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Some multilevel methods on graded meshes

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Abstract

We consider Yserentant's hierarchical basis method and multilevel diagonal scaling method on a class of refined meshes used in the numerical approximation of boundary value problems on polygonal domains in the presence of singularities. We show, as in the uniform case, that the stiffness matrix of the first method has a condition number bounded by $(\ln(1/h))^2$, where h is the meshsize of the triangulation. For the second method, we show that the condition number of the iteration operator is bounded by $\ln(1/h)$, which is worse than in the uniform case but better than the hierarchical basis method. As usual, we deduce that the condition number of the BPX iteration operator is bounded by $\ln(1/h)$. Finally, graded meshes fulfilling the general conditions are presented and numerical tests are given which confirm the theoretical bounds. © 2002 Elsevier Science B.V. All rights reserved.

Keywords: Multilevel methods; Mesh refinement; Graded meshes; Finite element discretizations

1. Introduction

The solution of boundary value problems (b.v.p.) in nonsmooth domains presents singularities in the neighbourhood of singular points of the boundary, e.g., in the neighbourhood of re-entrant corners. Consequently, the use of uniform finite element meshes yields a poor rate of convergence. Many authors proposed to build graded meshes in the neighbourhood of these singular points in order to restore the optimal convergence order (see, e.g., [13,16]). Roughly speaking, such meshes consist in moving the nodal points by some coordinate transformation in order to compensate the singular behaviour of the solution, i.e., that the nodes accumulate near the singular point.

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As usual the finite element discretization leads to the resolution of large-scale systems of linear algebraic equations, where the system matrices in the nodal basis have a large condition number. This implies that the resolution by iterative methods requires a large number of iterations. Using preconditioners based on multilevel techniques one can reduce this number of iterations drastically. For uniform meshes standard multilevel methods, e.g., the hierarchical basis method [20] and BPX-like preconditioners [14] (see also [3–5,8,10,15,19,22]) allow the reduction of the condition number to the order $\mathcal{O}((\ln h^{-1})^2)$ and $\mathcal{O}(1)$, respectively, for two-dimensional problems and in the three-dimensional case to $\mathcal{O}(h^{-1})$ and $\mathcal{O}(1)$, respectively, where h is the largest diameter of the elements. Similar results were obtained in the case of nonuniformly refined meshes (see, e.g., [5,4,8,15,19,20]).

In our case, the first obstacle is that the graded meshes proposed in [13,16] are actually not nested. Consequently, we propose here to build a sequence of nested graded meshes $\mathcal{T}_0, \mathcal{T}_1, \dots, \mathcal{T}_j$ in two-dimensional domains which are also appropriate for the approximation of singularities. A similar algorithm was proposed in [12]. In [21] multigrid methods for solving elliptic boundary value problems discretized by means of nonnested graded meshes are studied. It is shown that in this case the multigrid V-cycle has only optimal convergence properties if the number of smoothing steps on each level is sufficiently large, depending on the coupling of the meshes. So, we cannot expect that one can construct for Raugel's meshes additive multilevel methods like Yserentant's hierarchical basis approach or the MDS preconditioner which lead to convergent algorithms. Especially for small grading parameters Raugel's meshes are strongly nonnested.

The meshes used in [5,4,8,15,19,20] are different from the above graded meshes. Therefore, our goal is to extend the kind of results obtained in these papers to our new meshes. The main idea is to prove that our graded meshes satisfy the conditions

$$\kappa_1 \beta^{k-l} \leqslant \frac{h_{K_k}}{h_{K_l}} \leqslant \kappa_2 \gamma^{k-l} \tag{1}$$

with positive constants κ_1, κ_2, β, and γ; h_{K_k} and h_{K_l} are the exterior diameter of the triangles $K_k \in \mathcal{T}_k$ and $K_l \in \mathcal{T}_l$ with $K_k \subset K_l$, $k \geqslant l$. Using this property, we can prove that the condition number of the stiffness matrix in the hierarchical basis is of the order $\mathcal{O}((\ln h^{-1})^2)$ and that the condition number of a $(j+1)$-level additive Schwarz operator with multilevel diagonal scaling (MDS method) is of the order $\mathcal{O}(\ln h^{-1})$.

The outline of the paper is the following one: In Section 2, we present our model problem and describe its finite element discretization. In Section 3, we analyse the condition number of the stiffness matrix in the hierarchical basis by showing the equivalence between the H^1-norm and the standard discrete one, and in Section 4, we derive estimates of the condition number of the MDS method by adapting Zhang's arguments [22]. Section 5 is devoted to the building of the nested graded meshes. We also check that these meshes are regular and fulfil conditions (1). Finally, numerical tests are presented in Section 6 which confirm our theoretical estimates.

2. The model problem

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain of the plane with a polygonal boundary Γ (i.e. the union of a finite number of linear segments). On Ω, we shall consider usual Sobolev spaces $H^s(\Omega)$, with $s \in \mathbb{R}^+$, of norm and semi-norm denoted by $\|\cdot\|_{s,\Omega}$, $|\cdot|_{s,\Omega}$, respectively (we refer to [1,11] for more details). As usual, $\mathring{H}^s(\Omega)$ is the closure in $H^s(\Omega)$ of $C_0^\infty(\Omega)$, the space of C^∞ functions with compact support in Ω.

Consider the boundary value problem

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \Gamma, \end{aligned} \tag{2}$$

whose variational formulation is: Find $u \in \mathring{H}^1(\Omega)$ such that

$$a(u,v) = f(v), \quad \forall v \in \mathring{H}^1(\Omega), \tag{3}$$

where we have set

$$a(u,v) = \int_\Omega \nabla^{\mathrm{T}} u \nabla v \,\mathrm{d}x \quad \text{and} \quad f(v) = \int_\Omega f v \,\mathrm{d}x,$$

when $f \in L^2(\Omega)$. It is well known that if Ω is convex then $u \in H^2(\Omega)$ and consequently the use of uniform meshes in standard finite element methods yields an optimal order of convergence h. On the contrary, if Ω is not convex then $u \notin H^2(\Omega)$ in general and uniform meshes yield a poor rate of convergence. Many authors [18,16,13] have shown that local mesh grading allows to restore the optimal order. But such meshes are not uniform in the sense used in standard multilevel techniques. Hereabove and later on, by uniform meshes we mean either regular refinements (partition of triangles of level k into four congruent subtriangles of level $k+1$) or nonuniform refinements (PLTMG package of [2]), see for instance [15, Section 4] and the references cited there. For this reason, as in [20,22], we relax the conditions of the meshes in the following way (graded meshes that fulfil these conditions are built in Section 5). We suppose that we have a sequence of nested triangulations $\{\mathscr{T}_k\}_{k\in\mathbb{N}}$ such that each triangle of $\mathscr{T}_k$ is divided into four triangles of $\mathscr{T}_{k+1}$. We assume that the triangulations are regular in Ciarlet's sense [6], i.e., the ratios h_K/ρ_K between the exterior diameters h_K and the interior diameters ρ_K of elements $K \in \bigcup_{k\in\mathbb{N}} \mathscr{T}_k$ are uniformly bounded from above and the maximal mesh size $h_k = \max_{K\in\mathscr{T}_k} h_K$ tends to zero as k goes to infinity. We further assume (see [20, Section 3] and [22, Section 2]) that there exist positive constants $\beta, \gamma < 1$ and positive constants κ_1, κ_2 such that for all $k \geqslant l$, all triangles $K_k \in \mathscr{T}_k$ and $K_l \in \mathscr{T}_l$ with $K_k \subset K_l$, we have

$$\kappa_1 \beta^{k-l} \leqslant \frac{h_{K_k}}{h_{K_l}} \leqslant \kappa_2 \gamma^{k-l}. \tag{4}$$

For regular refinements we have $\beta = \gamma = \frac{1}{2}$ and $\kappa_1 = \kappa_2 = 1$. We shall see later on that our graded meshes satisfy (4) with $\beta = (\frac{1}{2})^{1/\mu}$ and $\gamma = \frac{1}{2}$, where $\mu \in (0,1]$ is the grading parameter.

