, then $d_{T_3'}(x_{i-1}) \leqslant d_{G'}(x_{i-1})$

$-2 = d_G(x_{i-1}) - 3 \leqslant 7$, and so, $\Delta(T_3) \leqslant 8$. Hence the existence of T_1, T_2, T_3 contradicts the choice of G. So x_{i-1} must be high.

Similarly, the forests $T_1 := (T_1' - x_i x_{i+2}) + \{x, xx_i, xx_{i+2}\}$, $T_2 := (T_2' - x_i x_{i-2}) + \{x, xx_{i-1}, xx_{i-2}\}$, and $T_3 := T_3' + \{x, xx_{i+1}\}$ allow us to conclude that x_{i+1} must be high.

Subcase 4.6: There is an x_i-x_{i-1} path in $T_1' - x_i x_{i+2}$, there is no x_i-x_{i+1} path in $T_1' - x_i x_{i+2}$, there is an x_i-x_{i+1} path in $T_2' - x_i x_{i-2}$, and there is no x_i-x_{i-1} path in $T_2' - x_i x_{i-2}$.

Then $T_1' - x_i x_{i+2}$ contains no x_{i-1}-x_{i+2} path, and $T_2' - x_i x_{i-2}$ contains no x_{i+1}-x_{i-2} path.

Let $T_1 := (T_1' - x_i x_{i+2}) + \{x, xx_{i-1}, xx_{i+2}\}$, $T_2 := (T_2' - x_i x_{i-2}) + \{x, xx_i, xx_{i-2}\}$, and $T_3 := T_3' + \{x, xx_{i+1}\}$. Then T_1, T_2, T_3 are forests and cover G. Note that $d_{T_3}(x) = 1$ and, for any $w \in V(T_3) - \{x, x_{i+1}\}$, $d_{T_3}(w) = d_{T_3'}(w) \leqslant 8$. If x_{i+1} is low, then $d_{T_3'}(x_{i+1}) \leqslant d_{G'}(x_{i+1}) - 2 = d_G(x_{i+1}) - 3 \leqslant 7$, and so, $\Delta(T_3) \leqslant 8$. Hence the existence of T_1, T_2, T_3 contradicts the choice of G. So x_{i+1} must be high.

Similarly, the forests $T_1 := (T_1' - x_i x_{i+2}) + \{x, xx_i, xx_{i+2}\}$, $T_2 := (T_2' - x_i x_{i-2}) + \{x, xx_{i+1}, xx_{i-2}\}$, and $T_3 := T_3' + \{x, xx_{i-1}\}$ allow us to conclude that x_{i-1} must be high.

Therefore x_{i-1} and x_{i+1} are high. $\square$

We can now complete the proof of Theorem 3 as follows.

Proof. By Theorem 2, there is a vertex x of G such that $d(x) \leqslant 5$ and x has at most two high neighbors. By Lemma 4 and Lemma 5, we see that $d(x) = 5$. Let $x_0, x_1, \ldots, x_4$ denote the neighbors of x such that $x_0 x_1 \ldots x_4 x_0$ is a facial cycle of $G - x$. By planarity, there exist $0 \leqslant i \neq j \leqslant 4$ such that $x_i x_{i-2}, x_i x_{i+2}, x_j x_{j-2}, x_j x_{j+2} \notin E(G)$. So by Lemma 6, $x_{i-1}, x_{i+1}, x_{j-1}, x_{j+1}$ are high vertices. Since $x_i \neq x_j$ and $x_0 x_1 x_2 x_3 x_4 x_0$ is a cycle, $|\{x_{i-1}, x_{i+1}, x_{j-1}, x_{j+1}\}| \geqslant 3$. But this means that x has at least three high neighbors, a contradiction.

It is not hard to see that we may further require T_1, T_2 be trees. $\square$

4. Special planar graphs

In this section, we shall see that Theorem 3 can be improved for some special classes of planar graphs, thereby providing further evidence for Conjecture 1. Recall that a graph is outerplanar if it can be embedded in the plane such that all vertices are incident with its infinite face.

Theorem 7. *Let G be a 2-connected outerplanar graph and let C be the cycle of an outerplanar embedding of G bounding the infinite face. Let $y \in V(C)$ and let $yx, yz \in E(C)$. Then there is a forest T in G such that $d_{G-E(T)}(y) = 0$, $d_{G-E(T)}(x) \leqslant 1$, $d_{G-E(T)}(z) \leqslant 2$, $\Delta(G - E(T)) \leqslant 3$, and $G - E(T)$ is a forest.*

Proof. We apply induction on $|V(G)|$. It is easy to see that the theorem holds when $|V(G)| = 3$. So assume that $|V(G)| \geqslant 4$. Without loss of generality, we may assume that x, y, z occur on C in the clockwise order listed.

