Structure of Solvable Lie Groups

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CHAPTER I. SPLITTING THEOREMS

1. Introduction.

In this paper we give an account of some of the techniques which lie at the foundation of solvable Lie group theory. Our main idea will be to associate to certain solvable Lie groups their semi-simple splitting. The semi-simple splitting will be groups of the form $N \cdot T$ where $N$ is a nilpotent group and $T$ is an abelian group. This idea has been exploited previously in [1, 2 and 12]. Our methods and definitions owe a great deal to these papers, especially in sections one through four. Our methods will include both [1] and [2] as special cases and show how their apparent differences are superficial. We will after these general considerations introduce the concept of the integer semi-simple splitting which is the basic tool used in [8]. In this setting we prove the nil-shadow theorem from which the results in the fundamental paper of G. D. Mostow [9] can be derived. We have also proved by these methods the following conjecture of Mostow: every solv-manifold, i.e., a quotient space of a solvable analytic group by a closed subgroup, is a vector bundle over a compact solv-manifold. The proofs, using these methods, will appear in [12]. We can also use these methods to prove quite simply the main results in [10, 11].

2. Preliminaries.

We shall assume that the reader is familiar with the theory of abelian Lie groups as developed in [6], and with the theory of nilpotent groups as developed in [3] and [4]. Since the results of these papers will play an important role we have included a brief description of their content.

By a vector group we mean the additive group of a real vector space. The following results can be found in [6].

1. Let $C$ be a closed subgroup of a vector group $V$. Then there is a vector space decomposition $V = X \oplus Y \oplus Z$, where $C = X \oplus (C \cap Y)$ and $C \cap Y$ is the free abelian group generated by an $R$-basis of $Y$. 

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2. By an elementary group we mean any group of the form $C'/C$ where $C$ and $C'$ are closed subgroups of some vector group $V$ with $C \subseteq C'$. An elementary group can be characterized as a group that can be written as the direct product of a vector group, a toroid, and a discrete finitely generated abelian group. Let $G = R^s \oplus Z^t \oplus K \oplus F$ be an elementary abelian group where $K$ is a compact connected abelian group, $F$ is a discrete finite group, $R$ denotes the real numbers and $Z$ denotes the integers. Then by the rank of $G$ we will mean $s + t$.

By a real nilpotent group we mean a simply connected nilpotent analytic group. Malcev [3] generalized the results about abelian Lie groups to nilpotent Lie groups. We will call a closed subgroup of a real nilpotent group a $CN$-group. When the group is a discrete subgroup of a real nilpotent group we will call it an $FN$-group.

1. A $CN$-group can be characterized as a torsion free nilpotent Lie group whose component group is finitely generated.

2. If $C$ is a $CN$-group then there exists a unique real nilpotent group $N(C)$, called the Lie hull of $C$, containing $C$ as a closed co-compact subgroup, i.e. $N(C)/C$ is compact. Moreover every automorphism $A$ of $C$ extends uniquely to an automorphism $A_N$ of $N(C)$. If $C$ is a closed subgroup of the real nilpotent group $N$ then $N(C) \cap C$ is a subgroup of $N$.

3. Let $C$ be a discrete co-compact subgroup of the real nilpotent group $N$. Then there exists a rational canonical basis $e_1, \ldots, e_n$ of the Lie algebra $L(N)$ of $N$ such that $\exp(\sum r_i e_i : r_i \in \mathbb{Q})$ is a subgroup of $N$ containing $C$, and $\exp(\sum n_i e_i : n_i \in \mathbb{Z})$ is a subgroup of $N$ containing $C$ as a subgroup of finite index. Subgroups of $N$ of the form $\exp(\sum r_i e_i : r_i \in \mathbb{Q})$ will be called rational nilpotent groups. Subgroups of $N$ of the form $\exp(\sum n_i e_i : n_i \in \mathbb{Z})$ will be called lattice nilpotent groups.

We can also prove the following result.

**Lemma 1.** Let $N$ be a real nilpotent group with Lie algebra $L(N)$. Let $L_k$ be the $k$th step in the lower central series of $L(N)$. Then if $e_1, \ldots, e_k, \ldots, e_{r+1}, \ldots, e_{r+t}$ is a rational canonical basis of $L$ where $e_{k+1}, \ldots, e_{r+1}$ is a basis of $L_{k+1}$, there exists integers $m_1 = 1, m_2, \ldots, m_{r+1}$ such that $\exp(\gamma)$ is a subgroup of $N$, where $\gamma$ is the lattice in $L(N)$ generated by $e_1, \ldots, e_k, \ldots, e_{r+1} | m_{r+1}, \ldots, e_{r+1} | m_{r+1}$.

**Proof.** We will use induction on the steps of nilpotency of $N$. If $N$ is abelian there is nothing to prove. Assume the lemma for all $k$-step nilpotent groups where $k \leq r$ and let $N$ be an $r + 1$ step nilpotent group. Denote the
natural map of $L(N)$ onto $L(N/N_{r+1})$ by $p$. Clearly $p(e_1), \ldots, p(e_r)$ is a rational canonical basis of $L(N/N_{r+1})$. By our induction hypothesis there are integers $m_1 = 1, m_2, \ldots, m_r$ such that where $\gamma$ is the lattice in $L(N/N_{r+1})$ generated by $p(e_1), \ldots, p(e_{r+1})$, we have that $\exp(\gamma)$ is a subgroup of $N/N_{r+1}$. Let $f_i = e_1, \ldots, f_{i-1} = e_i/m_i, f_{i+1} = e_{i+1}, \ldots, f_{r+1} = e_{r+1}$. By the Hausdorff–Campbell formula $\exp(\sum r_i f_i) \exp(\sum s_i f_i) = \exp(\sum t_i f_i)$ where $t_i = q_i(r_i, \ldots, r_{i-1}; s_i, \ldots, s_{i-1})$ and $q_i(X_1, \ldots, X_{i-1}; Y_1, \ldots, Y_{i-1})$ is a polynomial whose coefficients are rational polynomials in the structure constants of $f_i, \ldots, f_{r+1}$. Therefore $q_i$ is a rational polynomial since we have assumed the structure constants to be rational. Choose an integer $m$ such that $mq_i$ is an integer polynomial. By our induction hypothesis for any $r_i$ and $s_i$ in $Z$ we have that $\exp(y)$ is a subgroup of $N$, where $y$ is the lattice in $L(N)$ generated by $f_i, 1 \leq i \leq i_r$, and $f_{i+1}/m_i, i_r + 1 \leq j \leq i_{r+1}$ we have that $\exp(y)$ is a subgroup of $N$.

The fundamental result of [4] gives us a faithful matrix representation $\beta : N \to U(n, R)$, where $U(n, R)$ is the algebraic group of all upper triangular unipotent $n$ by $n$ matrices with real coefficients. It can also be proved that $\beta(N)$ is an algebraic subgroup of $U(n, R)$. Where $N_Q$ is a rational nilpotent subgroup of $N$, $\beta$ takes $N_Q$ onto an algebraic subgroup of $U(n, Q)$. Let $T$ be an abelian group of semi-simple automorphisms of $N$. We can extend $\beta$ to a faithful matrix representation, $\beta' : N \cdot T \to GL(n, R)$ where $\beta'(T)$ consists of semi-simple matrices. Let $X$ be a matrix subgroup of $GL(n, R)$. We denote the algebraic hull of $X$ over the reals by $\mathcal{A}_R(X)$. It is simple to show that $\beta(N)$ is the group of all unipotent matrices in $\mathcal{A}_R(\beta(N \cdot T))$ and that $\beta(T)$ is contained in a maximal completely reducible subgroup of $\mathcal{A}_R(\beta(N \cdot T))$.

**Lemma 2.** Using the notation above, $\beta(N \cdot T)$ contains the unipotent and semi-simple parts of each of its elements.

**Proof.** Let $x \in \beta(N \cdot T)$. Since an algebraic group contains the unipotent part of each of its elements, the unipotent part of $x$ is contained in $\beta(N)$. The lemma follows immediately from this fact.

Let $G$ be a simply connected analytic group. We denote the Lie algebra of $G$ by $L(G)$. It is a well known fact that the differential defines a group isomorphism from $\text{Aut}(G)$ onto $\text{Aut}(L(G))$. Given an automorphism $A$ of $G$ we will denote the differential of $A$ by $A_L$. We will say that an automorphism $A$ of $G$ is unipotent if and only if $A_L$ is a unipotent linear transformation of $L(G)$ considered as a vector space. We will say that an automorphism $A$ of $G$
is semi-simple if and only if \( A_L \) is semi-simple. In general every automorphism of a Lie algebra \( L \) can be written uniquely as the commuting product of a semi-simple automorphism of \( L \) with a unipotent automorphism of \( L \). Hence every automorphism \( A \) of \( G \) can be written uniquely as the commuting product of a semi-simple automorphism with a unipotent automorphism of \( G \). Let \( A = U \cdot S \) where \( U \) is a unipotent automorphism and \( S \) is a semi-simple automorphism such that \([U, S] = 1\). The decomposition \( A = U \cdot S \) will be called the canonical decomposition of \( A \). Let \( O \) be a group of automorphisms of \( G \). We will denote the induced group of automorphisms of \( L(G) \) by \( O_L \). We will say that \( O \) is an algebraic group if and only if \( O_L \) is an algebraic group. In particular since \( \text{Aut}(L(G)) \) is an algebraic group, \( \text{Aut}(G) \) is an algebraic group. Let \( g \in G \). We will denote the automorphism of \( G \) given by \( h \mapsto ghg^{-1} \) for \( h \in G \) by \( \text{ad}_g \). We will refer to \( \text{ad}_g \) as the adjoint action of \( g \) on \( G \). When \( H \) is a subgroup of \( G \), normalized by the element \( g \) then we will denote the restriction of \( \text{ad}_g \) to \( H \) by \( \text{ad}_h \). The differential of \( \text{ad}_g \) is denoted by \( \text{Ad}_g \).

3. The \( \epsilon \)-Category.

We now introduce the basic object of our study, the \( \epsilon \)-category. The objects of the \( \epsilon \)-category are short exact sequences of the form

\[
1 \rightarrow N \rightarrow C \xrightarrow{\rho} A \rightarrow 1,
\]

where \( N \) is a real nilpotent group and \( A \) is an elementary abelian group. We will denote this short exact sequence by \((C, \rho)\). When we speak about an \( \epsilon \)-group \( C \) it will be understood that we have associated with the group \( C \) a definite short exact sequence \( 1 \rightarrow N \rightarrow C \xrightarrow{\rho} A \rightarrow 1 \). Let \((C, \rho)\) and \((C', \rho')\) be two \( \epsilon \)-objects. A morphism \( \phi : (C, \rho) \rightarrow (C', \rho') \) is a homomorphism \( \phi : C \rightarrow C' \) such that \( \phi(\ker \rho) \subseteq \ker \rho' \). The rank of a group \( C \) satisfying a short exact sequence of the form

\[
1 \rightarrow N \rightarrow C \xrightarrow{\rho} A \rightarrow 1,
\]

where \( N \) is a real nilpotent group and \( A \) is an elementary abelian group is given by \( \text{rank}(C) = \dim(N) + \text{rank}(A) \).

It is important to note that all we do can also be carried out in the category of short exact sequences \((C, \rho)\) where \( \ker \rho \) is a rational nilpotent group and \( C/\ker \rho \) is \( Z^s \) for some integer \( s \).