In each triangulation $\mathscr{T}_k$, we use the approximation space

$$V_k = \{u \in \mathring{H}^1(\Omega) \colon u_{|K} \in \mathbb{P}_1(K), \forall K \in \mathscr{T}_k\},$$

where $\mathbb{P}_1(K)$ is the set of polynomials of order $\leqslant 1$ on K. We consider the Galerkin approximation $u_k \in V_k$, solution of

$$a(u_k, v_k) = f(v_k), \quad \forall v_k \in V_k. \tag{5}$$

Let us remark that with the mesh $\mathcal{T}_k$ built in Section 5 and an appropriate parameter μ, we have the error estimate (see [13,16])

$$\|u - u_k\|_{1,\Omega} \lesssim 2^{-k}\|f\|_{0,\Omega},$$

where here and in the sequel $a \lesssim b$ means that there exists a positive constant C independent of k and of the above constants β, γ such that $a \leqslant Cb$. In Section 5, the constant will also be independent of the grading parameter μ.

3. Yserentant's hierarchical basis method

The goal of this section is to show that the stiffness matrix of the Galerkin method in the hierarchical basis on meshes $\mathcal{T}_k$ of the previous section has a condition number bounded by $(\ln h_k^{-1})^2$ as in the uniform case. The same result was already underlined by Yserentant in [20, Section 3] for nonuniformly refined meshes (in the above sense). There it is explained shortly that the results for uniformly refined meshes proved in [20, Section 2] could be adapted to nonuniformly refined meshes satisfying (4). In this section, we present a more detailed derivation of the estimate of the condition number. We follow the arguments of [20, Section 2], underline the differences with the standard refinement rule and also give the dependence with respect to the parameters β and γ.

Let $\mathcal{N}_k$ be the set of vertices of the triangles of $\mathcal{T}_k$ and $\mathcal{S}_k$ be the space of continuous functions on $\bar{\Omega}$ and linear on the triangles of $\mathcal{T}_k$. For a continuous function u in $\bar{\Omega}$, let $I_k u$ be the function in $\mathcal{S}_k$ interpolating u at the nodes of $\mathcal{T}_k$, i.e.,

$$I_k u \in \mathcal{S}_k \quad \text{and} \quad I_k u(p) = u(p), \quad \forall p \in \mathcal{N}_k. \tag{6}$$

For further use, let us also denote by $\mathcal{V}_k$ the subspace of $\mathcal{S}_k$ of functions vanishing at the nodes of level $k-1$, in other words, $\mathcal{V}_k$ is the range of $I_k - I_{k-1}$.

On the finite element space $\mathcal{S}_j$, we define the semi-norm $|\cdot|$ as follows:

$$|u|^2 = \sum_{k=1}^{j} \sum_{p \in \mathcal{N}_k \setminus \mathcal{N}_{k-1}} |I_k u(p) - I_{k-1} u(p)|^2, \quad \forall u \in \mathcal{S}_j. \tag{7}$$

The proof of the equivalence of norms we have in mind is based on the two following preliminary lemmas. The first one concerns equivalence of semi-norms (cf. [20, Lemma 2.4]).

Lemma 3.1. *For all $u \in \mathcal{S}_j$, we have*

$$|u|^2 \lesssim \sum_{k=1}^{j} |I_k u - I_{k-1} u|_{1,\Omega}^2 \lesssim |u|^2. \tag{8}$$

Proof. In view of Lemma 2.4 of [20], we simply need to show that the following estimates hold:

$$\sum_{p \in K \cap \mathcal{N}_k \setminus \mathcal{N}_{k-1}} |v(p)|^2 \lesssim |v|_{1,K}^2 \lesssim \sum_{p \in K \cap \mathcal{N}_k \setminus \mathcal{N}_{k-1}} |v(p)|^2 \tag{9}$$

for all $K \in \mathcal{T}_{k-1}$ and all $v \in \mathcal{V}_k$. To prove this estimate, we remark that $K \in \mathcal{T}_{k-1}$ is divided into four triangles $K_l \in \mathcal{T}_k$, $l = 1,2,3,4$, such that v is linear in each K_l and satisfies $v(p_j) = 0$,

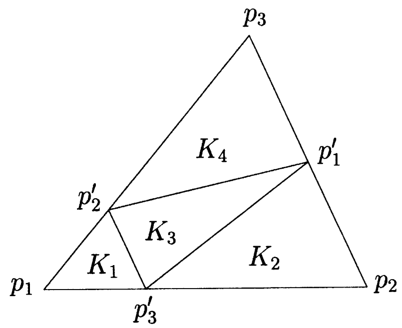

Fig. 1. Triangle $K \in \mathcal{T}_{k-1}$ divided in four subtriangles $K_l \in \mathcal{T}_k$, $l = 1,2,3,4$.

for all $j = 1,2,3$, where p_j, $j = 1,2,3$, are the vertices of K (see Fig. 1). Due to the fact that the triangulation $\mathcal{T}_k$ is regular, by an affine coordinate transformation (reducing to the reference element $\hat{K}$), we prove that

$$\sum_{j\in\mathcal{I}(K_l)} |v(p_j')|^2 \lesssim |v|_{1,K_l}^2, \quad l = 1,2,4,$$

$$|v|_{1,K_l}^2 \lesssim \sum_{j\in\mathcal{I}(K_l)} |v(p_j')|^2, \quad l = 1,2,3,4,$$

where $\mathcal{I}(K_l)$ is the set of vertices of K_l which are not vertices of K. Summing the first estimate on $l = 1,2,4$ and the second one on $l = 1,2,3,4$, we obtain (9). □

The second ingredient is a Cauchy–Schwarz type inequality already proved in Lemma 2.7 of [20] in the case of regularly refined meshes and that we easily extend to the case of our mesh as suggested in [20, Section 3].

Lemma 3.2. *For all $u \in \mathcal{V}_k$, $v \in \mathcal{V}_l$, we have*

$$a(u,v) \lesssim \gamma^{|k-l|/2} |u|_{1,\Omega} |v|_{1,\Omega}. \tag{10}$$

Proof. Similar to the proof of Lemma 2.7 of [20] with the following slight modification: if K is a fixed triangle of $\mathcal{T}_l$ and S the boundary strip of K consisting of all triangles of $\mathcal{T}_k$, with $l < k$, which are subsets of K and meet the boundary of K then due to (4), we have

$$\frac{\text{meas }(S)}{\text{meas }(K)} \lesssim \gamma^{k-l}.$$

In view to the proof of Lemma 2.7 of [20], this yields the assertion. □

Now we can formulate the equivalence between the H^1 norm and the discrete one (see Theorem 2.2 of [20]).

Theorem 3.3. *For all* $u \in \mathscr{S}_j$, *it holds*

$$\frac{1}{(1+\ln(\beta^{-1}))(j+1)^2}\{\|I_0u\|_{1,\Omega}^2+|u|^2\} \lesssim \|u\|_{1,\Omega}^2 \lesssim \frac{1+\gamma^2}{1-\gamma^2}\{\|I_0u\|_{1,\Omega}^2+|u|^2\}. \tag{11}$$

Proof. For the lower bound, we remark that assumption (4) and Lemmas 2.2 and 2.3 of [20] imply that

$$|I_ku|_{1,K}^2 \lesssim (1+\ln(\beta^{-1}))(j-k+1)|u|_{1,K}^2,$$

$$\|I_0u\|_{0,K}^2 \lesssim (1+\ln(\beta^{-1}))(j+1)\|u\|_{1,K}^2$$

for every $K \in \mathscr{T}_k,\ k \leqslant j$. Summing these inequalities on all $K \in \mathscr{T}_k$, we get

$$|I_ku|_{1,\Omega}^2 \lesssim (1+\ln(\beta^{-1}))(j-k+1)|u|_{1,\Omega}^2, \quad \forall k \leqslant j, \tag{12}$$

$$\|I_0u\|_{0,\Omega}^2 \lesssim (1+\ln(\beta^{-1}))(j+1)\|u\|_{1,\Omega}^2. \tag{13}$$

Therefore by Lemma 3.1 and the triangular inequality, we get

$$\|I_0u\|_{1,\Omega}^2+|u|^2 \lesssim \|I_0u\|_{1,\Omega}^2+\sum_{k=1}^{j}|I_ku-I_{k-1}u|_{1,\Omega}^2 \lesssim \|I_0u\|_{0,\Omega}^2+\sum_{k=0}^{j}|I_ku|_{1,\Omega}^2.$$

By estimates (12) and (13), we then obtain the lower bound in (11).

