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Eigenvalue and Dirichlet problem for fully-nonlinear operators in non-smooth domains

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ABSTRACT

We study the maximum principle, the existence of eigenvalue and the existence of solution for the Dirichlet problem relative to operators which are fully-nonlinear, elliptic but presenting some singularity or degeneracy which are similar to those of the p-Laplacian, we consider the equations in bounded domains which only satisfy the exterior cone condition.

1. Introduction

The aim of this paper is to extend the generalized concept of eigenvalue for fully-nonlinear operators, when the bounded domain involved satisfies only the uniform exterior cone condition; we shall also obtain regularity results, and maximum principle in this setting.

Before defining the precise notions described above let us recall that Berestycki, Nirenberg and Varadhan in [1], have proved maximum principle, principal eigenvalue and existence of solution for a Dirichlet problem involving linear uniformly elliptic operators $Lu = \operatorname{tr} A(x)D^2u + b(x)\cdot\nabla u + c(x)u$ in domains without any regularity condition on the boundary.

In order to do so, they need to define the concept of boundary condition. Hence, using Alexandrov–Bakelman–Pucci inequality and Krylov–Safonov–Harnack inequality they first prove the existence of u_o, a strong solution of

$$\operatorname{tr} A(x)D^2u_o + b(x)\cdot\nabla u_o = -1 \quad \text{in } \Omega,$$

which is zero on the points of the boundary that have some smoothness. Then they define the boundary condition for the Dirichlet problem associated to the full operator L through this function u_o. Their paper, which constructs the principal eigenvalue using only the maximum principle, has allowed to generalize the notion of eigenvalue to fully-nonlinear operators, see e.g. [3,4,6,9,11,12,15–18].

Here, as in [3,4,15] we shall consider operators that satisfy for some real $\alpha > -1$:

(H1) $F : \Omega \times \mathbb{R}^N \setminus \{0\} \times S \to \mathbb{R}$, and $\forall t \in \mathbb{R}^\star$, $\mu \geqslant 0$, $F(x, tp, \mu X) = |t|^\alpha \mu F(x, p, X)$.

(H2) There exist $0 < a < A$, such that for $x \in \overline{\Omega}$, $p \in \mathbb{R}^N \setminus \{0\}$, $M \in S$, $N \in S$, $N \geqslant 0$,

$$a|p|^\alpha \operatorname{tr}(N) \leqslant F(x, p, M + N) - F(x, p, M) \leqslant A|p|^\alpha \operatorname{tr}(N)$$

and other "regularity" conditions.

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For this class of operators it is not known whether the Alexandrov–Bakelman–Pucci inequality holds true (see [7]), hence in our previous works [3–5] we supposed that $\partial\Omega$ was $\mathcal{C}^2$. The regularity of the boundary in those papers played a crucial role because it allowed to use the distance function to construct sub- and super-solutions. This was the key step in the proof of the maximum principle. Here, instead, we shall suppose that Ω satisfies only the "uniform exterior" cone condition i.e.:

There exist $\psi > 0$ and $\bar r > 0$ such that for any $z \in \partial\Omega$ and for an axe through z of direction $\vec n$,

$$T_\psi = \left\{x:\ \frac{(x-z)\cdot\vec n}{|z-x|} \leqslant \cos\psi\right\}, \qquad T_\psi \cap \overline{\Omega} \cap B_{\bar r}(z) = \{z\}.$$

This cone condition allows to construct some barriers and consequently a function which will play the same role as u_o in [1]. In particular we can prove that there exists an eigenfunction $\varphi > 0$, solution of

$$\begin{cases} F(x, \nabla\varphi, D^2\varphi) + h(x)\cdot\nabla\varphi|\nabla\varphi|^\alpha + (V(x) + \bar\lambda(\Omega))\varphi^{1+\alpha} = 0 & \text{in } \Omega,\\ \varphi = 0 & \text{on } \partial\Omega,\end{cases}$$

for

$$\bar\lambda(\Omega) = \sup\{\lambda,\ \exists u > 0 \text{ in } \Omega,\ F(x, \nabla u, D^2u) + h(x)\cdot\nabla u|\nabla u|^\alpha + (V(x)+\lambda)u^{1+\alpha} \leqslant 0 \text{ in } \Omega\}.$$

Finally in the last section we also define

$$\lambda_e = \sup\{\bar\lambda(\Omega'),\ \Omega \Subset \Omega',\ \Omega' \text{ regular and bounded}\}$$

and

$$\tilde\lambda = \sup\{\lambda,\ \exists u > 0 \text{ in } \overline{\Omega},\ F(x, \nabla u, D^2u) + h(x)\cdot\nabla u|\nabla u|^\alpha + (V(x)+\lambda)u^{1+\alpha} \leqslant 0\}.$$

We prove that $\lambda_e = \tilde\lambda$ and that this value is an "eigenvalue" in the sense that there exists some $\phi_e > 0$, which satisfies

$$\begin{cases} F(x, \nabla\phi_e, D^2\phi_e) + h(x)\cdot\nabla\phi_e|\nabla\phi_e|^\alpha + (V(x) + \lambda_e(\Omega))\phi_e^{1+\alpha} = 0 & \text{in } \Omega,\\ \phi_e = 0 & \text{on } \partial\Omega.\end{cases}$$

We also prove that for any $\lambda < \lambda_e$ the maximum principle holds and there exists a solution of the Dirichlet problem when the right-hand side is negative.

Observe that $\lambda_e \leqslant \bar\lambda$, and furthermore if Ω is smooth, the equality holds. It is an open problem to know if the equality still holds when Ω satisfies only the exterior cone condition (see the example at the end of Section 5). Let us observe that the identity of these values is equivalent to the existence of a maximum principle for $\lambda < \bar\lambda$.

2. Assumptions on *F*

The following hypothesis will be considered. For $\alpha > -1$, F satisfies:

(H1) $F : \Omega \times \mathbb{R}^N \setminus \{0\} \times S \to \mathbb{R}$, and $\forall t \in \mathbb{R}^\star$, $\mu \geqslant 0$, $F(x, tp, \mu X) = |t|^\alpha \mu F(x, p, X)$.

(H2) There exist $0 < a < A$, such that for any $x \in \overline{\Omega}$, $p \in \mathbb{R}^N\backslash\{0\}$, $M \in S$, $N \in S$, $N \geqslant 0$,

$$a|p|^\alpha \operatorname{tr}(N) \leqslant F(x, p, M+N) - F(x, p, M) \leqslant A|p|^\alpha \operatorname{tr}(N). \tag{2.1}$$

(H3) There exists a continuous function $\tilde\omega$, $\tilde\omega(0) = 0$ such that for all $(x, y) \in \Omega^2$, $\forall p \neq 0$, $\forall X \in S$,

$$|F(x, p, X) - F(y, p, X)| \leqslant \tilde\omega(|x-y|)|p|^\alpha|X|.$$

(H4) There exists a continuous function ω with $\omega(0) = 0$, such that if $(X, Y) \in S^2$ and $\zeta \in \mathbb{R}^+$ satisfy

$$-\zeta\begin{pmatrix} I & 0\\ 0 & I\end{pmatrix} \leqslant \begin{pmatrix} X & 0\\ 0 & Y\end{pmatrix} \leqslant 4\zeta\begin{pmatrix} I & -I\\ -I & I\end{pmatrix}$$

and I is the identity matrix in $\mathbb{R}^N$, then for all $(x, y) \in \mathbb{R}^N$, $x \neq y$,

$$F(x, \zeta(x-y), X) - F(y, \zeta(x-y), -Y) \leqslant \omega(\zeta|x-y|^2).$$

Remark 2.1. When no ambiguity arises we shall sometime write $F[u]$ to signify $F(x, \nabla u, D^2u)$.

