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Calculation of fields of magnetic deflection systems with FEM using a vector potential approach - Part I: stationary fields

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Abstract

A procedure is presented for calculating stationary fields in magnetic deflection systems (especially saddle coils) in a rotationally symmetric ferromagnetic surrounding using the FEM method and a vector potential approach. The vector potential and the current distribution are expanded as Fourier series with respect to the azimuthal coordinate φ. Consequently each Fourier harmonic can be handled as a separate two-dimensional problem. Both the energy functional for the m^{th} harmonic and the local FEM equations are derived. The global FEM system of equations is given.

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1. Introduction

The computation of stationary fields of magnetic deflection systems (e.g. saddle coils) using the finite element method (FEM) has been described in various papers (e.g. [1], [2]). The use of the FEM is necessary for magnetic deflection systems in a ferromagnetic surrounding, e.g. for a deflector inside a magnetic lens. All the authors of the papers mentioned above ([1], [2]) tried to find a solution for the magnetic field strength of the form

$$\vec{H} = grad\,\Phi + \vec{F},$$

where Φ is the magnetic scalar potential and $\vec{F}$ is some vector function which is non-vanishing inside the coil windings only and which is closely related to the current in the coil. Therefore the task to find the magnetic field is essentially reduced to the solution of the FEM problem for the magnetic scalar potential Φ.

Moreover, since both the magnetic scalar potential and the current distribution are expanded into a Fourier series with respect to the azimuth angle φ of the cylindrical coordinate system, each Fourier harmonic of Φ can be handled as a separate two-dimensional FEM problem Therefore only few two-dimensional FEM problems have to be solved

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instead of solving a much more complicated three-dimensional FEM problem, as long as the distribution of ferromagnetic materials is rotationally symmetric.

The use of the scalar magnetic potential Φ is possible because $curl\,\vec{H}$ is zero outside the coil windings. However, this approach is no longer possible if time-dependent processes are considered because eddy currents are created and $curl\,\vec{H}$ is nonzero in eddy current regions. In these cases the magnetic vector potential has to be used. Time-dependent three-dimensional field approaches with the magnetic vector potential have been described elsewhere [3], and commercially available FEM software can be used to simulate three-dimensional eddy current problems. However, the fully three-dimensional approach is very time-consuming and requires a large amount of computer resources. In the specific case of a deflector inside a complicated magnetic lens surrounding with shielding ferrites etc. the three-dimensional approach is nearly impossible. Fortunately in electron optics only few field harmonics are of interest and therefore the three-dimensional problem can be replaced by few two-dimensional problems. Therefore the purpose of this paper is to modify the procedure prescribed in [1] and [2] by using the magnetic vector potential throughout. In the present paper, as a first step we focus on stationary problems, especially for saddle coils. In the subsequent paper of this series we will describe the time-dependant case. For very simple cases results can be achieved by analytical means, too [4].

2. Energy Functional for the Vector Potential Formulation

The Maxwell equations for a stationary magnetic field are

$$\begin{aligned} &curl\,\vec{H} = \vec{j} \\ &div\,\vec{B} = 0 \\ &\vec{B} = \mu_0\mu_r\vec{H}. \end{aligned} \tag{1}$$

The second equation is satisfied with the ansatz

$$\vec{B} = curl\,\vec{a}, \tag{2}$$

where $\vec{a}$ is the magnetic vector potential. Therefore the system of Maxwell equations reduces to the single equation

$$curl\left(\frac{1}{\mu_0\mu_r}curl\,\vec{a}\right) = \vec{j}. \tag{3}$$

Equation (3) can be derived as Euler-Lagrange equation from an energy functional containing the Lagrange density

$$£ = -\frac{1}{2\mu_0\mu_r}\left(curl\,\vec{a}\right)^2 + \vec{j}\vec{a}. \tag{4}$$

In the following we use cylindrical coordinates (r,φ,z). In these coordinates (4) can be written as

$$\begin{aligned} £ = -\frac{1}{2\mu_0\mu_r}&\left[\left(\frac{1}{r}\frac{\partial a_z}{\partial\varphi} - \frac{\partial a_\varphi}{\partial z}\right)^2 + \left(\frac{\partial a_r}{\partial z} - \frac{\partial a_z}{\partial r}\right)^2 + \frac{1}{r^2}\left(r\frac{\partial a_\varphi}{\partial r} + a_\varphi - \frac{\partial a_r}{\partial\varphi}\right)^2\right] \\ &+ \left(j_r a_r + j_\varphi a_\varphi + j_z a_z\right). \end{aligned} \tag{5}$$