**Example 1.** A group \( C \) may satisfy several short exact sequences of the form \( 1 \rightarrow N \rightarrow C \xrightarrow{\rho} A \rightarrow 1 \), where \( N \) is a real nilpotent group and \( A \) is an elementary abelian group. Consider the group \( C = R \oplus S \), where \( R \) denotes
the real numbers and $S$ is the circle group. Clearly $C$ satisfies the short exact sequence $1 \rightarrow R \rightarrow C \rightarrow S \rightarrow 1$. Let us denote the points of $C$ by $(r, e^{it})$ where $r, t \in R$. Consider the real abelian subgroup of $C$ given by all those points $x$ that can be written as $x = (r, e^{it})$ for $r \in R$. Denote this group by $R'$. We have $C = R' \oplus S$. Hence $C$ satisfies the short exact sequence

$$1 \rightarrow R' \rightarrow C \rightarrow S \rightarrow 1.$$ 

Let $(C, \rho)$ be an $\epsilon$-object. Denote the commutator subgroup of $C$ by $[C, C]$. Since $[C, C] \subseteq N = \ker \rho$, there exists in $N$ a unique real nilpotent group containing the closure of $[C, C]$ as a closed co-compact subgroup. Denote this group by $[C, C]^\#$. $[C, C]^\#$ does not only depend on $C$ but also on the short exact sequence associated with $C$.

**Example 2.** Let $H$ be the manifold $R^3 \times Z^2$, with the following group multiplication

$$(r, s, t, u)(w, x, y, z) = (r + w + tz, s + x, t + y, u + z).$$

Let $D = \{(0, q, 0, 0) : q \in Z\}$. Then $D$ is a discrete central subgroup of $H$, and $H/D$ is an $\epsilon$-group in the following two ways:

$$1 \rightarrow DN_1/D \rightarrow H/D \rightarrow H/DN_1 \rightarrow 1,$n

$$1 \rightarrow DN_2/D \rightarrow H/D \rightarrow H/DN_2 \rightarrow 1,$$

where $N_1 = \{(r, r, 0, 0) : r \in R\}$ and $N_2 = \{(r, 000) : r \in R\}$. Also $[H, H] = \{(p, 000) : p \in Z\}$. It directly followings that $[H/D, H/D]^\# = DN_1/D$ relative to the first short exact sequence and that $[H/D, H/D]^\# = DN_2/D$ relative to the second short exact sequence.

Let $1 \rightarrow N \rightarrow C \rightarrow A \rightarrow 1$ be an $\epsilon$-group.

**Definition 2.** By the nil-radical of $C$ we mean the collection of all elements of $C$ whose adjoint action on $[C, C]^\#$ is by unipotent automorphisms. Denote the nil-radical of $C$ by $\text{nil}(C)$.

Let $D$ be a group satisfying a short exact sequence of the form

$$1 \rightarrow M \rightarrow D \rightarrow B \rightarrow 1$$

where $M$ is a $CN$-group and $B$ is abelian. It is easy to show that $D$ is nilpotent if and only if the adjoint action of $D$ on $M$ is by unipotent automorphisms. Hence we can characterize $\text{nil}(C)$ as the unique maximal nilpotent normal subgroup of $C$ containing $[C, C]^\#$. Since $\text{nil}(C)/N$ is a closed subgroup of the elementary group $C/N$ it follows that it is also an elementary group. We now prove the following generalization of Lemma 9.2 of Wang [2].
Lemma 3. Let $N$ be a nilpotent $e$-group. Then we can write

$$N = M \oplus K$$

where $M$ is a $CN$-group and $K$ is a compact central subgroup of $N$. Moreover any surjective homomorphism $\alpha : N \to N$ has a finite kernel contained in $K$.

Proof. Let $N$ satisfy the short exact sequence

$$1 \to P \to N \to A \to 1,$$

where $P$ is a real nilpotent group and $A$ is an elementary group. Write $A = R^t \oplus Z^t \oplus K'$, where $K'$ is a compact group. Since a compact extension of a real nilpotent group splits, the short exact sequence

$$1 \to P \to \rho^{-1}(K') \to K' \to 1$$

splits. Hence there is a compact subgroup $K$ of $N$ such that $P \cap K = (1)$ and $\rho(K) = K'$. Let $M = \rho^{-1}(R^t \oplus Z^t)$. $M$ is a $CN$ group and $N = M \cdot K$. Since $N$ is nilpotent $\text{ad}_{N(M)}K$ consists of unipotent automorphisms. However since $K$ is compact, $\text{ad}_{N(M)}K$ is completely reducible. Therefore $K$ is a central subgroup of $N$.

Let $\alpha : N \to N$ be a surjective homomorphism of $N$. There is no loss in generality assuming $M$ to be a real nilpotent group. Let $p_1 : N \to M$ be the projection of $N$ onto $M$ relative to the representation $N = M + K$. Let $p_2 : N \to K$ be the projection of $N$ onto $K$ relative to the representation $N = M + K$. Since $K$ is a central subgroup of $N$, $p_1$ is a homomorphism. Hence $p_1(\alpha(K))$ is a compact subgroup of $M$. Since $M$ is a real nilpotent group, $p_1 \circ \alpha : M \to M$ is an isomorphism. Hence $\alpha | M : M \to M$ is an automorphism of $M$, $\alpha(K) = K$, and $N = \alpha(M) + K$. Take $x \in \ker \alpha$. Clearly $x \in K$. Therefore $\ker \alpha$ is a discrete subgroup of $K$. Since $K$ is compact we have that $\ker \alpha$ is finite.

Example 3. The following simple example shows that a surjective homomorphism need not be an isomorphism. Let $S^1$ be the circle group. Represent in the usual way the elements of $S^1$ by $e^{it}$ where $t \in R$. Let $\alpha : S^1 \to S^1$ be given by $\alpha(e^{it}) = e^{2it}$. Then $\alpha$ is a surjective homomorphism of $S^1$ with finite kernel.

Corollary 1. Let $C$ be an $e$-group satisfying the short exact sequence

$$1 \to N \to C \to A \to 1$$
where $N$ is a real nilpotent group and $A$ is an elementary group. Then \( \text{nil}(C) = M \oplus K \), where $M$ is a CN group and $K$ is a compact central subgroup of $C$.

**Proof.** By Lemma 3, we can write \( \text{nil}(C) = M \oplus K \), where $M$ is a CN group and $K$ is a compact central subgroup of \( \text{nil}(C) \). We must show that $K$ is central in $C$. Take \( k \in K \) with finite order and take any $c \in C$. Since $C/N$ is abelian we can write $\text{ad}(c) = \nu c$ where $\nu$ is in $N$. Then $\nu$ has torsion. But $N$ is torsion free, hence $\nu = 1$. But the set of all elements of $K$ with finite order is dense in $K$. Hence $K$ is a central subgroup of $C$.

Let $(C, \rho)$ and $(C', \rho')$ be $\epsilon$-groups.

**Definition 3.** An $\epsilon$-homomorphism $\alpha : C \to C'$ is called an $\epsilon^\#$-homomorphism if and only if $\alpha(\text{nil}(C)) \subset \text{nil}(C')$.

Not every $\epsilon$-homomorphism is an $\epsilon^\#$-homomorphism as the following example shows. Let

\[
N = \begin{pmatrix}
1 & 0 & 0 & x \\
0 & 1 & 0 & y \\
0 & 0 & 1 & z \\
0 & 0 & 0 & 1
\end{pmatrix}
\quad \text{for} \quad x, y, z \in \mathbb{R}.
\]

Let

\[
T = \begin{pmatrix}
\cos t & \sin t & 0 & 0 \\
-\sin t & \cos t & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.
\]

Form $S = N \cdot T$. Clearly the short exact sequence

\[
1 \to N \to S \to T \to 1
\]

is an $\epsilon$-group since $N$ is a vector group and $T$ is the circle group. Then

\[
[S, S]^\# = \begin{pmatrix}
1 & 0 & 0 & x \\
0 & 1 & 0 & y \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.
\]

Also the short exact sequence

\[
1 \to T \to T \to T \to 1
\]
is trivially an $e$-group, and the injection $T \to S$ is an $e$-homomorphism. We have that nil($T$) = $T$, and nil($S$) = $N$. Hence the injection $T \to S$ is not an $e^*$-homomorphism.

Clearly the collection of $e^*$-groups and $e^*$-homomorphisms forms a category. We will call this category the $e^*$-category.

4. Linear Algebra on $e$-Groups.

Let $C$ be an $e$-group.

**Definition 4.** $\text{Aut}_e(C)$ is the group of all automorphisms of $C$ which induce the identity on $C/[C, C]^e$. An automorphism $t$ in $\text{Aut}_e(C)$ is called semi-simple if the restriction of $t$ to $[C, C]^e$ is semi-simple and there is a section of $C$ modulo $[C, C]^e$ on which $t$ acts as the identity mapping. We shall say that such a section belongs to $t$. An automorphism $u$ in $\text{Aut}_e(C)$ is called unipotent if the restriction of $u$ to $[C, C]^e$ is unipotent. A canonical decomposition of an automorphism $\alpha$ in $\text{Aut}_e(C)$ is the writing of $\alpha$ as the commuting product of a semi-simple automorphism and a unipotent automorphism. The following lemma generalizes Lemma 6.1 of Wang [2].

**Lemma 4.** The canonical decomposition of an automorphism $\alpha$ of $C$ in $\text{Aut}_e(C)$ if it exists is unique.

Proof. Let $\alpha = us$ and $\alpha = u's'$ be two canonical decompositions of $\alpha$. Take section $f : A \to C$ and $f' : A \to C$ belonging to $s$ and $s'$ respectively. The vector space version of the lemma gives us that $s$ and $s'$ are equal on $[C, C]^e$. Choose any $a \in A$. Write $c = f(a)$ and $c' = f'(a)$. We have $c' = wc$ where $w \in [C, C]^e$. Since $\alpha$ induces the identity on $C/[C, C]^e$ we have $\alpha(c) = wc$ where $w \in [C, C]$. By definition $s(c') = c'$ and $s(c) = c$. Hence $s(c') c'^{-1} = s(w) c c^{-1} w^{-1} = s(w) w^{-1}$. Since $s$ and $s'$ are equal on $[C, C]^e$, we get that $s$ commutes with $adc'$ on $[C, C]^e$. Hence $s(c') c'^{-1}$ centralizes $[C, C]^e$. The commuting of $\alpha$ and $s$ gives $s\alpha(c) = s(v) c = s(s(c)) = s(c) = wc$. Thus $s(v) = v$. The commuting of $\alpha$ and $s'$ gives

$$\alpha'(c) = \alpha(s(w^{-1})w) \alpha(c) = s'(s(c)) = s(v) s'(c) = vs(w^{-1}) wc = s(w^{-1}) vs(c).$$

Therefore $\alpha$ fixes $s(w) w^{-1}$ as does the semi-simple part of $\alpha$. Hence identifying $[C, C]^e$ with its Lie algebra $(s - I)^2 w = 0$, where $I$ is the identity automorphism on $L([C, C]^e)$. Since $s$ is semi-simple, $s(w) = w$ and $s(c) = c$. Thus $s = s'$ on $C$.

5. Almost Algebraic Lie Groups.

In this section we begin developing the machinery for studying $e$-groups. Our plan is the following. We define a sub-category of the $e^*$-category and
a map from the $\varepsilon^*$-category to this subcategory. Under this map homomorphisms in the $\varepsilon^*$-category induce homomorphisms in the subcategory. However the induced homomorphism is not uniquely determined. A suitable restriction of the subcategory forces uniqueness. Under this restriction the map is a functor.

Let $(C, \rho)$ be an $\varepsilon$-group. Let $C'$ be a normal $\varepsilon$-subgroup of $C$. An element $c \in C$ will be called $\text{ad-}C'$ reductive if $\text{ad}_{C'}c$ is semi-simple. An element $c \in C$ will be called $\text{ad-}C'$ unipotent if $\text{ad}_{C'}c$ is unipotent. A subset of $C$ each of whose elements are $\text{ad-}C'$ reductive will be called $\text{ad-}C'$ reductive.