Let us now pass to the upper bound. First, Lemma 3.2 and the arguments of Lemma 2.8 of [20] yield

$$|u|_{1,\Omega}^2 \lesssim \frac{1+\gamma^2}{1-\gamma^2}|u|^2. \tag{14}$$

On the other hand, assumption (4), the fact that our triangulation is regular and the arguments of Lemma 2.9 of [20] lead to

$$\|u\|_{0,\Omega}^2 \lesssim \|I_0u\|_{0,\Omega}^2+\frac{\gamma^2}{1-\gamma^2}|u|^2. \tag{15}$$

The sum of the two above estimates gives the upper bound in (11). □

Using a hierarchical basis of V_j and the former results, we directly get the

Corollary 3.4. *The Galerkin stiffness matrix* A_j *of the approximated problem* (5) *in the hierarchical basis has a spectral condition number* $\kappa(A_j)$ *which grows at most quadratically with the number of levels* j, *more precisely*,

$$\kappa(A_j) \lesssim \frac{1+\gamma^2}{1-\gamma^2}(1+\ln(\beta^{-1}))(j+1)^2.$$

4. Multilevel diagonal scaling method

In this section, we analyse the multilevel diagonal scaling method and the BPX algorithm in the spirit of [22]. Here the main difficulty relies on the fact that our meshes are not quasi-uniform (quasi-uniform meshes mean that $h_K \sim h_k$, for all triangles $K \in \mathscr{T}_k$, for all $k \in \mathbb{N}$), leading to the fact that Assumption 2.1.c of [22] is violated.

Let us recall that the multilevel diagonal scaling method consists in the following algorithm: First we represent V_j as a sum

$$V_j = \sum_{k=0}^{j}\sum_{i=1}^{N_k} V_i^k,$$

where $V_i^k = \mathrm{Span}\{\phi_i^k\}$, when ϕ_i^k is the nodal basis function of V_k associated with the interior vertex p_i^k of $\mathscr{T}_k$, $N_k = \mathrm{card}\,\mathscr{N}_k$ being the number of interior vertices of $\mathscr{T}_k$. Define the operator A from V_j to V_j by

$$(Au, \phi) = a(u, \phi), \quad \forall \phi \in V_j,$$

where $(\cdot,\cdot)$ means the $L^2(\Omega)$ inner product. Let us further define the preconditioner B_{MDS}^{-1} and the $(j+1)$-level multilevel diagonal scaling operator P_{MDS} by

$$B_{\mathrm{MDS}}^{-1} v = \sum_{k=0}^{j}\sum_{i=1}^{N_k} \frac{(v, \phi_i^k)}{a(\phi_i^k, \phi_i^k)} \phi_i^k,$$

$$P_{\mathrm{MDS}} v = B_{\mathrm{MDS}}^{-1} A v = \sum_{k=0}^{j}\sum_{i=1}^{N_k} \frac{a(v, \phi_i^k)}{a(\phi_i^k, \phi_i^k)} \phi_i^k.$$

The multilevel diagonal scaling algorithm consists in finding $u_j \in V_j$ of the Galerkin problem (5) by solving iteratively (using for instance the conjugate gradient method) the equation

$$P_{\mathrm{MDS}} u_j = f_{\mathrm{MDS}} := B_{\mathrm{MDS}}^{-1} f. \tag{16}$$

As usual to solve iteratively (16), the crucial point is to estimate the condition number of the iteration operator P_{MDS}. For quasi-uniform meshes, it was shown by Zhang in Theorem 3.1 and Section 4 of [22] that this condition number is uniformly bounded (with respect to the level j). The same result was extended to the case of nonuniformly refined meshes [8, Section 5]; [15, Section 4.2.2]. Our goal is to extend this type of results to meshes satisfying only (4) which can be nonquasi-uniform. Analysing carefully the proof of Theorem 3.1 of [22] we remark that the upper bound of the condition number is valid under assumption (4) and is fully independent of the quasi-uniformity of the meshes. On the contrary the proof of the lower bound uses this last property. The key point in our proof of this lower bound is the use of an interpolation operator with appropriate properties.

For a fixed $k \in \{0,\dots,j\}$, with each $i \in \{1,\dots,N_k\}$, we associate the macro-element

$$S_i^k = \bigcup\{K \in \mathscr{T}_k;\, p_i^k \in K\},$$

which is actually the support of ϕ_i^k. For any triangle $K \in \mathscr{T}_k$, let us further denote by $S(K)$ the union of all macro-elements containing K, i.e.,

$$S(K) = \bigcup\{S_i^k;\, K \subset S_i^k\}.$$

The following well known facts result from the regularity of the family $\{\mathcal{T}_k\}_{k\in\mathbb{N}}$: There exists a positive integer M (independent of k) such that

$$\mathrm{card}\{K' \subset S(K); K' \in \mathcal{T}_k\} \leqslant M, \tag{17}$$

$$h_K \lesssim h_{K'} \quad \text{for any } K,\ K' \in \mathcal{T}_k \text{ such that } K \cap K' \neq \emptyset. \tag{18}$$

A direct consequence of these two properties is that the diameter of $S(K)$ is equivalent to h_K, indeed from the triangular inequality we have

$$\mathrm{diam}\, S(K) \leqslant \max_{K_1,K_2,K_3 \subset S(K)} \{h_{K_1} + h_{K_2} + h_{K_3}\}.$$

Using properties (18) and (17), we get

$$h_K \leqslant \mathrm{diam}\, S(K) \lesssim h_K. \tag{19}$$

Let us now fix a linear continuous operator π_k from $\mathring{H}^1(\Omega)$ into V_k, which is a projection on V_k (i.e., $\pi_k v = v$, for all $v \in V_k$) and that enjoys the following local interpolation property: for all triangles $K \in \mathcal{T}_k$ and $q = 0$ or 1, we have

$$|u - \pi_k u|_{q,K} \lesssim h_K^{1-q} |u|_{1,S(K)}, \quad \forall u \in \mathring{H}^1(\Omega). \tag{20}$$

Let us notice that Scott–Zhang's interpolation operator as well as the operator $\tilde{Q}_k$ defined by (5.15) in [8] satisfy such properties (see Section 4 of [17] for the first one and Section 5 of [8] for the second one). Note further that Clément's interpolation operator [7,9] also satisfies (20) but unfortunately is not a projection on V_k.

Now, we are able to prove the estimate of the condition number $\kappa(P_{\mathrm{MDS}})$ of the iteration operator P_{MDS}.

Theorem 4.1. *The multilevel diagonal scaling operator P_{MDS} satisfies*

$$\frac{1}{j+1} a(u,u) \lesssim a(P_{\mathrm{MDS}} u, u) \lesssim \frac{1}{1-\sqrt{\gamma}} a(u,u), \quad \forall u \in V_j. \tag{21}$$

Consequently, we have

$$\kappa(P_{\mathrm{MDS}}) \lesssim \frac{j+1}{1-\sqrt{\gamma}},$$

which means that $\kappa(P_{\mathrm{MDS}})$ grows at most linearly with the number of levels $j+1$.

Proof. As already mentioned, the upper bound was proved by Zhang [22, Lemmas 3.2–3.5]. To prove the lower bound instead of using the H^1-projection on V_k, for $k \in \{0,\dots,j\}$ having a global approximation property which is not convenient for nonquasi-uniform meshes (see Remark 4.2 below), we take advantage of the local interpolation property (20) of the interpolation operator π_k. Indeed for any $u \in \mathring{H}^1(\Omega)$, we set

$$u^k = \pi_k u - \pi_{k-1} u \in V_k, \quad \forall k \in \mathbb{N}, \tag{22}$$

with the convention $\pi_{-1} u = 0$. Consequently any $u \in V_j$ may be written

$$u = \pi_j u = \sum_{k=0}^{j} u^k. \tag{23}$$

Then for all triangles $K \in \mathscr{T}_k$ and $q = 0$ or 1, we have

$$\begin{aligned} |u^k|_{q,K} &\leqslant |\pi_k u - u|_{q,K} + |u - \pi_{k-1} u|_{q,K}, \\ &\leqslant |\pi_k u - u|_{q,K} + |u - \pi_{k-1} u|_{q,M(K)}, \end{aligned}$$

where $M(K)$ is the unique triangle in $\mathscr{T}_{k-1}$ containing K if $k \geqslant 1$ and $M(K) = \emptyset$ if $k = 0$. Owing to (20) and (18), we deduce that

$$|u^k|_{q,K} \lesssim h_K^{1-q} \{ |u|_{1,S(K)} + |u|_{1,S(M(K))} \}, \quad q = 0, 1. \tag{24}$$