We assume that h and V are some continuous and bounded functions on $\overline{\Omega}$ and h satisfies the following condition:

(H5) Either $\alpha \leqslant 0$ and h is Hölder continuous of exponent $1+\alpha$, or $\alpha > 0$ and $(h(x) - h(y)\cdot x - y) \leqslant 0$.

The solutions that we consider will be taken in the sense of viscosity. For convenience of the reader we state the precise definition.

Definition 2.2. Let Ω be a bounded domain in $\mathbb{R}^N$, then v, bounded and continuous on $\overline{\Omega}$ is called a viscosity super-solution (respectively sub-solution) of $F(x, \nabla u, D^2 u) + h(x) \cdot \nabla u |\nabla u|^\alpha = f(x, u)$ if for all $x_0 \in \Omega$:

- Either there exists an open ball $B(x_0, \delta)$, $\delta > 0$ in Ω on which $v = cte = c$ and $0 \leqslant f(x, c)$, for all $x \in B(x_0, \delta)$ (respectively $0 \geqslant f(x, c)$).
- Or $\forall \varphi \in \mathcal{C}^2(\Omega)$, such that $v - \varphi$ has a local minimum on x_0 (respectively a local maximum) and $\nabla\varphi(x_0) \neq 0$, one has

$$F\big(x_0, \nabla\varphi(x_0), D^2\varphi(x_0)\big) + h(x_0) \cdot \nabla\varphi(x_0)\big|\nabla\varphi(x_0)\big|^\alpha \leqslant f\big(x_0, v(x_0)\big),$$

respectively

$$F\big(x_0, \nabla\varphi(x_0), D^2\varphi(x_0)\big) + h(x_0) \cdot \nabla\varphi(x_0)\big|\nabla\varphi(x_0)\big|^\alpha \geqslant f\big(x_0, v(x_0)\big).$$

We now recall what we mean by first eigenvalue and some of the properties of this eigenvalue.

For Ω a bounded domain, let

$$\bar\lambda(\Omega) := \sup\big\{\lambda,\ \exists\varphi > 0 \text{ in } \Omega,\ F[\varphi] + h(x) \cdot \nabla\varphi|\nabla\varphi|^\alpha + \big(V(x) + \lambda\big)\varphi^{1+\alpha} \leqslant 0\big\}.$$

When Ω is a bounded regular set, we proved in [4] that:

Theorem 2.3. *Suppose that F satisfies* (H1)–(H4), *that h satisfies* (H5), *and that V is continuous and bounded. Suppose that Ω is a bounded C^2 domain.*

Then there exists φ which is a solution of

$$\begin{cases} F[\varphi] + h(x) \cdot \nabla\varphi|\nabla\varphi|^\alpha + \big(V(x) + \bar\lambda\big)\varphi^{1+\alpha} = 0 & \text{in } \Omega, \\ \varphi = 0 & \text{on } \partial\Omega. \end{cases}$$

Moreover φ is strictly positive inside Ω and it is Hölder continuous.

We now recall some properties of the eigenvalue:

Theorem 2.4. *Suppose that Ω is a bounded C^2 domain, and that F, h, and V satisfy the previous assumptions. Suppose that $\lambda < \bar\lambda$ and that u satisfies*

$$\begin{cases} F\big(x, \nabla u, D^2 u\big) + h(x) \cdot \nabla u|\nabla u|^\alpha + \big(V(x) + \lambda\big)|u|^\alpha u \geqslant 0 & \text{in } \Omega, \\ u \leqslant 0 & \text{on } \partial\Omega. \end{cases}$$

Then $u \leqslant 0$ in Ω.

We now recall the following comparison principle which holds without assumptions on the regularity of the bounded domain Ω:

Proposition 2.5. *Suppose that $\beta(x, .)$ is nondecreasing and $\beta(x, 0) = 0$, that w is an upper semicontinuous sub-solution of*

$$F\big(x, \nabla w, D^2 w\big) + h(x) \cdot \nabla w|\nabla w|^\alpha - \beta\big(x, w(x)\big) \geqslant g$$

and u is a lower semicontinuous supersolution of

$$F\big(x, \nabla u, D^2 u\big) + h(x) \cdot \nabla u|\nabla u|^\alpha - \beta\big(x, u(x)\big) \leqslant f$$

with g lower semicontinuous, f upper semicontinuous, $f < g$ in Ω and

$$\limsup\big(w(x_j) - u(x_j)\big) \leqslant 0,$$

for all $x_j \to \partial\Omega$. Then $w \leqslant u$ in Ω.

Remark 2.6. The result still holds if β is increasing and $f \leqslant g$ in Ω.

The proof is as in [2].

We also recall the following weak comparison principle.

Theorem 2.7. *Suppose that Ω is some bounded open set. Suppose that F satisfies* (H1), (H2), *and* (H4), *that h satisfies* (H5) *and V is continuous and bounded. Suppose that $f \leqslant 0$, f is upper semi-continuous and g is lower semi-continuous with $f \leqslant g$.*

Suppose that there exist u and v continuous, $v \geqslant 0$, satisfying

$$F\big(x, \nabla u, D^2 u\big) + h(x) \cdot \nabla u |\nabla u|^\alpha + V(x)|u|^\alpha u \geqslant g \quad \text{in } \Omega,$$
$$F\big(x, \nabla v, D^2 v\big) + h(x) \cdot \nabla v |\nabla v|^\alpha + V(x) v^{1+\alpha} \leqslant f \quad \text{in } \Omega,$$
$$u \leqslant v \quad \text{on } \partial\Omega.$$

Then $u \leqslant v$ in Ω in each of these two cases:

(1) *If $v > 0$ on $\overline{\Omega}$ and either $f < 0$ in Ω, or $g(\bar{x}) > 0$ on every point $\bar{x}$ such that $f(\bar{x}) = 0$.*
(2) *If $v > 0$ in Ω, $f < 0$ in $\overline{\Omega}$ and $f < g$ on $\overline{\Omega}$.*

3. Barriers in non-smooth domains

In this section we assume that Ω satisfies the exterior cone condition. More precisely we assume that there exist $\bar{r}$ and $\psi \in]0, \pi[$ such that for each $z \in \partial\Omega$ the set $\Omega \cap B(z, \bar{r})$ is included in the open cone which, up to change of coordinates can be given by

$$T_\psi = \left\{ r \in]0, \bar{r}[,\ 0 \leqslant \arccos\left(\frac{x_N}{r}\right) \leqslant \psi \right\}$$

choosing the main direction of the cone to be e_N. Indeed, in that case, the exterior of Ω contains at least the set of (x', x_N) with $-1 \leqslant \frac{x_N}{r} \leqslant \cos\psi$, $r < \bar{r}$.

On the operator F we suppose that it satisfies conditions (H1)–(H4), while h satisfies (H5).

3.1. Local barriers

Under the exterior cone condition we are going to construct a local barrier i.e. for any $z \in \partial\Omega$, a super-solution in a neighborhood of z, of $F[v] + h(x) \cdot \nabla v |\nabla v|^\alpha \leqslant -1$, such that $c|x - z|^\gamma \leqslant v(x) \leqslant C|x - z|^\gamma$ for some $\gamma \in (0, 1]$ and for some constant c and C which depend on $\psi, a, A, \gamma, \bar{r}$. This barrier is constructed on the model of those given by Miller for the Pucci operators in [13,14].

We define

$$v(x) = |x - z|^\gamma \varphi(\theta)$$

where $\theta = \arccos(\frac{x_N - z_N}{|x - z|})$. Without loss of generality, we suppose that $z = 0$.

We suppose first that $h \equiv 0$ and, at the end of the proof, we shall say which are the changes that need to be done when $h \not\equiv 0$. We shall first show that there exists φ a solution of some differential linear equation such that v is a super-solution of

$$F\big(x, \nabla v, D^2 v\big) \leqslant -b$$

where b is a positive constant that depends only on ψ, γ, r_o and the structural constant of the operator. It will be useful for the following to observe that $1 \geqslant \frac{x_N}{r} \geqslant \cos\psi$ on the considered set.