**Definition 5.** An $\varepsilon$-group $E$ is called almost algebraic if and only if there exists an $\text{ad-}E$ reductive subgroup $T$ of $E$ such that $E = N \cdot T$ where $N = \text{nil}(E)$. We call $T$ a Malcev factor of $E$ and the decomposition of $E$ into a semi-direct product $\text{nil}(E) \cdot T$ a Malcev decomposition of $E$. (Note that $T$ acts faithfully on $\text{nil}(E)$ under the adjoint map since $T \cap \text{nil}(E) = (1)$.) The projection of $E$ onto $\text{nil}(E)$ relative to the decomposition

$$E = \text{nil}(E) \cdot T$$

is denoted by $n_T$. The projection of $E$ onto $T$ relative to the decomposition $E = \text{nil}(E) \cdot T$ is denoted by $S_T$. Note $n_T$ is not a homomorphism but $s_T$ is a homomorphism.

**Definition 6.** Let $E$ and $E'$ be almost algebraic groups. An $\varepsilon^*$-homomorphism $\phi : E \rightarrow E'$ is called a rational $\varepsilon^*$-homomorphism if and only if for each $\text{ad-}E$ reductive subgroup $S$ of $E$, we have that $\phi(S)$ is an $\text{ad-}E'$ reductive subgroup of $E'$.

The almost algebraic groups and rational $\varepsilon^*$-homomorphisms determine a subcategory of the $\varepsilon^*$-category. We will call this category the $\mathcal{O}^*$-category.

Let us now briefly outline the role of the Birkhoff imbedding theorem in the investigation of almost algebraic groups. Let $E$ be an almost algebraic group. Choose a Malcev decomposition $E = \text{nil}(E) \cdot T$. By Lemma 3, $\text{nil}(E) = M \oplus K$ where $M$ is a $\text{CN}$-group and $K$ is a compact central subgroup of $E$. Let $N(M)$ be the Lie hull of $M$. By the Birkhoff imbedding theorem there is a faithful matrix representation $\beta : N(M) \cdot T \rightarrow GL(n, R)$ where $\beta(N(M))$ is the group of all unipotent matrices in $\mathcal{O}_n(N(M) \cdot T)$ and $\beta(T)$ consists of semi-simple matrices. Let $H$ be a subgroup of an arbitrary group $G$. Two subgroups $L$ and $L'$ of $G$ are said to be conjugate over $H$ if and only if there is an $x \in H$ such that $xLx^{-1} = L'$.

**Theorem 1.** Let $E$ be an almost algebraic group. Then any two maximal abelian $\text{ad-}E$ reductive subgroups of $E$ are conjugate over $[E, E]^*$. 
**Proof.** Let $E = \text{nil}(E) \cdot T$ be a Malcev decomposition. Write $\text{nil}(E) = M \oplus K$ where $M$ is a CN group and $K$ is a central compact subgroup of $E$. Without any loss of generality we can assume $K = (1)$, since if $K \neq (1)$ we can consider $E/K$. Let $N(M)$ be the Lie hull of $M$. Denote the center of $N(M) \cdot T$ by $D$. Clearly $D \oplus T$ is a maximal abelian $ad\cdot N(M) \cdot T$ reductive subgroup of $N(M) \cdot T$. We shall prove that any maximal abelian $ad\cdot E$ reductive subgroup of $E$ is conjugate to $(D \cap M) \oplus T$ over $[E, E]^\#$. This is equivalent to proving that any maximal abelian $ad\cdot N(M) \cdot T$ reductive subgroup of $N(M) \cdot T$ is conjugate to $D \oplus T$ over $[E, E]^\#$. We shall give two proofs. The first proof depends upon the conjugacy theorem of algebraic groups found in [13]: any two maximal completely reducible subgroups of a solvable algebraic group $G$ are conjugate over $U^\#$, the group of all unipotent matrices in $G$. Identify $N(M) \cdot T$ with its image in $\text{GL}(n, \mathbb{R})$ given by the Birkhoff imbedding. Let $N = N(M)$ and let $T'$ be a maximal completely reducible subgroup of $\phi_N(N \cdot T)$. Let $B$ be a maximal $ad\cdot N \cdot T$ reductive subgroup of $N \cdot T$. Take $b \in B$. By Lemma 2 we can write $b = u \cdot s$ where $u$ is a unipotent matrix, $s$ is a semi-simple matrix and $[u, s] = 1$. By the conjugacy theorem there is an $x \in N$ such that $xxx^{-1} \in T'$. Since $N \cdot T/[E, E]^\#$ is abelian we can assume that $x$ is in $[E, E]^\#$. Thus $xxx^{-1} = xx^{-1}xx^{-1}$ where $xx^{-1} \in N \cdot T$. Therefore $xxx^{-1} \in T$. Since $T$ is $ad\cdot N \cdot T$ reductive, $s$ is $ad\cdot N \cdot T$ reductive. By Lemma 4 we have that $ad\cdot x\cdot T \cdot u = 1$. Let $B_1$ be the collection of all semi-simple matrices in $B$. Then $B = D \oplus B_1$. By the conjugacy there is an $x' \in [E, E]^\#$ such that $x'B_1x'^{-1} \subset T'$. However $x'B_1x'^{-1} \subset N \cdot T$. Hence $x'Bx'^{-1} = D \oplus T$.

Our second proof is independent of algebraic group theory and the Birkhoff imbedding theorem. It is modeled after the conjugacy theorem in [1]. We shall use induction on the dimension of $[N, N]$. Consider the case when $N$ is abelian. Let $N_1$ be the eigenvalue one space of $B$ acting on $N$. Choose a subspace $N_2$ of $N$ such that $N = N_1 \oplus N_2$ and $N_2$ is invariant under $B$. There exists a $b' \in B$ such that $(b' - I)$ is an automorphism of $N_2$, where $I$ is the identity automorphism on $N_2$. Note we are implicitly identifying $N$ with its Lie algebra. For each $b \in B$ we can write $b = n_1(b) n_2(b) t(b)$ where $n_1(b) \in N_1$, $n_2(b) \in N_2$ and $t(b) \in T$. We will consider the map $n_2 : B \rightarrow N_2$. Since $(b' - I)$ is an automorphism, a simple computation shows that $(b' - I) n_2(b) = (b - I) n_2(b')$ for all $b \in B$. Hence $n_2(b) = (b - I)x' \forall b \in B$. Now $x'Bx'^{-1}$ is a maximal $ad\cdot N \cdot T$ reductive subgroup of $N \cdot T$. Take $b \in B$. From $n_2(b) = (b - I)x'$ it is immediate that $x'Bx'^{-1} = n_2(b) t(b)$. But $[n_1(b), t(b)] = [n_1(b), b] = 1$. By Lemma 4, $ad\cdot x\cdot T \cdot n_1(b) = 1$ and $x'Bx'^{-1} = D \oplus T$. Assume that the theorem has been proved for dim$[N, N] < k$ and assume that dim$[N, N] = k$. $N \cdot T/[N, N]$ is an almost algebraic $e$-group with nil-radical $N/[N, N]$ which is abelian. By what we have just proved any maximal $ad\cdot N \cdot T$ reductive subgroup $B$ of $E$ can be moved by inner
conjugation into $D' \oplus T$ where $D'$ is the subgroup spanned by $D$ and $[N, N]$ in $N$. Since $[D', D']$ is strictly contained in $[N, N]$ we can apply induction to give the theorem.

Let $E$ be an almost algebraic group.

**Corollary 1.1.** Every $x \in E$ has a decomposition $x = u \cdot s$, where $u$ and $s$ are in $E$, $[u, s] = 1$, $u$ is ad-$E$ unipotent and $s$ is ad-$E$ semi-simple.

**Proof.** Follows immediately from Lemma 2 and from the fact that given a Birkhoff imbedding of $E$ any semi-simple matrix in $E$ is ad-$E$ reductive by the conjugacy theorem.

**Corollary 1.2.** Any maximal ad-reductive subgroup of $E$ is abelian.

**Proof.** Let $B$ be a maximal ad-$E$ reductive subgroup of $E$. We will use the notation of Theorem 1. Take $b \in B$. Write $b = u \cdot s$ where $u$ and $s$ are in $E$, $[u, s] = 1$, $u$ is unipotent and $s$ is semi-simple. Then $ad_E u = 1$ and $u \in D$. Let $B_1$ be the collection of all semi-simple matrices in $B$. We are done as soon as we prove $B_1$ is a group. Let $s$ and $s'$ be two elements in $B_1$. Then $ss's^{-1} = ds'$ where $d \in D$. But $ss's^{-1}$ is a semi-simple matrix hence $s = 1$. Therefore $ss' \in B_1$.


Let $C$ be an $e$-group. We associate with $C$ the category $\mathcal{C}(C)$ defined as follows. An object of $\mathcal{C}(C)$ is an $e$-isomorphism $\alpha : C \rightarrow E$, of $C$ into an almost algebraic group $E$ satisfying the following properties:

1. $E/\alpha((C, C)^e)$ is abelian.
2. $\alpha(C)$ and nil($E$) generate $E$.
3. For any Malcev factor $T$ of $E$, $\alpha(C)$ and $T$ generate $E$.

Let $\alpha : C \rightarrow E$ and $\beta : C \rightarrow F$ be two objects in $\mathcal{C}(C)$. A morphism $\phi : \alpha \rightarrow \beta$ of $\mathcal{C}(C)$ is a rational $e$-homomorphism $\phi : E \rightarrow F$ such that $\phi \circ \alpha = \beta$.

Let $\alpha : C \rightarrow E$ be an object of $\mathcal{C}(C)$. Any automorphisms $\rho$ of $C$ induces an automorphism $\rho^e$ of $\alpha(C)$ given by $\rho^e(\alpha(c)) = \alpha(\rho(c))$. The map $\rho \rightarrow \rho^e$ is an automorphism of $Aut(C)$ onto $Aut(\alpha(C))$. If $\rho$ is a semi-simple automorphism in $Aut(C)$ then $\rho^e$ is a semi-simple automorphism in $Aut(\alpha(C))$. If $\rho$ is a unipotent automorphism in $Aut(C)$ then $\rho^e$ is a unipotent automorphism in $Aut(\alpha(C))$. Also for $c \in C$, $ad_{\alpha(c)}(\alpha(c)) = (ad_c)^e$. In many instances these facts enable us to regard $C$ as a subgroup of $E$ and $\alpha$ as the inclusion map. When we are regarding $C$ as a subgroup of $E$ and $\alpha$ as the inclusion map we will denote this by $C \rightarrow E$. 
Lemma 7. Let $C \hookrightarrow E$ be an object of $\mathcal{E}(C)$. Then $ad_C E$ is the group generated by the unipotent and semi-simple parts of $ad_C C$. In particular for every $c \in C$, $ad_C c$ has a canonical decomposition in the sense of section 4.

Proof. By Corollary 1.1 each $c \in C$ can be written as $c = us$, where $u \in E, s \in E, [u, s] = 1$, $u$ is $ad-E$ unipotent and $s$ is $ad-E$ semi-simple. Therefore $ad_C E$ contains the group generated by the unipotent parts and the semi-simple parts of $ad_C C$. Let $E = \text{nil}(E) \cdot T$ be a Malcev decomposition. Without loss of generality we can assume that $\text{nil}(E)$ is a CN-group. Let $M = \text{nil}(E)$ and $N(M)$ be the Lie hull of $M$. We will be done if we can show that for each $t \in T$ there is an $n \in N(M)$ such that $[n, t] = 1$ and $n \cdot t \in C$. Since $E/[C, C]^a$ is abelian and $ad_N(M)T$ is completely reducible there is a vector space decomposition $N(M) = [C, C]^a \oplus X$ where $t(x) = x$ for all $t \in T$ and $x \in X$. Take $t \in T$. There is a $c \in C$ such that $c = nt$ where $n \in N(M)$. Write

$$n = n_1n_2$$

where $n_1 \in [C, C]^a$ and $n_2 \in X$. Hence $n_1^{-1}c = n_2t \in C$ and $[n_2, t] = 1$.