Now we decompose u^k in the nodal basis, in other words we write

$$u^k = \sum_{i=1}^{N_k} u_i^k, \tag{25}$$

where $u_i^k = u^k(p_i^k)\phi_i^k$. Consequently, we get

$$|u_i^k|_{1,\Omega}^2 = |u_i^k|_{1,S_i^k}^2 \lesssim |u^k(p_i^k)|^2 \lesssim \sum_{K \subset S_i^k} \{ |u^k|_{1,K}^2 + h_K^{-2} |u^k|_{0,K}^2 \}.$$

This last estimate being obtained using the equivalence of norms in finite-dimensional spaces on the reference element $\hat{K}$ and an affine coordinate transformation. Using now estimate (24) we arrive at

$$|u_i^k|_{1,\Omega}^2 \lesssim \sum_{K \subset S_i^k} \{ |u|_{1,S(K)}^2 + |u|_{1,S(M(K))}^2 \}.$$

Summing this last estimate on $i = 1, \ldots, N_k$ and using property (17), we obtain

$$\sum_{i=1}^{N_k} |u_i^k|_{1,\Omega}^2 \lesssim \sum_{K \in \mathscr{T}_k} |u|_{1,K}^2 \lesssim |u|_{1,\Omega}^2. \tag{26}$$

The sum on $k = 0, \ldots, j$ yields

$$\sum_{k=0}^{j} \sum_{i=1}^{N_k} |u_i^k|_{1,\Omega}^2 \lesssim (j+1) |u|_{1,\Omega}^2. \tag{27}$$

With the help of Lemma 3.1 of [22] (see also [22, Remark 3.1]) and the definition of the bilinear form a, we conclude that

$$\frac{1}{j+1} \lesssim \lambda_{\min}(P_{\mathrm{MDS}}).$$

The lower bound directly follows. □

Remark 4.2. If we would have taken in the above proof

$$u^k = P_{V_k}u - P_{V_{k-1}}u,$$

where P_{V_k} is the H^1-projection in $\mathring{H}^1(\Omega)$ on V_k, as it is made in [22], then we would get for the graded meshes of Section 5

$$\sum_{i=1}^{N_k} |u_i^k|_{1,\Omega}^2 \lesssim h_k^{2(1-1/\mu)} |u^k|_{1,\Omega}^2, \tag{28}$$

instead of (26), since the (global) error estimate

$$\|u - P_{V_k}u\|_{0,\Omega} \lesssim h_k |u|_{1,\Omega}, \quad \forall u \in \mathring{H}^1(\Omega),$$

holds for our meshes [16], while due to the nonquasi-uniformity of the meshes, we only have

$$|\nabla \theta_i^k| \lesssim h_k^{-1/\mu}, \quad \forall i = 1, \ldots, N_k,$$

for the partition of unity $\{\theta_i^k\}_{i=1}^{N_k}$ introduced in [22]. This estimate (28) implies (instead of (27))

$$\sum_{k=0}^{j} \sum_{i=1}^{N_k} |u_i^k|_{1,\Omega}^2 \lesssim h_j^{2(1-1/\mu)} |u|_{1,\Omega}^2.$$

This estimate is too rough since $h_j^{2(1-1/\mu)}$ blows up exponentially as $j \to \infty$ if $\mu < 1$.

We have chosen the splitting (22) since then we have the local error estimate (24) but leads to a factor j in (27) (which is quite better than $h_j^{2(1-1/\mu)}$ for $\mu < 1$). For our graded meshes, this factor j is confirmed by numerical tests.

Another alternative would be the approach of [3,8] but unfortunately the quasi-uniformity of the meshes (see [8, equivalence (5.22)]; [3, Lemma 3.1]), namely

$$\operatorname{diam} S(K) \sim h_k, \quad \forall K \in \mathscr{T}_k,$$

fails for our meshes. □

Let us finish this section by looking at the BPX algorithm. As the BPX preconditioner is defined by

$$B^{-1}v = \sum_{k=0}^{j} \sum_{i=1}^{N_k} (v, \phi_i^k)\phi_i^k,$$

the BPX operator $P_{\mathrm{BPX}} = B^{-1}Av$ is given by

$$P_{\mathrm{BPX}}v = \sum_{k=0}^{j} \sum_{i=1}^{N_k} a(v, \phi_i^k)\phi_i^k.$$

Since $a(\phi_i^k, \phi_i^k)$ is equivalent to 1 (uniformly with respect to k), the condition numbers of P_{BPX} and P_{MDS} are equivalent. This means that the following holds.

Corollary 4.3. *The BPX operator enjoys the property*

$$\kappa(P_{\mathrm{BPX}}) \lesssim \frac{j+1}{1-\sqrt{\gamma}}.$$

5. Graded nested meshes

The triangulations $\mathscr{T}_k$ of Ω which will be used are graded according to Raugel's procedure [16,11]. But here since we need a nested sequence of triangulations this procedure is slightly modified. As a consequence we need to check the regularity of the meshes. In a second step we shall show that this family satisfies condition (4).

Let us first describe the construction of the meshes:

(i) Divide Ω into a coarse triangular mesh $\mathscr{T}_0$ such that each triangle has either one or no singular point (of Ω) as vertex. If a triangle has a singular point as vertex (i.e. the interior angle at this point is $> \pi$), it is called a singular triangle and we suppose that all its angles are acute and the edges hitting the singular point have the same length (this is always possible by eventual subdivisions).

(ii) Any nonsingular triangle T of $\mathscr{T}_0$ is divided using the regular refinement procedure, i.e., divide any triangle of $\mathscr{T}_k$ included in T into four congruent subtriangles of $\mathscr{T}_{k+1}$, see Fig. 2.

(iii) Any singular triangle T of $\mathscr{T}_0$ is refined iteratively as follows: Fix a grading parameter $\mu \in (0,1]$ (that for simplicity we take identical for all singular triangles; if there exists more than one singular point, then we simply need to take the same parameter for triangles containing the same singular point). In order to make understandable our procedure we describe $T \cap \mathscr{T}_1$ and $T \cap \mathscr{T}_2$ and then explain how to pass from $T \cap \mathscr{T}_k$ to $T \cap \mathscr{T}_{k+1}$. For convenience we first recall Raugel's grading procedure.

Introduce barycentric coordinates $\lambda_0, \lambda_1, \lambda_2$ in T such that the singular point of T has the coordinate $\lambda_0 = 1$. For all $n \in \mathbb{N}^*$, define vertices $p_{i,j}^{(n)}$, $0 \leqslant i+j \leqslant n$ in T whose coordinates are

$$\lambda_1 = \frac{i}{n}\left(\frac{i+j}{n}\right)^{-1+1/\mu}, \qquad \lambda_2 = \frac{j}{n}\left(\frac{i+j}{n}\right)^{-1+1/\mu}.$$

Raugel's grading procedure consists in defining $T \cap \mathscr{T}_k$ as the set of triangles described by their three vertices as follows:

$$\begin{aligned} &(p_{i,j}^{(2^k)}, p_{i+1,j}^{(2^k)}, p_{i,j+1}^{(2^k)}), \quad 0 \leqslant i+j \leqslant 2^k - 1, \\ &(p_{i+1,j}^{(2^k)}, p_{i,j+1}^{(2^k)}, p_{i+1,j+1}^{(2^k)}), \quad 0 \leqslant i+j \leqslant 2^k - 2. \end{aligned} \tag{29}$$

First $T \cap \mathscr{T}_1$ is simply defined by Raugel's procedure, i.e., it is the set of four triangles described by (29) with $k = 1$ (see Fig. 3).

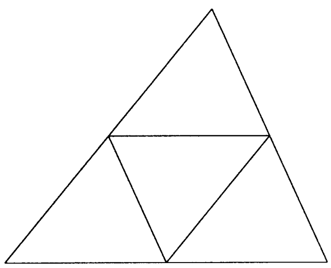

Fig. 2. Triangle $K \in \mathscr{T}_k$ divided into four congruent subtriangles.