Let $x = (x_1, \ldots, x_N) = (x', x_N)$. Let $r = |x|$ and $r' = |x'|$. We shall also use the following notation $X' = (x', 0)$.

One has:

$$\nabla v = \gamma r^{\gamma-2} x \varphi(\theta) + r^\gamma \varphi'(\theta) \nabla\theta$$

and

$$D^2 v = r^{\gamma-2} \varphi\gamma \left(I + \frac{(\gamma - 2)}{r^2} x \otimes x \right) + r^{\gamma-2} \varphi' \big(r^2 D^2\theta + \gamma(\nabla\theta \otimes x + x \otimes \nabla\theta) \big) + r^{\gamma-2} \varphi'' \big(r^2 \nabla\theta \otimes \nabla\theta \big).$$

We now suppose that $\varphi \geqslant 0$, $\varphi' \leqslant 0$ and $\varphi'' \leqslant 0$ then

$$\mathcal{M}^+_{a,A}\big(D^2 v\big) \leqslant r^{\gamma-2} \bigg(\varphi\gamma \mathcal{M}^+_{a,A}\left(I + \frac{(\gamma - 2)}{r^2} x \otimes x \right) + \varphi' \mathcal{M}^-_{a,A}\big(r^2 D^2\theta + \gamma(\nabla\theta \otimes x + x \otimes \nabla\theta) \big)$$
$$+ \varphi'' \mathcal{M}^-_{a,A}\big(r^2 \nabla\theta \otimes \nabla\theta \big) \bigg).$$

Since we need to find the eigenvalues of the above matrices let us remark that

$$\nabla\theta = \frac{1}{r'} \left(\frac{x_N x}{r^2} - e_N \right) = \frac{x^\perp}{r^2}$$

with

$$x^{\perp} = \frac{x_N}{r'} X' - r' e_N = \cot\theta x - \frac{r^2}{r'} e_N.$$

In particular $x^{\perp} \cdot x = 0$ and $|x^{\perp}| = r$. We obtain

$$\mathcal{M}^+_{a,A}\left(I + \frac{(\gamma-2)}{r^2} x \otimes x\right) = A(N-1) + a(\gamma-1),$$
$$\mathcal{M}^-_{a,A}\big(r^2 \nabla\theta \otimes \nabla\theta\big) = a r^2 |\nabla\theta|^2 = a,$$
$$\mathcal{M}^-_{a,A}\big(\gamma(\nabla\theta \otimes x + x \otimes \nabla\theta)\big) = \gamma |\nabla\theta| r(a-A) = \gamma(a-A).$$

To complete the calculation we need to compute

$$\begin{aligned} D^2\theta &= -\frac{1}{r'^2}\frac{X'}{r'} \otimes \left(\frac{x_N x}{r^2} - e_N\right) + \frac{1}{r'}\left[\frac{1}{r^2} e_N \otimes x - \frac{2x_N}{r^4} x \otimes x + \frac{x_N}{r^2} I\right] \\ &= -\frac{x_N}{(r')^3 r^2} X' \otimes X' + \frac{1}{r' r^2}\left[X' \otimes e_N + e_N \otimes X' + x_N e_N \otimes e_N - 2\frac{x_N}{r^2} x \otimes x + x_N I\right]. \end{aligned}$$

To estimates the eigenvalues of $r^2 D^2\theta$ we shall use the following facts and notations:

I_{N-1} indicates the identity $(N-1)\times(N-1)$ matrix,

$$I' = \begin{pmatrix} I_{N-1} & 0 \\ 0 & 0 \end{pmatrix}, \qquad I = I' + e_N \otimes e_N,$$

$$x \otimes x = X' \otimes X' + x_N(X' \otimes e_N + e_N \otimes X') + x_N^2 e_N \otimes e_N.$$

Then

$$r^2 D^2\theta = \frac{x_N}{r'}\left(-\frac{1}{(r')^2} X' \otimes X' + I'\right) + \frac{x_N}{r'}\left(2 - 2\frac{x_N^2}{r^2}\right) e_N \otimes e_N + \frac{1}{r'}\left(1 - 2\frac{x_N^2}{r^2}\right)(X' \otimes e_N + e_N \otimes X') - 2\frac{x_N}{r'}\frac{1}{r^2} X' \otimes X'.$$

One has

$$\mathcal{M}^-_{a,A}\left(\frac{x_N}{r'}\left(-\frac{1}{r'^2} X' \otimes X' + I'\right) + \frac{x_N}{r'}\left(2 - 2\frac{x_N^2}{r^2}\right) e_N \otimes e_N\right) \geqslant -A\frac{x_N^-}{r'}(N-1) \geqslant -A(N-1)(\cot\psi)^-$$

and, using $\frac{|2x_N r'|}{r^2} \leqslant 1$,

$$\mathcal{M}^-_{a,A}\left(-2\frac{x_N}{r'}\frac{1}{r^2} X' \otimes X'\right) \geqslant -2\frac{A|x_N| r'}{r^2} \geqslant -A.$$

From this one gets that

$$\begin{aligned} \mathcal{M}^-_{a,A}\big(r^2 D^2\theta\big) &\geqslant -A\big((N-1)(\cot\psi)^- + 1\big) + \mathcal{M}^-_{a,A}\Big(\frac{1}{r'}\left(1 - 2\frac{x_N^2}{r^2}\right)(X' \otimes e_N + e_N \otimes X')\Big) \\ &\geqslant -\left|1 - 2\frac{x_N^2}{r^2}\right| A - A\big((N-1)(\cot\psi)^- + 1\big) \geqslant -A - A\big((N-1)(\cot\psi)^- + 1\big) \\ &\geqslant -A\big((N-1)(\cot\psi)^- + 2\big) \end{aligned}$$

where we have used that $|1 - 2\frac{x_N^2}{r^2}| \leqslant 1$.

Putting everything together we have obtained:

$$\begin{aligned} \mathcal{M}^+_{a,A}\big(D^2 v\big) &\leqslant r^{\gamma-2}\big(\varphi\gamma\big(A(N-1) + a(\gamma-1)\big) - \varphi'\big(A(N-1)(\cot\psi)^- + 2\big) + \gamma(A-a) + a\varphi''\big) \\ &\leqslant r^{\gamma-2}\big(\varphi\gamma\big(A(N-1) + a(\gamma-1)\big) - \varphi'\big(A\big((N-1)(\cot\psi)^- + 2\big) + \gamma(A-a) + a\varphi''\big). \end{aligned}$$

Defining $\beta = A((N-1)(\cot\psi)^- + 2) + \gamma(A-a)$ we shall choose φ such that

$$a\varphi'' - \beta\varphi' + \varphi\gamma A(N-1) = 0$$

and such that for θ in some interval $[0,\psi]$:

$$\varphi > 0, \qquad \varphi' \leqslant 0, \qquad \varphi'' \leqslant 0.$$

Indeed, for γ sufficiently close to zero, in order that $\beta^2 > 4\gamma(\frac{(N-1)A}{a})$, the solutions are given by

$$\varphi = C_1 e^{\sigma_1\theta} + C_2 e^{\sigma_2\theta}$$

with σ_1 and σ_2 being the positive constants $\sigma_1 = \frac{1}{2}(\beta + \sqrt{\beta^2 - 4\gamma(\frac{(N-1)A}{a})})$, $\sigma_2 = \frac{1}{2}(\beta - \sqrt{\beta^2 - 4\gamma(\frac{(N-1)A}{a})})$. Observe that σ_1 and σ_2 are both positive, one also has $\sigma_1 > \sigma_2$. We prove that for γ small enough, one can find a solution φ such that on $[0,\psi]$, $\varphi \geqslant 1$, $\varphi' \leqslant 0$ and $\varphi'' \leqslant 0$.