We now expand the components of the vector potential as a Fourier series of m^{th} harmonics in the azimuthal coordinate φ, i.e.

$$a_\alpha(r,\varphi,z) = \sum_{m=1,3,5...}\left[a_\alpha^m(r,z)\cdot\cos(m\varphi) + b_\alpha^m(r,z)\cdot\sin(m\varphi)\right], \tag{6}$$

where the Greek indices $\alpha,\beta,\gamma,... = 0,1,2$ are running through the coordinates r,φ,z throughout the paper. In the case of deflection systems without tolerancing errors only odd Fourier harmonics contribute to the expansion (6).

In the following we confine ourselves to the case of magnetic saddle coils, although the treatment of toroidal coils can be handled similarly. As shown by Munro and Chu [1] and Lencova et al [2] the current density $\vec{j} = (j_r, j_\varphi, j_z)$ of a saddle coil can be derived from a single function $F_r(r, \varphi, z)$ which is non-vanishing only inside the coil windings, i.e.

$$j_r = 0 \qquad j_\varphi = \frac{\partial F_r}{\partial z} \qquad j_z = -\frac{1}{r} \cdot \frac{\partial F_r}{\partial \varphi}. \tag{7}$$

The function $F_r(r, \varphi, z)$ can be written as a product

$$F_r(r, \varphi, z) = g(r, z) \cdot f(\varphi) \tag{8}$$

where $g(r, z)$ is given by ([1], [2])

$$g(r, z) = \begin{Bmatrix} 0 \ \ \textit{outside the coil windings} \\ 1 \ \ \textit{inside the coil windings} \end{Bmatrix}, \tag{9}$$

and $f(\varphi)$ is the “current loading function”, which can also be expanded as a Fourier series

$$f(\varphi) = \sum_{m=1,3,5,\ldots} f_m \cos m\varphi \tag{10}$$

with the coefficients

$$f_m = \frac{4I}{\pi m \Delta R} \cdot \sin m\Theta, \tag{11}$$

where I is the current inside the coil, ΔR is the wire thickness and Θ is the semi-angle of the saddle winding. The definition of “outside the coil windings” and “inside the coil windings” is given in [1].

Now the energy functional

$$W = \int \pounds \, r \, dr \, d\varphi \, dz \tag{12}$$

can also be expanded as a series

$$W = \sum_{m=1,3,5\ldots} W^m. \tag{13}$$

Integrating over φ and using the orthonormalization relations of the trigonometric functions we finally obtain the energy functional of the m^{th} harmonic

$$W^m = -\frac{\pi^2}{2} \cdot \int \frac{1}{\mu_0 \mu_r} \cdot \left\{ \begin{aligned} & \left(\frac{m}{r} a_2^m + \frac{\partial b_1^m}{\partial z} \right)^2 + \left(\frac{m}{r} b_2^m - \frac{\partial a_1^m}{\partial z} \right)^2 + \left(\frac{\partial a_0^m}{\partial z} - \frac{\partial a_2^m}{\partial r} \right)^2 + \left(\frac{\partial b_0^m}{\partial z} - \frac{\partial b_2^m}{\partial r} \right)^2 \\ & + \left(\frac{\partial a_1^m}{\partial r} + \frac{1}{r} \cdot a_1^m - \frac{m}{r} b_0^m \right)^2 + \left(\frac{\partial b_1^m}{\partial r} + \frac{1}{r} b_1^m + \frac{m}{r} a_0^m \right)^2 \\ & - 2 \mu_0 \mu_r \frac{\partial g}{\partial z} f_m a_1^m - 2 \mu_0 \mu_r \frac{1}{r} g f_m m b_2^m \end{aligned} \right\} r dr dz \tag{14}$$

where the range of integration extends over the whole region in the r-z-plane considered. Note that the last two terms in the brackets correspond to the fields of the arc-like and the straight wires of the saddle coil, respectively. It becomes evident that it was important in equation (6) to include not only the cos terms but also the sin terms. Otherwise the fields of the straight wires would have been neglected. The last but one term in the brackets in (14) containing $\partial g / \partial z$ can be rewritten by using integration by parts.