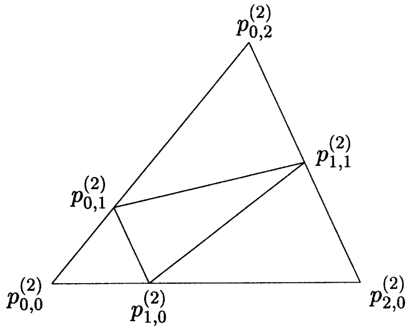

Fig. 3. Defining $T \cap \mathscr{T}_1$ by Raugel's procedure.

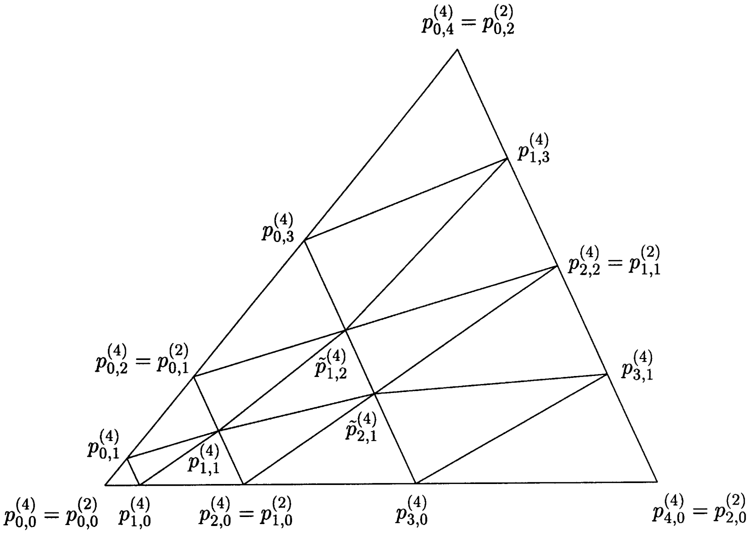

Fig. 4. Defining $T \cap \mathscr{T}_2$ by our procedure.

Secondly, the triangulation $T \cap \mathscr{T}_2$ is built as follows (see Fig. 4): The part below the line $\lambda_1 + \lambda_2 = (\frac{1}{2})^{1/\mu}$ is identical with Raugel's one, namely it is described by the four triangles of vertices:

$$(p^{(4)}_{i,j}, p^{(4)}_{i+1,j}, p^{(4)}_{i,j+1}), \quad 0 \leqslant i+j \leqslant 1, \quad (p^{(4)}_{1,0}, p^{(4)}_{0,1}, p^{(4)}_{1,1}).$$

On the contrary the part above the line $\lambda_1 + \lambda_2 = (\frac{1}{2})^{1/\mu}$ is modified in order to guarantee the nestedness. More precisely, the set of triangles in this zone is described by

$$(\tilde{p}^{(4)}_{i,j}, \tilde{p}^{(4)}_{i+1,j}, \tilde{p}^{(4)}_{i,j+1}), \quad 2 \leqslant i+j \leqslant 3,$$

$$(\tilde{p}^{(4)}_{i+1,j}, \tilde{p}^{(4)}_{i,j+1}, \tilde{p}^{(4)}_{i+1,j+1}), \quad 1 \leqslant i+j \leqslant 2,$$

where for $i+j \geqslant 1$, the points $\tilde{p}^{(4)}_{i,j}$ are identical with $p^{(4)}_{i,j}$ except in the case $(i,j) = (2,1)$ and $(i,j) = (1,2)$ where we take $\tilde{p}^{(4)}_{2,1}$ (resp. $\tilde{p}^{(4)}_{1,2}$) as the intersection between the line $\lambda_1 + \lambda_2 = (\frac{3}{4})^{1/\mu}$

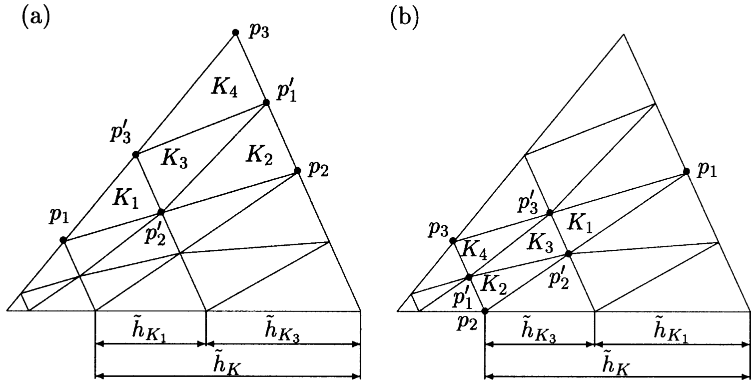

Fig. 5. Definition of the nodes p_i and p'_i.

and the line joining the points $p_{1,0}^{(2)}$ (resp. $p_{0,1}^{(2)}$) and $p_{1,1}^{(2)}$, see Fig. 4. Notice that these points $\tilde{p}_{i,j}^{(4)}$ are actually on one edge of a triangle of $T \cap \mathscr{T}_1$. We now remark that in this procedure the three triangles K_l, $l = 2, 3, 4$, of $T \cap \mathscr{T}_1$ above the line $\lambda_1 + \lambda_2 = (\frac{1}{2})^{1/\mu}$ are divided into four triangles in the following way: determine the two points which are intersection between the line $\lambda_1 + \lambda_2 = (\frac{3}{4})^{1/\mu}$ and the edges of K_l; determine the mid-point of the third edge (uniform subdivision in two parts). Using these three points on the edges of K_l and the vertices of K_l, we divide K_l into four triangles in a standard way (see Fig. 1). This will be the general rule.

Now we can describe the passage from $T \cap \mathscr{T}_k$ to $T \cap \mathscr{T}_{k+1}$. The triangle of $T \cap \mathscr{T}_k$ containing a singular corner is divided into four triangles in Raugel's way: these triangles are described by their three vertices

$$(p_{i,j}^{(2^{k+1})}, p_{i+1,j}^{(2^{k+1})}, p_{i,j+1}^{(2^{k+1})}), \quad 0 \leqslant i + j \leqslant 1, \quad (p_{1,0}^{(2^{k+1})}, p_{0,1}^{(2^{k+1})}, p_{1,1}^{(2^{k+1})}).$$

Any triangle $K \in T \cap \mathscr{T}_k$ above the line $\lambda_1 + \lambda_2 = (1/2^k)^{1/\mu}$ is divided into four triangles in the following way: First there exists $i \geqslant 1$ such that K is between the lines $\lambda_1 + \lambda_2 = (i/2^k)^{1/\mu}$ and $\lambda_1 + \lambda_2 = ((i+1)/2^k)^{1/\mu}$. Two vertices are on one line that we denote by p_2, p_3 and the third one denoted by p_1 is on the other line. Secondly determine the two points p'_2, p'_3 which are intersection between the line $\lambda_1 + \lambda_2 = ((2i+1)/2^{k+1})^{1/\mu}$ and the edges of K; determine the mid-point p'_1 of the third edge. Now the four triangles K_l, $l = 1, 2, 3, 4$, of $K \cap \mathscr{T}_{k+1}$ are described by their three vertices (see Fig. 5):

$$K_1 \equiv (p_1, p'_2, p'_3),$$
$$K_2 \equiv (p'_2, p_2, p'_1),$$
$$K_3 \equiv (p'_3, p'_1, p'_2),$$
$$K_4 \equiv (p'_3, p'_1, p_3).$$

Let us finally notice that the above procedure guarantees the conformity of the meshes. Now we want to show that this family of meshes is regular.

Lemma 5.1. *The above family is regular in the sense that*

$$h_K/\rho_K \lesssim e^{9(1/\mu-1)}, \quad \forall K \in \bigcup_{k\in\mathbb{N}} \mathcal{T}_k. \tag{30}$$

Proof. To prove the assertion it suffices to look at the triangles of $T \cap \mathcal{T}_k$ for any singular triangle T of $\mathcal{T}_0$. If we show that for all $K \in T \cap \mathcal{T}_k$, we have

$$h_i(K) \lesssim e^{6(1/\mu-1)} h_1(K), \quad \forall i = 1,2,3, \tag{31}$$

$$e^{-3(1/\mu-1)} \lesssim \sin \alpha_{\max}(K), \tag{32}$$

where $h_i(K)$ are the lengths of the edges of K in increasing order and $\alpha_{\max}(K)$ is the largest angle of K, then we deduce that the smallest angle α_K of K satisfies

$$\frac{1}{\sin \alpha_K} \lesssim \frac{h_K}{h_1(K)} \frac{1}{\sin \alpha_{\max}(K)}.$$

By Zlámal's condition [23], we then deduce

$$\frac{h_K}{\rho_K} \leqslant \frac{2}{\sin \alpha_K} \lesssim e^{9(1/\mu-1)},$$

which yields (30).