We choose $C_1 < 0$ and $C_2 > 0$ with

$$\begin{cases} C_1\sigma_1 + C_2\sigma_2 = 0, \\ C_1 e^{\sigma_1\psi} + C_2 e^{\sigma_2\psi} = 1. \end{cases}$$

This system has a solution because, for γ small enough

$$e^{(\sigma_2-\sigma_1)\psi} \geqslant e^{-\beta\psi} \geqslant \frac{4\gamma(N-1)A}{a\beta^2} \geqslant 1 - \sqrt{1 - \frac{4\gamma(N-1)A}{a\beta^2}} \geqslant \frac{\beta - \sqrt{\beta^2 - \frac{4\gamma(N-1)A}{a}}}{\beta} \geqslant \frac{\beta - \sqrt{\beta^2 - \frac{4\gamma(N-1)A}{a}}}{\beta + \sqrt{\beta^2 - \frac{4\gamma(N-1)A}{a}}} = \frac{\sigma_2}{\sigma_1}.$$

We now deduce from this that $\varphi' \leqslant 0$, and $\varphi'' \leqslant 0$ on $[0,\psi]$.

Indeed the assumption implies that $\varphi'(0) = 0$. Then, for $\theta > 0$,

$$\varphi'(\theta) = C_1\sigma_1 e^{\sigma_1\theta} + C_2\sigma_2 e^{\sigma_2\theta} \leqslant (C_1\sigma_1 + C_2\sigma_2)e^{\sigma_1\theta} = 0.$$

One also has

$$\varphi''(0) = C_1\sigma_1^2 + C_2\sigma_2^2 = -C_2\sigma_1\sigma_2 + C_2\sigma_2^2 \leqslant 0$$

and for $\theta > 0$,

$$\varphi''(\theta) = C_1\sigma_1^2 e^{\sigma_1\theta} + C_2\sigma_2^2 e^{\sigma_2\theta} \leqslant e^{\sigma_1\theta}\bigl(C_1\sigma_1^2 + C_2\sigma_2^2\bigr).$$

Let us note that

$$1 \leqslant \varphi(\theta) \leqslant \varphi(0) =: C_1 + C_2 = C_2\left(1 - \frac{\sigma_2}{\sigma_1}\right)$$

and

$$\bigl|\varphi'(\theta)\bigr| \leqslant \bigl|\varphi'(\psi)\bigr| = C_2\sigma_2\bigl(e^{\sigma_1\psi - \sigma_2\psi}\bigr).$$

Let $C_\psi = \sup(\varphi^2 + (\varphi')^2)^{\frac{\alpha}{2}}$. We have obtained that

$$\begin{aligned} F\bigl(x, \nabla v, D^2 v\bigr) &\leqslant |\nabla v|^\alpha \mathcal{M}^+_{a,A}\bigl(D^2 v\bigr) \leqslant \gamma^\alpha r^{(\gamma-1)\alpha}\bigl(\varphi^2 + (\varphi')^2\bigr)^{\frac{\alpha}{2}} \mathcal{M}^+_{a,A}\bigl(D^2 v\bigr) \\ &\leqslant -a\gamma^{2+\alpha}(1-\gamma)\varphi r^{(\gamma-1)\alpha+\gamma-2}\bigl(\varphi^2 + (\varphi')^2\bigr)^{\frac{\alpha}{2}} \leqslant -a\gamma^{2+\alpha} C_\psi r^{\gamma(\alpha+1)-\alpha-2}. \end{aligned}$$

We now consider the case $h \neq 0$. The above computations give

$$\begin{aligned} F\bigl(x, \nabla v, D^2 v\bigr) + h(x)\cdot\nabla v|\nabla v|^\alpha &\leqslant -C_\psi r^{\gamma(\alpha+1)-\alpha-2}\gamma^{2+\alpha} a\varphi + |h|_\infty\bigl(\gamma r^{\gamma-1}\bigr)^{1+\alpha}\sup\bigl(|\varphi|^2 + (\varphi')^2\bigr)^{\frac{1+\alpha}{2}} \\ &< -\frac{C_\psi r^{\gamma(\alpha+1)-\alpha-2}\gamma^{2+\alpha} a}{2} \leqslant -\frac{C_\psi r_o^{\gamma(\alpha+1)-\alpha-2}\gamma^{2+\alpha} a}{2} \\ &:= -b \end{aligned}$$

for $r \leqslant r_o := \inf(\bar r, \frac{\gamma a}{C_\psi^{\frac{1}{\alpha}}|h|_\infty})$. This ends the proof.

Remark 3.1. In the same manner one can construct a local barrier by below, i.e. some continuous non-positive function w'_z such that $w'_z(z) = 0$ which in the cone is a sub-solution of

$$F\bigl[w'_z\bigr] + h(x)\cdot\nabla w'_z\bigl|\nabla w'_z\bigr|^\alpha \geqslant 1.$$

3.2. Global barriers and existence

In all this section we shall suppose that Ω satisfies the exterior cone condition, F satisfies conditions (H1) to (H4) and h satisfies (H5). The global barrier constructed below will allow to prove the following existence result.

Proposition 3.2. *There exists u_o a non-negative viscosity solution of*

$$\begin{cases} F\bigl(x, \nabla u_o, D^2 u_o\bigr) + h(x)\cdot\nabla u_o|\nabla u_o|^\alpha = -1 & \text{in } \Omega, \\ u_o = 0 & \text{on } \partial\Omega \end{cases} \tag{3.1}$$

which is γ-Hölder continuous.

Proposition 3.2 will be the first step in the proof of the maximum principle and the construction of the principal eigenfunction in non-smooth bounded domains.

The global barrier is given in

Proposition 3.3. *For all $z \in \partial\Omega$, there exists a continuous function W_z on $\overline{\Omega}$, such that $W_z(z) = 0$, $W_z > 0$ in $\Omega \setminus \{z\}$ which is a super-solution of* (3.1)*, or equivalently*

$$F\big(x, \nabla W_z, D^2 W_z\big) + h(x) \cdot \nabla W_z |\nabla W_z|^\alpha \leqslant -1 \quad \text{in } \Omega.$$

Proof. We argue on the model of [8]. Choose any point $y \notin \Omega$ and r_1 such that $2r_1 < d(y, \partial\Omega)$. Let $G_1(x) = \frac{1}{r_1^\sigma} - \frac{1}{|x-y|^\sigma}$ then

$$F[G_1] + h(x) \cdot \nabla G_1 |\nabla G_1|^\alpha \leqslant \sigma^{1+\alpha} |x-y|^{-(\sigma+1)\alpha-\sigma-2}\big(AN - (\sigma+2)a + |h|_\infty |x-y|\big) \leqslant -(r_1)^{-\sigma(\alpha+1)-\alpha-2}\sigma^{1+\alpha}\frac{AN}{4}$$

as soon as

$$\sigma + 2 > \sup\left(\frac{4AN}{a}, \frac{2|h|_\infty \operatorname{diam}\Omega}{a}\right).$$

Moreover

$$\frac{1}{r_1^\sigma} \geqslant G_1(x) \geqslant \frac{2^\sigma - 1}{(2r_1)^\sigma} \quad \text{in } \Omega.$$

Defining $G = \frac{r_o^\gamma r_1^\sigma}{2} G_1$, one gets that $G \leqslant \frac{r_o^\gamma}{2}$.