3. Discretization of the r-z plane and local FEM equations

We utilize the first order FEM method (FOFEM) and subdivide the region in the r-z plane into quadrilaterals, see Fig. 1.

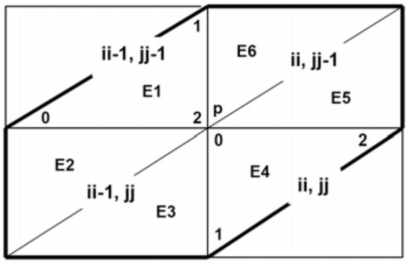

Fig.1: Part of a finite element mesh. The *r-z*-plane is subdivided into quadrilaterals. The quadrilaterals are numbered by *ii* = 0…*nez*-2, *jj* = 0…*ner*–2, where *nez* is the number of mesh points in *z* direction and *ner* is the number of mesh points in *r* direction. Each quadrilateral is subdivided into a *left upper triangle* and *right lower triangle*. There are six triangles E1…E6 surrounding the mesh point p. The corners of the triangles are numbered i = 0…2 as shown for the left upper triangle E4 of the quadrilateral *ii*, *jj* and the left lower triangle E1 of the quadrilateral *ii*–1, *jj*–1.

In each triangle we make the linear ansatz for the vector potential components

$$a_\alpha^m(r,z) = A_\alpha^m \cdot r + B_\alpha^m \cdot z + C_\alpha^m$$
$$b_\alpha^m(r,z) = D_\alpha^m \cdot r + E_\alpha^m \cdot z + F_\alpha^m . \tag{15}$$

Let $a_{\alpha,i}^m$ and $b_{\alpha,i}^m$ be the values of the components of the vector potential of the m^{th} harmonic in the i^{th} point (node) (i = 0…2) of the triangle considered. The coefficients $A_\alpha^m, B_\alpha^m, C_\alpha^m, D_\alpha^m, E_\alpha^m, F_\alpha^m$ are determined by the condition that the potentials in the corners of the triangle coincide with the potentials $a_{\alpha,i}^m$ in the mesh points. The coefficients are given by

$$A_\alpha^m = \left(1/D\right) \cdot \left[c_0 a_{\alpha,0}^m + c_1 a_{\alpha,1}^m + c_2 a_{\alpha,2}^m\right]$$

$$B_\alpha^m = \left(1/D\right) \cdot \left[b_0 a_{\alpha,0}^m + b_1 a_{\alpha,1}^m + b_2 a_{\alpha,2}^m\right]$$

$$C_\alpha^m = -\left(1/D\right) \cdot \left[\left(r_1 z_2 - r_2 z_1\right) \cdot a_{\alpha,0}^m + \left(r_2 z_0 - r_0 z_2\right) \cdot a_{\alpha,1}^m + \left(r_0 z_1 - r_1 z_0\right) \cdot a_{\alpha,2}^m\right]$$

$$D_\alpha^m = \left(1/D\right) \cdot \left[c_0 b_{\alpha,0}^m + c_1 b_{\alpha,1}^m + c_2 b_{\alpha,2}^m\right]$$

$$E_\alpha^m = \left(1/D\right) \cdot \left[b_0 b_{\alpha,0}^m + b_1 b_{\alpha,1}^m + b_2 b_{\alpha,2}^m\right]$$

$$F_\alpha^m = -\left(1/D\right) \cdot \left[\left(r_1 z_2 - r_2 z_1\right) \cdot b_{\alpha,0}^m + \left(r_2 z_0 - r_0 z_2\right) \cdot b_{\alpha,1}^m + \left(r_0 z_1 - r_1 z_0\right) \cdot b_{\alpha,2}^m\right] \tag{16}$$

where

$$D = b_0 c_1 - b_1 c_0 = b_1 c_2 - b_2 c_1 = b_2 c_0 - b_0 c_2 \tag{17}$$

and $\left(r_0, z_0\right)$, $\left(r_1, z_1\right)$ and $\left(r_2, z_2\right)$ are the coordinates of the corners of the triangle. $|D|$ is double the area of the triangle and b_i, c_i are the coordinate differences

$$b_0 = r_1 - r_2 \qquad b_1 = r_2 - r_0 \qquad b_2 = r_0 - r_1$$
$$c_0 = z_2 - z_1 \qquad c_1 = z_0 - z_2 \qquad c_2 = z_1 - z_0 . \tag{18}$$