It then remains to prove (31) and (32). We now remark that if we apply a similarity of centre at the singular point and of ratio $2^{-1/\mu}$ to the triangulation $T \cap \mathcal{T}_k$, we obtain the part of the triangulation of $T \cap \mathcal{T}_{k+1}$ below the line $\lambda_1 + \lambda_2 = (\frac{1}{2})^{1/\mu}$. This means that we are reduced to prove (31) and (32) for the triangles above that line $\lambda_1 + \lambda_2 = (\frac{1}{2})^{1/\mu}$. Therefore, we say that $K \in \tilde{T} \cap \mathcal{T}_k$ if and only if K is between the lines $\lambda_1 + \lambda_2 = (i/2^k)^{1/\mu}$ and $\lambda_1 + \lambda_2 = ((i+1)/2^k)^{1/\mu}$ with $i \geqslant 2^{k-1}$.

For any triangle $K \in \tilde{T} \cap \mathcal{T}_k$, let us denote by p_K the length of the edge parallel to the line $\lambda_1 + \lambda_2 = 1$ and by

$$\tilde{h}_K = \left(\frac{i+1}{2^k}\right)^{1/\mu} - \left(\frac{i}{2^k}\right)^{1/\mu},$$

when K is between the lines $\lambda_1 + \lambda_2 = (i/2^k)^{1/\mu}$ and $\lambda_1 + \lambda_2 = ((i+1)/2^k)^{1/\mu}$.

We first prove that

$$e^{-3(1/\mu-1)} \tilde{h}_K \lesssim p_K \lesssim e^{3(1/\mu-1)} \tilde{h}_K, \quad \forall K \in \tilde{T} \cap \mathcal{T}_k. \tag{33}$$

Indeed we shall establish inductively that

$$\left(\prod_{l=1}^{k+1} r_l\right) \tilde{h}_K \lesssim p_K \lesssim \left(\prod_{l=1}^{k+1} r_l^{-1}\right) \tilde{h}_K, \quad \forall K \in \tilde{T} \cap \mathcal{T}_k, \tag{34}$$

where $r_l = (1 - 1/(2^{l-3}+1))^{1/\mu-1}$ for $l \geqslant 2$ and $r_1 = 1$. It is clear that (34) holds for $k = 1$. Consequently, to prove (34) for all k, it suffices to show that if (34) holds for k, it also holds for $k+1$. Fix any $K \in \tilde{T} \cap \mathcal{T}_k$, then as already explained it is divided into four triangles K_l, $l = 1,2,3,4$, of $\tilde{T} \cap \mathscr{T}_{k+1}$. Two geometrical cases can be distinguished: either p_1 is on the line $\lambda_1 + \lambda_2 = (i/2^k)^{1/\mu}$ or p_1 is on the line $\lambda_1 + \lambda_2 = ((i+1)/2^k)^{1/\mu}$. Let us first show that (34) holds for the triangles K_l, $l = 1,2,3,4$, in the first case. With the notation from Fig. 5, we deduce from the construction of the mesh that $p'_j = h(p_j)$ for $j = 2,3$, when h is the similarity of centre p_1 and ratio

$$r = \frac{\tilde{h}_{K_1}}{\tilde{h}_K}.$$

This implies that

$$p_{K_1} = p_{K_3} = r p_K.$$

Since by assumption K satisfies (34), K_1 and K_3 directly satisfies

$$\left(\prod_{l=1}^{k+1} r_l\right) \tilde{h}_{K_1} \lesssim p_{K_1} = p_{K_3} \lesssim \left(\prod_{l=1}^{k+1} r_l^{-1}\right) \tilde{h}_{K_1},$$

leading to (34) for K_1 (with $k+1$ instead of k) because $r_{k+2} \leqslant 1$. For the triangle K_3, the above estimate yields

$$r_K \left(\prod_{l=1}^{k+1} r_l\right) \tilde{h}_{K_3} \lesssim p_{K_3} \lesssim r_K \left(\prod_{l=1}^{k+1} r_l^{-1}\right) \tilde{h}_{K_3},$$

where $r_K = \tilde{h}_{K_1}/\tilde{h}_{K_3}$. This leads to (34) for K_3 because

$$r_{k+2} \leqslant r_K \leqslant 1$$

due to the fact that $i \geqslant 2^{k-1}$.

For K_2 and K_4, we have $p_{K_2} = p_{K_4} = p_K/2$. Therefore by the inductive assumption and the fact that $\tilde{h}_{K_2} = \tilde{h}_{K_4} = (1-r)\tilde{h}_K$, we get

$$\frac{1}{2(1-r)} \left(\prod_{l=1}^{k+1} r_l\right) \tilde{h}_{K_q} \lesssim p_{K_q} \lesssim \frac{1}{2(1-r)} \left(\prod_{l=1}^{k+1} r_l^{-1}\right) \tilde{h}_{K_q} \quad \text{for } q = 2,4.$$

Again this leads to (34) for K_2 and K_4 because we easily check that (note that $r \leqslant 1/2$)

$$r_{k+2} \leqslant \frac{1}{2(1-r)} \leqslant 1.$$

The second case is treated similarly, for K_3 we have the same estimate than before with r_K^{-1} instead of r_K that is the reason of the factor r_{k+2}^{-1} on the right-hand side. For K_2 and K_4, we simply remark that the ratio $\tilde{r}$ of the second similarity is $1-r$ and use the fact that $r_{k+2} \leqslant 2\tilde{r}$.

The proof of (34) is then complete.

Now (33) follows from (34) because using the fact that

$$-\log_a(1-x) \leqslant x, \quad \forall x \in [0, \tfrac{1}{2}]$$

with $a = \mathrm{e}^2$, we can estimate

$$\prod_{l=2}^{k+1} r_l^{-1} = r_2^{-1} \prod_{l=3}^{k+1} r_l^{-1} \leqslant \left(\frac{1}{3}\right)^{1/\mu - 1} \prod_{l=3}^{\infty} r_l^{-1} \leqslant \mathrm{e}^{3(1/\mu - 1)}.$$

Let us now come back to (31). For any $K \in \tilde{T} \cap \mathscr{T}_k$ by construction of the mesh, we clearly have

$$\tilde{h}_K \lesssim |p_1 p_q|, \quad q = 2, 3 \tag{35}$$

with the above notation for the vertices of K. Let us now show by induction on k that

$$|p_1 p_q| \lesssim \left(\prod_{l=1}^{k+1} r_l^{-1}\right)^2 p_K, \quad q = 2, 3. \tag{36}$$

Since this estimate clearly holds for $k = 1$, we only need to show the inductive implication. As before, we start with the first geometrical case: p_1 on the line $\lambda_1 + \lambda_2 = (i/2^k)^{1/\mu}$ (Fig. 5a). To this end, assume that (36) holds for $K \in \mathscr{T}_k$ and let us show that it holds for $K_q, q = 1, 2, 3, 4$. For K_1, we use the similarity of centre p_1 and ratio r to conclude that

$$|p_1 p_q'| = r|p_1 p_q| \lesssim r\left(\prod_{l=1}^{k+1} r_l^{-1}\right)^2 p_K \lesssim \left(\prod_{l=1}^{k+1} r_l^{-1}\right)^2 p_{K_1} \lesssim \left(\prod_{l=1}^{k+2} r_l^{-1}\right)^2 p_{K_1}, \quad q = 2, 3,$$

since $r_{k+2} \leqslant 1$. This shows (36) for K_1. Similarly, we have

$$|p_3 p_3'| = (1 - r)|p_1 p_3| \lesssim 2(1 - r)\left(\prod_{l=1}^{k+1} r_l^{-1}\right)^2 p_{K_4} \lesssim \left(\prod_{l=1}^{k+2} r_l^{-1}\right)^2 p_{K_4}, \tag{37}$$

since we have already noticed that $2(1 - r) \leqslant r_{k+2}^{-1}$. The same arguments lead to

$$|p_2 p_2'| \lesssim \left(\prod_{l=1}^{k+2} r_l^{-1}\right)^2 p_{K_2}. \tag{38}$$