We denote by $w_z(x) = |z-x|^\gamma \varphi(\theta)$ some local barrier associated to the point $z \in \partial\Omega$ as constructed in the previous section. Let

$$V_z(x) = \min\big(G(x), w_z\big).$$

Since the infimum of two super-solutions is a super-solution, V_z is a super-solution of

$$F[V_z] + h(x) \cdot \nabla V_z |\nabla V_z|^\alpha \leqslant \sup\left(-\frac{C_\psi r_o^{\gamma(\alpha+1)-\alpha-2}\gamma^{2+\alpha} a}{2}, -\frac{r_o^\gamma}{2}(r_1)^{-\sigma(\alpha+1)-\alpha-2}\sigma^{1+\alpha}\frac{AN}{4}\right) \equiv -\kappa^{1+\alpha}.$$

Hence $W_z = \frac{V_z}{\kappa}$ will be some super-solution of (3.1). □

Remark 3.4. Observe that, since $G > 0$ in $\overline{\Omega}$ there exists δ such that $W_z(x) = \frac{w_z(x)}{\kappa}$ if $|x-z| < \delta$. Furthermore, by the uniform exterior cone condition there exists $C_w > 0$ such that if $|x-z| < \delta$,

$$W_z(x) \leqslant C_w |x-z|^\gamma,$$

where C_w depends on γ, r_o and ψ and is independent of $z \in \partial\Omega$.

Remark 3.5. Using Remark 3.1 one can also construct a continuous function W'_z on $\overline{\Omega}$, such that $W'_z(z) = 0$, $W'_z < 0$ in $\Omega \setminus \{z\}$ which is a sub-solution of

$$F\big(x, \nabla W'_z, D^2 W'_z\big) + h(x) \cdot \nabla W'_z \big|\nabla W'_z\big|^\alpha \geqslant 1 \quad \text{in } \Omega. \tag{3.2}$$

In the next proposition we shall see that existence of global barriers allows to prove Hölder's regularity for solutions in non-smooth domains:

Proposition 3.6. *Let H_j be a sequence of bounded open regular sets such that $H_j \subset \overline{H_j} \subset H_{j+1}$, $j \geqslant 1$, whose union equals Ω. Let u_j be a sequence of bounded solutions of*

$$\begin{cases} F\big(x, \nabla u_j, D^2 u_j\big) + h(x) \cdot \nabla u_j |\nabla u_j|^\alpha = f_j & \text{in } H_j, \\ u_j = 0 & \text{on } \partial H_j, \end{cases}$$

with f_j uniformly bounded. Then, for $\gamma \in (0,1)$ given in the previous construction, there exists C independent of j such that

$$\big|u_j(x) - u_j(y)\big| \leqslant C|x-y|^\gamma$$

for all $x, y \in \Omega$.

Proof. Since ∂H_j is $\mathcal{C}^2$, it satisfies the exterior sphere condition and a fortiori the exterior cone condition. Since the H_j converge to Ω which satisfies the exterior cone condition, we can choose exterior cones with opening ψ and height r_o which do not depend on j.

Using the global barriers of Proposition 3.3 and the comparison principle in H_j, one easily has that, for any $z \in \partial H_j$,

$$u_j \leqslant |f_j|_\infty^{\frac{1}{1+\alpha}} W_z \quad \text{in } H_j.$$

Let

$$\Delta_\delta = \{(x, y) \in H_j^2 \text{ such that } |x-y| < \delta\}.$$

Let $C = \max\{\frac{2|u|_\infty}{\delta^\gamma}, C_w |f_j|_\infty^{\frac{1}{1+\alpha}}\}$, we want to prove that for δ small enough, and for any $(x, y) \in \Delta_\delta$,

$$u_j(x) - u_j(y) \leqslant C|x-y|^\gamma. \tag{3.3}$$

In the first step we prove it on the boundary of Δ_δ. Indeed if $|x-y| = \delta$ it is immediate from the definition of C. Suppose hence that $x \in H_j$ and $y \in \partial H_j$, with $|x-y| \leqslant \delta$. Then, using Remark 3.4, for δ sufficiently small

$$u_j(x) \leqslant |f_j|_\infty^{\frac{1}{1+\alpha}} W_y(x) \leqslant C_w |f_j|_\infty^{\frac{1}{1+\alpha}} |x-y|^\gamma.$$

The second step is to check that the inequality (3.3) holds inside Δ_δ. This is done exactly as in the smooth case (see [3,10]) using hypotheses (H2) and (H3). □

Proof of Proposition 3.2. Let H_j be a sequence of bounded open regular sets such that $H_j \subset \overline{H_j} \subset H_{j+1}$, $j \geqslant 1$, with the union equals to Ω.

Let u_j for $j \geqslant 1$ be the solution of

$$\begin{cases} F(x, \nabla u_j, D^2 u_j) + h(x) \cdot \nabla u_j |\nabla u_j|^\alpha = -1 & \text{in } H_j, \\ u_j = 0 & \text{on } \partial H_j. \end{cases}$$

Using the global barriers of Proposition 3.3 and the comparison principle in H_j one easily has that

$$u_j \leqslant W_z \quad \text{in } H_j.$$

As a consequence, $(u_j)_{j \geqslant 1}$ is a bounded and increasing sequence, that is $u_j \geqslant u_{j-1}$ on H_{j-1}. Using Proposition 3.6, the sequence $(u_j)_j$ is uniformly γ-Hölder continuous. As a consequence, on any compact set $J \subset \Omega$, one gets that $(u_j)_j$ converges uniformly to some u_o which satisfies

$$F(x, \nabla u_o, D^2 u_o) + h(x) \cdot \nabla u_o |\nabla u_o|^\alpha = -1.$$

Furthermore u_o equals 0 on the boundary since, by passing to the limit in the previous inequality

$$u_o \leqslant W_z,$$

for all $z \in \partial\Omega$. We have also obtained that u_o is γ-Hölder continuous. □

Remark 3.7. In the same manner it is possible to prove that there exists u'_o a non-positive γ-Hölder continuous solution of

$$\begin{cases} F(x, \nabla u'_o, D^2 u'_o) + h(x) \cdot \nabla u'_o |\nabla u'_o|^\alpha = 1 & \text{in } \Omega, \\ u'_o = 0 & \text{on } \partial\Omega, \end{cases}$$

with $u'_o \geqslant W'_z$ for all $z \in \partial\Omega$.

Corollary 3.8. *Given $f \in \mathcal{C}(\overline{\Omega})$ there exists u, a γ-Hölder continuous viscosity solution of*

$$\begin{cases} F(x, \nabla u, D^2 u) + h(x) \cdot \nabla u |\nabla u|^\alpha = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.4}$$

with

$$|u(x)| \leqslant |f|_\infty^{\frac{1}{1+\alpha}} \sup(u_o(x), -u'_o(x)).$$

Furthermore if $f \leqslant 0$, $u \geqslant 0$, and if $f \geqslant 0$, $u \leqslant 0$.

Proof. Let $(z_j)_j$ be a sequence of solutions on H_j of

$$\begin{cases} F\big(x, \nabla z_j, D^2 z_j\big) + h(x)\cdot \nabla z_j |\nabla z_j|^\alpha = f & \text{in } H_j, \\ z_j = 0 & \text{on } \partial H_j. \end{cases}$$

By the comparison principle on H_j, $u_o' |f|_\infty^{\frac{1}{1+\alpha}} \leqslant z_j \leqslant u_o |f|_\infty^{\frac{1}{1+\alpha}}$. Using Proposition 3.6, the sequence $(z_j)_j$ is uniformly γ-Hölder continuous and then $(z_j)_j$ converges on every compact set in Ω to a solution z which is γ-Hölder continuous.