We denote by $\Delta W^{m,ii,jj,lower}$ the approximation of the energy functional of the m^{th} Fourier harmonic integrated over the right lower triangle of the quadrilateral (ii, jj). (For a left upper triangle the energy functional is $\Delta W^{m,ii,jj,upper}$.) Performing integration by parts in equation (14) and carrying out the usual discretization procedure we obtain the energy functional integrated over one triangle:

$$\Delta W^{m,ii,jj,lower} = -\frac{\pi^2}{4\mu_0\mu_r}\cdot|D|\cdot r_c\cdot\left\{\begin{array}{l}\left(\frac{m}{r_c}\cdot\frac{1}{3}\sum_{i=0}^{2}a_{2,i}^m+\frac{1}{D}\cdot\sum_{i=0}^{2}b_ib_{1,i}^m\right)^2+\left(\frac{m}{r_c}\cdot\frac{1}{3}\sum_{i=0}^{2}b_{2,i}^m-\frac{1}{D}\cdot\sum_{i=0}^{2}b_ia_{1,i}^m\right)^2\\ +\left(\frac{1}{D}\cdot\sum_{i=0}^{2}b_ia_{0,i}^m-\frac{1}{D}\cdot\sum_{i=0}^{2}c_ia_{2,i}^m\right)^2+\left(\frac{1}{D}\cdot\sum_{i=0}^{2}c_ib_{0,i}^m-\frac{1}{D}\cdot\sum_{i=0}^{2}c_ib_{2,i}^m\right)^2\\ +\left(\frac{1}{D}\cdot\sum_{i=0}^{2}c_ia_{1,i}^m+\frac{1}{r_c}\cdot\frac{1}{3}\sum_{i=0}^{2}a_{1,i}^m-\frac{m}{r_c}\cdot\frac{1}{3}\sum_{i=0}^{2}b_{0,i}^m\right)^2\\ +\left(\frac{1}{D}\cdot\sum_{i=0}^{2}c_ib_{1,i}^m+\frac{1}{r_c}\cdot\frac{1}{3}\sum_{i=0}^{2}b_{1,i}^m+\frac{m}{r_c}\cdot\frac{1}{3}\sum_{i=0}^{2}a_{0,i}^m\right)^2\\ +2\mu_0\mu_r\cdot f_m g(r_c,z_c)\cdot\frac{1}{D}\sum_{i=0}^{2}b_ia_{1,i}^m\\ -2\mu_0\mu_r\cdot f_m m g(r_c,z_c)\cdot\frac{1}{3r_c}\sum_{i=0}^{2}b_{2,i}^m\end{array}\right\} \tag{19}$$

(Similarly the procedure is executed for the left upper triangles). Here $|D|/2$ is the area of the triangle and (r_c, z_c) denotes the coordinates of the centroid of the triangle.

The local FEM equations, which state how the value of $\Delta W^{m,ii,jj,lower}$ changes if the potentials at the corners of the triangles change, are

$$\frac{\partial \Delta W^{m,ii,jj,lower}}{\partial a_{\alpha,i}^m} = \sum_{\beta=0}^{2}\sum_{j=0}^{2} F_{\alpha,\beta,i,j}^{m,ii,jj,lower}\cdot a_{\beta,j}^m + \sum_{\beta=0}^{2}\sum_{j=0}^{2} H_{\alpha,\beta,i,j}^{m,ii,jj,lower}\cdot b_{\beta,j}^m + G_{\alpha,i}^{m,ii,jj,lower} \tag{20a}$$

$$\frac{\partial \Delta W^{m,ii,jj,lower}}{\partial b_{\alpha,i}^m} = \sum_{\beta=0}^{2}\sum_{j=0}^{2} R_{\alpha,\beta,i,j}^{m,ii,jj,lower}\cdot a_{\beta,j}^m + \sum_{\beta=0}^{2}\sum_{j=0}^{2} S_{\alpha,\beta,i,j}^{m,ii,jj,lower}\cdot b_{\beta,j}^m + T_{\alpha,i}^{m,ii,jj,lower} \tag{20b}$$