It remains to show that

$$|p_1' p_3'| \lesssim \left(\prod_{l=1}^{k+2} r_l^{-1}\right)^2 p_{K_3}, \tag{39}$$

$$|p_1' p_2'| \lesssim \left(\prod_{l=1}^{k+2} r_l^{-1}\right)^2 p_{K_3}, \tag{40}$$

because these estimates imply

$$|p_1' p_3'| \lesssim \left(\prod_{l=1}^{k+2} r_l^{-1}\right)^2 p_{K_4}, \tag{41}$$

$$|p_1' p_2'| \lesssim \left(\prod_{l=1}^{k+2} r_l^{-1}\right)^2 p_{K_2}, \tag{42}$$

due to the estimate $p_{K_3} \leqslant p_{K_4}$ and $p_{K_3} \leqslant p_{K_2}$. Indeed estimates (39) and (40) yield (36) for K_3, while (38) and (42) (resp. (37) and (41)) lead to (36) for K_2 (resp. for K_4).

To prove (39), we remark that the triangular inequality and the fact that $p_K = p_{K_3}/r$ yield

$$|p'_1 p'_3| \leqslant |p'_1 p_3| + |p_3 p'_3| \lesssim \frac{p_K}{2} + (1-r)\left(\prod_{l=1}^{k+1} r_l^{-1}\right)^2 p_K$$

$$\lesssim p_{K_3}\left(\frac{1}{2r} + r^{-1}(1-r)\left(\prod_{l=1}^{k+1} r_l^{-1}\right)^2\right).$$

Since $r_{k+2} \leqslant r$ and $r_{k+2} \leqslant 1/(2(1-r))$, we obtain (39). Estimate (40) is proved similarly.

The second geometrical case is treated similarly using the easily checked estimates

$$r_{k+2}^2 \leqslant 1-r, \quad \frac{r}{1-r} \leqslant \frac{1}{2} r_{k+2}^{-2}.$$

Using estimates (36), (33) and (35), we conclude that

$$|p_1 p_q| \lesssim e^{6(1/\mu - 1)} |p_2 p_3|, \quad q = 2, 3,$$

$$|p_2 p_3| \lesssim e^{3(1/\mu - 1)} |p_1 p_q|, \quad q = 2, 3.$$

This yields (31).

To prove (32), we denote by $\alpha_i(K)$ the interior angle of K at p_i, $i = 1, 2, 3$ and first show by induction that

$$\prod_{l=1}^{k+1} r_l \lesssim \sin \alpha_i(K), \quad i = 2, 3. \tag{43}$$

Since this estimate is direct for $k = 1$, we only need to show the inductive implication. As before, we consider the two geometrical cases: p_1 on the line $\lambda_1 + \lambda_2 = (i/2^k)^{1/\mu}$ (Fig. 5a) or not (Fig. 5b). In the first case, we show that

$$\sin \alpha_i(K_j) \geqslant \min(\sin \alpha_2(K), \sin \alpha_3(K)), \quad i = 2, 3; \ j = 1, 2, 3, 4, \tag{44}$$

which implies (43) by the inductive assumption and the fact that $r_{k+2} \leqslant 1$.

Estimate (44) is direct for K_1. For K_2, we only need to minorate the interior angle α'_3 at the node p'_1 (since the interior angle at p_2 is $\alpha_2(K)$). If $\alpha'_3 \geqslant \pi/2$, then α'_3 is the largest angle of K_2, and therefore

$$\sin \alpha'_3 \geqslant \sin \alpha_2(K).$$

If $\alpha'_3 \leqslant \pi/2$, then we conclude that

$$\sin \alpha'_3 \geqslant \sin \alpha_3(K),$$

since by construction $\alpha'_3 \geqslant \alpha_3(K)$. The two above estimates prove (44) for K_2.

We use similar arguments for K_4, while the conclusion for K_3 follows from the results for K_2 and K_4, since the interior angle of K_3 at p'_2 is equal to α'_3 and the interior angle of K_3 at p'_3 is equal to the interior angle of K_4 at p'_1.

In the second geometrical case, we show that

$$\sin\alpha_i(K_j) \geqslant 2r\min(\sin\alpha_2(K), \sin\alpha_3(K)), \quad i=2,3;\ j=1,2,3,4, \tag{45}$$

which implies (43) by the inductive assumption and the fact that $r_{k+2} \leqslant 2r$.

As above, we only need to show (45) for K_2. If we denote by $\tilde{\alpha}_3$ the interior angle at the node p_1', the sin rule implies that

$$\frac{\sin\tilde{\alpha}_3}{|p_2p_2'|} = \frac{\sin\alpha_2(K)}{|p_1'p_2'|}.$$

Since by construction $|p_1'p_2'| \leqslant \frac{1}{2}|p_1p_3|$ and $|p_2p_2'| = r|p_1p_2|$, we deduce that

$$\sin\tilde{\alpha}_3 \geqslant 2r\sin\alpha_2(K)\frac{|p_1p_2|}{|p_1p_3|} = 2r\sin\alpha_3(K),$$

by the sin rule. This yields the statement for K_2.

Estimate (43) leads to (32) since

$$\sin\alpha_{\max}(K) \geqslant \sin\alpha_i(K), \quad i=2,3$$

and we have already shown that

$$e^{-3(1/\mu-1)} \lesssim \prod_{l=1}^{k+1} r_l. \qquad \square$$

Remark 5.2. It was shown by Raugel in [16, p. 96] that Raugel's graded meshes satisfy

$$h_K/\rho_K \lesssim \frac{1}{\mu}, \quad \forall K \in \bigcup_{k\in\mathbb{N}} \mathscr{T}_k.$$

Let us now show that our meshes satisfy condition (4).

Lemma 5.3. *The above family satisfies condition* (4) *with* $\beta = (\frac{1}{2})^{1/\mu}$, $\gamma = \frac{1}{2}$, $\kappa_2 = Ce^{6(1/\mu-1)}2^{1/\mu-1}$ *and* $\kappa_1 = \kappa_2^{-1}$, *with some positive constant* C *independent of* μ.

Proof. As before it suffices to prove the assertion for the triangles in a fixed singular triangle T of $\mathscr{T}_0$ (since the remainder of the triangulation is quasi-uniform). By estimates (33), (35) and (36), we can claim that

$$e^{-3(1/\mu-1)}\tilde{h}_K \lesssim h_K \lesssim e^{3(1/\mu-1)}\tilde{h}_K, \quad \forall K \in T\cap\mathscr{T}_k.$$

Consequently, we are reduced to estimate the quotient

$$\frac{\tilde{h}_{K_k}}{\tilde{h}_{K_l}}$$

when $k \geqslant l$, for any triangle $K_k \in T\cap\mathscr{T}_k$ and $K_l \in T\cap\mathscr{T}_l$ with $K_k \subset K_l$. This quotient is now easily estimated from above and from below by using the mean value theorem and by distinguishing the case when K_l contains the singular corner or not.

Remark 5.4. With our meshes, we have by Corollaries 3.4 and 4.3 and the two above Lemmas that

$$\kappa(A_j) \leqslant \frac{C(\mu)}{\mu}(j+1)^2,$$

$$\kappa(P_{\mathrm{MDS}}) \leqslant C(\mu)(j+1),$$

where $C(\mu)$ is a positive constant which depends on $e^{6(1/\mu-1)}$ and $2^{1/\mu-1}$ and then can blow up as μ tends to 0. This fact is confirmed by the numerical tests given in the next section.

6. Numerical results

In this Section, we present some numerical results which confirm our theoretical results derived in Sections 3 and 4.

Let us consider the boundary value problem (2) in a domain Ω with a re-entrant corner (see Fig. 6). Fig. 6 shows the mesh $\mathcal{T}_0$ and the mesh $\mathcal{T}_3$ resulting from the mesh generation procedure described in Section 5 with the grading parameter $\mu = 0.5$.