Lemma 8. Let $C \hookrightarrow E$ be an object of $\mathcal{E}(C)$. Choose a Malcev decomposition $E = \text{nil}(E) \cdot T$ for $E$. Let $S$ be an abelian group of semi-simple automorphisms of $C$ contained in $ad_C E$. Then $S$ is conjugate to a subgroup of $ad_C T$ over $ad_C [C, C]^a$.

Proof. Let $S'$ be the inverse image in $E$ of $S$ under the adjoint map. Applying Corollary 1.2 and Theorem 1, the lemma immediately follows.

Let $C$ be an $\mathcal{E}$-group.

Definition 6. An $\mathcal{E}$-semi-simple splitting of $C$ is an object $\alpha : C \rightarrow E$ of $\mathcal{E}(C)$ such that for any object $\beta : C \rightarrow F$ there is a morphism $\phi : \alpha \rightarrow \beta$.

Example 5. Let $N$ be a nilpotent $\mathcal{E}$-group. We claim that $N$ is an $\mathcal{E}$-semi-simple splitting of itself. Let $\rho : N \rightarrow E$ be an object of $\mathcal{E}(N)$. Let $E = M \cdot B$ be a Malcev decomposition of $E$ where $M$ nil$(E)$. Since $\text{nil}(N) = N$ we have $N \subset M$. But since $N$ and $M$ generate $E$, $E = M$. Hence $N = M$ and essentially the only objects of $\mathcal{E}(N)$ are isomorphisms of $N$ onto itself.

Example 6. Let $E$ be an almost algebraic group. Clearly the group $E$ itself is an $\mathcal{E}$-semi-simple splitting of $E$.

Lemma 7. Let $S$ be an elementary group of semi-simple automorphisms of an $\mathcal{E}$-group $C$. Let $E = C \cdot S$ (semi-direct product). Assume that the following properties are satisfied.

1. $C$ and $\text{nil}(E)$ generate $E$.
2. $S$ and $\text{nil}(E)$ generate $E$.

Then $C \hookrightarrow E$ is an $\mathcal{E}$-semi-simple splitting of $C$. 
Proof. Clearly $E/[C, C]^e$ is abelian. Hence $ad_C$ nil($E$) consists of unipotent automorphisms. Thus $S \cap \text{nil}(E) = 1$. From this it follows that $E$ is an almost algebraic group and $C \rightarrow E$ is an object of $e^s(C)$. Let $\alpha : C \rightarrow F$ be an object of $e^s(C)$. Without loss of generality we can assume that $C$ is a subgroup of $F$ and $\alpha$ is the inclusion map. Let $F = \text{nil}(F) \cdot T$ be a Malcev decomposition of $F$. Then $S \subseteq ad_CF$. By Lemma 8 there is an $x \in [C, C]^e$ such that $S = ad_CxTt^{-1}$. Hence we can assume that $T$ has been chosen such that $S = ad_CT$. Therefore $C \cdot S = C \cdot ad_CT$. Let $\theta = ad_C : T \rightarrow ad_CT$. Since $T$ acts faithfully on $C$, $\theta$ is an isomorphism. Let $\phi : E \rightarrow F$ be given by $\phi(c) = c$ for $c \in C$ and $\phi(s) = \theta^{-1}(s)$ for $s \in S$. Then $\phi$ is a rational homomorphism of $E$ onto $F$ which is the identity map on $C$. Hence $C \cdot S$ is an $e^s$-semi-simple splitting of $C$.

**Lemma 8.** Let $C$ be an $e^s$-semi-simple splitting $C \hookrightarrow E$. Then

1. If $C$ is torsion free then for any Malcev decomposition $E = \text{nil}(E) \cdot T$ we have $C \cap T = (1)$.

2. There exists an $e^s$-semi-simple splitting $C \hookrightarrow F$ and a Malcev decomposition $F = \text{nil}(F) \cdot B$ such that $C \cap B = (1)$.

Proof. Let $E = \text{nil}(E) \cdot T$ be a Malcev decomposition of $E$. Form the semi-direct product $C \cdot T = F$ where the action of $T$ on $C$ is given by the adjoint action of $T$ on $C$ as a subgroup of $E$. Let $M$ be the subgroup of $F$ generated by elements of the form $(c, t(c)^{-1})$ where $t$ is the projection of $E$ onto $T$. It is easy to see that $M = \text{nil}(F)$. By Lemma 9, $C \rightarrow F$ is an $e^s$-semi-simple splitting of $C$. This proves statement (2). Now define $\phi : F \rightarrow E$ by $\phi((c, t)) = ct$. By definition there is a rational homomorphism $\psi : E \rightarrow F$ whose restriction to $C$ is the identity map. Take $t \in T$ such that $\phi(\psi(t)) \in \text{nil}(E)$. Since $\phi\psi$ restricted to $C$ is the identity map, $t = 1$. Thus $\psi(T) \cap M = (1)$. Moreover $C$ and $\psi(T)$ generate $M \cdot T$. Hence $\psi(T)$ is a Malcev factor for $M \cdot T$. It follows that $\phi\psi$ restricted to $M$ is a surjective homomorphism of $M$. Hence $\ker(\psi \circ \phi)$ is finite. Thus $\ker \phi$ is a finite subgroup of $M$. Take $c \in C \cap T$ in $E$. In $M \cdot T$ write $c = mt$ where $m \in M$ and $t \in T$. Since $m \in \ker \phi$, $m^\ell = 1$ for some integer $\ell$. Also $ad_{\phi}m = 1$. Therefore $c^\ell = t^\ell$ in $M \cdot T$. Hence $c^\ell = 1$. Assuming that $C$ is torsion free proves statement (1).

We will restrict ourselves to those $e^s$-semi-simple splittings $C \hookrightarrow E$ for which there is a Malcev decomposition $E = \text{nil}(E) \cdot T$ where $C \cap T = (1)$. This assumption is satisfied in the rest of this paper.

7. Extensions of Automorphisms.

In this section and the next section we shall assume that the $e$-groups we are dealing with have $e^s$-semi-simple splittings. Later we will prove the existence of the semi-simple splitting of all $e$-groups.
LEMMA 9. Let $C$ be an $e$-group. Then any two $e^*$-semi-simple splittings of $C$ are isomorphic in $e^*(C)$. We mean by this the following. If $\rho : C \to E$ and $\phi : C \to F$ are two $e^*$-semi-simple splittings of $C$, then there is a rational isomorphism $\phi : E \to F$ of $E$ onto $F$ such that $\phi \rho = \phi$.

Proof. Choose a Malcev decomposition $E = N \cdot T$ of $E$ where $N = \text{nil}(E)$ and $\rho(C) \cap T = (1)$. Choose a Malcev decomposition $F = M \cdot B$ where $M = \text{nil}(F)$ and $\phi(C) \cap B = (1)$. The isomorphism $\phi^{-1} : \rho(C) \to \phi(C)$ induces an isomorphism $\gamma$ of $\text{Aut}(\rho(C))$ onto $\text{Aut}(\phi(C))$ in a natural way. Let $\alpha \in \text{Aut}(\rho(C))$. Then $\gamma(\alpha)(\phi(c)) = q \circ \rho^{-1}(\alpha(\rho(c)))$. Clearly $\gamma(ad_{\rho(C)} T)$ is an abelian group of semi-simple automorphisms contained in $ad_{\phi(C)} F$. Hence by Lemma 8 there is no loss in generality assuming that $\gamma(ad_{\rho(C)} T) = ad_{\phi(C)} B$. Also the isomorphism $q \circ \rho^{-1} : \rho(C) \to \phi(C)$ induces an isomorphism $\lambda$ of $T$ onto $B$. Let $t \in T$. Then there is an $n \in N$ such that $\rho(c) = nt$ for some $c$ in $C$. Write $\phi(c) = mb$ where $m \in M$ and $b \in B$. Then define $\lambda(t) = b$. Since $\rho(\text{nil}(C)) \subset N$ and $\phi(\text{nil}(C)) \subset M$ it is easy to see that $\lambda$ is a well defined isomorphism of $T$ onto $B$. Let $\phi : E \to F$ be defined by $\phi(\rho(c)t) = (\phi(c)\lambda(t))$. Clearly $\phi$ is a rational isomorphism of $E$ onto $F$ such that $\phi \rho = \phi$.

We will denote the $e^*$-semi-simple splitting of $C$ by $C \to \phi^*(C)$.

LEMMA 10. Every automorphism of $C$ extends to a automorphism of $B^*(C)$. In particular if we choose a Malcev factor $T$ of $B^*(C)$, then for any automorphism $\alpha$ of $C$ there exists a unique automorphism $B^*(\alpha)$ of $B^*(C)$ such that $B^*(\alpha)(T)$ is conjugate to $T$ over $[C, C]^*$.

Proof. Let $\alpha : C \to C$ be an automorphism of $C$. Then the isomorphism $\alpha : C \to B^*(C)$ is an $e^*$-semi-simple splitting of $C$. In the same way the isomorphism $\alpha^{-1} : C \to B^*(C)$ is an $e^*$-semi-simple splitting of $C$. By Lemma 9 there exists an automorphism $\phi : B^*(C) \to B^*(C)$ such that $\phi$ extends $\alpha$. If $T$ is a Malcev factor of $B^*(C)$ then $\phi$ can be chosen such that $\phi(T)$ is conjugate to $T$ over $[C, C]^*$. The Birkhoff imbedding theorem enables us to regard $B^*(C)$ as a matrix group where $N = \text{nil}(B^*(C))$ is a closed co-compact subgroup in the group of all unipotent matrices in $\mathfrak{U}_R(B^*(C))$ and $T$ is a group of semi-simple matrices. Then any conjugate of $T$ is a group of semi-simple matrices. Hence the decomposition of an element in $B^*(C)$ into the commuting product of an element in $N$ and an element in a conjugate of $T$ is unique. Let $\phi'$ be another extension of $\alpha$ such that $\phi'(T)$ is conjugate to $T$ over $[C, C]^*$. Take $t \in T$. There is an $n \in N$ such that $c = nt \in C$ and $[n, t] = 1$. From $\phi'(c) = \phi(c) = \alpha(c) = \phi'(n) \phi'(t) = \phi(n) \phi(t)$ it follows that $\phi'(t) = \phi(t)$.

THEOREM 2. Let $\text{Aut}(C)$ be the automorphism group of $C$, and $\text{Aut}(E, C)$ be the group of all automorphisms of $E$ which leave $C$ invariant, where we have
denoted $B^*(C)$ by $E$. Then the restriction map $r : \text{Aut}(E, C) \to \text{Aut}(C)$ is onto $\text{Aut}(C)$. Automorphisms do not extend uniquely, however we can measure the non-uniqueness by the split exact sequence

$$1 \to H^1(C, D) \to \text{Aut}(E, C) \xrightarrow{r} \text{Aut}(C) \to 1$$

where $H^1(C, E)$ is the collection of all homomorphisms of $C$ into the center $D$ of $E$.

Because of the splitting of this sequence any group of automorphisms of $C$ extends to a group of automorphisms of $E$.