(similarly for the left upper triangles). The coefficients are given by

$$\begin{aligned}
F_{0,0,i,j}^{m,ii,jj,lower} &= S_{0,0,i,j}^{m,ii,jj,lower} = -f\cdot r_c\cdot\left(\frac{1}{D^2}\cdot b_ib_j+\frac{m^2}{9r_c^2}\right)\\
F_{1,1,i,j}^{m,ii,jj,lower} &= S_{1,1,i,j}^{m,ii,jj,lower} = -f\cdot r_c\cdot\left(\frac{1}{D^2}\cdot b_ib_j+\left(\frac{1}{D}c_i+\frac{1}{3r_c}\right)\cdot\left(\frac{1}{D}\cdot c_j+\frac{1}{3r_c}\right)\right)\\
F_{2,2,i,j}^{m,ii,jj,lower} &= S_{2,2,i,j}^{m,ii,jj,lower} = -f\cdot r_c\cdot\left(\frac{1}{D^2}\cdot c_ic_j+\frac{m^2}{9r_c^2}\right)\\
F_{0,2,i,j}^{m,ii,jj,lower} &= S_{0,2,i,j}^{m,ii,jj,lower} = f\cdot\frac{r_c}{D^2}\cdot b_ic_j\\
F_{2,0,i,j}^{m,ii,jj,lower} &= S_{2,0,i,j}^{m,ii,jj,lower} = f\cdot\frac{r_c}{D^2}\cdot c_ib_j
\end{aligned} \tag{21a}$$

$$\begin{aligned}
&H_{0,1,i,j}^{m,ii,jj,lower} = -R_{0,1,i,j}^{m,ii,jj,lower} = -\frac{1}{3} f \cdot m \cdot \left(\frac{1}{D} \cdot c_j + \frac{1}{3r_c} \right) \\
&H_{1,0,i,j}^{m,ii,jj,lower} = -R_{1,0,i,j}^{m,ii,jj,lower} = \frac{1}{3} f \cdot m \cdot \left(\frac{1}{D} \cdot c_i + \frac{1}{3r_c} \right) \\
&H_{1,2,i,j}^{m,ii,jj,lower} = -R_{1,2,i,j}^{m,ii,jj,lower} = \frac{1}{3} f \cdot m \cdot \frac{1}{D} . b_i \\
&H_{2,1,i,j}^{m,ii,jj,lower} = -R_{2,1,i,j}^{m,ii,jj,lower} = -\frac{1}{3} f \cdot m \cdot \frac{1}{D} \cdot b_j \\
&F_{0,1,i,j}^{m,ii,jj,lower} = F_{1,0,i,j}^{m,ii,jj,lower} = F_{1,2,i,j}^{m,ii,jj,lower} = F_{2,1,i,j}^{m,ii,jj,lower} = 0 \\
&S_{0,1,i,j}^{m,ii,jj,lower} = S_{1,0,i,j}^{m,ii,jj,lower} = S_{1,2,i,j}^{m,ii,jj,lower} = S_{2,1,i,j}^{m,ii,jj,lower} = 0 \\
&H_{0,0,i,j}^{m,ii,jj,lower} = H_{1,1,i,j}^{m,ii,jj,lower} = H_{2,2,i,j}^{m,ii,jj,lower} = H_{0,2,i,j}^{m,ii,jj,lower} = H_{2,0,i,j}^{m,ii,jj,lower} = 0 \\
&R_{0,0,i,j}^{m,ii,jj,lower} = R_{1,1,i,j}^{m,ii,jj,lower} = R_{2,2,i,j}^{m,ii,jj,lower} = R_{0,2,i,j}^{m,ii,jj,lower} = R_{2,0,i,j}^{m,ii,jj,lower} = 0
\end{aligned} \tag{21a}$$