If $f \leqslant 0$, each z_j is non-negative, which implies that $z \geqslant 0$. Using the inequality

$$|z_j|_\infty \leqslant |f|_\infty^{\frac{1}{1+\alpha}} u_o$$

in H_j, one gets the final inequality by passing to the limit. □

Remark 3.9. Observe that the existence of u_o and z solutions of (3.1) and (3.5) can be done via Perron's method adapted to viscosity solutions. In particular choosing

$$u = \sup\big\{v,\ \text{subsolution of (3.5) satisfying},\ |f|_\infty^{\frac{1}{1+\alpha}} (W')^\star \leqslant v \leqslant |f|_\infty^{\frac{1}{1+\alpha}} W_\star\big\}$$

where $W_\star$ is the lower semi-continuous envelope of $\inf_{z\in\partial\Omega} W_z$ and $(W')^\star$ is the upper semi-continuous envelope of $\sup_{z\in\partial\Omega} W_z'$. (The definition of viscosity solution is then intended in the sense of semi-continuous viscosity solutions, see [3].)

Remark 3.10. When V is some continuous, bounded and non-positive function in Ω then u_o is a super-solution of

$$F\big(x, \nabla u_o, D^2 u_o\big) + h(x)\cdot \nabla u_o |\nabla u_o|^\alpha + V(x) u_o |u_o|^\alpha = -1.$$

This implies that for any $f \leqslant 0$ there exists u solution of

$$\begin{cases} F\big(x, \nabla u, D^2 u\big) + h(x)\cdot \nabla u |\nabla u|^\alpha + V(x) u |u|^\alpha = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.5}$$

and

$$u(x) \leqslant |f|_\infty^{\frac{1}{1+\alpha}} u_o(x).$$

Proposition 3.11. *The function u_o in Proposition 3.2 satisfies also: $\forall\delta$, there exists K, a compact set in Ω such that*

$$\sup_{\Omega\setminus\overline{K}} |u_o| \leqslant \delta.$$

Proof. For each $z \in \partial\Omega$ we know that

$$u_o \leqslant W_z \quad \text{in } \Omega.$$

Let $\delta > 0$ then for all $z \in \partial\Omega$ there exists r_z such that for $x \in B(z, r_z) \cap \Omega$,

$$W_z(x) \leqslant \delta.$$

Since $\partial\Omega$ is compact one can extract from $\cup B(z, r_z)$ a finite covering, say $\bigcup_{i\leqslant k} B(z_i, r_{z_i})$. Let then K be a compact set such that

$$\Omega \setminus K \subset \bigcup_{i\leqslant k} B(z_i, r_{z_i}).$$

We have

$$u_o \leqslant W = \inf_{z_i, i\leqslant k} W_{z_i},$$

and then $u_o \leqslant \delta$ in $\Omega \setminus K$. This ends the proof. □

Corollary 3.12. $\forall M > 0$, *there exists K some compact subset of Ω, large enough, such that*

$$\bar\lambda(\Omega \setminus K) > M.$$

Proof. Let δ be such that $(\frac{1}{\delta})^{1+\alpha} \geqslant M + |V|_\infty$, and let K be large enough in order that

$$\sup_{\Omega\setminus K} |u_o| \leqslant \delta.$$

Then

$$F[u_o] + h(x)\cdot \nabla u_o |\nabla u_o|^\alpha + \big(M + V(x)\big)u_o^{1+\alpha} = -1 + \big(M + V(x)\big)u_o^{1+\alpha} \leqslant 0$$

in $\Omega \setminus K$, and since u_o is positive one gets that $\bar\lambda(\Omega \setminus K) \geqslant M$. $\square$

3.3. Maximum principle

Definition 3.13. We shall say that $\limsup_{x\to\partial\Omega} w(x) \leqslant 0$ if for all $\epsilon > 0$ there exists K compact in Ω, large enough in order that $\sup_{\Omega\setminus K} w \leqslant \epsilon$.

Proposition 3.14. *Let $\beta(x,\cdot)$ be a nondecreasing continuous function such that $\beta(x,0) = 0$. Suppose that w is upper semicontinuous and bounded by above and satisfies*

$$F\big(x, \nabla w, D^2 w\big) + h(x)\cdot \nabla w |\nabla w|^\alpha - \beta(x, w) \geqslant 0$$

with

$$\limsup w(x_j) \leqslant 0$$

for all $x_j \to \partial\Omega$. Then $w \leqslant 0$ in Ω.

Remark 3.15. If β is increasing then the result holds without requiring any regularity on the bounded domain Ω. In that case one can use comparison principle in Proposition 2.5.

Proof. We assume by contradiction that $w > 0$ somewhere in Ω. Let $\bar x$ be a point in Ω such that $w(\bar x) > 0$, and let $\gamma > 0$ be such that $\gamma u_o(\bar x) < w(\bar x)$. The function

$$w - \gamma u_o$$

is upper semicontinuous, bounded by above and it admits a strictly positive maximum value in Ω. Indeed, let $\epsilon < \frac{w(\bar x)-\gamma u_o(\bar x)}{2}$. Let K be compact and large enough, in order that $\bar x \in K$ and such that $w(x) \leqslant \epsilon$ in $\Omega \setminus K$. Then $(w - \gamma u_o)(x) \leqslant \epsilon$ in $\Omega \setminus K$. As a consequence $w - \gamma u_o$ achieves its maximum inside K. The end of the proof is the same as in the case of regular domains.

Introduce $\psi_j(x, y) = w(x) - \gamma u_o(y) - \frac{j}{q}|x - y|^q$, then one can prove as in [2], that for j large enough, ψ_j achieves its maximum on (x_j, y_j) inside $\Omega \times \Omega$, (more precisely in $K \times K$), and that there exists (X_j, Y_j) in $\mathcal{S}^2$ such that

$$\big(j|x_j - y_j|^{q-2}(x_j - y_j), X_j\big) \in J^{2,+} w(x_j), \qquad \big(j|x_j - y_j|^{q-2}(x_j - y_j), -Y_j\big) \in J^{2,-} \gamma u_o(y_j).$$

Moreover one can choose $x_j \neq y_j$ for j large enough, as it is done in [2].

One has then using (H2), (H4) and the decreasing properties of β,

$$\begin{aligned}
0 &\leqslant F\big(x_j, j|x_j - y_j|^{q-2}(x_j - y_j), X_j\big) + h(x_j)\cdot |x_j - y_j|^{(q-1)(\alpha+1)-1}(x_j - y_j) - \beta\big(x_j, w(x_j)\big)\\
&\leqslant F\big(y_j, j|x_j - y_j|^{q-2}(x_j - y_j), -Y_j\big) + h(y_j)\cdot |x_j - y_j|^{(q-1)(\alpha+1)-1}(x_j - y_j) + o(1)\\
&\leqslant -\gamma^{1+\alpha} + o(1)
\end{aligned}$$

a contradiction since $\gamma > 0$. $\square$

4. Existence of an eigenfunction

We recall that V is some bounded and continuous function and that $\bar\lambda(\Omega)$ is defined as:

$$\bar\lambda(\Omega) = \sup\big\{\lambda,\ \exists\varphi > 0 \text{ in } \Omega,\ F[\varphi] + h(x)\cdot\nabla\varphi|\nabla\varphi|^\alpha + \big(V(x) + \lambda\big)\varphi^{1+\alpha} \leqslant 0\big\}.$$

Theorem 4.1. *Let Ω be a bounded domain which satisfies the uniform exterior cone condition, F satisfies conditions* (H1) *to* (H4) *and h satisfies* (H5). *There exists a positive function ϕ solution of*

$$\begin{cases} F\big(x, \nabla\phi, D^2\phi\big) + h(x)\cdot\nabla\phi|\nabla\phi|^\alpha + \big(V(x) + \bar\lambda(\Omega)\big)\phi^{1+\alpha} = 0 & \text{in } \Omega,\\ \phi = 0 & \text{on } \partial\Omega, \end{cases}$$

which is γ-Hölder continuous.