**Proof.** Fix a Malcev factor $T$ of $E$ and lift each automorphism of $C$ to an automorphism of $E$ such that $T$ is taken onto a conjugate of $T$ by an inner automorphism from $[C, C]^\neq$. The theorem follows directly.

8. **Extensions of Homomorphisms.**

In this section we develop the functorial properties of the splitting map. Let $C$ and $C'$ be $\epsilon$-groups. Let $\alpha : C \to C'$ be an $\epsilon^\ast$-homomorphism. We will show that $\alpha$ extends to a rational homomorphism $\alpha : B^\ast(C) \to B^\ast(C')$. However there does not exist a unique extension. We will correct this deficiency in the following way. Choose for each $\epsilon$-group $C$ a Malcev factor $T(C)$ of its semi-simple splitting. Let $\alpha : C \to C'$ be an $\epsilon^\ast$-homomorphism. Then there will exist a unique rational homomorphism $B^\ast_{T, T'}(\alpha) : B^\ast(C) \to B^\ast(C')$ extending $\alpha$ such that $B^\ast_{T, T'}(\alpha)(T(C))$ is conjugate to a subgroup of $T(C')$ over $[C', C']^\neq$. The maps $C \to B^\ast(C)$ and $\alpha \to B^\ast_{T, T'}(\alpha)$ define a functor from the $\epsilon$-category to the $A^\ast$-category.

Let $C$ and $C'$ be $\epsilon$-groups.

**Lemma 11.** Every $\epsilon^\ast$-homomorphism of $C$ onto $C'$ extends to a rational homomorphism of $B^\ast(C)$ onto $B^\ast(C')$. Let $T$ be a Malcev factor of $B^\ast(C)$ and let $T'$ be a Malcev factor of $B^\ast(C')$. If $\alpha : C \to C'$ is an $\epsilon^\ast$-homomorphism of $C$ onto $C'$ then there exists a unique extension of $\alpha$ to a rational homomorphism $B^\ast_{T, T'}(\alpha)$ of $B^\ast(C)$ onto $B^\ast(C')$ such that $B^\ast_{T, T'}(\alpha)(T)$ is conjugate to $T'$ over $[C', C']^\neq$.

**Proof.** Let $K = \ker(\alpha)$. By Lemma 10 it is sufficient to prove the theorem for the case where $C' = C/K$ and $\alpha$ is the natural homomorphism of $C$ onto $C/K$. Let $T$ be a Malcev factor for $B^\ast(C)$. It is easy to see that $K$ is normalized by $T$. Hence $T$ induces a group of automorphisms of $C/K$. Denote this induced group of automorphisms by $S$. Clearly $S$ is an abelian group of semi-simple automorphisms of $C/K$ and $((C/K \cdot S)/(\alpha(C), \alpha(C))^\neq$ is abelian. It follows that the induced homomorphism $\alpha^\neq : C \cdot T \to (C/K) \cdot S$ takes
ad-C · T} unipotent elements into \( ad-C/K \cdot S \) unipotent elements and \( ad-C \cdot T \) semi-simple elements into \( ad-C/K \cdot S \) semi-simple elements. Hence \( \alpha(\text{nil}(C \cdot T)) \subseteq \text{nil}(C/K \cdot S) \), from which we can conclude that \( \text{nil}((C/K \cdot S) \text{ and } C/K \text{ generate } (C/K) \cdot S \text{ and } \text{nil}((C/K) \cdot S) \text{ and } S \) generate \( (C/K) \cdot S \). Applying Lemma 7 we get that \( (C/K) \cdot S - B^*(C/K) \), and that \( \alpha^* \) is a rational homomorphism of \( B^*(C) \) onto \( B^*(C/K) \). The second statement of the lemma follows exactly as in Lemma 10.

**Lemma 12.** Every \( \epsilon^* \) isomorphism of \( C \) into \( C' \) extends to an \( \epsilon^* \)-isomorphism of \( B^*(C) \) into \( B^*(C') \). Let \( T \) be a Malcev factor of \( B^*(C) \). Let \( T' \) be a Malcev factor of \( B^*(C') \). If \( \alpha : C \to C' \) is an \( \epsilon^* \)-isomorphism of \( C \) into \( C' \) then there exists a unique extension of \( \alpha \) to a rational isomorphism \( B_{T',T}(\alpha) \) of \( B^*(C) \) into \( B^*(C') \) such that \( B_{T',T}(\alpha(T)) \) is conjugate to a subgroup of \( T' \) over \( [C', C']^\epsilon \).

*Proof.* By the Birkhoff imbedding theorem we can assume that there is a Malcev decomposition of \( B^*(C') = N' \cdot T' \) such that where \( t' \) is the projection of \( B^*(C') \) onto \( T' \), we have that \( t'(C) \) normalizes \( C \) and that \( C \cdot t'(C)/[C, C]^\epsilon \) is abelian. Since \( \text{nil}(C) \subseteq \text{nil}(C') \) it follows that \( t'(C) \) acts faithfully on \( C \) under the adjoint representation. Moreover it follows from \( C \cdot t'(C)/[C, C]^\epsilon \) being abelian that \( n_{T'}(C) \) is a subgroup of \( N' \) and that

\[
C \cdot t'(C) = n_{T'}(C) \cdot t'(C).
\]

Hence \( n_{T'}(C) \) is the nil-radical of \( C \cdot t'(C) \) and \( B^*(C) = C \cdot t'(C) \). Now let \( S \) be a Malcev factor of \( B^*(C) \) and \( S' \) a Malcev factor of \( B^*(C') \). There exists an automorphism of \( C \cdot S \) onto \( C \cdot t'(C) \) which acts by the identity on \( C \) and takes \( S \) onto \( xt'(C)x^{-1} \). Now \( \alpha^* \) takes \( xt'(C)x^{-1} \) into \( xt'x^{-1} \) which is a Malcev factor of \( B^*(C') \). But again there is an automorphism of \( B^*(C') \) acting by the identity on \( C' \) taking \( xt'x^{-1} \) onto a conjugate of \( S' \).

Let us suppose that for each \( \epsilon \)-group \( C \) there has been chosen a Malcev factor \( T(C) \) of \( B^*(C) \). The above lemmas give us the following theorem.

**Theorem 3.** Let \( C \) and \( C' \) be \( \epsilon \)-groups. Let \( \alpha : C \to C' \) be an \( \epsilon^* \) homomorphism. Then there exists a unique rational homomorphism

\[
B_{T'}^*(\alpha) : B^*(C) \to B^*(C')
\]

extending \( \alpha \) where \( B_{T'}^*(\alpha)(T(C)) \) is conjugate to a subgroup of \( T(C') \) over \( [C', C']^\epsilon \).

The maps \( C \to B^*(C) \) and \( \alpha \to B_{T'}^*(\alpha) \) define an exact functor from the \( \epsilon^* \) category to the category of almost algebraic groups and rational homomorphisms.
We now take up the question of the existence of the semi-simple splitting of an \(\epsilon\)-group. Let \(C\) satisfy the short exact sequence

\[ 1 \to N \to C \xrightarrow{\rho} A \to 1 \]

where \(N\) is a real nilpotent group and \(A\) is an elementary abelian group. Write \(A = R^s \oplus Z^t \oplus K'\), where \(K'\) is a compact group. Arguing as in lemma 3 there is a compact subgroup \(K\) of \(C\) such that \(\rho(K) = K'\) and \(K \cap N = (1)\). Then \(C = C_1 \cdot K\) where \(C_1\) satisfies the short exact sequence

\[ 1 \to N \to C_1 \to R^s \oplus Z^t \to 1. \]

Let \(C_2 = \rho^{-1}(R^s)\). Then \(C_2\) satisfies the short exact sequence

\[ L \to N \to C_2 \xrightarrow{\rho} R^s \to 1 \]

and \(C_1\) satisfies the short exact sequence

\[ 1 \to C_2 \to C_1 \xrightarrow{\epsilon} Z^t \to 1. \]

**Theorem 3.** Every \(\epsilon\)-group has an \(\epsilon^s\)-semi-simple splitting.

**Proof.** We will prove the theorem for \(\epsilon\)-groups \(C\) defined by short exact sequences of the form

\[ 1 \to N \to C \xrightarrow{\rho} R^s \to 1 \]

where \(N\) is a real nilpotent group. It will be clear from the proof of this case and from the previous discussion that we may use our proof to prove the existence of the semi-simple splitting of arbitrary \(\epsilon\)-groups. We will use induction on \(s\) the dimension of \(C/N\). Assume that the theorem has been proved for all \(\epsilon\)-groups \((C, \rho)\) where \(C/\ker \rho = R^t\) and \(t < s\). Let \((C, \rho)\) be an \(\epsilon\)-group where \(C/\ker \rho = R^s\). Denote the kernel of \(\rho\) by \(N\). Choose a section \(f : R^s \to C\) for \((C, \rho)\). Let \(C_1 = \rho^{-1}(R^{s-1})\). Hence \(C_1\) satisfies the short exact sequence \(1 \to N \to C_1 \xrightarrow{\rho} R^{s-1} \to 1\). Moreover \(C\) and \(C_1\) satisfy the short exact sequence

\[ 1 \to C_1 \to C \xrightarrow{\epsilon} R^s \to 1 \]

and \(f\) induces a section \(f_1 : R \to C\) of \((C, \rho)\) where we have the relation \(f_1(f)(r_1 + r_2) = f_1(f)(r_1) + f_1(f)(r_2) \mod(N)\). By our induction hypothesis the semi-simple splitting of \(C_1\) exists. Let \(B^s(C_1) = N_1 \cdot T_1\) be a Malcev decomposition of \(B^s(C_1)\) where \(N_1 = \text{nil}(B^s(C_1))\), \(T_1\) is a Malcev factor of \(B^s(C_1)\) and
We can write for all $r \in R$, $f_1(r)^*(T_1) = n(r) T_1 n(r)^{-1}$, for some $n(r) \in [C_1, C_2]^e$. Let $d : R \to C$ be the section of $(C, q)$ given by $d(r) = n(r)^{-1} f_1(r)$. Clearly $d(r) d(s) = d(r + s) \mod(n_1)$. Then $d(r)^*(t) = t$ for all $t \in T_1$. Let $N(N_1)$ be the Lie hull of $N_1$. For each $r \in R$ we can write $d(r)^* = u(r) s(r)$ restricted to $N(N_1)$ where $u(r)$ is unipotent, $s(r)$ is semi-simple, and $[u(r), s(r)] = 1$. Since $d(r) d(s) = d(r + s) \mod(n_1)$ the map $s : R \to \text{Aut}(N(N_1))$ is a homomorphism. We can extend $s(r)$ to all of $B^e(C_1)$ by having it act as the identity map on $T_1$. We will denote the extension to all of $B^e(C_1)$ by $s(r)$ also. The map $s : R \to \text{Aut}(B^e(C))$ is a homomorphism.

Let $T$ be the group of automorphisms of $C_1$ generated by $ad_{C_1} T_1$ and $ad_{C_1} s(R)$. $T$ can be extended to a group of semi-simple automorphisms of $C$ by having it act by the identity map on $d(R)$. Using Lemma 7 it is clear that $C \cdot T$ is a semi-simple splitting of $C$.

10. The $\epsilon$-Semi-Simple Splitting.

In this section we return to the study of the $\epsilon$-category. We will define a semi-simple splitting in this category called the $\epsilon$-semi-simple splitting. Let $(E, \rho)$ be an $\epsilon$-group.

**Definition 8.** We say that $E$ is an $\epsilon$-almost algebraic group if and only if $E$ has a semi-direct product decomposition $E = N \cdot A$, where $N$ is a nilpotent $\epsilon$-group containing $\ker \rho$ and $A$ is an elementary $ad$-$E$ reductive subgroup of $E$. We will call $A$ a Malcev factor of $E$ and the decomposition $E = N \cdot A$ a Malcev decomposition of $E$.