We discretize the boundary value problem (2) by means of piecewise linear ansatz functions on level j and solve the corresponding system of linear algebraic finite element equations by means of the preconditioned conjugate gradient (pcg) method. The iteration process is stopped when a relative error of 10^{-8} measured in the A_j-energy norm is achieved. To be able to compute this norm, i.e., $(A_j(u_j^{(m)} - u_j), (u_j^{(m)} - u_j))^{1/2}$ ($u_j^{(m)}$ the mth iterate of the pcg method, u_j the exact solution of the system of finite element equations), we choose $f = 0$. Then the exact solution of the system of finite element equations is known, namely it is the zero vector. As initial guess in the pcg method a vector is used whose components which correspond to inner nodes are equal to one and components which corresponds to boundary nodes are equal to zero. The change from the nodal basis to the hierarchical basis is realized as a preconditioner as it is described in [20]. Numerical experiments with the MDS preconditioner are also performed. We show by our experiments the dependence of the number of iterations #it on the number $j + 1$ of levels used. In the experiments we use different values of the grading parameter μ.

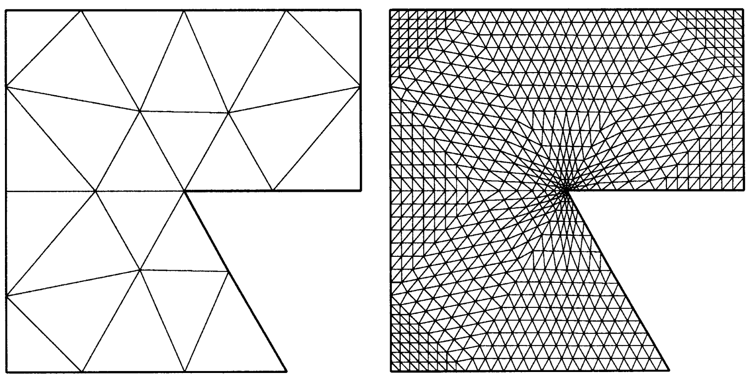

Fig. 6. Domain Ω with mesh $\mathcal{T}_0$ and $\mathcal{T}_3$.

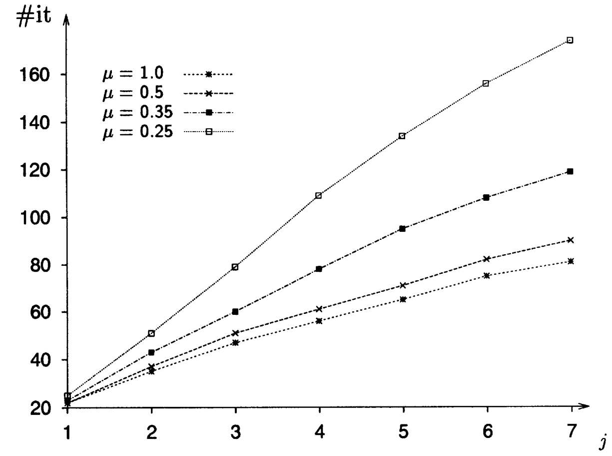

Fig. 7. Number of iterations (Yserentant's hierarchical basis method).

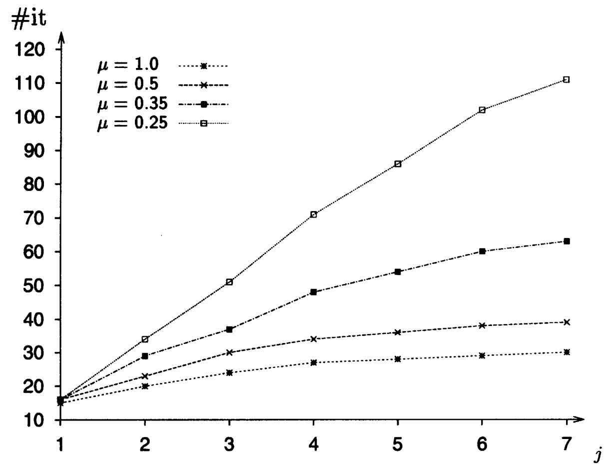

Fig. 8. Number of iterations (MDS preconditioner).

It is well known that the number of iterations of the preconditioned conjugate gradient method grows as $\sqrt{\kappa(B_j^{-1}A_j)}$, where B_j is the preconditioner and A_j the stiffness matrix in the nodal basis of piecewise linear functions on level j. Therefore, according to Corollary 3.4 we expect in the case of Yserentant's hierarchical basis method that the number of iterations grows as $j+1$. The numerical experiments illustrated in Fig. 7 confirm this theoretical result.

Fig. 8 shows the behaviour of the number of pcg iterations in the case of the MDS preconditioner. From the statement of Theorem 4.1 we expect that the number of iterations grows as $\sqrt{j+1}$. The numerical experiments presented in Fig. 8 corroborate this statement.

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References

[1] R.A. Adams, Sobolev Spaces, Academic Press, New York, 1975.
[2] R.E. Bank, in: A Software Package for Solving Elliptic Partial Differential Equations—Users' Guide 7.0., Frontiers in Applied Mathematics, Vol. 15, SIAM, Philadelphia, PA, 1994.
[3] F.A. Bornemann, H. Yserentant, A basic norm equivalence for the theory of multilevel methods, Numer. Math. 64 (1993) 455–476.
[4] J.H. Bramble, J.E. Pasciak, New estimates for multilevel algorithms including the V-cycle, Math. Comp. 60 (202) (1993) 447–471.
[5] J.H. Bramble, J.E. Pasciak, J. Xu, Parallel multilevel preconditioners, Math. Comp. 55 (191) (1990) 1–22.
[6] P.G. Ciarlet, in: The Finite Element Method for Elliptic Problems, St. in Math. and its Appl., Vol. 4, North-Holland, Amsterdam, 1978.
[7] P. Clément, Approximation by finite element functions using local regularization, RAIRO Anal. Numer. 9R2 (1975) 77–84.
[8] W. Dahmen, A. Kunoth, Multilevel preconditioning, Numer. Math. 63 (1992) 315–344.
[9] V. Girault, P.A. Raviart, Finite Element Methods for Navier–Stokes Equations: Theory and Algorithms, Springer, Berlin, 1986.
[10] M. Griebel. Multilevelmethoden als Iterationsverfahren über Erzeugendensystemen. Teubner Skripten zur Numerik. B.G. Teubner, Stuttgart, 1994.
[11] P. Grisvard, in: Elliptic Problems in Nonsmooth Domains, Monographs and Studies in Mathematics, Vol. 21, Pitman, Boston, 1985.
[12] M. Jung, On adaptive grids in multilevel methods, in: S. Hengst (Ed.), GAMM–Seminar on Multigrid-Methods, Gosen, Germany, September 21–25, 1992, Berlin, IAAS, Report No. 5, 1993, pp. 67–80.
[13] L.A. Oganesyan, L.A. Rukhovets, Variational-difference methods for the solution of elliptic equations, Izd. Akad. Nauk Armianskoi SSR, Jerevan, 1979 (in Russian).
[14] P. Oswald, On discrete norm estimates related to multilevel preconditioners in the finite element method, Proceedings of the International Conference on the Constructive Theory of Functions, Varna, 1991.
[15] P. Oswald. Multilevel Finite Element Approximation: Theory and Applications, Teubner Skripten zur Numerik. B.G. Teubner Stuttgart, 1994.
[16] G. Raugel, Résolution numérique de problèmes elliptiques dans des domaines avec coins, Ph.D. Thesis, Université de Rennes, France, 1978.
[17] L.R. Scott, S. Zhang, Finite element interpolation of nonsmooth functions satisfying boundary conditions, Math. Comp. 54 (1990) 483–493.
[18] G. Strang, G. Fix, An Analysis of the Finite Element Method, Prentice-Hall Inc., Englewood Cliffs, NJ 1973.
[19] J. Xu, Iterative methods by space decomposition and subspace correction, SIAM Rev. 34 (4) (1992) 581–613.
[20] H. Yserentant, On the multi-level splitting of finite element spaces, Numer. Math. 49 (4) (1986) 379–412.
[21] S. Zhang, Optimal-order nonnested multigrid methods for solving finite element equations II: on non-quasi-uniform meshes, Math. Comp. 55 (1990) 439–450.
[22] X. Zhang, Multilevel Schwarz methods, Numer. Math. 63 (1992) 521–539.
[23] M. Zlámal, On the finite element method, Numer. Math. 12 (1968) 394–408.