First, we consider the case when $d(y) = 2$. Let $H := (G - y) + xz$ and $D := (C - y) + xz$. Then H can be embedded in the plane so that H is an outerplanar graph with D bounding its infinite face. Let $xx' \in E(D)$ with $x' \neq z$ (because $|V(G)| \geqslant 4$). We apply induction to H, D, z, x, x' (as G, C, x, y, z, respectively). There is a forest S in H such that $d_{H-E(S)}(x) = 0$, $d_{H-E(S)}(z) \leqslant 1$, $d_{H-E(S)}(x') \leqslant 2$, $\Delta(H - E(S)) \leqslant 3$, and $H - E(S)$ is a forest. Now let T be the forest in G obtained from S by replacing the edge xz of S with the path xyz in G. It is easy to see that $d_{G-E(T)}(y) = 0$. Because $d_{H-E(S)}(x) = 0$, $d_{G-E(T)}(x) \leqslant 1$. Because $d_{H-E(S)}(z) \leqslant 1$, $d_{G-E(T)}(z) \leqslant 2$. The possible increase of 1 in the degrees comes from the edge xz. Therefore, because $\Delta(H - E(S)) \leqslant 3$, we have $\Delta(G - E(T)) \leqslant 3$. Since $G - E(T) = (H - E(S)) + xz$ and $d_{H-E(S)}(x) = 0$, we see that $G - E(T)$ is also a forest.

So we may assume that $d(y) \geqslant 3$. We label the neighbors of y as $y_1, \ldots, y_{k+1}$ in counterclockwise order on C. Then $k \geqslant 2$. Without loss of generality, assume that $y_1 = x$ and $y_{k+1} = z$. For $i = 1, \ldots, k$, let C_i denote the cycle which is the union of $y_{i+1}yy_i$ and the counterclockwise subpath of C from y_i to y_{i+1}, and let H_i denote the subgraph of G contained in the closed disc bounded by C_i. Then H_i is an outerplanar graph and C_i bounds its infinite face. For each $1 \leqslant i \leqslant k$, we apply induction to $H_i, C_i, y_i, y, y_{i+1}$ (as G, C, x, y, z, respectively). Therefore, for each $1 \leqslant i \leqslant k$, H_i has a forest T_i such that $d_{H_i-E(T_i)}(y) = 0$, $d_{H_i-E(T_i)}(y_i) \leqslant 1$, $d_{H_i-E(T_i)}(y_{i+1}) \leqslant 2$, $\Delta(H_i - E(T_i)) \leqslant 3$, and $H_i - E(T_i)$ is a forest. Let $T := \bigcup_{i=1}^{k} T_i$. Then T is a forest in G. It is easy to see that $d_{G-E(T)}(y) = 0$, $d_{G-E(T)}(x) \leqslant 1$, and $d_{G-E(T)}(z) \leqslant 2$. Note that for $i = 1, \ldots, k$, $d_{H_i-E(T_i)}(y_i) \leqslant 1$ and $d_{H_i-E(T_i)}(y_{i+1}) \leqslant 2$. Hence, $d_{G-E(T)}(y_i) \leqslant 3$ for $i = 2, \ldots, k-1$. Thus, $\Delta(G - E(T)) \leqslant 3$. It is also easy to see that $G - E(T) = \bigcup_{i=1}^{k}(H_i - E(T_i))$. Since $d_{G-E(T)}(y) = 0$, $G - E(T)$ is a forest. □

The following example gives a family of outerplanar graphs which are not $(1, 2)$-coverable. Take a long cycle $C = v_0v_1 \ldots v_{2n+1}v_0$ and add the following edges: v_0v_{2i+1} for $i = 1, \ldots, n-1$ and $v_{2i-1}v_{2i+1}$ for $i = 1, \ldots, n$.

Next, we show that all 4-connected planar graphs are $(2, 6)$-coverable. But first, we consider Hamiltonian planar graphs.

Corollary 8. *If G is a Hamiltonian planar graph, then it is* $(2, 6)$*-coverable.*

Proof. Take a plane embedding of G and let C be a Hamiltonian cycle in G. Let G_1 (respectively, G_2) denote the subgraph of G inside (respectively, outside) the closed disc bounded by C. Then G_1 and G_2 are outer planar graphs (with C as the boundary cycle). Pick a vertex $y \in V(C)$, and apply Theorem 7 to G_i, $i = 1, 2$, we find a forest T_i in G_i such that $d_{G_i-E(T_i)}(y) = 0$ and $\Delta(G_i - E(T_i)) \leqslant 3$. It is easy to verify that $\Delta(G - E(T_1 \cup T_2)) \leqslant 6$. □

Tutte [6] proved that every 4-connected planar graph contains a Hamilton cycle. Thus, by Corollary 8, we have the following result.

Corollary 9. *If G is a* 4*-connected planar graph, then it is* $(2, 6)$*-coverable.*

It is well known that a graph is outerplanar if and only if it contains no K_4-subdivision or $K_{3,2}$-subdivision [1, Proposition 7.3.1]. In view of Theorem 7, it is natural to consider the class of graphs containing no K_4-subdivisions and the class of graphs containing no $K_{3,2}$-subdivisions.