**Definition 9.** Let $E$ and $E'$ be $\epsilon$-almost algebraic groups. An $\epsilon$-homomorphism $\phi : E \to E'$ is called an $\epsilon$-rational homomorphism if and only if for each $ad$-$E$ reductive subgroup $A$ of $E$, $\phi(A)$ is an $ad$-$E'$ reductive subgroup of $E'$.

Let $(C, \rho)$ be an $\epsilon$-group. We define the category $\epsilon(C)$ as follows. An object of $\epsilon(C)$ is an $\epsilon$-isomorphism $\phi : C \to E$ where $E$ is an $\epsilon$-almost algebraic group satisfying the following properties: There exists a Malcev decomposition $E = N \cdot A$ where $N$ is a nilpotent $\epsilon$-group containing $[E, E]^e$ and $A$ is an elementary $ad$-$E$ reductive group such that

1. $E/\phi(C), \phi(C))^e$ is abelian.
2. $N$ and $\phi(C)$ generate $E$.
3. $A$ and $\phi(C)$ generate $E$. 
Such a Malcev decomposition \( E = N \cdot A \) of \( E \) is said to be related to \( C \). Given two objects \( \phi : C \to E \) and \( \theta : C \to F \) of \( \epsilon(C) \), a morphism \( \psi : \phi \to \theta \) is an \( \epsilon \)-rational homomorphism \( \psi : E \to F \) such that \( \psi\phi = \theta \).

As before, in many instances, we can regard \( C \) as a subgroup of \( E \) and \( \phi \) as the inclusion map. We will write in this case that \( C \subseteq E \) is an object of \( \epsilon(C) \).

**Lemma 13.** Given an object \( C \to E \) of \( \epsilon(C) \), let \( E = N \cdot A \) be a Malcev decomposition of \( E \) related to \( C \). Then \( C \cdot \text{ad}_C A = B(C) \).

**Proof.** The lemma follows immediately from lemma 7.

Let \( C \) be an \( \epsilon \)-group.

**Definition 10.** An \( \epsilon \)-semi-simple splitting of \( C \) is an object \( \phi : C \to E \) of \( \epsilon(C) \) such that for any object \( \theta : C \to F \) of \( \epsilon(C) \) there is a morphism \( \psi : \phi \to \theta \) of \( \epsilon(C) \).

**Example 7.** Let us consider an example to see how \( B(C) \) and a semi-simple splitting differ. Let \( N \) be a nilpotent \( \epsilon \)-group. Then \( B(N) = N \).

**Lemma 14.** Let \( (C, \rho) \) be an \( \epsilon \)-group. Denote \( \ker \rho \) by \( N \), and \( C/N \) by \( A \). Denote the natural map of \( C \) onto \( A \) by \( \eta \). Let \( \alpha : A \to \text{Aut}(C) \) be a homomorphism such that \( \alpha(A) \) consists of semi-simple automorphisms of \( C \). Form \( E = C \cdot A \) (semi-direct product) where \( \text{ad}_C a = \alpha(a) \) for \( a \in A \). Then if

\[
M = \{(c, \eta(c)^{-1}) : c \in C\}
\]

is nilpotent, we have that \( C \subseteq E \) is a semi-simple splitting of \( C \).

**Proof.** Clearly \( E = M \cdot A \) and \( 1 \to N \to E \to E/N \to 1 \) is an \( \epsilon \)-group. Since \( M \) is a nilpotent \( \epsilon \)-group, \( C \subseteq E \) is an object of \( \epsilon(C) \) and \( E = M \cdot A \) is a Malcev decomposition of \( E \) related to \( C \). Let \( C \subseteq F \) be an object of \( \epsilon(C) \). Choose a Malcev decomposition \( F = N \cdot B \) of \( F \) related to \( C \). By lemma 13 \( C \cdot \text{ad}_C B = B(C) \equiv C \cdot \text{ad}_C A \). Hence by Theorem 1 there is no loss in generality assuming that \( \text{ad}_C A = \text{ad}_C B \). The identity map on \( C \) induces a homomorphism \( \beta \) of \( A \) onto \( B \). This homomorphism can be extended to a rational homomorphism also denoted by \( \beta \), of \( C \cdot A \) onto \( C \cdot B = F \) given by \( \beta((c, a)) = c\beta(a) \). Hence \( C \subseteq E \) is a semi-simple splitting of \( C \).

**Lemma 15.** If an \( \epsilon \)-group \( C \) has a semi-simple splitting, then there exists a semi-simple splitting \( C \subseteq E \) for which there is a Malcev decomposition of \( E \), \( E = M \cdot A \) such that \( C \cap M = N \) and \( C \cap A = (1) \).
Proof. Let $B = C/N$. Since $N \subseteq M$, there is a natural homomorphism $\varphi : B \to A$. Form $C \cdot B$ where $ad_c b = ad_c \varphi(a)$. Let $M = \{(e, \eta(c)^{-1} : c \in C\}$. Then $C \cdot B = M \cdot B$. By Lemma 14 $C \hookrightarrow M \cdot B$ is an $\varepsilon$-semi-simple splitting of $C$. Clearly $C \cap M = N$ and $C \cap B = (1)$.

We will restrict ourselves to $\varepsilon$-semi-simple splittings of $C$, $C \hookrightarrow E$ for which there exists a Malcev decomposition $E = M \cdot A$ of $E$ such that $C \cap M = N$ and $C \cap A = (1)$. Given an $\varepsilon$-semi-simple splitting $C \hookrightarrow E$, of $C$ we will now say that a Malcev decomposition $E = M \cdot A$ of $E$ is related to $C$ if $C \cap M = N$ and $C \cap A = (1)$. These assumptions are satisfied throughout the rest of the paper.

Lemma 15. Let $C$ be an $\varepsilon$-group. Then any two semi-simple splittings of $C$ are isomorphic in $\varepsilon(C)$. We mean by this the following. Let $\varphi : C \to E$ and $\theta : C \to F$ be two semi-simple splittings of $C$. Then there exists an isomorphism $\psi : E \to F$ of $E$ onto $F$ such that $\psi \varphi = \theta$.

Proof. The proof is exactly the same as before.

Let $(C, \rho)$ be an $\varepsilon$-group. Let $N = \ker \rho$.

Theorem 5. $C$ has an $\varepsilon$-semi-simple splitting. We will denote the $\varepsilon$-semi-simple splitting of $C$ by $B(C)$.

Proof. Choose a Malcev factor of $B^\varepsilon(C)$, denoted by $T$. Let $A = C/N$. Since $N \subseteq \text{nil}(C)$ there is a naturally induced homomorphism from $A$ onto $T$. Form the semi-direct product $C \cdot A$ where the action of $A$ on $C$ is given by the action of $T$ on $C$. Let $\eta : C \to C/N$ be the natural map. In $C \cdot A$ consider the subgroup generated by elements of the form $(c, \eta(c)^{-1})$. Denote this subgroup by $M$. It is easy to see that $M$ is a nilpotent $\varepsilon$-group and that $C \cdot A = M \cdot A$. From $C \cap M = N$ and from Theorem 1 it is clear that $C \cdot A$ is an $\varepsilon$-semi-simple splitting of $C$.

Corollary 5.1. Let $B(C) = M \cdot A$ be a Malcev decomposition of the semi-simple splitting of $C$ related to $C$. Then $\text{rank}(C) = \text{rank}(M)$.

Proof. The projection $\rho : B^\varepsilon(C) \to M$ of $B^\varepsilon(C)$ onto $M$ relative to the Malcev decomposition $B^\varepsilon(C) = M \cdot A$ maps $C$ homeomorphically onto $M$. Moreover $\rho$ induces an isomorphism of $C/N$ onto $M/N$.

11. Extensions of Homomorphisms.

Let $(C, \rho)$ and $(C', \rho')$ be $\varepsilon$-groups. Let $M = \ker \rho$ and $M' = \ker \rho'$.

Lemma 16. Every $\varepsilon$-homomorphism $\alpha : C \to C'$ of $C$ onto $C'$ extends to an $\varepsilon$-rational homomorphism $\alpha^* : B(C) \to B(C')$ of $B(C)$ onto $B(C')$. Let
Let $B(C) = N \cdot A'$ and $B(C') = N' \cdot A'$ be Malcev decompositions of $B(C)$ and $B(C')$ relative to $C$ and $C'$ respectively. Then there exists a unique extension of $\alpha$ to an $e$-rational homomorphism $B(\alpha) : B(C) \to B(C')$ such that $B(\alpha)(N) = N'$ and $B(\alpha)(A)$ is conjugate to $A'$ over $M'$.

**Proof.** Let $B(C) = N \cdot A$ be a Malcev decomposition of $B(C)$, related to $C$. Then $\alpha = C/M$. Let $A' = C'/M'$. Since $\alpha(M) \subset M'$, $\alpha$ induces a homomorphism $\alpha_1 : A \to A'$ of $A$ onto $A'$. Arguing as before $ad_C A$ induces a group $S$ of semi-simple automorphisms of $C'$. Let $\beta : ad_C A \to S$ be the corresponding homomorphism. Define $\alpha_2 : A \to S$ by $\alpha_2 = B \circ ad_C$. It is easy to see that if $\alpha_2(0) = 1$ then $\alpha(0) = 1$. Hence there is a homomorphism $\alpha^* : A' \to S$ such that $\alpha^* \circ \alpha_1 = \alpha_2$. Then $C' \cdot A' = B(C')$ where $ad_C A' = \alpha^*(a')$. The map $\alpha^* : C \cdot A \to C' \cdot A'$ given by $\alpha^*((c, a)) = (\alpha(c), \alpha(a))$ is a rational homomorphism extending $\alpha$. The rest of the lemma follows as before, in Lemma 11.

**Lemma 17.** Every $e$-isomorphism $\alpha : C \to C'$ of $C$ into $C'$ extends to an $e$-rational homomorphism $\alpha^* : B(C) \to B(C')$ of $B(C)$ into $B(C')$. We note that $\alpha^*$ is not necessarily an isomorphism. Let $B(C) = N \cdot A$ and $B(C') = N' \cdot A'$ be Malcev decompositions of $B(C)$ and $B(C')$ related to $C$ and $C'$. Then there exists a unique $e$-rational homomorphism $B(\alpha) : B(C) \to B(C')$ such that $B(\alpha)$ restricted to $N$ is an isomorphism of $N$ into $N'$ and $B(\alpha)(A)$ is conjugate to a subgroup of $A'$ over $M'$.

**Proof.** Let $B(C') = N' \cdot A'$ be a Malcev decomposition of $B(C')$ related to $C'$. As in Lemma 12 we can choose $A'$ such that the projection $\alpha(C)$ of $C$ into $A'$ normalizes $\alpha(C)$. Since $\alpha(M) \subset M'$, $\alpha$ induces a homomorphism also denoted by $\alpha$ from $A$ into $A'$. Form $C \cdot A$ where $ad_C a = ad_{\alpha(a)} a$. Clearly $C \cdot A = B(C)$ and $\alpha^* : B(C) \to B(C')$ given by $\alpha^*((c, a)) = (\alpha(c), \alpha(a))$ is an $e$-rational homomorphism of $B(C)$ into $B(C')$. The rest of the lemma follows as in Lemma 12.

Note that the extension $\alpha^*$ of $\alpha$ need not be an isomorphism.

For each object $\varphi : C \to F$ of $\mathcal{E}(C)$ choose a Malcev decomposition of $F$ related to $C$. In particular for each $C$ we have a Malcev decomposition $B(C) = N(C) \cdot A(C)$.