$$\begin{aligned}
&G_{0,i}^{m,ii,jj,lower} = 0 \\
&G_{1,i}^{m,ii,jj,lower} = -\frac{\pi^2}{2} \cdot \frac{|D|}{D} \cdot r_c \cdot f_m g(r_c, z_c) b_i \\
&G_{2,i}^{m,,ii,jj,lower} = 0 \\
&T_{0,i}^{m,ii,jj,lower} = 0 \\
&T_{1,i}^{m,ii,jj,lower} = 0 \\
&T_{2,i}^{m,ii,jj,lower} = \frac{\pi^2}{6} \cdot |D| \cdot f_m \cdot m \cdot g(r_c, z_c)
\end{aligned} \tag{21b}$$

(similarly for the left upper triangles), where the abbreviation

$$f = \frac{\pi^2}{2} \cdot \frac{1}{\mu_0 \mu_r} \cdot |D| \tag{22}$$

has been used. Generally speaking, the terms $G_{1,i}^{m,ii,jj,lower}$ come from the arc-like wires of the saddle coil, whereas the terms $T_{2,i}^{m,ii,jj,lower}$ come from the straight wires. The local FEM equations (20a, b) are the basis for the FEM software. Obviously, the local FEM coefficients have the following symmetry:

$$\begin{aligned}
&F_{\alpha,\beta,i,j}^{m,ii,jj,lower} = F_{\beta,\alpha,j,i}^{m,ii,jj,lower} \\
&H_{\alpha,\beta,i,j}^{m,ii,jj,lower} = -H_{\beta,\alpha,j,i}^{m,ii,jj,lower} \\
&R_{\alpha,\beta,i,j}^{m,ii,jj,lower} = -R_{\beta,\alpha,j,i}^{m,ii,jj,lower} \\
&S_{\alpha,\beta,i,j}^{m,ii,jj,lower} = S_{\beta,\alpha,j,i}^{m,ii,jj,lower} .
\end{aligned} \tag{23}$$

This is important because it causes the matrix of the global FEM system of equations to be symmetric (see next section).

4. Global FEM system of equations

The Maxwell equations are fulfilled if the energy functional is minimized. This leads to a linear system of equations. These equations can be written as

$$\begin{aligned}
&\left(\frac{\partial \Delta W^m}{\partial a^m_{\alpha,p}}\right)_{E1}+\left(\frac{\partial \Delta W^m}{\partial a^m_{\alpha,p}}\right)_{E2}+\left(\frac{\partial \Delta W^m}{\partial a^m_{\alpha,p}}\right)_{E3}+\left(\frac{\partial \Delta W^m}{\partial a^m_{\alpha,p}}\right)_{E4}+\left(\frac{\partial \Delta W^m}{\partial a^m_{\alpha,p}}\right)_{E5}+\left(\frac{\partial \Delta W^m}{\partial a^m_{\alpha,p}}\right)_{E6}=0\\
&\left(\frac{\partial \Delta W^m}{\partial b^m_{\alpha,p}}\right)_{E1}+\left(\frac{\partial \Delta W^m}{\partial b^m_{\alpha,p}}\right)_{E2}+\left(\frac{\partial \Delta W^m}{\partial b^m_{\alpha,p}}\right)_{E3}+\left(\frac{\partial \Delta W^m}{\partial b^m_{\alpha,p}}\right)_{E4}+\left(\frac{\partial \Delta W^m}{\partial b^m_{\alpha,p}}\right)_{E5}+\left(\frac{\partial \Delta W^m}{\partial b^m_{\alpha,p}}\right)_{E6}=0,
\end{aligned} \tag{24}$$

where p denotes a point in the finite element mesh, see Fig. 1, and the ΔW^m are the energy functionals of the adjacent triangles E1…E6 of p.