Proof. Let $(H_j)_j$ be a sequence of regular subsets of Ω, strictly increasing, with union Ω. One has for $\mu_j = \bar{\lambda}(H_j)$ the existence of an eigenfunction $\phi_j > 0$ in H_j, assume that $\sup \phi_j = 1$. Let $\mu = \lim \mu_j \geqslant \bar{\lambda}(\Omega)$. (Note that the sequence $(\mu_j)_j$ is decreasing.)

Since the ϕ_j are uniformly bounded, we can apply Proposition 3.6 with $f_j = (V(x) + \mu_j)\phi_j^{1+\alpha}$ and we obtain that the sequence $(\phi_j)_j$ is uniformly Hölder continuous. Up to a subsequence, ϕ_j converges to ϕ a non-negative solution of

$$F\big(x, \nabla\phi, D^2\phi\big) + h(x) \cdot \nabla\phi|\nabla\phi|^\alpha + \big(V(x) + \mu\big)\phi^{1+\alpha} = 0.$$

We have to prove that ϕ is not identically zero.

Let K_1 be a compact set of Ω, such that $\bar{\lambda}(\Omega \setminus K_1) > \mu_1 = \bar{\lambda}(H_1) > \bar{\lambda}(\Omega)$, this is possible according to Corollary 3.12.

Let δ be small enough in order that

$$\big(\bar{\lambda}(\Omega \setminus K_1) + |V|_\infty\big)\delta^{1+\alpha} < 1.$$

According to Proposition 3.11, there exists K_2, a compact regular set, such that $K_1 \subset K_2$ and $\sup_{\Omega\setminus K_2} u_o < \delta$.

Furthermore $\bar{\lambda}(H_j \setminus K_2) \geqslant \bar{\lambda}(\Omega \setminus K_1) > \bar{\lambda}(\Omega)$.

We observe that u_o satisfies in $\Omega \setminus K_2$ (hence also in $H_j \setminus K_2$):

$$F\big(x, \nabla u_o, D^2 u_o\big) + h(x) \cdot \nabla u_o|\nabla u_o|^\alpha + \big(\bar{\lambda}(\Omega \setminus K_1) + |V|_\infty\big)u_o^{1+\alpha} \leqslant 0$$

which implies in particular that

$$F\big(x, \nabla u_o, D^2 u_o\big) + h(x) \cdot \nabla u_o|\nabla u_o|^\alpha + \big(\bar{\lambda}(H_j) + V\big)u_o^{1+\alpha} \leqslant 0.$$

On $\partial(H_j \setminus K_2)$,

$$\phi_j \leqslant 1 \leqslant \frac{1}{\inf_{K_2} u_o} u_o,$$

hence using the comparison principle Theorem 2.7 on the set $H_j \setminus K_2$, one gets that

$$\phi_j \leqslant \frac{1}{\inf_{K_2} u_o} u_o \quad \text{in } H_j \setminus K_2. \tag{4.1}$$

Let K_3 which contains K_2 such that in $\Omega \setminus K_3$, $u_o \leqslant \frac{\inf_{K_2} u_o}{2}$, then

$$\phi_j \leqslant \frac{1}{2} \quad \text{in } \Omega \setminus K_3.$$

This implies that $\sup_{K_3} \phi_j = 1$, and hence $\sup \phi = 1$.

In particular we have obtained that ϕ is not zero and by the strict maximum principle $\phi > 0$ in Ω. Furthermore $\mu \leqslant \bar{\lambda}(\Omega)$ and hence $\mu = \bar{\lambda}(\Omega)$.

Passing to the limit in (4.1) we also get that ϕ is zero on the boundary. □

5. Other maximum principle and eigenvalues

In all the results of this section we still assume that Ω satisfies the uniform exterior cone condition and F and h satisfy (H1) to (H5).

We define

$$\lambda_e = \sup\big\{\bar{\lambda}(\Omega'),\ \Omega \Subset \Omega',\ \Omega' \text{ is } C^2 \text{ and bounded}\big\}$$

and

$$\tilde{\lambda} = \sup\big\{\lambda,\ \exists\varphi > 0 \text{ in } \overline{\Omega},\ F[\varphi] + h(x) \cdot \nabla\varphi|\nabla\varphi|^\alpha + \big(V(x) + \lambda\big)\varphi^{1+\alpha} \leqslant 0\big\}.$$

In this section we are going to prove that $\lambda_e = \tilde{\lambda}$ and that it is an "eigenvalue" in the sense that there exists some $\phi_e > 0$, which satisfies

$$\begin{cases} F\big(x, \nabla\phi_e, D^2\phi_e\big) + h(x) \cdot \nabla\phi_e|\nabla\phi_e|^\alpha + \big(V(x) + \lambda_e\big)\phi_e^{1+\alpha} = 0 & \text{in } \Omega, \\ \phi_e = 0 & \text{on } \partial\Omega. \end{cases}$$

Observe that by definition $\lambda_e \leqslant \bar{\lambda}$, and furthermore if Ω is smooth, the equality holds. When Ω is not smooth it is not known if the two eigenvalues are the same even though we expect this to be true, comforted in this thought by the remark at the end of this section. The identity of these values is equivalent to the existence of a maximum principle for $\lambda < \bar{\lambda}$.

Let us start with the following maximum principle:

Proposition 5.1. *For $\lambda < \tilde{\lambda}$, if w is a subsolution of*

$$F\big(x, \nabla w, D^2 w\big) + h(x) \cdot \nabla w |\nabla w|^\alpha + \big(V(x) + \lambda\big) w^{1+\alpha} \geqslant 0$$

satisfying

$$w(x) \leqslant 0 \quad \text{on } \partial\Omega,$$

then $w \leqslant 0$ in Ω.

Sketch of the proof: Let $\varphi > 0$ on $\overline{\Omega}$, such that

$$F[\varphi] + h(x) \cdot \nabla\varphi |\nabla\varphi|^\alpha + \big(V(x) + \lambda\big)\varphi^{1+\alpha} \leqslant 0.$$

Suppose that $w > 0$ somewhere, since $\varphi > 0$ on $\overline{\Omega}$ one can define $\gamma' = \sup_{x\in\Omega} \frac{w}{\varphi}$ and follow the proof of [4] to derive a contradiction.

Remark 5.2. Observe that for $\lambda < \bar{\lambda}$, we do not know if the maximum principle holds when Ω is not smooth because for supersolutions satisfying $\varphi > 0$ in Ω we do not know if $\sup_{x\in\Omega} \frac{w}{\varphi}$ is bounded.

Proposition 5.3. *There exists $\phi_e > 0$ which satisfies*

$$\begin{cases} F[\phi_e] + h(x) \cdot \nabla\phi_e |\nabla\phi_e|^\alpha + \big(V(x) + \lambda_e\big)\phi_e^{1+\alpha} = 0 & \text{in } \Omega, \\ \phi_e = 0 & \text{on } \partial\Omega. \end{cases}$$

Proof. Let $(\Omega_j)_j$ be a decreasing sequence of regular open bounded domains which contain Ω. Let ϕ_j be some positive eigenfunction for Ω_j such that $|\phi_j|_\infty = 1$, which exists according to the results in [4].

Using the comparison principle Proposition 2.5, one has that for all $z \in \partial\Omega_j$

$$\phi_j \leqslant \big(|V|_\infty + \bar{\lambda}(\Omega_j)\big) W_z^j$$

where W_z^j is a global barrier for Ω_j. As in Remark 3.4, W_z^j satisfies

$$W_z^j(x) \leqslant C|x - z|^\gamma$$

with C independent of j and z, since the Ω_j converge to Ω which satisfies the uniform exterior cone condition. This implies that for $\epsilon > 0$ there exists K compact in Ω, large enough in order that

$$\sup_j \sup_{\Omega_j \setminus K} \phi_j \leqslant \epsilon.$$

In particular ϕ_j has the property that if $d(K, \partial\Omega_j) < (\frac{\epsilon}{C})^{\frac{1}{\gamma}}$,

$$\sup_{\Omega_j \setminus K} \phi_j(x) \leqslant \epsilon.$$

Let K be a compact set in Ω such that $d(K, \partial\Omega) < \epsilon$. Since the distance is continuous, for j large enough $d(K, \partial\Omega_j) < \epsilon$ and then

$$\sup_{\Omega_j \setminus K} \phi_j(x) \leqslant \epsilon.$$

In particular one can take a compact K large enough in Ω in order that

$$\sup_{\Omega_j \setminus K} \phi_j \leqslant \frac{1}{2}$$

and then the supremum of ϕ_j is achieved in K.