**Theorem 6.** For each $e$-homomorphism $\alpha : C \to C'$ there exists a unique $e$-rational homomorphism $B(\alpha) : B(C) \to B(C')$ taking $N(C)$ homomorphically into $N(C')$ and $A(C)$ into a subgroup contained in a conjugate of $A(C')$ over $M'$. Moreover the maps $C \to B(C)$ and $\alpha \to B(\alpha)$ define a right exact functor. The induced maps $C \to N(C)$ and $\alpha \to B(\alpha)/N(C)$ define an exact functor.

**Proof.** Follows directly from the previous two lemmas.
12. The Integer Splitting.

In the study of closed subgroups of solvable Lie groups we can not restrict ourselves solely to the study of \( e \)-groups. The refinement necessary is given in the following definition.

**Definition 11.** By a \( CS \)-group we mean a short exact sequence of the form

\[ 1 \rightarrow N \rightarrow C \rightarrow A \rightarrow 1, \]

where \( N \) is a \( CN \)-group and \( A \) is a torsion free elementary group. When we speak of a \( CS \)-group \( C \) we will implicitly understand that \( C \) has been given by a short sequence of this type. Let \( N_R \) be the Lie hull of \( N \). Then there exists a unique group \( C_R \) satisfying the commutative diagram

\[
\begin{array}{c}
1 \\ \downarrow \\ 1
\end{array}
\begin{array}{c}
N \\ C \\ A \\ 1
\end{array}
\begin{array}{c}
1 \\ \downarrow \\ 1
\end{array}
\begin{array}{c}
N_R \\ C_R \\ A \\ 1
\end{array}
\]

Clearly \( C_R \) is an \( e \)-group and will be called the Lie hull of the \( CS \)-group \( C \).

**Theorem 7.** We will use the notation above. Let \( B^\#(C_R) = M \cdot T \) be a Malcev decomposition of the \( e^\# \)-semi-simple splitting of the \( e \)-group \( C_R \). Then there exists a subgroup \( N' \) of \( N_R \) containing \( N \) as a subgroup of finite index such that

1. \( N' \) is normalized by \( C \). Let \( C' = N' C \) in \( C_R \).
2. \( n_T(C') \) is a lattice subgroup of \( M_R \), where \( n_T : B^\#(C) \rightarrow M \) is the projection onto the first factor in the decomposition \( B^\#(C) = M \cdot T \).
3. \( T \) normalizes \( N' \) and \( C' \).
4. \( C' \cdot T|N' \) is abelian.

We will call \( C' \cdot T \) an \( e^\# \)-semi-simple splitting of the \( CS \)-group \( C \) relative to the Malcev factor \( T \).

**Proof.** It is clear that all our results remain valid for the category of short exact sequences

\[ 1 \rightarrow N^* \rightarrow C^* \rightarrow Z^* \rightarrow 1, \]

where \( N^* \) is a rational nilpotent group. Let us consider the case of the theorem where the \( CS \)-group \( C \) is given by the short exact sequence

\[ 1 \rightarrow N \rightarrow C \rightarrow Z^* \rightarrow 1, \]
where $N$ is an $FN$-group. Let $N_0$ be the rational Lie hull of $N$ and let $C_0$ be the unique group satisfying the commutative diagram

$$
\begin{array}{cccccc}
1 & \longrightarrow & N & \longrightarrow & C & \longrightarrow & Z^* & \longrightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
1 & \longrightarrow & N_0 & \longrightarrow & C_0 & \longrightarrow & Z^* & \longrightarrow & 1.
\end{array}
$$

Let $F$ denote the rational Lie hull of $M$. It is easy to show that for the sake of the theorem we can assume without loss of generality that there exists preimages $c_1, \ldots, c_n$ in $C$, of a basis of $Z^*$, on which $T$ will act as the identity map. We use induction on the steps of nilpotency of $F$. If $F$ is abelian we are done. Assume the theorem for all $F$ with steps of nilpotency $< c$, and assume that $F$ is $c$-step nilpotent. Since we are working over the rationals it is easy to see that we can choose a basis $e_{t+1}, \ldots, e_n$ of $L(F_e)$ such that

$$\exp \left( \sum_{t+1}^n n_t e_t : n_t \in \mathbb{Z} \right) = N \cap F_e.$$ 

Clearly $N \cap F_e$ is invariant under both $C$ and $F$. It follows that we can apply our hypothesis to the diagram

$$
\begin{array}{cccccc}
1 & \longrightarrow & N/N \cap F_e & \longrightarrow & C/N \cap F_e & \longrightarrow & Z^* & \longrightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
1 & \longrightarrow & N_0/F_e & \longrightarrow & C_0/F_e & \longrightarrow & Z^* & \longrightarrow & 1,
\end{array}
$$

since $B^*(C_0/F_e) = (M/F_e) \cdot T^*$, where $T^*$ is the group of automorphisms of $M/F_e$ induced by $T$. In general we shall denote by $x^*$ the image of an $x$ in $B^*(C_0)$ under the natural map of $B^*(C_0)$ onto $B^*(C_0)/F_e$. We will denote by $X^*$ the automorphism of $B^*(C_0)/F_e$ induced by the automorphism $X$ of $B^*(C_0)$. By induction there exists a group $N^*$ of $N_0/F_e$ containing $F_e N/F_e$ as a subgroup of finite index, such that $C^*$ the image of $C$ in $B^*(C_0)/F_e$ normalizes $N^*$, and $n_T(N^* C^*)$ is a lattice subgroup of $F^*$. Choose $e_1, \ldots, e_t$ in $L(F)$ such that $e_1, \ldots, e_n$ is a canonical basis of $L(F)$ and $\exp(\sum_{t} n_t e_t^*) : n_t \in \mathbb{Z} = n(N^* C^*)$. The Hausdorff-Campbell formula applied to nilpotent groups gives us an integer such that the lattice $\Delta$ generated by $e_1, \ldots, e_t, e_{t+1} \ldots, e_n, m$ in $L(F)$ has the property that $\exp(\Delta)$ is a subgroup of $F$. Since $N$ is finitely generated and its image in $B^*(C_0)/F_e$ is contained in the corresponding image of $\exp(\Delta)$ it is clear that we can choose $m$ large enough so that $\exp(\Delta)$ contains $N$. From the fact that $C$ is finitely generated, $C^*$ preserves the image of $\exp(\Delta)$ in $F/F_e$, and $C$ preserves $N \cap F_e$, it follows that we can choose $m$ large enough so that
exp(Δ) is preserved by C. Letting \( N' \) be equal to \( \exp(Δ) \cap N_0 \) this case of the theorem has been proved.

Returning to the general case we note that \( N_0 \) the identity component of \( N \) is normal in both \( N_R \) and \( C \) and hence in \( C_R \). Dividing out by it we see that to complete the theorem it is sufficient to consider the following case. Let \( C \) be a group which satisfies the short exact sequence

\[
1 \to N \to C \xrightarrow{\psi} R^t \oplus Z^s \to 1,
\]

where \( N \) is an \( FN \)-group. Let \( C_1 = \rho^{-1}(Z^s) \). Let \( C_0 \) be the identity component of \( C \). Clearly \( C = C_1 \oplus C_0 \), where \( C_0 \) is a central subgroup. Let \( C_1^* \cdot T \) be a semi-simple splitting of \( C_1 \) as determined by the above discussion. Extending \( T \) to \( C^* = C_1^*C_0 \) by having it act by the identity map on \( C_0 \) gives us a semi-simple splitting of \( C \) relative to \( T \).

In terms of the \( \epsilon \)-semi-simple splitting our theorem is as follows.

**Theorem 8.** Let \( B(C_R) = M \cdot A \) be a Malcev decomposition relative to \( C_R \). Then there is a subgroup \( N' \) of \( N_R \) containing \( N \) as a subgroup of finite index such that

1. \( N' \) is normalized by \( C \). Let \( C' = N'C \) in \( C_R \).
2. \( n_B(C') \) is a lattice subgroup of \( M_R \), where \( n_B : B(C) \to M \) is the projection onto the first factor relative to the decomposition \( B(C) = M \cdot B \).
3. \( B \) normalizes \( N' \) and \( C' \).
4. \( C' \cdot B/N' \) is abelian.

We will call \( C' \cdot B \) an \( \epsilon \)-semi-simple splitting of the CS-group \( C \). We will call the decomposition \( C' \cdot B = n_B(C') \cdot B \) a Malcev decomposition of \( C' \cdot B \).


Let \( C \) be a closed subgroup of a simply connected solvable analytic group \( S \). Let \( H \) be the connected nil-radical of \( S \). \( S \) is an \( \epsilon \)-group given by the short exact sequence

\[
1 \to H \to S \to S/H \to 1.
\]

Our next lemma shows that the short exact sequence

\[
1 \to C \cap H \to C \to C/C \cap H \to 1,
\]

is a CS-group. Although we shall not do so it can be easily proved that every CS-group is a closed subgroup of a simply connected solvable analytic group. For a proof of this theorem see of [2].
Lemma 18. We will use the above notation. Let \( C_0 \) be the identity component of \( C \). Then \( C/C_0 \) is finitely generated. \( C/C \cap H \) is a torsion free elementary abelian group.

Proof. Let \( C_R = (C \cap H)_R C \) in \( S \), where \( (C \cap H)_R \) is the Lie hull of \( C \cap H \) in \( H \). Then \( C_R \) is a closed subgroup of \( S \) and satisfies the commutative diagram

\[
1 \longrightarrow C \cap H \longrightarrow C \longrightarrow A \longrightarrow 1
\]

\[
1 \longrightarrow (C \cap H)_R \longrightarrow C_R \longrightarrow A \longrightarrow 1,
\]

where \( A = C/C \cap H \). Let \( B(S) = M \cdot T \) be a Malcev decomposition of the \( \epsilon \)-semi-simple splitting of \( S \). Then \( n_R(C_R) \) is a closed subgroup of \( M \). Hence \( C_R/(C_R)_0 \) is homeomorphic to \( n_R(C_R)/(n(C_R))_0 \). But the latter group is a discrete subgroup of a real nilpotent group and therefore is finitely generated. This implies that \( C_R/(C_R)_0 \) is finitely generated. The lemma follows from this fact.

Theorem 9. Let \( C \) be a closed subgroup of a simply connected solvable analytic group \( S \). Consider \( C \) as a CS-group given by the short exact sequence

\[
1 \rightarrow C \cap H \rightarrow C \rightarrow C/C \cap H \rightarrow 1.
\]

Let \( C' \cdot B \) be a semi-simple splitting of \( C \). Then there is a homomorphism \( \varphi : C' \cdot B \rightarrow B(S) \), extending the injection of \( C' \) into \( S \). Let \( C' \cdot B = P' \cdot B \) be a Malcev decomposition of \( C' \cdot B \). Then there exists a Malcev decomposition \( B(S) = N_S \cdot B_S \) of \( B(S) \) such that \( \phi \) maps \( P' \) isomorphically onto a closed subgroup of \( N_S \).

Proof. The theorem follows directly from the functorial properties of \( B \) and from the previous theorem.