Let $M^m_{6p+3i+\alpha,6q+3j+\beta}$ be the Matrix of the linear system of equations (24a, b). (p, q = 0…nez*ner-1 denote the point in the finite element mesh, i, j = 0, 1 correspond to the cos and sin terms, α, β = 0…2 denote the component of the vector potential at a mesh point). The potential components $a^m_{\alpha,i}$ and $b^m_{\alpha,i}$ (α = 0…2, i = 0…nez*ner-1) are arranged in the solution vector as follows:

$$\begin{pmatrix} a^m_{0,0} & a^m_{1,0} & a^m_{2,0} & b^m_{0,0} & b^m_{1,0} & b^m_{2,0} & a^m_{0,1} & a^m_{1,1} & a^m_{2,1} & b^m_{0,1} & b^m_{1,1} & b^m_{2,1} & \ldots \\ a^m_{0,nez*ner-1} & a^m_{1,nez*ner-1} & a^m_{2,nez*ner-1} & b^m_{0,nez*ner-1} & b^m_{1,nez*ner-1} & b^m_{2,nez*ner-1} \end{pmatrix}^T. \tag{25}$$

The total number of unknowns is $nez \cdot ner \cdot 2 \cdot 3$. Now the global system of equations takes the form:

$$\sum_{q=0}^{nez*ner-1}\left[\sum_{\beta=0}^{2} M^m_{6p+3i+\alpha,6q+3\cdot 0+\beta}\cdot a^m_{\beta,q}+\sum_{\beta=0}^{2} M^m_{6p+3i+\alpha,6q+3\cdot 1+\beta}\cdot b^m_{\beta,q}\right]=RS^m_{6p+3i+\alpha}. \tag{26}$$

The system matrix M represents a $(nez \cdot ner)\times(nez \cdot ner)$ block matrix consisting of 2×2 matrices which are block matrices themselves consisting of 3×3 matrices.

As an example, one block of the system matrix can be expressed as

$$\begin{aligned}
M_{6i+3\cdot 0+\alpha,6(i-ner)+3\cdot 0+\beta} &= F^{m,ii-1,jj-1,lower}_{\alpha,\beta,2,0}+F^{m,ii-1,jj,upper}_{\alpha,\beta,2,0}\\
M_{6i+3\cdot 0+\alpha,6(i-ner)+3\cdot 1+\beta} &= H^{m,ii-1,jj-1,lower}_{\alpha,\beta,2,0}+H^{m,ii-1,jj,upper}_{\alpha,\beta,2,0}\\
M_{6i+3\cdot 1+\alpha,6(i-ner)+3\cdot 0+\beta} &= R^{m,ii-1,jj-1,lower}_{\alpha,\beta,2,0}+R^{m,ii-1,jj,upper}_{\alpha,\beta,2,0}\\
M_{6i+3\cdot 1+\alpha,6(i-ner)+3\cdot 1+\beta} &= S^{m,ii-1,jj-1,lower}_{\alpha,\beta,2,0}+S^{m,ii-1,jj,upper}_{\alpha,\beta,2,0} \qquad \alpha,\beta=0...2.
\end{aligned} \tag{27}$$

The other blocks have a similar structure. For convenience, only the matrix elements containing $F^{m,ii,jj,lower}_{\alpha,b,i,j}$ and $F^{m,ii,jj,upper}_{\alpha,b,i,j}$ are given, the elements containing $H^{m,ii,jj,lower}_{\alpha,b,i,j}, H^{m,ii,jj,upper}_{\alpha,b,i,j}, R^{m,ii,jj,lower}_{\alpha,b,i,j}, R^{m,ii,jj,upper}_{\alpha,b,i,j}, S^{m,ii,jj,lower}_{\alpha,b,i,j}, S^{m,ii,jj,upper}_{\alpha,b,i,j}$ can be deduced from (27):