By the uniform estimates in Proposition 3.6 the sequence $(\phi_j)_j$ is uniformly γ-Hölder on K and one can then extract from $(\phi_j)_j$ a subsequence such that ϕ_j converges to some function ϕ_e which is such that $|\phi_e|_{L^\infty(K)} = 1$. By compacity ϕ_e is a solution of

$$F\big(x, \nabla\phi_e, D^2\phi_e\big) + h(x) \cdot \nabla\phi_e |\nabla\phi_e|^\alpha + \big(V(x) + \lambda_e(\Omega)\big)\phi_e^{1+\alpha} = 0 \quad \text{in } \Omega.$$

Moreover $\phi_e > 0$ in Ω, and the estimate

$$\phi_j \leqslant C \inf_{z\in\partial\Omega_j} W_z^j$$

gives, by passing to the limit, that $\phi_e = 0$ on the boundary of Ω. $\square$

Corollary 5.4.

$$\lambda_e = \tilde{\lambda}.$$

Proof. Suppose by contradiction that $\lambda_e < \tilde{\lambda}$, then by the maximum principle one would obtain that $\phi_e \leqslant 0$. □

We now present some existence result for the Dirichlet problem.

Proposition 5.5. *Let $\lambda < \lambda_e$ then for any function $f \leqslant 0$ and continuous there exists u a viscosity solution of*

$$\begin{cases} F(x, \nabla u, D^2 u) + h(x) \cdot \nabla u |\nabla u|^\alpha + (V(x) + \lambda) u^{1+\alpha} = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.1}$$

Furthermore $u \geqslant 0$ and it is γ-Hölder continuous.

Proof. For $K = 2|V|_\infty + |\lambda|$, let u_n be the sequence of solutions of

$$\begin{cases} F[u_{n+1}] + h(x) \cdot \nabla u_{n+1} |\nabla u_{n+1}|^\alpha + (V(x) + \lambda - K) u_{n+1}^{1+\alpha} = f - K u_n^{\alpha+1} & \text{in } \Omega, \\ u_{n+1} = 0 & \text{on } \partial\Omega, \end{cases}$$

with $u_1 = 0$; u_n exists by Remark 3.10. The sequence $(u_n)_n$ is increasing by the comparison principle in Proposition 2.5. Arguing as in [3] one can prove that the sequence is bounded, using the maximum principle of Proposition 3.14. Furthermore there exists a constant C such that

$$u_n \leqslant C u_o.$$

Passing to the limit, which we can do thanks to the Hölder's regularity given in Proposition 3.6, we get the required solution. □

Remark 5.6. The validity of the maximum principle for $\lambda < \bar{\lambda}(\Omega)$ is equivalent to $\lambda_e = \bar{\lambda}(\Omega)$ and to the existence of a solution for the Dirichlet problem (5.1) for any $\lambda < \bar{\lambda}(\Omega)$.

We present a rather large class of operators and domains for which one has $\lambda_e = \bar{\lambda}$.

Proposition 5.7. *Suppose that Ω is some bounded domain having the cone property and starshaped with respect to some point x_o. Suppose in addition that F, h, and V do not depend on x. Then $\lambda_e = \bar{\lambda}$.*

Proof. Let Ω'_t be some open smooth domain such that

$$\Omega \Subset \Omega'_t \Subset \Omega_t.$$

Applying the homotetie on the super-solutions that define $\bar{\lambda}(\Omega)$ it is easy to see that $\bar{\lambda}(\Omega_t) = (\frac{1}{t})^{\alpha+2} \bar{\lambda}(\Omega)$. Furthermore

$$\lambda_e(\Omega) \geqslant \bar{\lambda}(\Omega'_t) \geqslant \bar{\lambda}(\Omega_t) = \left(\frac{1}{t}\right)^{\alpha+2} \bar{\lambda}(\Omega)$$

for any $t > 1$. Hence $\lambda_e(\Omega) \geqslant \bar{\lambda}(\Omega)$. This ends the proof. □

References

[1] H. Berestycki, L. Nirenberg, S.R.S. Varadhan, The principal eigenvalue and maximum principle for second-order elliptic operators in general domains, Comm. Pure Appl. Math. 47 (1994) 47–92.

[2] I. Birindelli, F. Demengel, Comparison principle and Liouville type results for singular fully nonlinear operators, Ann. Fac. Sci. Toulouse Math. 13 (2004) 261–287.

[3] I. Birindelli, F. Demengel, Eigenvalue and maximum principle for fully nonlinear singular operators, Adv. Partial Differ. Equ. (Basel) 11 (2006) 91–119.

[4] I. Birindelli, F. Demengel, Eigenvalue, maximum principle and regularity for fully nonlinear homogeneous operators, Commun. Pure Appl. Anal. 6 (2007) 335–366.

[5] I. Birindelli, F. Demengel, The Dirichlet problem for singular fully nonlinear operators, Discrete Contin. Dyn. Syst. (Special vol.) (2007) 110–121.

[6] J. Busca, M.J. Esteban, A. Quaas, Nonlinear eigenvalues and bifurcation problems for Pucci's operator, Ann. Inst. H. Poincaré Anal. Non Linéaire 22 (2005) 187–206.

[7] L. Caffarelli, X. Cabré, Fully-Nonlinear Equations, Colloquium Publications, vol. 43, American Mathematical Society, Providence, RI, 1995.

[8] M.G. Crandall, M. Kocan, P.L. Lions, A. Swiech, Existence results for boundary problems for uniformly elliptic and parabolic fully-nonlinear equation, Electron. J. Differential Equations 24 (1999) 1–20.

[9] H. Ishii, Y. Yoshimura, Demi-eigen values for uniformly elliptic Isaacs operators, preprint.

[10] H. Ishii, P.L. Lions, Viscosity solutions of fully-nonlinear second order elliptic partial differential equations, J. Differential Equations 83 (1990) 26–78.

[11] P. Juutinen, Principal eigenvalue of a very badly degenerate equation, J. Differential Equations 236 (2007) 532–550.

[12] P.-L. Lions, Bifurcation and optimal stochastic control, Nonlinear Anal. 7 (1983) 177–207.
[13] K. Miller, Barriers on cones for uniformly elliptic operators, Ann. Mat. Pura Appl. 76 (1967) 93–106.
[14] K. Miller, Extremal barriers on cones with Phragmèn–Lindelöf theorems and other applications, Ann. Mat. Pura Appl. 90 (1971) 297–329.
[15] S. Patrizi, The Neumann problem for singular fully nonlinear operators, J. Math. Pure Appl., in press.
[16] A. Quaas, B. Sirakov, Existence results for non-proper elliptic equations involving the Pucci operator, Comm. Partial Differential Equations 31 (2006) 987–1003.
[17] A. Quaas, B. Sirakov, On the principal eigenvalues and the Dirichlet problem for fully nonlinear operators, C. R. Math. Acad. Sci. Paris 342 (2006) 115–118.
[18] A. Quaas, B. Sirakov, On the principal eigenvalues and the Dirichlet problem for fully nonlinear operators, Adv. Math. 218 (1) (2008) 105–135.