Chapter II. Matrix Representations

1. Introduction.

In this section we will relate our semi-simple splitting functor to the theory of matrix representation, and to the computation of the Hochschild-Mostow functor introduced in [5]. By a vector space we will mean unless otherwise specified, a finite dimensional complex vector space. Let \( V \) be a vector space. By \( E(V) \) we mean the algebra of all linear endomorphisms of \( V \) topologized by the coarsest topology making all linear maps of \( E(V) \) into \( C \) continuous.
Let $\text{Aut}(V)$ be the group of all linear isomorphisms of $V$, topologized by the topology induced by the topology on $E(V)$. Let $G$ be an arbitrary topological group. A representation of $G$ is a continuous homomorphism of $G$ into $\text{Aut}(V)$, for some vector space $V$. We will use the following notation. Let $\rho : G \rightarrow \text{Aut}(V)$ be a representation of $G$. Then we denote $V$ by $V_\rho$. The algebraic hull of $\rho(G)$ will be denoted by $\rho(G)^a$. The set of all unipotent elements in the radical of $\rho(G)^a$ will be denoted by $U_\rho$. Note that when $G$ is solvable, $U_\rho$ is the algebraic group of all unipotent elements in $\rho(G)^a$. Let $V_1$ be a subspace of the vector space $V_2$. Let $E(V_2 ; V_1)$ be the algebra of all linear endomorphisms $A$ of $V_2$ such that $A(V_1) \subset V_1$. Let $\text{Aut}(V_2 ; V_1)$ be the group of all linear automorphisms $A$ of $V_2$ such that $A(V_1) = V_1$. Define $r_{V_1}^V : \text{Aut}(V_2 ; V_1) \rightarrow \text{Aut}(V_1)$ by $r_{V_1}^V(A) = \text{the restriction of } A \text{ to } V_1$. Then $r_{V_1}^V$ is an everywhere defined rational representation of $\text{Aut}(V_2 ; V_1)$ in $\text{Aut}(V_1)$. Hence from the general theory of algebraic groups over an algebraically closed field the following statements are true.

1. If $X$ is an algebraic subgroup of $\text{Aut}(V_2 ; V_1)$, then $r_{V_1}^V(X)$ is an algebraic subgroup of $\text{Aut}(V_1)$.

2. $r_{V_1}^V$ takes unipotent elements into unipotent elements, semi-simple elements into semi-simple elements and maximal completely reducible subgroups onto maximal completely reducible subgroups.

2. Inverse Limit Systems of Representations.

We will assume that the reader has knowledge of the basic facts of inverse limit systems. We give [14] as reference. Let $G$ be a topological group. Let $\rho$ and $\varphi$ be representations of $G$. We say that $\rho \leq \varphi$ if and only if $V_\rho$ is a subspace of $V_\varphi$ and $r_{\varphi}^\rho(\varphi(G)) = \rho(G)$. We denote $r_{\varphi}^\rho$ by $r_{\varphi}^\rho$. It is clear that $r_{\varphi}^\rho(\varphi(G)^a) = \rho(G)^a$. Hence we have an inverse system $\{\rho(G)^a, r_{\varphi}^\rho\}$ of algebraic linear groups $\rho(G)^a$ and rational epimorphisms $r_{\varphi}^\rho : \varphi(G)^a \rightarrow \rho(G)^a$ for $\rho \leq \varphi$. This inverse system defines in the category of topological groups a limit group $A(G) = \lim \rho(G)^a$. We denote the projection map of $A(G)$ into $\rho(G)^a$ by $r_{\rho}$. Let $G$ and $G'$ be topological groups and let $\varphi : G \rightarrow G'$ be a homomorphism. Let $\rho'$ be a representation of $G'$. Then $\rho' \varphi$ defines a representation of $G$ and $(\rho' \varphi)(G)^a \subset \rho'(G')$. Hence $\varphi$ induces a morphism $\{\rho' \varphi\}$ of the inverse limit system $\{\rho(G)^a, r_{\varphi}^\rho\}$ into the inverse limit system $\{\rho'(G')^a, r'_{\varphi}^\rho\}$. Let $A(\varphi) = \lim \varphi$. From the general theory of inverse limit systems the maps $G \rightarrow A(G)$ and $\varphi \rightarrow A(\varphi)$ define a right exact covariant functor on the category of topological groups. The functor will be called the Hochschild–Mostow functor. In general the inverse limit of algebraic groups and rational homomorphisms is not an algebraic group. We can however define a concept of dimension for the inverse limit. Let $\{G_\alpha, r_\alpha^\beta\}$ be an inverse system of Lie groups $G_\alpha$ and analytic homomorphisms $r_\alpha^\beta : G_\beta \rightarrow G_\alpha$. Let $L(G_\alpha)$ be the
Lie algebra of $G_a$ and let $L(r_x^y) : L(G_y) \to L(G_x)$ be the differential of $r_x^y$. Then $\{L(G_x), L(r_x^y)\}$ is an inverse system of Lie algebras $L(G_x)$ and Lie algebra homomorphisms $L(r_x^y)$. We define

$$\dim(\lim_{\to} G_x) = \dim(\lim_{\to} L(G_x)).$$

Let $G$ be a solvable Lie group. Let $\rho$ be a representation of $G$. Then $U_\rho$ is the algebraic group of all unipotent elements of $\rho(G)^\#$. Also if $\rho$ and $\varphi$ are representations of $G$ such that $\rho \leq \varphi$ then $r^\varphi_\rho(U_\rho) = U_\rho$. Let $G$ and $G'$ be solvable Lie groups and let $\varphi : G \to G'$ be a homomorphism. Then for every representation $\rho'$ of $G'$, we have $U_{\rho'} \subseteq U_{\rho'}$. Therefore $\varphi$ induces a morphism of the inverse limit system $\{U_{\rho'}, r_{\rho'}\}$ into $\{U_{\rho'}, r_{\rho'}\}$. Let $U(\varphi) = \lim_{\to} \varphi_n$. Then the map $G \to U(G)$ and $\varphi \to U(\varphi)$ is a right exact covariant functor.

3. Representation of $e$-Groups.

Let $A$ be an elementary group. Write $A = R^s \oplus Z^t \oplus K$ where $K$ is a compact group. Then we have defined the rank of $A$ as $s + t$. Let $C$ be a group given by a short sequence

$$1 \to N \to C \to A \to 1,$$

where $N$ is a real nilpotent group and $A$ is an elementary group. Then we have defined the rank of $C$ to be equal to $\dim(N) + \operatorname{rank}(A)$.

**Lemma 1.** Let $1 \to N \to C \to A \to 1$ be an $e$-group where $N$ is a real nilpotent group and $A$ is an elementary group. Then there exists a faithful matrix representation of $C$, $\beta : C \to GL(n, \mathbb{C})$ such that $\dim(U_\beta) = \operatorname{rank}(C)$.

**Proof.** Let $\beta$ be the faithful matrix representation defined by the Birkhoff imbedding theorem as in Lemma 2 of Chapter I. Then Lemma 1 follows directly from Corollary 5.1.

**Lemma 2.** Let $1 \to N \to C \to A \to 1$ be an $e$-group where $N$ is a real nilpotent group and $A$ is an elementary group. Let $\rho : C \to \operatorname{Aut}(V)$ be a representation of $C$. Then $\operatorname{rank}(C) \geq \dim(U_\rho)$.

**Proof.** Let $\rho(N)^\# = V^\# : S^\#$ where $V^\#$ is the group of all unipotent elements in $\rho(N)^\#$. Since $\rho(N)^\#$ is nilpotent and connected, $S^\#$ is a central subgroup of semi-simple automorphisms of $\rho(N)^\#$. Let $\rho(C)^\# = U^\# : T^\#$, where $U^\#$ is the group of all unipotent elements in $\rho(C)^\#$ and $T^\#$ is a maximal completely reducible subgroup of $\rho(C)^\#$ containing $S^\#$. Let $\rho_1$ be the projection onto the
first factor of $\rho(C)^\#$ relative to the decomposition $\rho(C)^\# = U^* T^*$. Since $\rho(N, N)$ is an analytic subgroup of $V^*$, it is an algebraic subgroup of $V^*$, and hence $(N)^* / \rho(N, N)$ is abelian. Hence $p_1(\rho(N))$ is an algebraic subgroup of $V^*$. Therefore $p_1(\rho(N)) = V^*$ and $\dim V^* \leq \dim N$. Now since $\rho(N)^\#$ is a normal algebraic subgroup of $\rho(C)^\#$, there is a rational representation of $\rho(C)^\#$, whose kernel is $\rho(N)^\#$. Hence we may consider $B^* = \rho(C)^\#/\rho(N)^\#$ as an abelian linear group. Write $B^* = W \cdot D$, where $W$ is the group of all unipotent elements in $B^*$ and $D$ is a maximal completely reducible subgroup of $B^*$. Since $B^*$ is abelian $D$ is the central subgroup of all semi-simple automorphisms in $B^*$. Clearly $W = U^*/V^*$. Denote the image of $\rho(C)$ in $B^*$ by $B$. Then $B$ is a Zariski dense elementary subgroup of $B^*$ where rank $B \leq \text{rank } C/N$. In this case it is clear that $\dim W \leq \text{rank } B$.

**Theorem 1.** Let $\rho : C \to \GL(n, \mathbb{C})$ be a faithful representation of $C$ such that $\dim U_\rho = \text{rank}(C)$. Then $U(C) = U_\rho$. In particular if $C$ is a torsion free $e$-group and if $B(C) = M \cdot T$ is a Malcev decomposition of $B(C)$, then $U(C) = \text{the Lie hull of } M$.

**Proof.** Since we can add representations the representation $\rho$ of $C$ such that $\rho \leq \rho'$ define a cofinal subset of the collection of all representations of $C$. Hence by the general theory of inverse systems we have $U(C) = \lim_{\rho \leq \rho'} U_\rho$. But by the previous lemma $r_\rho : U^\# \to U_\rho$ for $\rho \leq \rho'$ is an isomorphism. Therefore $r_\rho : U(C) \to U_\rho$ for $\rho \leq \rho'$ is an isomorphism. The second conclusion of the theorem follows from this case since when $C$ is torsion free, $M$ is a torsion free nilpotent $e$-group. This implies that under the Birkhoff imbedding theorem $M$ is mapped onto a closed co-compact subgroup of $U_\rho$.

**Corollary 1.1.** For every $e$-group $C$, $\dim(U(C)) = \text{rank}(C)$.

**Corollary 1.2.** $U(C)$ is an exact functor on the category of $e$-groups and arbitrary homomorphisms.

**Proof.** Consider the exact sequence $1 \to G' \to G \to G^* \to 1$ of $e$-groups. We know that $U(C') \to U(C) \to U(C^*) \to 1$ is exact. Since $\dim U(C) = \text{rank}(C)$ the corollary immediately follows.

We have used our methods to compute the unipotent hull of the Hochschild–Mostow functor. The following theorem proved in [5] gives us information about the action of $A(C)$ on $U(C)$.

**Theorem 2.** For every representation $\rho$ of $C$, $r_\rho : A(C) \to \rho(C)^\#$ is surjective.

This theorem immediately gives the following corollary.
Corollary 2.1. Let $C$ be an $e$-group given by the short exact sequence

$$1 \to N \to C \to A \to 1,$$

where $N$ is a real nilpotent group and $A$ is a torsion free elementary group. Let $B(C) = M \cdot B$ be a Malcev decomposition of the $e$-semi-simple splitting of $C$ related to $C$. Then

$$ad_{(U(C))} A(C) = (ad_{N(M)} B(C))^e.$$

Note that in general $A(C)$ is infinite dimensional.

Example. Let $N = \left( \begin{smallmatrix} x & 0 \\ 0 & 1 \end{smallmatrix} \right)$, $x \in R$. Let $\rho_n : N \to GL(n, R)$ be given by $\rho_n \left( \begin{smallmatrix} x \\ 0 \end{smallmatrix} \right) = \left( \begin{smallmatrix} e^{x_1} & 0 & \cdots & 0 \\ 0 & e^{x_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{x_n} \end{smallmatrix} \right)$, $\lambda_1, \ldots, \lambda_n$ distinct element of $R$. Then $\rho_n(N)^e$ has dimension equal to $n$.

References