$$\begin{aligned}
M_{6i+3\cdot 0+\alpha,6(i-ner)+3\cdot 0+\beta} &= F^{m,ii-1,jj-1,lower}_{\alpha,\beta,2,0}+F^{m,ii-1,jj,upper}_{\alpha,\beta,2,0}\\
M_{6i+3\cdot 0+\alpha,6(i-ner+1)+3\cdot 0+\beta} &= F^{m,ii-1,jj,upper}_{\alpha,\beta,2,1}+F^{m,ii-1,jj,lower}_{\alpha,\beta,1,0}\\
M_{6i+3\cdot 0+\alpha,6(i-1)+3\cdot 0+\beta} &= F^{m,ii-1,jj-1,lower}_{\alpha,\beta,2,1}+F^{m,ii,jj-1,upper}_{\alpha,\beta,1,0}\\
M_{6i+3\cdot 0+\alpha,6i+3\cdot 0+\beta} &= F^{m,ii-1,jj-1,lower}_{\alpha,\beta,2,2}+F^{m,ii-1,jj,upper}_{\alpha,\beta,2,2}+F^{m,ii-1,jj,lower}_{\alpha,\beta,1,1}+F^{m,ii,jj,upper}_{\alpha,\beta,0,0}+F^{m,ii,jj-1,lower}_{\alpha,\beta,0,0}+F^{m,ii,jj-1,upper}_{\alpha,\beta,1,1}\\
M_{6i+3\cdot 0+\alpha,6(i+1)+3\cdot 0+\beta} &= F^{m,ii-1,jj,lower}_{\alpha,\beta,1,2}+F^{m,ii,jj,upper}_{\alpha,\beta,0,1}\\
M_{6i+3\cdot 0+\alpha,6(i+ner-1)+3\cdot 0+\beta} &= F^{m,ii,jj-1,lower}_{\alpha,\beta,0,1}+F^{m,ii,jj-1,upper}_{\alpha,\beta,1,2}\\
M_{6i+3\cdot 0+\alpha,6(i+ner)+3\cdot 0+\beta} &= F^{m,ii,jj,upper}_{\alpha,\beta,0,2}+F^{m,ii,jj-1,lower}_{\alpha,\beta,0,2}
\end{aligned} \tag{28}$$

whereas the right hand side is given by:

$$\begin{aligned} RS^m_{6i+3\cdot 0+\alpha} &= -G^{m,ii-1,jj-1,lower}_{\alpha,2} - G^{m,ii-1,jj,upper}_{\alpha,2} - G^{m,ii-1,jj,lower}_{\alpha,1} \\ &\quad - G^{m,ii,jj,upper}_{\alpha,0} - G^{m,ii,jj-1,lower}_{\alpha,0} - G^{m,ii,jj-1,upper}_{\alpha,1} \\ RS^m_{6i+3\cdot 1+\alpha} &= -T^{m,ii-1,jj-1,lower}_{\alpha,2} - T^{m,ii-1,jj,upper}_{\alpha,2} - T^{m,ii-1,jj,lower}_{\alpha,1} \\ &\quad - T^{m,ii,jj,upper}_{\alpha,0} - T^{m,ii,jj-1,lower}_{\alpha,0} - T^{m,ii,jj-1,upper}_{\alpha,1}. \end{aligned} \tag{29}$$

The relation between the global node index i and the indices ii, jj of the triangles is given by

$$\begin{aligned} i &= ii \cdot ner + jj \\ ii &= i \,/\, ner \\ jj &= i \,\%\, ner \end{aligned} \tag{30}$$

where % is the remainder of the integer division and / is the integer division (disregarding the remainder). The number of upper / lower co-diagonals of the matrix M is $nl = 6 \cdot ner + 5$, the band width is $2 \cdot nl + 1 = 2 \cdot (6 \cdot ner + 5) + 1$. In the FEM software instead of the matrix (27), (28) we use the corresponding matrix in "band symmetric storage mode":

$$SM^m_{i-j+nl,j} = M^m_{i,j}. \tag{31}$$

The system of equations (26)-(29) has to be solved using Dirichlet boundary conditions (flux-parallel boundary conditions) at the outer boundary of the region in the r-z plane. In the case of saddle coils, only the components a_1^m, b_0^m, b_2^m of the vector potential (6) are non-vanishing, the components a_0^m, a_2^m, b_1^m are zero. This is a consequence of the special structure of the current density (7) and the local FEM coefficients (21b), respectively. Thus for saddle coils we could have reduced the number of degrees of freedom per node from six to three from the very beginning. However, for other coil geometries such as toroidal coils or tapered saddle coils [2], the radial component of the current density is nonzero. Therefore, components which vanish in the case of saddle coils are non-vanishing for other coil geometries.

5. Concluding Remarks

For calculating the magnetic field of a deflection coil a set of formulas was derived suitable to be implemented in an available FEM-program. Existing algorithms were modified by using a vector potential $\vec{a}$ instead of the scalar potential Φ in order to prepare the extension to time dependent fields in conductive materials. This extension will be discussed in a subsequent paper.

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