The graphs containing no K_4-subdivisions are also called *series-parallel* graphs. It is known that any simple series-parallel graph has a vertex of degree at most two (see [2]). Therefore, by applying induction on the number of vertices, we can show that any simple series-parallel graph is $(2, 0)$-coverable.

On the other hand, the graph $K_{n,2}$ is series-parallel, but is not $(1, \lfloor \frac{n}{2} - 2 \rfloor)$-coverable. So it is natural to consider graphs containing no $K_{n,2}$-subdivisions. An easier question is to determine the smallest t and D so that every simple graph with no $K_{n,2}$-minors is (t, D)-coverable, for $n \geqslant 2$. To this end, we consider the cases $n = 2, 3$. We note that when $n = 2, 3$, a graph contains a $K_{n,2}$-minor if, and only if, it contains a $K_{n,2}$-subdivision.

Note that if G is a simple graph containing no $K_{2,2}$-minor, then every block of G is either a triangle or induced by an edge. So it is easy to see that any simple graph containing no $K_{2,2}$-minor is $(1, 1)$-coverable.

For graphs with no $K_{3,2}$-minor, we have the following result.

Proposition 10. *If G is a simple graph containing no $K_{3,2}$-subdivision, then G is both* $(1, 3)$-*coverable and* $(2, 0)$-*coverable*.

Proof. First we shall prove the existence of a $(1, 3)$-cover. To do this, we prove the following stronger result.

(1) For any vertex v of G there is a forest T in G such that $d_{G-E(T)}(v) = 0$ and $\Delta(G - E(T)) \leqslant 3$.

We use induction on the number of K_4-subdivisions contained in G. If G contains no K_4-subdivision, then it is outerplanar, and (1) follows from Theorem 7. So assume that G contains a K_4-subdivision. In fact, every K_4-subdivision in G must be isomorphic to K_4, since any K_4-subdivision not isomorphic to K_4 is also a $K_{3,2}$-subdivision.

Let $\{v_1, v_2, v_3, v_4\} \subseteq V(G)$ induce a K_4 in G. Since G has no $K_{3,2}$-subdivision, $G - \{v_i v_j : 1 \leqslant i \neq j \leqslant 4\}$ has exactly four components C_i with $v_i \in V(C_i)$, $i = 1, 2, 3, 4$. Without loss of generality, we may assume that $v \in V(C_1)$. By applying induction to C_1, we conclude that C_1 contains a forest T_1 such that $d_{C_1-E(T_1)}(v) = 0$ and $\Delta(C_1 - E(T_1)) \leqslant 3$. Similarly, by applying induction to C_i, $i = 2, 3, 4$, C_i contains a forest T_i such that $d_{C_i-E(T_i)}(v_i) = 0$ and $\Delta(C_i - E(T_i)) \leqslant 3$. Let $T := (\bigcup_{i=1}^4 T_i) + \{v_1v_2, v_1v_3, v_1v_4\}$. It is easy to check that T is a forest, $d_{G-E(T)}(v) = 0$, and $\Delta(G - E(T)) \leqslant 3$.

To prove that G is $(2, 0)$-coverable, it suffices to prove the following result (by using Nash-Williams' theorem).

(2) If G is a graph containing no $K_{3,2}$-subdivision, then G contains at most $2|V(G)| - 2$ edges.

It is easy to check that (2) holds when $|V(G)| \leqslant 4$. So assume that $|V(G)| \geqslant 5$. Then G is not a complete graph. Further, G is not 3-connected. For otherwise, there are three internally disjoint paths in G between two non-adjacent vertices, and they would form a $K_{3,2}$-subdivision in G.

So let $\{u, v\}$ be a 2-cut of G and let C be a component of $G - \{u, v\}$. We choose $\{u, v\}$ and C so that $|V(C)|$ is minimum (among all choices of 2-cuts of G). Assume for the moment that $|V(C)| = 1$. Let $V(C) = \{x\}$. Then $d_G(x) = 2$. By applying induction to $G - x$, we see that $|E(G-x)| \leqslant 2|V(G-x)|-2$. Thus, $|E(G)| \leqslant 2|V(G)|-2$. Hence we may assume $|V(C)| \geqslant 2$. Let S denote the set of edges of G with one endpoint in $\{u, v\}$ and one endpoint in $V(C)$, and let $C^* := C + (\{u, v, uv\} \cup S)$. By the choice of $\{u, v\}$ and C, we can prove that C^* is 3-connected. Therefore, $C^* - uv$ contains two internally disjoint paths P, Q between u and v. On the other hand, $G - V(C)$ contains a path R from u to v and containing at least three vertices. Now $P \cup Q \cup R$ gives a $K_{3,2}$-subdivision in G, a contradiction. $\square$

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